Stress Analysis of Tapered Sandwich Panels with Isotropic or Laminated Composite Facings

Huyue Zhao

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STRESS ANALYSIS OF TAPERED SANDWICH PANELS
WITH ISOTROPIC OR LAMINATED COMPOSITE
FACINGS

By

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A THESIS
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science
(in Mechanical Engineering)

The Graduate School
The University of Maine
December, 2002

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STRESS ANALYSIS OF TAPERED SANDWICH PANELS
WITH ISOTROPIC OR LAMINATED COMPOSITE FACINGS

By Huyue Zhao

Thesis Co-Advisors: Dr. Senthil Vel and Dr. Vincent Caccese


Structural sandwich construction is used in many air and space vehicles, cargo containers, boats and ships. Connection of the sandwich construction component to a framework or substructure is a critical issue in the detail design for sandwich construction. The tapered connection where the facings are drawn together at the support is one of the most efficient types of connections in composite sandwich construction.

We present a tapered sandwich theory that is simple to use, yet accurately predicts the stresses and deflection of both symmetric and non-symmetric tapered sections. In this investigation we assume that the facings are relatively thin and therefore in a state of plane stress. The core is assumed to be inextensible in the thickness direction and carry only transverse shear stress. Since the facings are tapered, the plane stress conditions for each facing are established in a local coordinate system with axes oriented parallel and
normal to the facing to obtain the reduced stiffness for the tapered laminae. The force and moment resultants are obtained by integrating the stresses in the facings and core.

The resulting resultants are related to the reference surface strains and curvatures through the familiar $[A]$, $[B]$ and $[D]$ matrices. The deflections are computed using an energy method approach. The shear and peeling stresses at the interface between the core and the facings, which may cause delamination at the interface, are computed by integrating the three-dimensional equilibrium equations along a straight path that is perpendicular to the facings.

We also have systematically derived a total of 12 elastic stiffnesses that couple the force and moment resultants to the transverse shear deformation. Six of the twelve elastic couplings are due to the tapered sandwich construction itself, irrespective of whether the facings are isotropic or anisotropic, whereas the remaining six elastic couplings are present only for anisotropic laminated facings. Their influence on the behavior of tapered sandwich beams of the stiffnesses is investigated. Analytical models have shown that the behavior of the tapered section is counterintuitive and that, for a tapered cantilever sandwich beam with fixed dimensions at the clamped edge, there is an optimum taper angle where the tip deflection is a minimum. This decrease in deformation with increasing taper angle is due to the participation of the facings in resisting transverse shear loads. Results from the tapered sandwich theory show very good comparison with finite element models for several case studies. The theory enhances our understanding of tapered sandwich beams and clarifies the causes of premature failure encountered at the interfaces between the core and facings.
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Chapter 1

INTRODUCTION

This chapter presents an introduction to the research conducted on tapered sandwich composite structures. In the first part, the motivation for this study is given followed by a review of the literature on sandwich composites. An overview of this thesis, including its contributions and approach, are summarized in Section 1.3.

1.1. Motivation

Sandwich construction is one of the most functional forms of composite structures developed by the composite industry. It is widely employed in aircraft and space vehicles, ships, boats, cargo containers and residential construction (Vinson, 1999). Honeycomb sandwich panels revolutionized the aerospace industry over 40 years ago, making aircraft lighter, stronger and faster and allowing them to carry more weight and improve fuel efficiency. Every aircraft in production today with two or more engines utilizes some honeycomb sandwich (Bitzer, 1992). In the Boeing 747, the fuselage shell, floors, side panels, overhead bins and ceiling are of sandwich construction. A major portion of the space shuttle is made of honeycomb-core sandwich construction with composite facings. The U.S. Navy uses honeycomb-sandwich bulkheads to reduce the weight of ships above the waterline. Fiberglass sandwich construction with foam core is widely used in the construction of recreational boat hulls. Sandwich composite construction offers great potential for large civil infrastructure projects such as industrial buildings and vehicular bridges (Karbhari, 1997). Sandwich-cladding panels composed of
metallic facings and a lightweight insulating core are being increasingly used for roof cladding due to their outstanding thermal performance (Davies, 1997).

A bolted connection to an aluminum framework

Tapered sandwich construction

Figure 1.1 Photograph of tapered sandwich construction and a connection to a framework (Caccese, 1998)

Sandwich construction provides several key benefits over conventional structures, such as very high bending stiffness, low weight, cost effectiveness and durability. A typical sandwich beam or panel usually consists of honeycomb, polymer foams or low-density wood cores sandwiched between isotropic or laminated anisotropic facings. When laminate composite facings are used, they may be designed to have quasi-isotropic or anisotropic properties. The facings are designed to carry primarily the bending stresses while the core resists the shear loads (analogous to the web of an I-beam) and increases the stiffness of the structure by holding the facings apart. The major advantage of this structural type is the very large stiffness-to-weight ratio and high bending strength. With proper design the sandwich increases the flexural rigidity while adding very little weight. Sandwich composite structures are easier to manufacture than laminated skin panels with bonded or mechanically fastened stiffeners.
In some applications, such as in the design of aerospace vehicles, it is necessary to use variable-thickness sandwich construction, either locally or globally, for functional and/or aerodynamic reasons. The tapered connection where the facings are drawn together at the support is one of the most functionally efficient types of connections in composite sandwich construction. Use of a tapered connection in sandwich structural components typically leads to a substantial reduction in construction depth. This type of connection, as shown in Figure 1.1, was studied experimentally and analytically by Caccese and Gauthier (1998a, 1998b) for its potential use in the aeroshell structure of the NASA X-38. The tapering of the core can also be used to improve the load transfer mechanism between a sandwich laminate and a monolithic laminated skin panel. In the Boeing's Model 360 helicopter, the frame and longeron spars are attached to a sandwich-to-laminate tapered member (Llorente, 1989).

In general, experience has shown that the tapered region is the weak link in tapered sandwich structures. Tensile tests indicate that the initial damage occurs at the root of the taper in the form of delamination at the interface between the core and the facings (Kuczma and Vizzini, 1999). In order to design and use tapered sandwich composite construction in practical applications, it is essential to accurately compute the stresses and deflections. Reliable estimates of the stresses, including the shear and peeling stresses at the interface between the core and the facings, in conjunction with a good failure theory, are needed to predict the maximum load carrying capacity of a tapered sandwich structure. One way to approach the problem is to treat the tapered sandwich structure as a three-dimensional anisotropic elastic composite continuum and utilize the three dimensional equations of equilibrium and associated point wise boundary
conditions to compute the stresses and deflections. However, solving a three-dimensional boundary value problem is tedious and time consuming. Therefore, there is a need to develop a simplified theory that is easy to use and able to capture the salient features of the displacement and stress fields in tapered sandwich structures.

1.2. Literature Review

1.2.1. Sandwich Structures with Uniform Thickness

A great deal of research has been conducted on the analysis of sandwich members of uniform depth. Considering the vast number of papers on the topic of sandwich construction, the following review is meant to be a brief overview of the literature and it is by no means complete.

The first research paper concerning sandwich construction was due to Marguerre, and it dealt with in-plane compressive loads (Marguerre, 1944). Libove and Batdorf (1948) published a general small deflection theory for sandwich plates. Hoff (1950) derived the differential equations and boundary conditions for the bending and buckling of sandwich plates using the principle of virtual displacements. In all cases studied, the materials were isotropic and the edges were free or simply supported. Bijlaard (1951) approached the subject of sandwich optimization by considering plates with given weight per unit surface area, and computing the ratio of the elastic moduli of core and faces, which lead to a maximum buckling load (Bijlaard, 1951). He carried out the optimization for a given ratio between thickness of core and face. In 1952, Flugge published a paper on the structural optimization of sandwich panels (Flugge, 1952).
Ericksen of U.S Forest Products Laboratory (USFPL) issued a report in 1956 accounting for the effects of shear deformation on deflections of the sandwich panels with isotropic core and facings. He presented general expressions for the strains in a sandwich panel with orthotropic facings and core. Eringen (1952) used the theorem of minimum potential energy to obtain four partial differential equations for the bending and buckling of rectangular sandwich plates with isotropic cores and faces under various loading and edge conditions. The early theoretical work on the behavior of rectangular sandwich panels subjected to lateral loads was restricted to uniform loads and simply support edge conditions. During the early post-World-War-II period, the USFPL was the primary group in the development of analysis and design methods for sandwich structures.

By the mid 1960s, efforts in sandwich construction research had spread widely. In 1966, Plantema published his famous, and the first, book on sandwich structures (Plantema, 1966). In 1969, this was followed by the book by H.G. Allen (Allen, 1969). In 1989, Ha provided an overview of finite elements applied to sandwich plates (Ha, 1989). Noor and Burton (1995) also reviewed the computational models for sandwich panels and shells. The review by Noor, Burton, and Bert (1996) provides over 800 references discussed in the review and another 559 references as a supplemental bibliography.

1.2.2. Tapered Sandwich Structures

Although there are hundreds of papers related to sandwich composites of uniform thickness, only a handful of them deal with tapered sandwich construction. When dealing with homogeneous beams of variable thickness, it is usually assumed that the constant-
thickness moment-curvature relationships of beam theory are still valid, provided we use
the bending rigidity based on the local thickness. Huang and Aspaugh (1974) used a
constant-thickness sandwich theory, with stiffnesses that varied in accordance with the
local thickness, to study sandwich beams of variable thickness. It has been shown that
this approach can lead to significant errors since the membrane stresses in the facings
have a transverse shear component which alters the transverse shear load in the core and
hence the transverse shear deformation (Libove and Lu, 1989; Lu and Libove, 1991). Lu
(1994) has analyzed symmetric tapered sandwich beams consisting of laminated fiber-
reinforced anisotropic facings and honeycomb core. Paydar and Libove (1986, 1988)
aires the general bending of sandwich plates of variable thickness with isotropic
facings. In order to design and use tapered sandwich composite construction in practical
applications, it is essential to accurately compute the stresses and deflections. Peled and
Frostig (1994) have rigorously developed a theory for tapered sandwich beams with
transversely flexible core. Their analysis accounts for higher-order effects in the form of
nonlinear displacements fields through the thickness of the sandwich beam which are
pronounced in the vicinity of concentrated loads or supports as well as at the ends of
tapered transition zones.
Kuczma and Vizzini (1999) have investigated the failure modes and load distributions in tapered sandwich-to-laminate specimens under tensile, compressive and bending loads and the experimental data were correlated with three-dimensional finite element models. In general, experiments have shown that initial damage in tapered sandwich connections occurs at the root of the taper resulting in delamination of the facings from the core, as shown in Figure 1.2 (Caccese and Malm, 1999).

1.3. Overview of Thesis

1.3.1. Contribution

The significance of this thesis is in the development of a new tapered sandwich theory that is simple to use, yet accurately predicts the stresses and deflection of both symmetric and non-symmetric tapered sections. Results from the tapered sandwich theory show very good comparison with plane strain finite element models for several case studies. The significant contributions of this thesis are as follows:

1. The analysis of laminated composite materials has been developed since the 1960s and it is now well established and widely used. There are many books devoted to the analysis of laminated and sandwich composite structures (Jones, 1998; Hyer, 1997;
Herakovich, 1997; Gibson, 1994; Whitney, 1987; Vinson, 1999; Reddy, 1997). Thus, the analysis and design methodology is well known and the notation has been standardized. For example, the elastic stiffness matrices of a laminated or sandwich composite structure are denoted by \([A]\), \([B]\) and \([D]\), which are the extensional stiffness, bending-extension coupling stiffness and bending stiffness matrices, respectively. Commercial computer codes that have been developed for the analysis and design of composite structures also employ the standardized notation. However, the analysis of tapered sandwich panels has not yet been cast in this standard notation. In this thesis, we have developed a tapered sandwich theory in which the force and moment resultants are related to the reference surface strains and curvatures through the familiar \([A]\), \([B]\) and \([D]\) matrices. This facilitates implementation in the standard finite element codes.

2. Unlike sandwich panels of uniform thickness, tapered sandwich structures exhibit bending-shear and extension-shear elastic couplings. For example, the bending-shear coupling implies that a bending moment will cause shear deformation of the core in a tapered sandwich beam. This is because the longitudinal force in the plane of facing, caused by the bending moment, has a vertical component that alters the shear force in the core. Although, the bending-shear coupling effect in tapered sandwich beams is well known (Paydar and Libove, 1988, Libove and Lu, 1989), there are additional elastic couplings in tapered sandwich beams. In our tapered sandwich formulation, we have systematically derived a total of 12 elastic stiffnesses that couple the force and moment resultants to the transverse shear deformation. Six of the twelve elastic couplings are due to the tapered sandwich construction itself, irrespective of whether
the facings are isotropic or anisotropic, whereas the remaining six elastic couplings are present only for anisotropic laminated facings.

3. Stresses and displacements from the tapered sandwich theory show good comparison with plane strain finite element models for several case studies. The cases include symmetric and unsymmetric tapered sandwich beams composed of either isotropic or laminated anisotropic facings.

4. It is important to quantify the shear and peeling stresses at the interface between the core and the facings since excessive interfacial stresses could cause delamination followed by debonding of the facings. By integrating the three-dimensional equilibrium equations, we are able to obtain the transverse shear and peeling stresses at the interfaces that compare well with results from plane strain finite element analysis.

1.3.2. Approach

In this investigation, we assume that the facings are relatively thin and therefore in a state of plane stress. The core is assumed to be inextensible in the thickness direction and carry only transverse shear stress. Since the facings are tapered, the plane stress conditions for each facing are established in a local coordinate system with axes oriented parallel and normal to the facing to obtain the reduced stiffness for the tapered laminae. The force and moment resultants are obtained by integrating the stresses in the facings and core. The resultants are related to the reference surface strains and curvatures through the familiar \([A]\), \([B]\) and \([D]\) matrices. The deflections are computed using an energy method approach. The shear and peeling stresses at the interface between the core and the
facings is computed by integrating the three-dimensional equilibrium equations along a straight path that is perpendicular to the facings.

1.3.3. Outline

The analysis of sandwich composite beams and plates of uniform thickness are briefly reviewed in Chapter 2. Chapter 3 deals with the analysis of tapered sandwich panels composed of isotropic facings and the theory is extended to laminated anisotropic facings in Chapter 4. The displacements and stresses from the tapered sandwich theory are compared with plane strain finite element analyses for several cases in Chapter 5. Chapter 6 is the conclusion of this thesis.
Chapter 2

BACKGROUND ON ANALYSIS OF SANDWICH PLATES
OF CONSTANT THICKNESS

This chapter provides a brief overview of sandwich members of uniform thickness. The mechanics of sandwich structures presented here is well known and can be found in the books by Whitney (1987) and Vinson (1999). This chapter is included so that one may fully understand and appreciate the subsequent analysis of tapered sandwich members and to delineate the differences between tapered sandwich members and sandwich members of constant depth.

2.1. Problem Formulation

The analytical development of a sandwich composite with constant depth starts with the general depiction of a sandwich beam as shown in Figure 2.1. It is assumed that

![Diagram of sandwich structure with layers labeled and z-axis indicated](image)

Figure 2.1 Layer numbering for a sandwich structure with constant thickness
the thickness of the core is constant and the laminated facings are relatively thin compared to the core and therefore behave as membranes. The primary function of the core is to stabilize the facings and resist the shear load. The coordinate system is placed at the center of the core.

The laminated facings consist of orthotropic laminae and each lamina has a distinct fiber orientation \( \theta_k \) relative to the \( x \)-axis. The sandwich member is composed of \( N \) distinct layers numbered from bottom to top as shown in Figure 2.1. The core is denoted as layer number \( C \) and the bottom and top facings are composed of \( C - 1 \) and \( N - C \) distinct laminae, respectively. The \( z \)-coordinates of the bottom and top surfaces of the \( k \)th layer are designated as \( z_k \) and \( z_{k+1} \), respectively.

## 2.2. Analytical Model

Since the facings are relatively thin compared to the core, we assume that the facings are in a state of plane stress:

\[
\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0. \tag{2.1}
\]

The plane stress reduced constitutive relations for an orthotropic lamina in the material coordinate system are

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{bmatrix}, \tag{2.2}
\]

where \( Q_{ij} \) are the reduced stiffnesses for an orthotropic material in the principal material coordinate system (Hyer, 1998). The reduced stiffnesses are defined in terms of the engineering constants as follows
where $E$, $\nu$ and $G$ denote the Young's modulus, Poisson's ratio and shear modulus respectively. The principal material direction of the $k^{th}$ lamina is oriented at angle $\theta_k$ with respect to the $x$-coordinate direction. The plane stress reduced constitutive relations for an orthotropic lamina in the $(x, y, z)$ coordinate system are

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix},
\]

(2.3)

where the reduced stiffness $\overline{Q}_{ij}$ components for the laminae in terms of the engineering constants are

\[
\begin{align*}
\overline{Q}_{11} &= Q_{11}m^4 + 2m^2n^2(Q_{12} + 2Q_{66}) + Q_{22}n^4, \\
\overline{Q}_{12} &= m^2n^2(Q_{11} + Q_{22} - 4Q_{66}) + Q_{12}(m^4 + n^4), \\
\overline{Q}_{16} &= mn[Q_{11}m^2 - Q_{22}n^2 - (Q_{12} + 2Q_{66})(m^2 - n^2)], \\
\overline{Q}_{22} &= Q_{11}n^4 + 2m^2n^2(Q_{12} + 2Q_{66}) + Q_{22}m^4, \\
\overline{Q}_{26} &= mn[Q_{11}n^2 - Q_{22}m^2 + (Q_{12} + 2Q_{66})(m^2 - n^2)], \\
\overline{Q}_{66} &= m^2n^2(Q_{11} + Q_{22} - 2Q_{12}) + Q_{66}(m^2 - n^2)^2,
\end{align*}
\]

(2.4)

where $m = \cos \theta_k$ and $n = \sin \theta_k$.

We assume that the strains in the facings are linear functions of the thickness coordinate. The strains in the facing $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\gamma_{xy}$ are taken to be linear functions of the $z$-coordinate as follows:

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^0(x) + z\kappa_{xx}^0(x), \\
\varepsilon_{yy} &= \varepsilon_{yy}^0(x) + z\kappa_{yy}^0(x), \\
\gamma_{xy} &= \gamma_{xy}^0(x) + z\kappa_{xy}^0(x).
\end{align*}
\]

(2.5)
where $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$ are strain in the reference surface, and $\kappa_{xx}, \kappa_{yy}$ and $\kappa_{xy}$ are bending curvature in the reference surface.

The core is made of an orthotropic material and its primary function is to space and stabilize the facings and transfer shear between them. The in-plane stresses $\sigma_{xx}, \sigma_{yy}$ and $\tau_{xy}$ of the core are assumed to be negligible compared to that in the facings. The transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$ in the core are assumed to be constant throughout the thickness and they are related to the core shear strains by

$$
\tau_{xz} = G^c_{xz} \gamma^c_{xz}, \\
\tau_{yz} = G^c_{yz} \gamma^c_{yz},
$$

(2.6)

where $G^c_{xz}$ and $G^c_{yz}$ are the transverse shear moduli, and $\gamma^c_{xz}$ and $\gamma^c_{yz}$ are the transverse shear strains of the core. The stress resultants are defined as

$$
\begin{align*}
[N_x, N_y, N_{xy}] &= \int_{z_1(x)}^{z_2(x)} [\sigma_{xx}, \sigma_{yy}, \tau_{xy}] \, dz, \\
[M_x, M_y, M_{xy}] &= \int_{z_1(x)}^{z_2(x)} z [\sigma_{xx}, \sigma_{yy}, \tau_{xy}] \, dz, \\
[Q_x, Q_y] &= \int_{z_1(x)}^{z_2(x)} \tau_{xz}, \tau_{yz} \, dz,
\end{align*}
$$

(2.7)

where the quantities $N_x, N_y$ and $N_{xy}$ are the in-plane force resultants, $Q_x$ and $Q_y$ are the transverse shear load resultants and $M_x, M_y$ and $M_{xy}$ are the moment resultants, as depicted in Figure 2.2.
Figure 2.2 Force and moment resultants for a sandwich panel with constant thickness (a) force resultants (b) moment resultants

Substitution of the facing stresses from (2.3) and the core stresses from (2.6) into (2.7) leads to the following matrix equation for the resultant forces and moments in terms of the core shear strains and reference surface strains and curvatures,

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0
\end{bmatrix},
\]

(2.8)

\[
\begin{bmatrix}
Q_y \\
Q_x
\end{bmatrix}
=
\begin{bmatrix}
A_{44} & 0 \\
0 & A_{55}
\end{bmatrix}
\begin{bmatrix}
\gamma_{yx}^c \\
\gamma_{xx}^c
\end{bmatrix}.
\]
where $A_{ij}$ are the extensional and shear rigidities, $D_{ij}$ are the bending rigidities and $B_{ij}$ are the bending-extension rigidities defined by

$$
\begin{align*}
[A_y(x), B_y(x), D_y(x)] &= \sum_{k=1}^{N} \int_{z_i(x)}^{z_{i+1}(x)} Q_{ij}^{(k)} [1, z, z^2] dz, 
\end{align*}
$$

$$
\begin{align*}
[A_{44}(x), A_{55}(x)] &= [G_{zz}^c, G_{zz}^c] [z_{C+1} - z_C].
\end{align*}
$$

(2.9)

The most important feature of Equation (2.8) is the elastic coupling that exists between extension and bending. The rigidities $B_{ij}$ relate the bending moments $M^i$'s to the reference surface strains $\varepsilon_0^i$'s, and the resultant forces $N^i$'s to the reference surface curvatures $\kappa_0^i$'s. Therefore, the matrix $[B_{ij}]$ is known as the bending-extension coupling matrix. An examination of equation (2.9) reveals that bending-extension coupling vanishes for sandwich plates that are symmetric about the middle surface (Whitney, 1999).
Chapter 3

ANALYTICAL MODEL OF ISOTROPIC FACINGS

A new tapered sandwich theory that accurately predicts the stresses and deflection of both symmetric and non-symmetric tapered sections of sandwich composites with isotropic facings is presented. We have obtained explicit expressions for 6 elastic rigidities that couple the reference surface strains and curvatures to the transverse shear forces. The physical interpretations of the elastic couplings are given in this chapter. The elastic couplings are intrinsic to tapered sandwich members and they are identically zero for sandwich members of uniform depth as one would expect. The interface shear and peeling stresses between the core and the facings are computed by integrating the three-dimensional equilibrium equations in a direction perpendicular to the facing surfaces. For sandwich members that are highly tapered, the bending and extension rigidities can be negative. We give physical explanations for the negative rigidities and discuss its ramifications on the response of tapered sandwich members.

3.1. Problem Formulation

The analytical development of tapered composite members starts with the general depiction of a tapered sandwich beam as shown in Figure 3.1. We use a rectangular Cartesian coordinate system, denoted by \( x-y-z \) in Figure 3.1(a), to describe the deformation of a tapered sandwich beam with isotropic facings. The thickness is assumed to vary linearly along the span (x-direction). It is assumed that the facings are relatively thin compared to the core and therefore behave as membranes. The core is assumed to be
Figure 3.1 Coordinate system and layer numbering for a tapered section

Inextensible in the thickness direction and to carry only transverse shear stresses. Since the facings are tapered, it is advantageous to use a local coordinate system, denoted by $s$-$y$-$n$ in Figure 3.1(a), such that the $s$- and $n$-axes are parallel and normal to the facing, respectively. The taper angle of the facing, $\phi$, is taken as positive counter-clockwise with respect to the $x$-axis.

The tapered beam is assumed to be composed of three distinct layers numbered from bottom to top as shown in Figure 3.1(b). The core is denoted as layer number 2, the bottom facing as layer number 1 and top facing as layer number 3. The $z$-coordinate of the bottom surface of layer number $k$ is designated $z_k$ with the top surface of the layer being $z_{k+1}$. In general, the $z_k$'s are functions of the $x$-coordinate since the sandwich
member is tapered. The taper angle of the top and bottom facings are $\phi = \phi_t$ and $\phi = \phi_b$, respectively. The sandwich member is said to be symmetric if $\phi_t = - \phi_b$ and the material properties and thickness of the facings are identical. If the laminate is not symmetric, it is referred to as an unsymmetric laminate.

3.2. Analytical Model

Since the facings are relatively thin compared to the core, we assume that the facings are in a state of plane stress. This is a reasonable assumption since the stress components in the plane of the facing are generally much larger than the stress components perpendicular to the plane. For tapered sandwich structures, the plane stress conditions are

$$
\sigma_{nn} = \tau_{sn} = \tau_{yn} = 0,
$$

where the normal stress components are denoted by $\sigma$ and the shear stress components by $\tau$. The facings are made of an isotropic material, such as aluminum. The stress-strain relations for an isotropic facing in the $(s, y, n)$ coordinate system are,

$$
\begin{bmatrix}
\sigma_{ss} \\
\sigma_{yy} \\
\sigma_{nn} \\
\tau_{yn} \\
\tau_{sn} \\
\tau_{sy}
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{ss} \\
\varepsilon_{yy} \\
\varepsilon_{nn} \\
\gamma_{yn} \\
\gamma_{sn} \\
\gamma_{sy}
\end{bmatrix},
$$

where $C_{ij}$ are the components of the elastic stiffness matrix in contracted notation. The elastic stiffness components for the isotropic facings in terms of the Young’s moduli $E$ and Poisson’s ratio $\nu$ are
\[ C_{11} = C_{22} = C_{33} = E (1-\nu)/(1+\nu)(1-2\nu), \]
\[ C_{12} = C_{13} = C_{23} = E\nu(1+\nu)(1-2\nu), \]
\[ C_{44} = C_{55} = C_{66} = G = E/2(1+\nu), \]

where \( G \) is the shear modulus of the facings. The stresses and strains in the \((x, y, z)\) and \((s, y, n)\) coordinate systems are related as

\[
\left\{ \sigma \right\}_{xyz} = \left[ R^+ \right] \left\{ \sigma \right\}_{syn},
\]
\[
\left\{ \varepsilon \right\}_{xyz} = \left[ R^- \right] \left\{ \varepsilon \right\}_{syn}, \tag{3.3}
\]

\[
\left\{ \varepsilon \right\}_{syn} = \left[ R^+ \right]^T \left\{ \varepsilon \right\}_{xyz},
\]
\[
\left\{ \varepsilon \right\}_{xyz} = \left[ R^- \right]^T \left\{ \varepsilon \right\}_{syn}, \tag{3.4}
\]

where \( \left\{ \sigma \right\}_{xyz} = \left\{ \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{zx}, \tau_{xy} \right\}^T \), \( \left\{ \varepsilon \right\}_{xyz} = \left\{ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{zx}, \gamma_{xy} \right\}^T \), \( \left\{ \sigma \right\}_{syn} = \left\{ \sigma_{ss}, \sigma_{yy}, \sigma_{nn}, \tau_{yn}, \tau_{yn}, \tau_{yn} \right\}^T \) and \( \left\{ \varepsilon \right\}_{syn} = \left\{ \varepsilon_{ss}, \varepsilon_{yy}, \varepsilon_{nn}, \gamma_{yn}, \gamma_{yn}, \gamma_{yn} \right\}^T \) are column vectors of stresses and strains in the \((x, y, z)\) and \((s, y, n)\) coordinate systems, respectively. Matrices \([R^+]\) and \([R^-]\) are the transformation matrices defined by

\[
[R^+] = \begin{bmatrix}
\cos^2 \phi & 0 & \sin^2 \phi & 0 & 2 \cos \phi \sin \phi & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin^2 \phi & 0 & \cos^2 \phi & 0 & -2 \cos \phi \sin \phi & 0 \\
0 & 0 & 0 & \cos \phi & 0 & -\sin \phi \\
-\cos \phi \sin \phi & 0 & \cos \phi \sin \phi & \cos^2 \phi - \sin^2 \phi & 0 & 0 \\
0 & 0 & 0 & \sin \phi & 0 & \cos \phi
\end{bmatrix}, \tag{3.5}
\]

\[
[R^-] = \begin{bmatrix}
\cos^2 \phi & 0 & \sin^2 \phi & 0 & -2 \cos \phi \sin \phi & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin^2 \phi & 0 & \cos^2 \phi & 0 & 2 \cos \phi \sin \phi & 0 \\
0 & 0 & 0 & \cos \phi & 0 & \sin \phi \\
\cos \phi \sin \phi & 0 & -\cos \phi \sin \phi & \cos^2 \phi - \sin^2 \phi & 0 & 0 \\
0 & 0 & 0 & -\sin \phi & 0 & \cos \phi
\end{bmatrix} \tag{3.6}
\]
As in the case of uniform sandwich beams of constant depth (see Whitney, 1987; Vinson, 1999), an analytical model for tapered members is developed by assuming that the strains in the facings \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \gamma_{xy} \) are linear functions of the \( z \)-coordinate as follows

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^0(x) + z\kappa_{xx}^0(x), \\
\varepsilon_{yy} &= \varepsilon_{yy}^0(x) + z\kappa_{yy}^0(x), \\
\gamma_{xy} &= \gamma_{xy}^0(x) + z\kappa_{xy}^0(x),
\end{align*}
\] (3.7)

where \( \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0 \) and \( \kappa_{xx}^0, \kappa_{yy}^0, \kappa_{xy}^0 \) are the strains and curvatures of the reference surfaces \( z = 0 \). The transverse normal strain \( \varepsilon_{zz} \), and the transverse shear strains \( \gamma_{xz} \) and \( \gamma_{yz} \) in the facings are obtained from \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \gamma_{xy} \) by recognizing that the facings are in a state of plane stress. The stresses in the \((s, y, n)\) coordinate system are obtained through \( \{\sigma\}_{syn} = [C] \{\varepsilon\}_{syn} = [C][R]^{T} \{\varepsilon\}_{xyz} \). The three plane stress equations (3.1) are enforced to obtain \( \varepsilon_{zz}, \gamma_{xz} \) and \( \gamma_{yz} \) in terms of \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \gamma_{xy} \) as

\[
\begin{align*}
\varepsilon_{zz} &= \frac{(1-2\nu-\cos 2\phi)\varepsilon_{xx} - 2\nu\varepsilon_{yy} \cos 2\phi}{1-2\nu + \cos 2\phi}, \\
\gamma_{xz} &= \frac{2(\varepsilon_{xx} + \nu\varepsilon_{yy})\sin 2\phi}{1-2\nu + \cos 2\phi}, \\
\gamma_{yz} &= \gamma_{xy} \tan \phi.
\end{align*}
\] (3.8)

The stress-strain relations in the \((x, y, z)\) coordinate system for a tapered facing lamina, computed using (3.8), (3.4)\(_1\), (3.2), (3.3)\(_2\), are

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xz} \\
\tau_{yz} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & 0 \\
\tilde{Q}_{21} & \tilde{Q}_{22} & 0 \\
\tilde{Q}_{31} & \tilde{Q}_{32} & 0 \\
0 & 0 & \tilde{Q}_{46} \\
0 & 0 & \tilde{Q}_{56} \\
0 & 0 & \tilde{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix},
\] (3.9)

where \( \tilde{Q}_{ij} \) are the reduced stiffnesses for a tapered isotropic facings, defined as
\[
\begin{align*}
\tilde{Q}_{11} &= \frac{2E\cos^2\phi}{D}, \quad \tilde{Q}_{12} = \frac{2Ev\cos^2\phi}{D}, \\
\tilde{Q}_{21} &= \frac{2Ev}{D}, \quad \tilde{Q}_{22} = E + \frac{2Ev^2}{D}, \\
\tilde{Q}_{31} &= \frac{2E\sin^2\phi}{D}, \quad \tilde{Q}_{32} = \frac{2Ev\sin^2\phi}{D}, \\
\tilde{Q}_{46} &= G\tan\phi, \quad \tilde{Q}_{51} = \frac{E\sin2\phi}{D}, \\
\tilde{Q}_{52} &= \frac{Ev\sin2\phi}{D}, \quad \tilde{Q}_{60} = G, \\
D &= (1+\nu)(1-2\nu+\cos2\phi).
\end{align*}
\] (3.10)

The core is made of an orthotropic material and its primary function is to space and stabilize the facings and transfers shear between them. The in-plane stresses \(\sigma_{xx}, \sigma_{yy}\) and \(\tau_{xy}\) of the core are assumed to be negligible compared to the stresses in the facings. The transverse shear stresses \(\tau_{xz}\) and \(\tau_{yz}\) in the core are assumed to be constant throughout the thickness and they are related to the core shear strains by

\[
\begin{align*}
\tau_{xz}^{(2)} &= G_{xz}^{c} \gamma_{xz}^{c}, \\
\tau_{yz}^{(2)} &= G_{yz}^{c} \gamma_{yz}^{c},
\end{align*}
\] (3.11)

where \(G_{xz}^{c}\) and \(G_{yz}^{c}\) are the transverse shear moduli of the core, and \(\gamma_{xz}^{c}\) and \(\gamma_{yz}^{c}\) are the transverse shear strains of the core. The stress resultants are defined as

\[
\begin{aligned}
[N_{x}, N_{y}, N_{xy}] &= \int_{z_{1}(x)}^{z_{4}(x)} [\sigma_{xx}, \sigma_{yy}, \tau_{xy}] dz, \\
[M_{x}, M_{y}, M_{xy}] &= \int_{z_{1}(x)}^{z_{4}(x)} z[\sigma_{xx}, \sigma_{yy}, \tau_{xy}] dz, \\
[Q_{x}, Q_{y}] &= \int_{z_{1}(x)}^{z_{4}(x)} [\tau_{xz}, \tau_{yz}] dz,
\end{aligned}
\] (3.12)

where the quantities \(N_{x}, N_{y}\) and \(N_{xy}\) are the in-plane force resultants, \(Q_{x}\) and \(Q_{y}\) are the transverse shear load resultants and \(M_{x}, M_{y}\) and \(M_{xy}\) are the moment resultants, as
Substitution of the stresses in the facings from (3.9) and the core stresses from (3.11) into (3.12) leads to the following matrix equation for the resultant forces and moments in terms of the core shear strains and reference surface strains and curvatures,
where $A_{ij}$ are the extensional and shear rigidities, $D_{ij}$ are the bending rigidities and $B_{ij}$ are the rigidities that couple bending to extension and shear. The rigidities in the $ABD$ matrix are defined by

\[
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & B_{11} & B_{12} & 0 \\
A_{21} & A_{22} & 0 & 0 & 0 & B_{21} & B_{22} & 0 \\
0 & 0 & A_{44} & 0 & A_{46} & 0 & 0 & B_{46} \\
A_{51} & A_{52} & 0 & A_{55} & 0 & B_{51} & B_{52} & 0 \\
0 & 0 & 0 & 0 & A_{66} & 0 & 0 & B_{66} \\
B_{11} & B_{12} & 0 & 0 & 0 & D_{11} & D_{12} & 0 \\
B_{21} & B_{22} & 0 & 0 & 0 & D_{21} & D_{22} & 0 \\
0 & 0 & 0 & 0 & B_{66} & 0 & 0 & D_{66}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{yz}^c \\
\gamma_{zx}^c \\
\gamma_{xy}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0
\end{bmatrix}
\]

(3.13)

The relation (3.14) can be inverted to obtain the following equation, from which the reference strains and core shear strains can be determined if the resultant forces and moments are known:

\[
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{yz}^c \\
\gamma_{zx}^c \\
\gamma_{xy}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & b_{11} & b_{12} & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & b_{21} & b_{22} & 0 \\
0 & 0 & a_{44} & 0 & a_{46} & 0 & 0 & b_{46} \\
a_{51} & a_{52} & 0 & a_{55} & 0 & b_{51} & b_{52} & 0 \\
0 & 0 & 0 & 0 & a_{66} & 0 & 0 & b_{66} \\
b_{11} & b_{12} & 0 & 0 & 0 & d_{11} & d_{12} & 0 \\
b_{21} & b_{22} & 0 & 0 & 0 & d_{21} & d_{22} & 0 \\
0 & 0 & 0 & 0 & b_{66} & 0 & 0 & d_{66}
\end{bmatrix}
\begin{bmatrix}
N_x \\
N_y \\
Q_y \\
Q_x \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
\]

(3.15)
3.3. Beam Bending and Cylindrical Bending

There are two cases of tapered sandwich plates that can be treated as one-dimensional problems: (1) tapered sandwich beams, and (2) cylindrical bending of tapered sandwich members.

When the width of the sandwich member along the y-axis is much smaller than the length along the x-axis, it is treated as a sandwich beam. For a sandwich beam in bending, the transverse load $Q_x$ and bending moment $M_x$ are assumed to be known, whereas the other loads vanish, i.e., $N_x = N_y = Q_y = N_{xy} = M_y = M_{xy} = 0$. By substituting the quantity of resultant force into Equation (3.15), the reference surface strains, curvatures and transverse shear strains of the core are

$$
\varepsilon^0_{xx} = b_{11}M_x, \quad \varepsilon^0_{yy} = b_{21}M_x,
$$

$$
\gamma^c_{xx} = a_{55}Q_x + b_{21}M_x,
$$

$$
\kappa^0_{xx} = d_{11}M_x, \quad \kappa^0_{yy} = d_{21}M_x,
$$

$$
\gamma^c_{yx} = 0, \quad \gamma^0_{xy} = 0, \quad \kappa^0_{xy} = 0.
$$

In cylindrical bending, the tapered sandwich plate is assumed to be a sandwich plate strip that is very long along y-axis and has a finite dimension along x- and z-axes. All derivatives with respect to y are zero and reference surface of the tapered member bends into a cylindrical shape. For cylindrical bending, transverse shear load $Q_x$ and bending moment $M_x$ are assumed to be known and $N_x = Q_y = N_{xy} = M_y = M_{xy} = 0$. The reaction forces resultant $N_y$ and moment resultant $M_y$ required to produce cylindrical bending can be determined from the second and seventh rows of matrix equation (3.15)
By substituting the quantity of resultant force into Equation (3.15), the reference surface strains and transverse shear strains of the core are

\[
\left\{ \begin{array}{c}
N_y \\
M_y
\end{array} \right\} = \begin{bmatrix}
a_{22} & b_{22} \\
b_{22} & d_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
b_{21} \\
d_{21}
\end{bmatrix} M_x,
\]

(3.17)

By substituting of the quantity of resultant force into Equation (3.15), the reference surface strains and transverse shear strains of the core are

\[
\varepsilon_{xx}^0 = \frac{a_{12}(-d_{22}b_{21} + b_{22}d_{21}) + b_{11}(a_{22}d_{22} - b_{22}^2) + b_{12}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x,
\]

\[
\gamma_{xz}^c = \frac{a_{22}(-d_{22}b_{21} + b_{22}d_{21}) + b_{31}(a_{22}d_{22} - b_{22}^2) + b_{32}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x + a_{55} \eta_x,
\]

\[
\kappa_{xx}^0 = \frac{b_{12}(-d_{22}b_{21} + b_{22}d_{21}) + d_{11}(a_{22}d_{22} - b_{22}^2) + d_{12}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x,
\]

\[
\gamma_{xy}^0 = 0, \gamma_{yz}^c = 0, \kappa_{xy}^0 = 0,
\]

\[
\varepsilon_{yy}^0 = 0, \kappa_{yy}^0 = 0.
\]

(3.18)

We can obtain the distributions of strains through the thickness of the tapered sandwich section from Equation (3.7) after all the core shear strains and reference surface strains and curvatures have been determined. The stresses in the tapered sandwich member are obtained from the stress-strain relations (3.9) and (3.11). By using the transformation relations (3.3), we can determine the stresses and strains in the local s-y-n coordinate system.

### 3.4. Deflections of Tapered Sandwich Structures

An energy method approach can be used to facilitate the calculation of deflections at some particular key point in the structure. After calculating all the stresses and strains in the structure, a general expression for the strain energies in the bottom facing, top facing and core at any location are respectively:
The general expression for the total strain energy $U_0$ is:

$$U_0 = \int_0^L \left[ U_{\text{bottom}}(x) + U_{\text{top}}(x) + U_{\text{core}}(x) \right] dx,$$

(3.20)

Let vertical forces $P_i$ ($i = 1, 2, 3, \ldots$) be applied at various locations along the tapered sandwich beam. Using Castigliano’s second theorem, the deflection $\delta_i$ at the location of concentrated load $P_i$ is

$$\delta_i = \partial U_0 / \partial P_i.$$

### 3.5. Transverse Shear Stresses and Transverse Normal Stresses

An important assumption in the development of our theory for tapered sandwich member is the plane stress conditions $\sigma_{nn} = \tau_{sn} = \tau_{yn} = 0$ for the facings stated earlier in Equation (3.1). Yet these three stress components, although small, are nonzero within the facings and they are usually largest at the interface between the core and the facings. Large interlaminar stresses are known to be the basis of a particular failure mechanism in laminated fiber-reinforced composite materials, namely free-edge delamination and subsequent delamination growth along the length. Some real applications and experiments have shown that these stresses can also cause failure of a tapered sandwich member, such as debonding of the facings from the core, as shown in Figure 1.3.
Therefore, in order to design tapered sandwich members, it is important that we are able to compute the stresses at the interface between the core and facings.

We compute the transverse shear and normal stresses by integrating the 3D equilibrium equations along a straight path that is perpendicular to the facings, as depicted in Figure 3.3. This is done after we have calculated the in-plane stresses \( \sigma_{zz}, \sigma_{yy} \) and \( \tau_{yz} \) in the facings. The transverse shear stress \( \tau_{sn} \) can be obtained by integrating the 3D equilibrium equation given in Equation (3.21),

\[
\sigma_{ss,s} + \tau_{sy,y} + \tau_{sn,n} = 0,
\]

\[
\tau_{sn}(s, n) = -\int_0^n (\sigma_{ss,s} + \tau_{sy,y}) dn.
\]  

Equation (3.21) not only allows us to compute the \( \tau_{ns} \) within the layers but also at the layer interface. Integrating the 3D equilibrium equations, as shown in Equation (3.22), leads to the transverse shear stresses \( \tau_{yn} \) in the \( s-y-n \) coordinate system

\[
\tau_{yr,s} + \sigma_{yy,y} + \tau_{yn,n} = 0,
\]

\[
\tau_{yn}(s, n) = -\int_0^n (\tau_{yr,s} + \sigma_{yy,y}) dn.
\]  

The transverse normal ('peeling') stress \( \sigma_{nn} \) is computed by integrating Equation (3.23), after \( \tau_{sn} \) and \( \tau_{yn} \) have been determined.
The integration constants in Equation (3.21), (3.22), and (3.23) are determined by recognizing that \( \tau_{sn} \), \( \tau_{mn} \) and \( \sigma_{nn} \) vanish on the top and bottom surfaces of the sandwich member.

### 3.6. Elastic Couplings

Whitney (1987, pp. 299-300) has discussed the structural analysis of constant-thickness sandwich composite structures. The constitutive equation that relates the resultant forces and moments to the reference surface strains and curvatures for a sandwich member of constant thickness is

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0
\end{bmatrix},
\]

\[
\begin{bmatrix}
Q_y \\
Q_x
\end{bmatrix} =
\begin{bmatrix}
A_{44} & 0 \\
0 & A_{55}
\end{bmatrix}
\begin{bmatrix}
\gamma_{yx}^c \\
\gamma_{xx}^c
\end{bmatrix}.
\]

Notice that the constitutive equation that relates the transverse shear forces \( Q_y \) and \( Q_x \) to the shear strains \( \gamma_{yx}^c \) and \( \gamma_{xx}^c \) of the core are uncoupled from the in-plane force and moment resultants. An important feature of tapered sandwich members is the elastic couplings between bending and transverse shear, and extension and transverse shear as shown in Equation. (3.13). For a tapered member with isotropic facings, there are 6 new
elastic couplings, namely $A_{51}$, $A_{52}$, $A_{46}$, $B_{51}$, $B_{52}$ and $B_{46}$. These elastic couplings vanish for sandwich members with constant depth. In this section, we give physical interpretations of the new elastic coupling coefficients.

3.6.1. Bending – Transverse Shear Coupling $B_{51}$

![Diagram of stress transformation](image)

Figure 3.4. Transforming extensional stresses in s-y-n coordinate system to x-y-z coordinate system

The elastic rigidity $B_{51}$ in Equation (3.13) is bending-shear coupling. To study the effect of $B_{51}$, consider the following deformation wherein the reference surface strains and curvatures of a tapered sandwich section are $\kappa_{xx}^0 > 0$ and $\varepsilon_{xx}^0 = \varepsilon_{yy}^0 = \gamma_{xy}^0 = \kappa_{yy}^0 = \kappa_{xy}^0 = \gamma_{yx}^0 = \gamma_{xz}^0 = 0$. Since the core shear strain $\gamma_{xz}$ has been prescribed to be zero, the core shear stress $\tau_{xz}$ is $0$ using (3.11). Strain $\varepsilon_{xx}$ in the $x$-direction of the facings is computed by Equation (3.7). Strains $\varepsilon_{ss}$, $\varepsilon_{mm}$ and $\gamma_{mn}$ in the s-y-n coordinate system are obtained by transforming the strain $\varepsilon_{xx}$ to the s-y-n coordinate system from Equations (3.8) and (3.4). Stresses $\sigma_{ss}$, $\sigma_{yy}$ and $\tau_{yy} = 0$ are obtained using the stress-strain relations in the s-y-n coordinate system (3.2). By transforming the stress $\sigma_{ss}$ to the x-y-z coordinate system, we would get the stresses $\sigma_{xx}$, $\sigma_{zz}$ and $\tau_{xz}$, as shown in Figure 3.4. The shear stress...
In other words, a transverse shear force $Q_x$ has to be applied in order to obtain the reference surface bending curvature $\kappa_{0x}$, although the transverse shear strain in the x-z plane is zero. Therefore we refer to the constant $B_{31}$ as the bending-transverse shear elastic coupling.

The physical meaning of the shear-bending elastic coupling can also be understood through a simple analysis of the forces on a cross-section of a symmetric tapered member as shown in Figure 3.5. The forces include the internal moment $M_x$ and shear force $Q_x$. The assumption is made that normal force due to the bending moment is transmitted through the facings only. A portion of the resultant shear force is transmitted through the facings due to their angle of inclination. The core resists the remainder of the shear force. As can be seen, even if the shear load $Q_x$ is absent, a shear force $Q_x^c$ is induced in the core due to the inclinations of the facings, thus causing transverse shear deformation of the core.

Figure 3.5 Shear-bending coupling in symmetric tapered sandwich

\begin{align*}
Q_x^c &= Q_x - 2M_x \tan \phi / h \\
F &= M_x h \cos \phi
\end{align*}
### 3.6.2. Bending – Transverse Shear Coupling $B_{s2}$

The elastic rigidity $B_{s2}$ in Equation (3.13) is a bending-transverse shear coupling caused by the Poisson’s effect. To study the effect of $B_{s2}$, consider the following situation wherein at a point on the reference surface of a tapered sandwich section, $\kappa_{yy}^0 > 0$ and $\varepsilon_{xx}^0 = \varepsilon_{yy}^0 = \gamma_{xy}^0 = \kappa_{xx}^0 = \kappa_{xy}^0 = \gamma_{yx}^0 = \gamma_{xx}^0 = 0$. From Equation (3.7), the facings would have a strain $\varepsilon_{yy}$ in the $y$-direction. Since the facings are inclined, the strain $\varepsilon_{xz}$ and $\gamma_{xz}$ from Equation (3.8) are nonzero due to the plane stress state in the facings. Transforming the strains using Equation (3.4) to obtain strains $\varepsilon_{ss}$, $\varepsilon_{mn}$ and $\gamma_{mn}$ in the $s$-$y$-$n$ coordinate system and making use of the stress-strain relations Equation (3.2) in the $s$-$y$-$n$ coordinate system, we obtain the stresses $\sigma_{ss}$, $\sigma_{yy}$ and $\tau_{yy} = 0$. The stress $\sigma_{ss}$, which is essentially a reaction stress component due to the constraint $\varepsilon_{xx} = 0$, gives rise to a transverse shear stress $\tau_{xz}$ and transverse shear force $Q_x$ as shown in Figure 3.4.

### 3.6.3. Extension – Transverse Shear Coupling $A_{s1}$

To study the effect of $A_{s1}$, consider the following situation wherein at every point

![Figure 3.6 Transforming extensional stresses in $s$-$y$-$n$ coordinate system to $x$-$y$-$z$ coordinate system](image)
Figure 3.7 Extension-shear coupling in unsymmetric tapered sandwich beams

on the reference surface of a sandwich members, $\varepsilon_{xx}^0 > 0$ and $\varepsilon_{yy}^0 = \gamma_{xy}^0 = \kappa_{xx}^0 = \kappa_{yy}^0 = \kappa_{xy}^0 = \gamma_{yx}^0 = \gamma_{zx}^0 = 0$. Strain $\varepsilon_{xx}$ in the $x$-direction of the facings causes the stress $\sigma_{xx}$, $\sigma_{zz}$ and $\tau_{xz}$ in the facings as shown in Figure 3.6. The transverse shear stress $\tau_{xz}$ in the facings contribute to the transverse shear force $Q_x$. The presence of the elastic coupling rigidity $A_{31}$ implies that a transverse shear force $Q_x$ is required to produce an extensional strain $\varepsilon_{xx}^0$. Note that if the tapered sandwich section is symmetric, the directions of transverse shear stress $\tau_{xz}$ of top and bottom facing are equal and opposite, and the transverse shear force $Q_x$ is zero. Therefore, $A_{31} = 0$ for symmetric tapered sandwich members. However, $A_{31}$ is nonzero for unsymmetric sandwich members.

The physical meaning of the extension–transverse shear elastic coupling can also be understood through a simple analysis of the forces on a cross-section of an unsymmetric tapered member shown in Figure 3.7. The forces acting on the section include the stress resultant $N_x$ and transverse shear force $Q_x$. The assumption is made that the normal force $N_x$ is transmitted through the facing only. A portion of the resultant shear force is transmitted through the facing due to the angle of inclination. The core
resists the remainder of the shear force. As can be seen, the transverse shear force carried by the core $Q_x$ is different from the transverse shear force resultant $Q_x$ due to $N_x$. That is, the stress resultant $N_x$ can cause shear deformation in the core even when $Q_x$ is zero.

3.6.4. Extension – Transverse Shear Coupling $A_{52}$

The elastic rigidity $A_{52}$ is an extension-transverse shear coupling caused by the Poisson’s effect. Consider the following deformation wherein a point on the reference surface of a sandwich member has the following strains and curvatures, $\varepsilon_{yy} > 0$ and $\varepsilon_{xx} = \gamma_{xy} = \kappa_{xx} = \kappa_{yy} = \kappa_{xy} = \gamma_{yx} = \gamma_{zz} = 0$. As explained in section 3.6.2, the strain $\varepsilon_{yy}$ causes a transverse shear stress $\tau_{zz}$ in the facings. Therefore, it is necessary to have $Q_x$ in order to obtain an extensional strain in the $y$-direction, although there is no shear deformation of the core in the $x$-$z$ plane. If the tapered sandwich section is symmetric, the shear stresses $\tau_{zz}$ of the both top and bottom facing are equal in magnitude and opposite in direction and therefore $Q_x = 0$. That is, the elastic coupling $A_{52} = 0$ for symmetric tapered sandwich members. It can be directly verified that $A_{52}$ in Equation (3.14) zero.
3.6.5. In-plane Shear – Transverse Shear Coupling $A_{46}$

The elastic rigidity $A_{46}$ in Equation (3.13) causes coupling between the shear strain in $x$-$y$ plane and the transverse shear force $Q_y$. Consider the following situation wherein points on the reference surface of sandwich members have the following strains and curvatures, $\gamma_{xy}^0 > 0$ and $\gamma_{xx}^0 = \gamma_{yy}^0 = \kappa_{xx}^0 = \kappa_{yy}^0 = \kappa_{xy}^0 = \gamma_{yz}^0 = \gamma_{xz}^0 = 0$. From Equation (3.7), the shear strain $\gamma_{xy}^0$ on reference surface causes shear strain $\gamma_{xy}$ in the facings. Transforming this shear strain $\gamma_{xy}$ using Equation (3.4) leads to shear strains $\gamma_{xy}$ and $\gamma_{yn}$ in the $s$-$y$-$n$ coordinate system. By using of the stress-strain relations (3.2) in the $s$-$y$-$n$ coordinate system, we see that the shear stress $\tau_{xy}$ is nonzero as depicted in Figure 3.8. The shear stress $\tau_{xy}$ acting on the $x$-$z$ surfaces of the facings causes a shear stress $\tau_{yz}$ and hence $Q_y$ when transformed to the $x$-$y$-$z$ coordinate system. The presence of the elastic
coupling rigidity $A_{46}$ implies that a transverse shear force $Q_y$ is required to produce an in-plane shear strain $\gamma_{xy}$.

Note that if the tapered sandwich section is symmetric, the directions of the shear stress $\tau_{yz}$ of top and bottom facings are equal in magnitude and opposite in direction, therefore $Q_y = 0$. Therefore, the elastic coupling $A_{46} = 0$ for symmetric sandwich members.

### 3.6.6 Twisting – Transverse Shear Coupling $B_{46}$

To understand the physical interpretation of $B_{46}$, consider the following situation wherein the strains and curvatures of a point on the reference surface of a sandwich member are $\kappa_{xy}^0 > 0$ and $\varepsilon_{xx}^0 = \varepsilon_{yy}^0 = \gamma_{xy}^0 = \kappa_{xx}^0 = \kappa_{yy}^0 = \gamma_{yz}^0 = \gamma_{xz}^0 = 0$. From Equation (3.7), the twisting curvature $\kappa_{xy}^0$ on the reference surface causes shear strain $\gamma_{xy}$ in both facings. Transforming this shear strain using Equation (3.4) and by using the constitutive
equation (3.2), we obtain a shear stress $\tau_{xz}$, and hence a shear force $Q_y$, that acts on the $x$-$z$ plane of the facings, as shown in Figure 3.9. The presence of the elastic coupling rigidity $B_{46}$ implies that a transverse shear force $Q_y$ is required to produce a twisting curvature $\kappa_{xy}$. Note that if the facings are parallel to each other, $B_{46} = 0$, which implies that this coupling vanishes.

![Reference surface](image)

Figure 3.10 Negative extension rigidity for sandwich members with large taper angles

### 3.7. Negative Rigidity for Steep Taper Angles

It is observed from Equation (3.9), that the reduced elastic stiffness of the facings is negative if the tapered angle $\phi$ is larger than the critical angle

$$\phi_c = \frac{1}{2} \cos^{-1}(2\nu - 1).$$  \hfill (3.25)

Since the critical taper angle depends only on the Poisson's ratio of the facing material and $0<\nu<0.5$ for the isotropic facings, $\phi_c$ lies in the range $45^\circ < \phi_c < 90^\circ$. If the facings are made of aluminum, the critical angle $\phi_c = 54.39^\circ$. If the taper angle of a symmetric sandwich member is greater than $\phi_c$, then both the extension rigidity $A_{11}$ and bending...
rigidity $D_{11}$ are negative. Since negative bending and extensional rigidities are rarely encountered in mechanics, we need to explore its physical interpretations and implications on the response of sandwich members with very large taper angles.

For the effects of the negative extension rigidity $A_{11}$, consider the following situation wherein a point on the reference surface is subjected to the axial load $N_x > 0$, and all other resultant forces and moments are zero. The negative extension rigidity $A_{11}$ implies that the strain in the $x$-direction is negative, although the axial load is positive. There is a physical explanation for this phenomenon. The axial force is transmitted through the facings only and the core doesn't resist any extension loading due to its negligible extension stiffness. Therefore, the stress $\sigma_{xx}$ and strain $\varepsilon_{xx}$ in the facings are positive. However, since the facings are in a state of plane stress, the normal strain $\varepsilon_{nn}$ in the facings is negative due to the Poisson's effect, as shown in Figure 3.10. The negative Poisson's contraction of the facings, in conjunction with the steep taper angle, outweighs the effect of the positive strain $\varepsilon_{xx}$ and results in a negative axial strain $\varepsilon_{xx}$. It is important to note that although $\varepsilon_{xx} < 0$, the sandwich member extends in the axial direction since the strain energy is positive.

The explanation for the occurrence of negative bending $D_{11}$ at large taper angles $\phi > \phi_{cr}$ is similar to the reasoning for the negative extension rigidity $A_{11}$. Consider a tapered member that is subjected to a positive bending moment $M_x$. The stress $\sigma_{ss}$ and strain $\varepsilon_{ss}$ are positive in the top facing and negative in the bottom facing. Due to the Poisson's effect and the steep taper angle, the corresponding transverse normal strain $\varepsilon_{nn}$ and axial strain $\varepsilon_{xx}$, are negative in the top facing and positive in the bottom facing as shown in
Figure 3.11 Negative bending rigidity for sandwich members with large taper angles

Figure 3.11. Since the reference surface strain $\varepsilon_{xx}^0 = 0$, Equation (3.7)_1 implies a negative reference surface curvature $\kappa_{xx}^0$. 
Chapter 4

ANALYTICAL MODEL OF ANISOTROPIC FACINGS

In this chapter, we extend the theory to tapered sandwich members with laminated fiber-reinforced facings. In addition to the 6 elastic couplings described in the previous chapter for isotropic facings, we obtain 6 additional elastic couplings due to the anisotropic facings. The physical interpretations of the elastic couplings are given in this chapter.

4.1. Problem Formulation

The analytical development of a tapered sandwich member with laminated anisotropic facings starts with the general depiction of the geometry as shown in Figure 4.1. We use a rectangular Cartesian coordinate system \(x-y-z\). The thickness is assumed to...
vary linearly along the span (x-direction). It is assumed that the facings are relatively thin compared to the core and therefore behave as membranes. The core is assumed to be inextensible in the thickness direction and to carry only transverse shear stresses. Since the facings are tapered, it is advantageous to use a local coordinate system, denoted by s-y-n, such that the s- and n-axes are parallel and normal to the facing, respectively. The taper angle of the facing, \( \phi \), is taken as positive counter-clockwise with respect to the x-axis. The facing laminae are orthotropic and each lamina has a distinct fiber orientation \( \theta_k \) relative to the s-axis. The tapered member is composed of \( N \) distinct layers numbered from bottom to top as shown in Figure 4.1. The core is denoted as layer number \( C \) and the bottom and top facings are composed of \( C - 1 \) and \( N - C \) distinct laminae, respectively. The z-coordinate of the bottom surface of the \( k \)th layer is designated \( z_k \) with the top surface of the layer being \( z_{k+1} \). In general, the \( z_k \)'s are functions of the x-coordinate since the sandwich beam is tapered. The taper angle of the top and bottom facings are \( \phi = \phi_t \) and \( \phi = \phi_b \), respectively.

4.2. Analytical Model

Since the facings are relatively thin compared to the core, we assume that the plane stress conditions stated in Equation 3.1 are applicable to the anisotropic laminae of the facings. The stress-strain relations for an orthotropic lamina in the principal material coordinate system are
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{23} \\
\tau_{13} \\
\tau_{11}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{11}
\end{bmatrix},
\]

(4.1)

The elastic stiffnesses \(C_{ij}\) in Equation (4.1) for an orthotropic material in terms of the engineering constants are

\[
\begin{align*}
C_{11} &= \frac{E_1 \left(1 - \nu_{13}^2 E_3 / E_2 \right)}{1 - \nu}, \\
C_{12} &= \frac{E_2 \left(\nu_{12} + \nu_{13} \nu_{23} E_3 / E_2 \right)}{1 - \nu}, \\
C_{13} &= \frac{E_3 \left(\nu_{13} + \nu_{12} \nu_{23} \right)}{1 - \nu}, \\
C_{22} &= \frac{E_2 \left(1 - \nu_{13}^2 E_3 / E_1 \right)}{1 - \nu}, \\
C_{23} &= \frac{E_3 \left(\nu_{23} + \nu_{12} \nu_{13} \right) E_2 / E_1}{1 - \nu}, \\
C_{33} &= \frac{E_3 \left(1 - \nu_{12}^2 E_2 / E_1 \right)}{1 - \nu}, \\
C_{44} &= G_{23}, C_{55} = G_{13}, C_{66} = G_{12}.
\end{align*}
\]

where

\[
\nu = \frac{\nu_{12}^2 E_2 + \nu_{13}^2 E_3 + \nu_{23}^2 E_3 - 2 \nu_{12} \nu_{13} \nu_{23} E_3}{E_1}.
\]

The principal material direction of the \(k^{th}\) lamina is oriented at angle \(\theta_k\) with respect to the \(s\)-coordinate direction. The stress-strain relations for an orthotropic lamina in the \((s, y, n)\) coordinate system are,
where the elastic stiffness components $\overline{C}_y$ in terms of the Young’s moduli, Poisson’s ratios, shear moduli and the angle $\theta_k$ are

$$\begin{align*}
\overline{C}_{11} & = C_{11} m^4 + 2 m^2 n^2 (C_{12} + 2 C_{66}) + C_{22} n^4, \\
\overline{C}_{12} & = m^2 n^2 (C_{11} + C_{22} - 4 C_{66}) + C_{12} (m^4 + n^4), \\
\overline{C}_{13} & = C_{13} m^2 + C_{23} n^2, \\
\overline{C}_{16} & = mn [C_{11} m^2 - C_{22} n^2 - (C_{12} + 2 C_{66}) (m^2 - n^2)], \\
\overline{C}_{22} & = C_{11} n^4 + 2 m^2 n^2 (C_{12} + 2 C_{66}) + C_{22} m^4, \\
\overline{C}_{23} & = C_{13} n^2 + C_{23} m^2, \\
\overline{C}_{26} & = mn [C_{11} n^2 - C_{22} m^2 + (C_{12} + 2 C_{66}) (m^2 - n^2)], \\
\overline{C}_{33} & = C_{33}, \\
\overline{C}_{36} & = mn (C_{23} - C_{13}), \\
\overline{C}_{44} & = C_{44} m^2 + C_{55} n^2, \\
\overline{C}_{45} & = mn (C_{44} - C_{55}), \\
\overline{C}_{55} & = C_{44} n^2 + C_{55} m^2, \\
\overline{C}_{66} & = m^2 n^2 (C_{11} + C_{22} - 2 C_{12}) + C_{66} (m^2 - n^2)^2,
\end{align*}$$

where $m = \cos \theta_k$ and $n = \sin \theta_k$. The expressions are also available in books on composite materials (e.g. see Herakovich, 1997, pp. 57-61). The transformation of the stresses and strains between the $(x, y, z)$ and $(s, y, n)$ coordinate systems given in Equation (3.3) and (3.4) remain unchanged.

As before, it is assumed that the strains in the facings are linear functions of the thickness coordinate, as shown in Equation (3.7). Likewise, the transverse normal strain $\varepsilon_{yz}$, and the transverse shear strains $\gamma_{xz}$ and $\gamma_{yz}$ in the facings are expressed in terms of $\varepsilon_{xz}$,
and $\gamma_{xy}$ by recognizing that the facings are in a state of plane stress. The stresses in the 
(s, y, n) coordinate system are obtained through $\{\sigma\}_{sny} = [C]\{\varepsilon\}_{sny} = [C][R]^{T}\{\varepsilon\}_{xyz}$. The three plane stress equations (3.1) are enforced to obtain $\varepsilon_{xz}$, $\gamma_{xz}$ and $\gamma_{yz}$ in terms of $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\gamma_{xy}$ as

$$
\varepsilon_{xx} = \frac{C_{13}\cos^{2}\phi - C_{33}\sin^{2}\phi}{D} \varepsilon_{xx} + \frac{C_{23}(\cos^{2}\phi - \sin^{2}\phi)}{D} \varepsilon_{yy} + \frac{C_{36}(\cos^{2}\phi - \sin^{2}\phi)}{D} \gamma_{xy},
$$

$$
\gamma_{xz} = \frac{2(C_{13} + C_{33})\cos\phi\sin\phi}{D} \varepsilon_{xx} - \frac{2C_{23}\cos\phi\sin\phi}{D} \varepsilon_{yy} - \frac{2C_{36}\sin\phi}{D} \gamma_{xy},
$$

$$
\gamma_{yz} = \gamma_{xy} \tan\phi,
$$

where,

$$
D = C_{13}\sin^{2}\phi - C_{33}\cos^{2}\phi.
$$

The stress-strain relations in the (x, y, z) coordinate system for a tapered facing lamina, computed using (4.4), (3.4)1, (4.2), (3.3)2, are

$$
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{16} \\
\tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{26} \\
\tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{36} \\
\tilde{Q}_{41} & \tilde{Q}_{42} & \tilde{Q}_{46} \\
\tilde{Q}_{51} & \tilde{Q}_{52} & \tilde{Q}_{56} \\
\tilde{Q}_{61} & \tilde{Q}_{62} & \tilde{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix},
$$

(4.5)

where $\tilde{Q}_{ij}$ are the reduced stiffnesses for an inclined facing lamina, defined as
\[ Q_{11} = \frac{(C_{13}^2 - C_{11}C_{33}) \cos^2 \phi}{D}, \]
\[ Q_{12} = \frac{(C_{13}C_{23} - C_{12}C_{33}) \cos^2 \phi + (C_{12}C_{13} - C_{11}C_{23}) \sin^2 \phi \cos^2 \phi}{D}, \]
\[ Q_{16} = \frac{(C_{13}C_{36} - C_{16}C_{33}) \cos^2 \phi + (C_{13}C_{16} - C_{11}C_{36}) \cos \phi \sin^2 \phi}{D}, \]
\[ Q_{21} = \frac{C_{13}C_{23} - C_{12}C_{33}}{D}, \]
\[ Q_{22} = \frac{(C_{23}^2 - C_{22}C_{33}) \cos^2 \phi + (C_{13}C_{22} - C_{12}C_{23}) \sin^2 \phi}{D}, \]
\[ Q_{26} = \frac{(C_{23}C_{36} - C_{26}C_{33}) \cos^2 \phi + (C_{13}C_{26} - C_{12}C_{36}) \sin^2 \phi}{D \cos \phi}, \]
\[ Q_{31} = \frac{(C_{13}^2 - C_{11}C_{33}) \sin^2 \phi}{D}, \]
\[ Q_{32} = \frac{(C_{13}C_{23} - C_{12}C_{33}) \cos^2 \phi + (C_{12}C_{13} - C_{11}C_{23}) \sin^2 \phi \sin^2 \phi}{D}, \]
\[ Q_{36} = \frac{(C_{13}C_{23} - C_{16}C_{33}) \cos^2 \phi + (C_{13}C_{16} - C_{11}C_{36}) \sin^2 \phi \sin^2 \phi}{D \cos \phi}, \]
\[ Q_{41} = \frac{(C_{16}C_{33} - C_{13}C_{36}) \sin \phi}{D}, \]
\[ Q_{42} = \frac{(C_{23}C_{36} - C_{26}C_{33}) \cos^2 \phi \sin \phi + (C_{13}C_{26} - C_{16}C_{23}) \sin^3 \phi}{D}, \]
\[ Q_{46} = \frac{(C_{36}^2 - C_{33}C_{66}) \cos^2 \phi \sin \phi + (C_{13}C_{66} - C_{16}C_{36}) \sin^3 \phi}{D \cos \phi}, \]
\[ Q_{51} = \frac{(C_{13}^2 - C_{11}C_{33}) \cos \phi \sin \phi}{D}, \]
\[ Q_{52} = \frac{(C_{13}C_{23} - C_{12}C_{33}) \cos^3 \phi \sin \phi + (C_{12}C_{13} - C_{11}C_{23}) \cos \phi \sin^3 \phi}{D}, \]
\[ Q_{56} = \frac{(C_{13}C_{36} - C_{16}C_{33}) \cos^2 \phi \sin \phi + (C_{13}C_{16} - C_{11}C_{36}) \sin^3 \phi}{D \cos \phi}, \]
\[ Q_{61} = \frac{(C_{13}C_{36} - C_{16}C_{33}) \cos \phi}{D}, \]
\[ Q_{62} = \frac{(C_{23}C_{36} - C_{26}C_{33}) \cos^3 \phi \sin \phi + (C_{13}C_{26} - C_{16}C_{23}) \cos \phi \sin^2 \phi}{D}, \]
\[ Q_{66} = \frac{(C_{36}^2 - C_{33}C_{66}) \cos^2 \phi \sin \phi + (C_{13}C_{66} - C_{16}C_{36}) \sin^2 \phi}{D}. \]
The in-plane stresses $\sigma_{xx}$, $\sigma_{yy}$ and $\tau_{xy}$ of the core are assumed to be negligible compared to that in the facings. The transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$ in the core are assumed to be constant throughout the thickness and they are related to the core shear strains by

$$
\tau_{xz}^{(C)} = G_{xz}^c \gamma_{xz}^c,
$$

$$
\tau_{yz}^{(C)} = G_{yz}^c \gamma_{yz}^c.
$$

The stress resultants are defined as

$$
\begin{align*}
[N_x, N_y, N_{xy}] &= \int_{z_1(x)}^{z_{N+1}(x)} \sigma_{xx}, \sigma_{yy}, \tau_{xy} \, dz, \\
[M_x, M_y, M_{xy}] &= \int_{z_1(x)}^{z_{N+1}(x)} z [\sigma_{xx}, \sigma_{yy}, \tau_{xy}] \, dz, \\
[Q_x, Q_y] &= \int_{z_1(x)}^{z_{N+1}(x)} \tau_{xx}, \tau_{yy} \, dz,
\end{align*}
$$

where the quantities $N_x$, $N_y$ and $N_{xy}$ are the in-plane force resultants, $Q_x$ and $Q_y$ are the transverse shear load resultants and $M_x$, $M_y$ and $M_{xy}$ are the moment resultants, as depicted in Figure 3.2. Substitution of the facing stresses from (4.5) and the core stresses from (4.7) into (4.8) leads to the following matrix equation for the resultant forces and moments in terms of the core shear strains and reference surface strains and curvatures,
where $A_{ij}$ are the extensional and shear rigidities, $D_{ij}$ are the bending rigidities and $B_{ij}$ are the bending-extension rigidities defined by

$$
[A_y(x), B_y(x), D_y(x)] = \sum_{k=1}^{N} \int_{z_1(x)}^{z_2(x)} \tilde{Q}_y(z) \left[1, z, z^2\right] dz; i, j = 1, 2, 6
$$

$$
[A_{44}(x), A_{55}(x)] = \left[G_{yz}, G_{yx}\right] z_{C+1}(x) - z_C(x). 
$$

The relation (4.9) can be inverted to obtain the following equation, from which the reference strains can be determined if the resultant forces and moments are known:

$$
\begin{bmatrix}
N_x \\
N_y \\
Q_y \\
Q_x \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & a_{16} & b_{11} & b_{12} & b_{16} \\
a_{21} & a_{22} & 0 & 0 & a_{26} & b_{21} & b_{22} & b_{26} \\
a_{41} & a_{42} & a_{44} & 0 & a_{46} & b_{41} & b_{42} & b_{46} \\
a_{51} & a_{52} & 0 & a_{55} & a_{56} & b_{51} & b_{52} & b_{56} \\
a_{61} & a_{62} & 0 & 0 & a_{66} & b_{61} & b_{62} & b_{66} \\
b_{11} & b_{12} & 0 & 0 & b_{16} & d_{11} & d_{12} & d_{16} \\
b_{21} & b_{22} & 0 & 0 & b_{26} & d_{21} & d_{22} & d_{26} \\
b_{61} & b_{62} & 0 & 0 & b_{66} & d_{61} & d_{62} & d_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{yz}^c \\
\gamma_{xz}^c \\
\gamma_{xy}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0
\end{bmatrix},
$$

(4.11)
4.3. Beam Bending and Cylindrical Bending

There are also two cases of laminated plates that can be treated as one-dimensional problems: (1) a tapered sandwich beam, and (2) cylindrical bending of a tapered member, as described in Section 3.3.

For a sandwich beam in bending, the transverse load $Q_x$ and bending moment $M_x$ are assumed to be known, whereas the other loads vanish, i.e., $N_x = N_y = Q_y = N_{xy} = M_y = M_{xy} = 0$. By substitution of the resultant force into Equation (4.11), the reference surface strains, curvatures and transverse shear strains are

$$
\varepsilon_{xx}^0 = b_{11} M_x, \varepsilon_{yy}^0 = b_{21} M_x, \\
\gamma_{yx}^c = b_{41} M_x, \gamma_{zx}^c = a_{55} Q_x + b_{51} M_x, \\
\kappa_{xx}^0 = d_{11} M_x, \kappa_{yy}^0 = d_{21} M_x, \\
\gamma_{xy}^0 = b_{61} M_x, \kappa_{xy}^0 = d_{61} M_x.
$$

(4.12)

In the case of cylindrical bending, the transverse shear load $Q_x$ and bending moment $M_x$ are assumed to be known, whereas the strains and curvature in the $y$-direction and the other resultant forces and moment vanish. That is, $\varepsilon_{yy}^0 = \kappa_{yy}^0 = 0$ and $N_x = Q_y = N_{xy} = M_{xy} = 0$. The force and moment resultants $N_y$ and $M_y$ are obtained in terms of $M_x$ using the second and seventh rows of the matrix equation (4.9) and the strains and curvatures $\varepsilon_{xx}, \gamma_{yx}, \gamma_{zx}, \gamma_{xy}, \kappa_{xx}, \kappa_{xy}$ are determined from the remaining six equations in (4.11).
The strain distributions through the thickness of a tapered sandwich section are determined for Equation (3.7) after all the core shear strains and reference surface strains and curvatures have been computed. The stress distributions in the global $x$-$y$-$z$ coordinate system are obtained using the constitutive relations (4.5) and the stresses and strains in the local $s$-$y$-$n$ coordinate system are determined using the transformation relations (3.3) and (3.4).

### 4.4. Elastic Couplings

In chapter three, six important couplings for tapered sandwich members with isotropic facings, namely $B_{51}$, $B_{52}$, $A_{51}$, $A_{52}$, $A_{46}$ and $B_{66}$, have been discussed. An examination of Equation (4.9) reveals that there are six additional elastic couplings, namely $A_{41}$, $A_{42}$, $B_{41}$, $B_{12}$, $A_{56}$ and $B_{56}$, for tapered sandwich members with anisotropic facings, as shown in Figure 4.2. In this section, we explore the physical interpretation of these additional coupling rigidities. We define two special cases of tapered sandwich members since they are relevant in our discussion of elastic couplings. When the

\[
\varepsilon_{xx}^0 = \frac{a_{12}(-d_{22}b_{21} + b_{22}d_{21}) + b_{11}(a_{22}d_{22} - b_{22}^2) + b_{12}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x, \\
\gamma_{yz}^e = \frac{a_{42}(-d_{22}b_{21} + b_{22}d_{21}) + b_{41}(a_{22}d_{22} - b_{22}^2) + b_{42}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x, \\
\gamma_{zx}^e = \frac{a_{52}(-d_{22}b_{21} + b_{22}d_{21}) + b_{51}(a_{22}d_{22} - b_{22}^2) + b_{52}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x + a_{55}Q_x, \\
\gamma_{xy}^0 = \frac{a_{62}(-d_{22}b_{21} + b_{22}d_{21}) + b_{61}(a_{22}d_{22} - b_{22}^2) + b_{62}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x, \\
\kappa_{xx}^0 = \frac{b_{12}(-d_{22}b_{21} + b_{22}d_{21}) + d_{11}(a_{22}d_{22} - b_{22}^2) + d_{12}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x, \\
\kappa_{xy}^0 = \frac{b_{62}(-d_{22}b_{21} + b_{22}d_{21}) + d_{61}(a_{22}d_{22} - b_{22}^2) + d_{62}(b_{22}b_{21} + a_{22}d_{21})}{a_{22}d_{22} - b_{22}^2} M_x.
\]

The strain distributions through the thickness of a tapered sandwich section are determined for Equation (3.7) after all the core shear strains and reference surface strains and curvatures have been computed. The stress distributions in the global $x$-$y$-$z$ coordinate system are obtained using the constitutive relations (4.5) and the stresses and strains in the local $s$-$y$-$n$ coordinate system are determined using the transformation relations (3.3) and (3.4).
inclination of the facings, laminae thicknesses and laminae fiber-orientations are symmetric about the mid-surface, the laminate is called a symmetric tapered sandwich member. However, if the inclination of the facings and laminae thicknesses are symmetric about the mid-surface, but the laminae fiber-orientations are antisymmetric about the mid-surface, it is known as an antisymmetric tapered sandwich member.

4.4.1. Extension -Transverse Shear Coupling $A_{ij}$

The elastic rigidity $A_{ij}$ in Equation (4.9) causes coupling between the extension strain in the $x$-direction and transverse shear load $Q_y$. Consider the situation wherein the strains and curvatures at points on the reference surface of a sandwich member are $\epsilon^0_{xx} > 0$ and $\gamma^0_{yy} = \gamma^0_{xy} = \kappa^0_{xx} = \kappa^0_{yy} = \kappa^0_{xy} = \gamma^0_{ye} = \gamma^0_{xe} = 0$. By substituting the assumed references strains and curvatures into Equation (3.7) and (4.4), transforming the strains from the $x$-$y$-$z$ coordinate system to the $s$-$y$-$n$ coordinate system using (3.4) and using the constitutive Equation (4.2), we obtain the stresses in the facings $\sigma_{ss}$, $\sigma_{nn}$ and $\tau_{ys}$ as shown in Figure 4.3. The shear stress $\tau_{ys}$ in the plane of the facings is nonzero because the
principal material directions of the laminae are oriented at an angle to the s-axis, \( \tau_{sy} = 0 \) for isotropic and cross-ply laminae (\( \theta_k = 0^\circ \) or \( 90^\circ \)). By transforming shear stresses \( \tau_{sy} \) to x-y-z coordinate system we obtain a vertical shear stress \( \tau_{yz} \) that contributes to a shear load \( Q_y \). That is, a transverse shear force \( Q_y \) has to be applied in order to obtain a reference surface strain \( \varepsilon_{xy}^0 \), although the transverse shear strain of the core in the y-z plane is zero.

![Figure 4.3 Stresses required to produce only \( \varepsilon_{xy}^0 \)](image)

(a) symmetric Case (b) antisymmetric Case

Note that if the tapered sandwich section is symmetric as shown in Figure 4.3a, the shear stress \( \tau_{yz} \) of top and bottom facings are equal in magnitude and opposite in direction, and therefore \( Q_y = 0 \). That is, \( A_{41} = 0 \) for symmetric sandwich members. We can also verify that \( \tilde{Q}_{41} \) in Equation (4.6) is an odd function of the taper angle \( \phi \), and therefore the contributions from a lamina in the top facing and the corresponding lamina
in the bottom facing of the symmetric sandwich member cancel out to give $A_{41} = 0$ in Equation (4.10). However, the contributions are additive for the antisymmetric sandwich members shown in Figure 4.3b.

4.4.2. Extension-Transverse Shear Coupling $A_{42}$

The elastic rigidity $A_{42}$ in Equation (4.9) causes coupling between the extension strain in $y$-direction and transverse shear load $Q_y$. Consider the following situation wherein at every point on the reference surface of a sandwich member, $\varepsilon^0_{yy} > 0$ and $\varepsilon^0_{xx} = \gamma^0_{xy} = \kappa^0_{xx} = \kappa^0_{yy} = \gamma^0_{yx} = \gamma^0_{xx} = 0$. From Equation (3.7), the extensional strain $\varepsilon^0_{yy}$ on reference surface causes extensional strain $\varepsilon_{yy}$ in both facings. We obtain the stresses $\sigma_{zz}$, $\sigma_{nn}$ and $\tau_{ny}$ in the facings by using Equation (4.4), the strain transformations Equation (3.4)_1 and constitutive equations Equation (4.2). The shear stress $\tau_{ny}$ is due to the combination of the Poisson's effect and the anisotropic material properties of the facings. The shear stress $\tau_{ny}$ contributes to a shear load $Q_y$. That is, a transverse shear force $Q_y$ has to be applied in order to obtain a reference surface strain $\varepsilon^0_{yy}$, although the transverse shear strain of the core in the $y$-$z$ plane is zero.

Note that if the tapered sandwich section is symmetric, the shear stress $\tau_{nz}$ of top and bottom facings are equal and opposite, as shown in Figure 4.4(a), and therefore $Q_y = 0$, In other words, $A_{42} = 0$ for symmetric sandwich members. This can be verified directly from (4.10) since $\tilde{Q}_{42}$ is an odd function of the taper angle $\phi$. 

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4.4.3. Bending-Transverse Shear Coupling $B_{41}$

The elastic rigidity $B_{41}$ in Equation (4.9) causes coupling between the bending curvature in the $x$-direction and transverse shear load $Q_y$. Consider the following situation wherein at every point on the reference surface of a sandwich members, $\kappa_{xx}^0 > 0$ and $\epsilon_{xx}^0 = \gamma_{xy}^0 = \kappa_{yy}^0 = \kappa_{xy}^0 = \gamma_{yz} = \gamma_{xz} = 0$. Due to reasons explained in Section 4.3.1, we obtain the stress state depicted in Figure 4.5. The shear stress $\tau_{xy}$ has a vertical component that contributes to the shear load $Q_y$. This implies that a transverse shear force $Q_y$ has to be applied in order to obtain a reference surface curvature $\kappa_{xx}^0$ although the transverse shear strain of the core in the $y$-$z$ plane is zero.
Note that if the tapered sandwich section is antisymmetric, the shear stress $\tau_{xy}$ of top and bottom facings are equal and opposite and therefore $Q_y = 0$. That is, $B_{41} = 0$ for antisymmetric tapered sandwich members. This can also be directly verified from Equation (4.10) since $\tilde{Q}_{41}$ is an odd function of the taper angle $\phi$ and material orientation angle $\theta$.

4.4.4. Bending-Transverse Shear Coupling $B_{42}$

The elastic rigidity $B_{42}$ in Equation (4.9) causes coupling between the bending curvature in $y$-direction and transverse shear load $Q_y$. Consider the following situation wherein at a point on the reference surface of a sandwich member, $\kappa_{yy} > 0$ and $\varepsilon_{xx} = \varepsilon_{yy} = \gamma_{xy} = \kappa_{xx} = \kappa_{xx} = \gamma_{yy} = \gamma_{xx} = 0$. For reasons explained in Section 4.4.2, the stress state is shown in Figure 4.5. Note that if the tapered sandwich section is antisymmetric, the shear stress $\tau_{xx}$
of top and bottom facings are equal and opposite, therefore $Q_y = 0$. Therefore, $B_{42} = 0$ for antisymmetric tapered sandwich members.

![Figure 4.6 Stresses required to produce only $\gamma^0_{xy}$](image)

4.4.5. In-plane Shear-Transverse Shear Coupling $A_{56}$

The elastic rigidity $A_{56}$ in Equation (4.9) causes coupling between in-plane shear strain and transverse shear load $Q_x$. Consider the following situation wherein at the reference surface strains and curvatures of a sandwich members, $\gamma^0_{xy} > 0$ and $\varepsilon^0_{xx} = \varepsilon^0_{yy} = \kappa^0_{xx} = \kappa^0_{yy} = \kappa^0_{xy} = \gamma^0_{yx} = \gamma^0_{xx} = 0$. The stresses in the facings are shown in Figure 4.6. As before, the stress $\sigma_{ss}$ has a vertical $\tau_{sz}$ component that contributes to $Q_x$. Note that if the tapered sandwich section is symmetric, the shear stress $\tau_{sz}$ of top and bottom facings equal and opposite and therefore $Q_x = 0$. That is, $A_{56} = 0$ for symmetric sandwich members. This
can also be verified directly from Equation (4.10) since $\tilde{Q}_{46}$ is an odd function of the taper angle.

### 4.4.6. Twisting-Transverse Shear Coupling $B_{56}$

The elastic rigidity $B_{56}$ in Equation (4.9) causes coupling between the twisting curvature $\kappa_{xy}$ and transverse shear load $Q_x$. Consider the following situation wherein $\kappa_{xy}^0 > 0$ and $\varepsilon_{xx}^0 = \varepsilon_{yy}^0 = \gamma_{xy}^0 = \kappa_{xx}^0 = \kappa_{yy}^0 = \gamma_{yx}^0 = \gamma_{xx}^0 = 0$. For this case, the stresses are shown in Figure 4.7. Here too, the vertical component of the shear stress $\sigma_{ss}$ contributes to $Q_x$. Note that if the tapered sandwich section is antisymmetric, the shear stress $r_{zx}$ of the top and bottom facings are equal and opposite. Therefore, $B_{56} = 0$ for antisymmetric sandwich members.
Chapter 5

CASE STUDIES AND COMPARISON WITH FINITE ELEMENT ANALYSIS

This section presents the results from the tapered sandwich theory and finite element analysis (FEA). Results for the following three configurations are presented, 1) Aluminum facing symmetric cases, 2) Graphite/epoxy facing symmetric cases and 3) Graphite/epoxy facing unsymmetric cases. It is demonstrated that the tapered sandwich theory developed in the third and fourth chapters accurately represents the response of tapered sandwich sections and gives excellent comparisons with the finite element method. These studies also further expand our understanding of the response of tapered sections to applied loads.

5.1. Finite Element Models

Finite element models of several different configurations were developed to validate the analytical model. For this purpose, linear finite element analyses were run in the ABAQUS/Standard (Hibbitt, Karlsson & Sorensen, Inc., 2002) finite element analysis software package. Although a tapered sandwich composite structure can be modeled as a three-dimensional structure, the number of elements required for the analysis is prohibitively large. This is especially true for sandwich panels with laminated facings with each facing having numerous individual laminae. Since at least three to four elements are required through the thickness of each lamina to accurately compute the stresses, the finite element mesh would be large and impractical for implementation on a
complex structure. Therefore, we have analyzed the tapered sandwich structure using a two-dimensional modeling approach. There are two cases of tapered sandwich structures that can be analyzed using a two-dimensional finite element model,

1) Plane Stress model

The plane stress condition can only be applied when the thickness of the structure along the y-axis is much smaller than the length along the x-axis. The stresses are functions of the planar x- and z-coordinates alone and the out-of-plane normal and shear stress are equal to zero. This case corresponds to the analysis of sandwich beams in our analytical solution.

2) Plane Strain model

The plane strain condition can be applied with the sandwich plate is very long along the y-axis and has a finite dimension along the x-axis. Under these conditions, we can assume that the displacements are functions of x and z coordinates alone. The strains are the functions of planar coordinate alone and the out-of-plane normal and shear strains are equal to zero. This case corresponds to the cylindrical bending of tapered sandwich members.

In order to compare analytical solutions, 2D continuum elements are used for the honeycomb core and facings in the finite element models. In ABAQUS/CAE package, the SOLID SECTION option is used to assign material orientation and thickness of each lamina and several continuum elements are used for each lamina to obtain better accuracy. The elements have two translational degrees of freedom in the x and z-directions. The laminate consists of several laminae that are perfectly bonded together.
Each lamina has four elements through its thickness. The honeycomb core is modeled as homogenous. The microstructure of the honeycomb is neglected and it is modeled as a homogeneous orthotropic continuum. This is a reasonable approach since we are interested in the macroscopic behavior of the tapered sandwich member and we are not concerned about the deformation and stresses in the individual cells of the honeycomb. This simplified modeling technique is widely used for sandwich composite structures since it reduces the computational cost tremendously. When defining the properties of the honeycomb core, we assume that the in-plane Young's moduli are negligible, i.e., $E_x$ and $E_y$ are very small. We do not set them equal to zero, but instead assign very small values to avoid a singularity. In addition, the in-plane shear modulus $G_{xy}$ and Poisson's ratios $\nu_{xy}$, $\nu_{xz}$ and $\nu_{yz}$ are negligible.

Consider tapered sandwich structures that are simply supported at the edges, such as the tapered member depicted in Figure 5.1. Note that the boundary condition is simply supported along the height at the right end. A concentrated load of magnitude $2P$ is applied on the top surface of the top facing at the mid span. All of the tapered members that we analyzed are symmetric about the centerline $x = 0$. Due to the symmetry about the
centerline, only the right half of the member is modeled using FEA and symmetry boundary conditions about the centerline is used. Eight-noded biquadratic plane strain full integration quadrilateral elements (CPE8) are used to obtain accurate FEA results. Several models with the increasing mesh densities were studied to establish convergence. Our studies show that the displacements and stresses gradually converge. We have used sufficient number of elements to obtain satisfactory results.

5.2. Tapered Sandwich Members with Aluminum Facings in Cylindrical Bending

In the cases studied, the facings of the sandwich composite are made of 7075 aluminum of 0.8 mm thickness, whose elastic modulus is 70 GPa, and Poisson’s ratio is 0.33. The core is made of aluminum (Hexcel HexWeb™ 5052 4.5-1/8-10) with physical properties given in Table 5.1.

<table>
<thead>
<tr>
<th>Hexcel HexWeb™ 5052 4.5-1/8-10</th>
<th>Modulus (MPa)</th>
<th>483</th>
</tr>
</thead>
<tbody>
<tr>
<td>L-direction Plate Shear (G_{xy})</td>
<td>Strength (MPa)</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>Modulus (MPa)</td>
<td>214</td>
</tr>
<tr>
<td>W-direction Plate Shear (G_{yz})</td>
<td>Strength (MPa)</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The tapered sandwich member is symmetric about the horizontal mid-surface as shown in Figure 5.1. The facings are of equal thickness and the angle of inclination of the top and bottom facings with respect to the horizontal axis are the same, i.e., \( \phi_0 = -\phi \). The mid-surface is chosen as the reference surface \( z = 0 \).
Due to the symmetry of the sandwich composite structure about the reference surface, the magnitude of the in-plane normal stress $\sigma_{n}$ in the bottom facing is the same as that in the upper facing. Experiments on tapered members have shown that in many cases, the facings delaminate from the core. Therefore we pay close attention to stresses at the interface between the facings and the core. We will compare the following three stress components between the analytical and finite element solutions: 1) extensional stresses on the bottom-most surface of the bottom facing $\sigma_{ss}$, 2) transverse shear stress at the interface between the bottom facing and core $\tau_{m}$, and 3) peeling stresses at the interface between the bottom facing and core $\sigma_{nm}$.

![Figure 5.2 Model of highly tapered sandwich construction with aluminum facing ($H = 1800\, \text{mm}$, $L = 500\, \text{mm}$, $t = 0.8\, \text{mm}$ and $\phi = 60^\circ$)](image-url)
5.2.1. Highly Tapered Sandwich Members

In order to validate the analytical model for highly tapered sandwich beams with negative bending rigidity and to understand the influence of the bending-shear elastic coupling on the deformation, we present results for a tapered sandwich construction with taper angle $\phi = 60^\circ$ which is greater than the critical tapered angle $\phi_{cr} = 54.93^\circ$. The core is 1800 mm thick at the centerline and the span $2L = 1000$ mm. The tapered member is simply supported at both ends. A downward concentrated load of $2P = 20$ KN is placed at a point that is located on the upper surface of the top facing at mid-span.

A finite element mesh consisting of 33,500 8-noded quadratic elements is used to model the tapered member. The stresses of analytical and FEA results are compared at point $A$ which is located at the top surface of the upper facing at $x = L/2$, as illustrated in Figure 5.2. The Deflection is compared at the original of the $x$-$y$-$z$ coordinate system. The numerical comparison in Table 5.2 shows that results of our analytical model are in very good agreement with finite element analysis.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Analytical</th>
<th>FEA</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{xx}$</td>
<td>-1.673 MPa</td>
<td>-1.671 MPa</td>
<td>0.12%</td>
</tr>
<tr>
<td>$\sigma_{zz}$</td>
<td>-5.019 MPa</td>
<td>-5.026 MPa</td>
<td>0.14%</td>
</tr>
<tr>
<td>$\tau_{xz}$</td>
<td>2.898 MPa</td>
<td>2.897 MPa</td>
<td>0.03%</td>
</tr>
<tr>
<td>$\sigma_{ss}$</td>
<td>-2.208 MPa</td>
<td>-2.210 MPa</td>
<td>0.09%</td>
</tr>
<tr>
<td>$\sigma_{ss}$</td>
<td>-6.692 MPa</td>
<td>-6.697 MPa</td>
<td>0.07%</td>
</tr>
<tr>
<td>$\varepsilon_{xx}$</td>
<td>10.172</td>
<td>10.235</td>
<td>0.62%</td>
</tr>
<tr>
<td>$\varepsilon_{ss}$</td>
<td>-85.187</td>
<td>-85.25</td>
<td>0.07%</td>
</tr>
<tr>
<td>$U_z(0,0)$</td>
<td>-8.548(10^{-5}) m</td>
<td>-8.509(10^{-5}) m</td>
<td>0.46%</td>
</tr>
</tbody>
</table>
According to our tapered sandwich theory, the extensional and bending rigidities at $x = L/2$ are $A_{11} = -530\text{MPa}$ and $D_{11} = -120\text{MPa}$. The strain $\varepsilon_{xx}$ at point $A$ is positive, although the bending moment $M_x$ at that location is negative. As previously discussed in Section 3.7, this behavior is due to the negative bending rigidity $D_{11}$. However, the strain $\varepsilon_{zz}$ in the top facing at point $A$ is negative as expected since the bending moment causes a compressive in-plane strain in the upper facing. The normal stress $\sigma_{xx}$ at point $A$ is also negative as expected although the strain $\varepsilon_{xx}$ at point $A$ is positive.

### 5.2.2. Results for Various Taper Angles

In order to understand the influence of the tapered angles on the deflection and stresses, the tapered angle is varied from $0, 5, 10, 15, 20, 25, 30$ and $33.5$ degrees while other geometry configurations was kept the same. The span of the tapered member is $2L$.
Figure 5.4 Stress distributions of $\sigma_z$, ($H = 60\, \text{mm}$, $L = 45\, \text{mm}$, $t = 0.8$ and $\phi = 20^\circ$)

=180 mm and the core is 60 mm thick at mid-span. The core thickness at the two ends would depend on the taper angle. The finite element model used in this case is presented in Figure 5.3. The edges are simply supported and a downward concentrated load of $2P=2\, \text{N}$ is placed on the top of the upper facing at mid-span. Totally 10,800 8-noded quadratic elements are used in this model. There are four elements through the thickness of each aluminum facing.

5.2.2.1. Stresses

Extensional stresses on the bottom surface of the bottom facing and transverse shear stresses on the bottom interface between facing and core will be studied in this section. All the stresses are presented in the local $s$-$y$-$n$ coordinate system. Figure 5.4 shows a contour plot of normal stress in $s$-direction, $\sigma_{ss}$. It should be noted that stress $\sigma_{ss}$
is in the local coordinate system. For the core, the local coordinate system is the x-y-z coordinate system. The peak tensile stress of 4,118 N/m², occurs at the mid-span in interface between the upper facing and the core. This stress concentration is due to two factors. The first one is that the concentrated force will stress concentration. Secondly, the abrupt change in the orientation of the facings at mid-span also cause stress variation and it cannot be captured by our simple tapered sandwich theory.

The comparison between the finite element and analytical models is shown in the Figure 5.5 (a,b) for $\phi = 0^\circ$. In this figure, $\sigma_n$ is the normal stresses at the bottom surface of the lower facing, and the $\tau_{sn}$ is the shear stress at the interface between the bottom facing and core. It clearly demonstrates that analytical and FEA normal stresses are in very good comparison at most points, except at the mid-span. The FEA normal stress $\sigma_n$ reaches a peak of 1,300 Pa near $x = 0.05L$, probably due to a singularity in the stress field due to the abrupt change in taper angle at mid-span. As to the analytical transverse shear stress $\tau_{sn}$ on the interface between the core and bottom facing, it remains constant at 17 Pa along the entire length. There is considerable deviation between the analytical and FEA values of $\tau_{sn}$ near the mid-span due to the abrupt change in taper angle.

The axial variation of the longitudinal stress $\sigma_n$ at the bottom-most surface of the lower facing is depicted in Figure 5.5(c) for $\phi = 15^\circ$. The maximum value of the tensile stress $\sigma_n$ from the analytical solution is 970 Pa, whereas the finite element solution reaches a peak of 1,400 Pa slightly to the right of mid-span. The transverse shear stresses increases from 11 Pa at $x = 0$ to 20 Pa at $x = L$. 

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Figure 5.5 Comparison of analytical and FEA ($H = 60$ mm, $L = 45$ mm) normal stress at bottom surface of the lower facing, $\sigma_{nt}$ and shear stress at the interface between bottom facing and core, $\tau_{sn}$, (a,b) $\phi = 0^\circ$, (c,d) $\phi = 15^\circ$ and (e,f) $\phi = 30^\circ$. 
For $\phi = 30^\circ$, the core thickness at the right edge is only 8.06 mm compared to 60 mm at the mid-span. Results for this taper angle are given in Figure 5.5 (e) and (f). The normal stress $\sigma_{ss}$ at the bottom surface remains nearly uniform, except near the right edge where it exhibits a large gradient. The analytical solution deviates from the FEA solution near the right edge for large taper angles. As the taper angle increases, the core thickness at the right edge decreases and the transverse shear stress $\tau_{sn}$ increases. Therefore, the plane stress condition $\tau_{sn} = 0$ is a poor assumption for the facings near the right edge for large $\phi$. The large shear stress could cause delamination to begin at the interface between the core and facing at the right edge for large taper angles and lead to the debonding of the facings. The delamination of the facings has been observed experimentally by Kuczma and Vizzini (1999).

![Diagram](image-url)

**Figure 5.6** Analytical solution of case with ($H = 60$ mm, $L = 45$ mm, $t = 0.8$ mm) a) normal stresses at the bottom surface of the lower facing, $\sigma_{ss}$ and b) shear stress at the interface between the bottom facing and core, $\tau_{sn}$ versus taper angles at three locations.
The normal stress $\sigma_n$ at three different locations on the bottom surface versus the taper angle is depicted in Figure 5.6a. It shows that the normal stress monotonically increases as the taper angle becomes larger. This is because of the decreasing core thickness and increasing participation of the facings in resisting the transverse shear force. The shear stress between the core and bottom facing versus tapered angles is depicted in Figure 5.6b. Interestingly, the interface shear stresses $\tau_{sn}$ at $x/L = 0.25$ and 0.5 decreases as the taper angle increases. However $\tau_{sn}$ at $x/L = 0.75$ initially increases and reaches a maximum at $\phi = 25^0$ before decreasing.

5.2.2.2. Deflection

The mid-span deflections are investigated as the tapered angle varies from $0^0$ to $30^0$, as shown in Figure 5.7. It helps us to understand how the taper angle affects the overall stiffness of the structure. It should be noted that the core depth at the mid-span is kept constant as the taper angle is increased. The deflection predicted by the tapered sandwich theory is roughly 5% larger than the FEA solution for small taper angles. Numerical results for the deflection from the analytical solution and FEA are given in Table 5.3. As the taper angle increases, the deflection initially decreases despite the fact that the core depth has decreased at locations other than the mid-span. It should be noted that the deflection of a sandwich structure is the sum of the flexural deformation in the facings and the shear deformation in the core. In a tapered sandwich member, the facings participate in resisting the transverse shear force, thereby reducing the transverse shear load in the core. This results in an overall reduction in the transverse shear deformation of the core. The decrease in shear deformation for small taper angles is larger than the
associated increase in flexural deformation. For the parameters used in this study, an optimum taper angle of approximately 20° exists where the deflections become a minimum. Beyond this taper angle, the deflection starts to increase as a result of the dramatic increase in flexural deformation caused by the significant reduction in the vertical distance between the facings.

![Figure 5.7 Comparison of analytical and FEA deflection for various taper angles $\phi$ ($H = 60$ mm, $L = 45$ mm and $t = 0.8$ mm)](image)

Table 5.3 Deflections of mid-span $u_x(0,0)/P$ ($10^{-9}$ m/N)

<table>
<thead>
<tr>
<th>Angle $\phi$</th>
<th>Analytical</th>
<th>FEM</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>-1.81</td>
<td>-1.72</td>
<td>4.93%</td>
</tr>
<tr>
<td>$5^\circ$</td>
<td>-1.74</td>
<td>-1.65</td>
<td>4.86%</td>
</tr>
<tr>
<td>$10^\circ$</td>
<td>-1.67</td>
<td>-1.59</td>
<td>4.78%</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>-1.62</td>
<td>-1.54</td>
<td>4.73%</td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>-1.59</td>
<td>-1.51</td>
<td>4.78%</td>
</tr>
<tr>
<td>$25^\circ$</td>
<td>-1.61</td>
<td>-1.52</td>
<td>5.26%</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>-1.75</td>
<td>-1.61</td>
<td>8.03%</td>
</tr>
</tbody>
</table>
5.3. Symmetric Sandwich Members with Laminated Anisotropic Facings

In the case studied next, the facings of sandwich composite are made of 6 layers graphite/epoxy fiber-reinforced laminae. The sandwich member is symmetric about the mid-surface and the orientation of the fibers in the top and bottom facings are \([0/90/0_2]\). The material properties of each lamina in the principal material coordinates are shown in Table 5.4. The orthotropic elastic constants shown in Table 5.5 are used to define the material properties in ABAQUS for the \(0^0\) and \(90^0\) laminae, respectively. Here \(D_{ijkl}\) are the components of the fourth-order elastic tensor. The thickness of each lamina is 0.15 mm and the thickness of the laminated facing is 0.9 mm. The core is made of Hexcel HRP-3/16-5.5 Fiberglass/Phenolic honeycomb core with the elastic properties shown in Table 5.6.

<table>
<thead>
<tr>
<th>Table 5.4 Facing lamina properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E1)</td>
</tr>
<tr>
<td>(GPa)</td>
</tr>
<tr>
<td>155</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5.5 Orthotropic material properties by the terms in the elastic stiffness matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_{1111})</td>
</tr>
<tr>
<td>(GPa)</td>
</tr>
<tr>
<td>Ply 0</td>
</tr>
<tr>
<td>Ply 90</td>
</tr>
</tbody>
</table>

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Table 5.6 Honeycomb core properties

<table>
<thead>
<tr>
<th></th>
<th>Hexcel HRP-3/16-5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>L-direction Plate Shear</td>
<td>Modulus (MPa)</td>
</tr>
<tr>
<td></td>
<td>Strength (MPa)</td>
</tr>
<tr>
<td>W-direction Plate Shear</td>
<td>Modulus (MPa)</td>
</tr>
<tr>
<td></td>
<td>Strength (MPa)</td>
</tr>
</tbody>
</table>

The geometric configuration of the symmetric tapered sandwich composite beam is depicted in Figure 5.1. The taper angle is varied from 0 to 20 degree. The tapered member is simply supported at both edges, the span is $2L = 160$ mm, the core depth at mid-span is $H = 60$ mm and the magnitude of the concentrated load is $2P = 2$ N. The

Figure 5.8 Model of tapered sandwich construction with Gr/Ep facings ($H = 60$ mm, $L = 80$ mm, $t = 0.9$ mm and $\phi = 20^\circ$)
finite element model contains 29,600 8-noded quadratic elements. The finite element mesh used in this case is presented in Figure 5.8. A total of 24 elements are used through the thickness of each facing, with four elements through the thickness of each lamina.

5.3.1. Stresses

When the tapered angle $\phi = 10^0$, the axial variation of the longitudinal stress $\sigma_{zz}$ in the facing at the bottom-most surface is depicted in Figure 5.9(a) and the interface shear stress $\tau_{zn}$ between the core and bottom facing versus normalized distance is shown in Figure 5.9(b). The maximum value the normal stress $\sigma_{zz}$ predicted by the analytical solution is approximately 2,200 Pa at the mid-span, while the FE solution reaches a peak value of over 3,000 Pa at $x = 0.1L$.

The analytical value of the transverse shear stresses at the interface between the core and the bottom facing increase from approximately 10 Pa at $x = 0$ to over 30 Pa at $x = L$. 

Figure 5.9 Comparison of analytical and FEA ($H = 60\text{mm}$, $L = 80\text{mm}$, $t = 0.9\text{mm}$, $\phi = 10^0$) a) normal stresses at the bottom surface of the lower facing, $\sigma_{zz}$ b) shear stress at the interface between the bottom facing and core, $\tau_{zn}$. 

5.3.1. Stresses

When the tapered angle $\phi = 10^0$, the axial variation of the longitudinal stress $\sigma_{zz}$ in the facing at the bottom-most surface is depicted in Figure 5.9(a) and the interface shear stress $\tau_{zn}$ between the core and bottom facing versus normalized distance is shown in Figure 5.9(b). The maximum value the normal stress $\sigma_{zz}$ predicted by the analytical solution is approximately 2,200 Pa at the mid-span, while the FE solution reaches a peak value of over 3,000 Pa at $x = 0.1L$.

The analytical value of the transverse shear stresses at the interface between the core and the bottom facing increase from approximately 10 Pa at $x = 0$ to over 30 Pa at $x = L$. 

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However, the FEA solution exhibits a very large negative shear stresses at \( x = 0 \), which can be attributed to the discontinuity in the taper angle at mid-span.

![Graph showing comparison of analytical and FEA normal stresses at interface](image)

Figure 5.10 Comparison of analytical and FEA: normal stresses at the interface between the bottom facing and core, \( \sigma_{nn} \). (\( H = 60\text{mm}, L = 80\text{mm}, \phi = 20^\circ \))

The transverse normal stress at the interface between the bottom facing and core, \( \sigma_{nn} \), versus normalized distance is depicted in Figure 5.10. It shows that the transverse normal stress is localized at the edges and it is negligible everywhere else. The large transverse normal stress near the edge could cause the facing to peel off from the core, especially at the interface between the top facing and the core, where it is tensile.

The axial variations of the longitudinal stress \( \sigma_{xx} \) and the shear stress \( \tau_{sm} \) are depicted in Figure 5.11 (a) and (b), respectively for taper angle \( \phi = 20^\circ \). Note that core depth at \( x = L \) is only 8.06 mm. The normal stress \( \sigma_{xx} \) is approximately 2,200 Pa at \( x = 0 \) and remains constant along the span except near the edges. It drops dramatically at \( x = 0.8L \) to 0 Pa at the right edge. There is considerable deviation between the analytical and FEA values of the normal stress \( \sigma_{xx} \) near the edges. A reason for the discrepancy is that the plane stress assumption for the facing laminae is not valid near the right edge due to
Figure 5.11 Comparison of analytical and FEA ($H = 60\text{mm}, L = 80\text{mm}, \phi = 20^\circ$)

(a) normal stresses at the bottom surface of the lower facing, $\sigma_{nn}$
(b) shear stress at the interface between the bottom facing and core, $\tau_{in}$.

the large interfacial shear stress $\tau_{in}$. The interface shear stress remains very small for points between $x = 0$ to $x = 0.8 L$, and increases dramatically after that to reach 500 Pa at the right edge. Comparing Figure 5.9b and Fig 5.11b, we see that the transverse shear stress at the interface for $\phi = 20^\circ$ is more than an order of magnitude larger than that for $\phi = 10^\circ$.

5.3.2. Deflection

The mid-span deflection versus the taper angle is shown in Figure 5.12 as the taper angle varies from $0^\circ$ to $20^\circ$. As the taper angle increases from $0^\circ$ to $20^\circ$, the analytical solution predicts that the deflection decreases due to the participation of the facings in resisting transverse shear load. However, the FEA predicts that beyond $20^\circ$, the reduction in cross-sectional height leads to a dramatic increase in the deflection. The inability of the analytical solution to capture the increase in deformation can be attributed to the complex state of stress near the ends where the core depth is very small. Overall,
deflection from the analytical solution is about 5% lower than the FEA solution for small
taper angles.

![Graph](image)

Figure 5.12 Comparison of analytical and FEA deflection for various
taper angles $\phi$ ($H = 60$ mm, $L = 80$ mm and $t = 0.9$ mm)

### 5.4. Graphite/Epoxy Facing Unsymmetric Cases

#### 5.4.1. Various Taper Angles

In practical applications, the unsymmetric tapered sandwich members are more
often used than symmetric ones, such as the case shown in Figure 5.13. In order to
understand the behavior of tapered unsymmetric cases, the inclination of the bottom
facing $\phi = \phi_b$ is varied from 0, 15, 30 and 45 degree while the top facing is kept
horizontal. The top surface of the honeycomb core is chosen as the reference surface $z = 0$. The facings are composed of 6 layers of graphite/epoxy laminate fiber-reinforced
composite with fiber orientations $[0_2/90_2/0_2]$. The elastic properties of the $0^0$ and $90^0$
Figure 5.13 Schematic representation of unsymmetric tapered sandwich beam laminae are given in Table 5.5. The thickness of each lamina is 0.15 mm and the core is made of Hexcel HRP-3/16-5.5 Fiberglass/Phenolic honeycomb core with elastic properties shown in Table 5.6. The span is $2L = 160$ mm long and the core depth at mid-span is $H = 85$ mm. The finite element model used in this case is presented in Figure 5.14. The finite element mesh consists of 29,600 8-noded quadratic elements with four elements through the thickness of each lamina.

Figure 5.14 Model of tapered sandwich construction with aluminum facings, $(H = 85$ mm, $L = 80$ mm, $t = 0.9$ mm, $P = 1$ N and $\phi = 30^\circ$)
Figure 5.15 Comparison of analytical and FEA a) normal stresses at the bottom surface of the lower facing, $\sigma_n$, b) shear stress at the interface between the bottom facing and core, $\tau_{sn}$. ($H = 85$ mm, $L = 80$ mm, $P = 1$N and $\phi = 30^0$)

Extensional stresses on the bottom surface and transverse shear stresses on the interface between the bottom facing and core are reported in this section. Note that all the stresses are presented in the local $s-y-n$ coordinate system. The axial variations of the stresses are depicted in Figure 5.15 for $\phi = 30^0$. The trends are qualitatively similar to the symmetric tapered sandwich members discussed earlier.
5.4.2. Core Thickness

In order to understand effects of the core thickness, cases with $L = 80$ mm and $\phi = 30^0$ are studied. The ratio of thickness of core at the right end to facing thickness is varied from 3 to 50.

The deflection comparison of FEA and analytical solution is shown in Figure 5.16. The deflection decreases dramatically as the core becomes thicker. The analytical solution overestimates the deflection when the core is very thin.

The analytical solutions predict that when the core depth at the right end is small, the shear stress and peeling stress between the core and facing are large, as shown in Figure 5.17. As the core becomes thicker, the shear stress and peeling stress drop dramatically. This is also the reason why the failures always happen to the cross section where the core is very thin. Note that the numbers in the legend in Figure 5.17 represent the ratio of thickness of core at the right end to facing thickness.

Figure 5.16 Comparison of analytical and FEA deflection for core thickness ($L = 80$ mm, $t = 0.9$ mm, and $\phi = 30^0$ mm)
Figure 5.17 Shear stress and peeling Stress between core and facing
\((H = 85 \text{ mm}, L = 80 \text{ mm and } \phi = 30^\circ)\)
5.4.3. Core Materials

In order to understand effects of the core material, cases with \( H = 85 \) mm, \( L = 80 \) mm and \( \phi = 30^0 \) are studied. Various core materials are selected, as shown in Table 5.7.

The deflection decreases as the shear stiffness of the cores increases, as shown in Figure 5.18. Comparison between FEA and analytical solution is very good.

Table 5.7 Core material properties

<table>
<thead>
<tr>
<th>Compression L direction</th>
<th>Plate shear W direction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ksi</td>
</tr>
<tr>
<td>HRP - 3/8 - 2.2</td>
<td>13</td>
</tr>
<tr>
<td>HRP - 3/8 - 3.2</td>
<td>38</td>
</tr>
<tr>
<td>HRP - 3/16 - 4.0</td>
<td>57</td>
</tr>
<tr>
<td>HRP - 3/16 - 5.5</td>
<td>95</td>
</tr>
<tr>
<td>HRP - 3/16 - 7.0</td>
<td>136</td>
</tr>
<tr>
<td>HRP - 3/16 - 12.0</td>
<td>260</td>
</tr>
</tbody>
</table>

Figure 5.18 Comparison of analytical and FEA deflection for various core materials (\( H = 60 \) mm, \( L = 80 \) mm and \( \phi = 30^0 \) mm)
The shear stress between the core and facing is shown in Figure 5.19. The shear stresses are the same for different core shear stiffnesses. Therefore, the core material has no influence on the shear stress at the interface between the core and facing.

5.4.4. Three-dimensional Continuum Finite Element Model

In order to verify stress distribution of the sandwich construction with the angle-ply laminate in a state of plane stress condition in the y-direction, a three-dimensional model was built, as shown in Figure 5.20. The top and bottom facings are made of six-layer \([45^\circ/-45^\circ/30^\circ/-30^\circ/45^\circ/-45^\circ]\) and \([-45^\circ/45^\circ/-30^\circ/30^\circ/-45^\circ/45^\circ]\) Gr/Ep laminates, respectively. The top and bottom facings are symmetric. The FE model contains 28,160 20-node quadratic hexahedral elements, which reaches the maximum capability of the workstation used for the analysis. Since the sandwich construction is relatively thin, a state of plane stress in y-direction is assumed for the analytical model. The stressed at location A and B would be compared with analytical solutions, as denoted in Table 5.8. Note that point A lies at the bottom surface on the lower facing and point B lies at the interface between the bottom facing and core when \(x = L/2\).
Figure 5.20 3D Continuum model of tapered sandwich construction with angle ply laminate facings ($H = 85$ mm, $L = 80$ mm, $W = 5$ mm and $\phi = 30^\circ$)

Table 5.8 Comparison of FEA and analytical solutions

<table>
<thead>
<tr>
<th>Location A</th>
<th>Location B</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEA</td>
<td>Analytical</td>
</tr>
<tr>
<td>$\sigma_{ss}$</td>
<td>366.0</td>
</tr>
<tr>
<td>$\sigma_{yy}$</td>
<td>-145.0</td>
</tr>
<tr>
<td>$\tau_{xy}$</td>
<td>85.2</td>
</tr>
</tbody>
</table>

Results demonstrated in Table 5.8 show that the FEA and analytical solutions match fairly well at the Point A. However there is some error for the shear stress $\tau_{xy}$ at the point B. There are two reasons which can contribute to this error. First, the FE model is three-dimensional whereas the analytical model is one-dimensional. Secondly, the three-
dimensional FE model needs meshes along the \( y \) direction, therefore the mesh density on \( x-z \) plane is much sparser than the two-dimensional FE models discussed earlier.
Chapter 6

SUMMARY AND CONCLUSIONS

In this thesis, we have developed a tapered sandwich theory in which the force and moment resultants are related to the reference surface strains and curvatures through the familiar $[A]$, $[B]$ and $[D]$ matrices. Since the analysis of tapered sandwich panels could be cast in this standard notation by our formulation, computer codes that have been developed for the analysis and design of composite structures can be modified to incorporate our taper sandwich theory. Our formulation reduces the dimensionality of the analysis and therefore simplifies the analysis of complex sandwich structures.

We also have systematically derived a total of 12 elastic stiffnesses that couple the force and moment resultants to the transverse shear deformation. Six of the twelve elastic couplings are due to the tapered sandwich construction itself, irrespective of whether the facings are isotropic or anisotropic, whereas the remaining six elastic couplings are present only for anisotropic laminated facings. Their influence on the behavior of tapered sandwich beams of the stiffnesses is investigated. It is shown that the extension and bending stiffness could be negative at large taper angles.

The deflection of a symmetric simply supported member initially decreases for the increasing taper angle and there is an optimum taper angle where the deflection is a minimum. This decrease in deformation with increasing taper angle is due to the participation of the facings in resisting transverse shear loads.

Results from the tapered sandwich theory show good comparison with finite element models for several case studies. The deflections are overestimated by the theory.
This is because the facing transverse shear stiffnesses $G_{13}$ and $G_{23}$ have been neglected. The error could also be caused by the concentrated point load which leads to thickness distention of the sandwich structure. While we have compared the analytical deflection to the transverse displacement at the center of the structure, better results could have been obtained for other locations.
REFERENCES

ABAQUS Online Documents, 2002.


Erickson, W. S., 1956, Supplement of Effects of Shear Deformation in the Core of a Flat Rectangular Sandwich Panel, *Forest Products Laboratory Report 1583-c*.


Hoff, N. J., 1950, Bending and Buckling of Rectangular Sandwich Plates, *NACA Technical Note 2225*.


**APPENDIX**

**MATHCAD Program**

**Simplified Formulation**

---

**Number of layers Nb at the bottom of Core**

\[ \text{Nb} := 6 \]

**Tapered angle of bottom facesheet**

\[ \phi_b := 10 \degree \]

**Thickness of each layer**

\[ h := 0.15 \cdot 10^{-3} \]

**Length of the composite beam**

\[ L := 80 \cdot 10^{-3} \]

**Material properties of each layer**

\[ \begin{align*}
E_1 & := 155 \cdot 10^9 \\
E_2 & := 12.1 \cdot 10^9 \\
E_3 & := E_2 \\
v_{23} & := 0.458 \\
v_{12} & := 0.248 \\
v_{13} & := v_{12} \\
G_{12} & := 4.4 \cdot 10^9 \\
G_{23} & := 3.2 \cdot 10^9 \\
G_{31} & := 4.4 \cdot 10^9
\end{align*} \]

**Material properties of the core**

\[ \begin{align*}
G_{e13} & := 131 \cdot 10^6 \\
G_{e23} & := 76 \cdot 10^6
\end{align*} \]

**Lamina fiber orientations in degrees from bottom to top,**

\[ \theta := (0 \ 0 \ 90 \ 90 \ 0 \ 0 \ 0 \ 0 \ 90 \ 90 \ 0 \ 0) \ T \cdot \text{deg} \]

**Location of the top and bottom surfaces of the core at the left edge**

\[ \begin{align*}
t_{c\text{Top}} & := 30 \cdot 10^{-3} \\
t_{c\text{Bottom}} & := 30 \cdot 10^{-3}
\end{align*} \]

**The applied loads for plane strain case**

\[ \begin{align*}
P & := 1 \\
M_x(x) & := -P \cdot (L - x) \\
Q_x(x) & := \frac{d}{dx} M_x(x)
\end{align*} \]
Reorder the Lamina fiber orientation in degree

$$\theta := \text{augment}(0, \theta^T)^T$$

$\theta^T = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$ deg

**Total Number of layers**

$$N := Nb + Nt + 1$$

$$Nc := Nb + 1$$

$$N = 13 \quad Nc = 7$$

**Location of the top surface, the interfaces between adjoining layers and the bottom surface**

$$H := \frac{h \cdot Nb}{\cos(\phi_b)} + hc + \frac{h \cdot Nt}{\cos(\phi_t)}$$

$$H = 0.0618 \quad \text{Total thickness on the left end}$$

$$Z(n, x) := \begin{cases} -tcBottom - \frac{h(Nt - n)}{\cos(\phi_b)} + x \cdot \tan(\phi_b) & \text{if } n < Nc \\ tcTop + \frac{h(n - Nc)}{\cos(\phi_t)} + x \cdot \tan(\phi_t) & \text{if } n \geq Nc \end{cases}$$

**Height of the core at the right end**

$$He_{\text{right}} := Z(Nc, L) - Z(Nb, L)$$

$$He_{\text{right}} = 0.0317877 \quad Z(Nb, L) = -0.0159$$

**Define the tapered angle for the different layers**

$$k := 0 \ldots N$$

$$\phi_k := \begin{cases} \phi_b & \text{if } k < Nc \\ 0 & \text{if } k = Nc \\ \phi_t & \text{if } k > Nc \end{cases}$$

$$\phi^T = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix} \text{ deg}$$
The compliant and stiffness matrix in the local system, 1.2.3 system

\[
S := \begin{pmatrix}
1 & -v_{12} & -v_{13} \\
\frac{-v_{12}}{E_1} & 1 & -v_{23} \\
\frac{-v_{13}}{E_1} & \frac{-v_{23}}{E_2} & 1 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{G_{23}} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{G_{12}}
\end{pmatrix}
\]

\[C_{123} := S^{-1}\]

The stiffness matrix in the local s-y-n system

\[m := \cos(\theta) \quad n := \sin(\theta)\]

\[k := 1..N \quad \text{k is a range variable from 1 to N}\]

\[
T_k := \begin{pmatrix}
(m_k)^2 & (n_k)^2 & 0 & 0 & 0 & -2m_k \cdot n_k \\
(n_k)^2 & (m_k)^2 & 0 & 0 & 0 & 2m_k \cdot n_k \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_k & n_k & 0 \\
0 & 0 & 0 & -n_k & m_k & 0 \\
m_k \cdot n_k & -m_k \cdot n_k & 0 & 0 & 0 & (m_k)^2 - (n_k)^2
\end{pmatrix}
\]

\[C_k := T_k \cdot C_{123} \cdot T_k^T \quad C_0 := C_1\]

\[k := 0..N\]

\[C_{11_k} := (C_k)_{0,0} \quad C_{12_k} := (C_k)_{0,1} \quad C_{13_k} := (C_k)_{0,2} \quad C_{16_k} := (C_k)_{0,5}\]
Reduced stiffnesses for a tapered facesheet lamina

\[ k := 1..N \]

\[ D_{bk} := C_{13k} \cdot \sin(\phi_k)^2 - C_{33k} \cdot \cos(\phi_k)^2 \quad D_{b0} := D_{b1} \]

\[ Q_{11bk} := \left( (C_{13k})^2 - C_{11k} C_{33k} \right) \cdot \frac{\cos(\phi_k)^2}{D_{bk}} \]

\[ Q_{12bk} := \left( (C_{13k} C_{23k} - C_{12k} C_{33k}) \cdot \cos(\phi_k)^2 + (C_{12k} C_{13k} - C_{11k} C_{23k}) \cdot \sin(\phi_k)^2 \right) \cdot \cos(\phi_k)^2 \]

\[ Q_{16bk} := \left( (C_{13k} C_{36k} - C_{16k} C_{33k}) \cdot \cos(\phi_k)^2 + (C_{13k} C_{16k} - C_{11k} C_{36k}) \cdot \sin(\phi_k)^2 \right) \cdot \cos(\phi_k) \]

\[ Q_{21bk} := \frac{(C_{13k} C_{23k} - C_{12k} C_{33k})}{D_{bk}} \]

\[ Q_{22bk} := \left( (C_{23k})^2 - C_{22k} C_{33k} \right) \cdot \frac{\cos(\phi_k)^2}{D_{bk}} + (C_{13k} C_{22k} - C_{12k} C_{23k}) \cdot \sin(\phi_k)^2 \]

\[ Q_{26bk} := \frac{(C_{23k} C_{36k} - C_{26k} C_{33k}) \cdot \cos(\phi_k)^2 + (C_{13k} C_{26k} - C_{12k} C_{36k}) \cdot \sin(\phi_k)^2}{D_{bk} \cdot \cos(\phi_k)} \]

\[ Q_{31bk} := \left( (C_{13k})^2 - C_{11k} C_{33k} \right) \cdot \frac{\sin(\phi_k)^2}{D_{bk}} \]

\[ Q_{32bk} := \left( (C_{13k} C_{23k} - C_{12k} C_{33k}) \cdot \cos(\phi_k)^2 + (C_{12k} C_{13k} - C_{11k} C_{23k}) \cdot \sin(\phi_k)^2 \right) \cdot \sin(\phi_k)^2 \]

\[ Q_{36bk} := \left( (C_{13k} C_{36k} - C_{16k} C_{33k}) \cdot \cos(\phi_k)^2 + (C_{13k} C_{16k} - C_{11k} C_{36k}) \cdot \sin(\phi_k)^2 \right) \cdot \sin(\phi_k)^2 \]

\[ Q_{41bk} := \frac{(C_{16k} C_{33k} - C_{13k} C_{36k}) \cdot \sin(\phi_k)}{D_{bk}} \]
\[ Q_{42b_k} := \frac{(C_{23k} \cdot C_{36k} - C_{26k} \cdot C_{33k}) \cdot \cos(\phi_k)^2 \cdot \sin(\phi_k) + (C_{13k} \cdot C_{26k} - C_{16k} \cdot C_{23k}) \cdot \sin(\phi_k)^3}{D_b_k} \]

\[ Q_{46b_k} := \frac{((C_{36k})^2 - C_{33k} \cdot C_{66k}) \cdot \cos(\phi_k)^2 \cdot \sin(\phi_k) + (C_{13k} \cdot C_{66k} - C_{16k} \cdot C_{36k}) \cdot \sin(\phi_k)^3}{D_b_k \cdot \cos(\phi_k)} \]

\[ Q_{51b_k} := \frac{((C_{13k})^2 - C_{11k} \cdot C_{33k}) \cdot \cos(\phi_k) \cdot \sin(\phi_k)}{D_b_k} \]

\[ Q_{52b_k} := \frac{(C_{13k} \cdot C_{23k} - C_{12k} \cdot C_{33k}) \cdot \cos(\phi_k)^3 \cdot \sin(\phi_k) + (C_{12k} \cdot C_{13k} - C_{11k} \cdot C_{23k}) \cdot \sin(\phi_k)^3 \cdot \cos(\phi_k)}{D_b_k} \]

\[ Q_{56b_k} := \frac{(C_{13k} \cdot C_{36k} - C_{16k} \cdot C_{33k}) \cdot \cos(\phi_k)^2 \cdot \sin(\phi_k) + (C_{13k} \cdot C_{16k} - C_{11k} \cdot C_{36k}) \cdot \sin(\phi_k)^3}{D_b_k} \]

\[ Q_{61b_k} := \frac{(C_{13k} \cdot C_{36k} - C_{16k} \cdot C_{33k}) \cdot \cos(\phi_k)}{D_b_k} \]

\[ Q_{62b_k} := \frac{(C_{23k} \cdot C_{36k} - C_{26k} \cdot C_{33k}) \cdot \cos(\phi_k)^3 + (C_{13k} \cdot C_{26k} - C_{16k} \cdot C_{23k}) \cdot \sin(\phi_k)^2 \cdot \cos(\phi_k)}{D_b_k} \]

\[ Q_{66b_k} := \frac{((C_{36k})^2 - C_{33k} \cdot C_{66k}) \cdot \cos(\phi_k)^2 + (C_{13k} \cdot C_{66k} - C_{16k} \cdot C_{36k}) \cdot \sin(\phi_k)^2}{D_b_k} \]

\[ Q_{11b_{Ne}} := 0 \quad Q_{21b_{Ne}} := 0 \quad Q_{41b_{Ne}} := 0 \quad Q_{51b_{Ne}} := 0 \quad Q_{61b_{Ne}} := 0 \]

\[ Q_{12b_{Ne}} := 0 \quad Q_{22b_{Ne}} := 0 \quad Q_{42b_{Ne}} := 0 \quad Q_{52b_{Ne}} := 0 \quad Q_{62b_{Ne}} := 0 \]

\[ Q_{16b_{Ne}} := 0 \quad Q_{26b_{Ne}} := 0 \quad Q_{46b_{Ne}} := 0 \quad Q_{56b_{Ne}} := 0 \quad Q_{66b_{Ne}} := 0 \]

\[ A_{11}(x) := \sum_{n=1}^{N} Q_{11b_{h}}(Z(n, x) - Z(n - 1, x)) \]

\[ A_{12}(x) := \sum_{n=1}^{N} Q_{12b_{h}}(Z(n, x) - Z(n - 1, x)) \]

\[ A_{16}(x) := \sum_{n=1}^{N} Q_{16b_{h}}(Z(n, x) - Z(n - 1, x)) \]

\[ A_{21}(x) := \sum_{n=1}^{N} Q_{21b_{h}}(Z(n, x) - Z(n - 1, x)) \]

\[ A_{22}(x) := \sum_{n=1}^{N} Q_{22b_{h}}(Z(n, x) - Z(n - 1, x)) \]

\[ A_{26}(x) := \sum_{n=1}^{N} Q_{26b_{h}}(Z(n, x) - Z(n - 1, x)) \]
\[ A41(x) := \sum_{n=1}^{N} Q41_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A51(x) := \sum_{n=1}^{N} Q51_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A42(x) := \sum_{n=1}^{N} Q42_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A52(x) := \sum_{n=1}^{N} Q52_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A46(x) := \sum_{n=1}^{N} Q46_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A56(x) := \sum_{n=1}^{N} Q56_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A61(x) := \sum_{n=1}^{N} Q61_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A44(x) := G_{23}(Z(Nc,x) - Z(Nb,x)) \]
\[ A55(x) := G_{13}(Z(Nb + 1,x) - Z(Nb,x)) \]
\[ A62(x) := \sum_{n=1}^{N} Q62_{bn}(Z(n,x) - Z(n-1,x)) \]
\[ A66(x) := \sum_{n=1}^{N} Q66_{bn}(Z(n,x) - Z(n-1,x)) \]

\[ B11(x) := \frac{1}{2} \sum_{n=1}^{N} Q11_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B21(x) := \frac{1}{2} \sum_{n=1}^{N} Q21_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B12(x) := \frac{1}{2} \sum_{n=1}^{N} Q12_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B22(x) := \frac{1}{2} \sum_{n=1}^{N} Q22_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B16(x) := \frac{1}{2} \sum_{n=1}^{N} Q16_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B26(x) := \frac{1}{2} \sum_{n=1}^{N} Q26_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B41(x) := \frac{1}{2} \sum_{n=1}^{N} Q41_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B51(x) := \frac{1}{2} \sum_{n=1}^{N} Q51_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B42(x) := \frac{1}{2} \sum_{n=1}^{N} Q42_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B52(x) := \frac{1}{2} \sum_{n=1}^{N} Q52_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B46(x) := \frac{1}{2} \sum_{n=1}^{N} Q46_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B56(x) := \frac{1}{2} \sum_{n=1}^{N} Q56_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B61(x) := \frac{1}{2} \sum_{n=1}^{N} Q61_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
\[ B66(x) := \frac{1}{2} \sum_{n=1}^{N} Q66_{bn}(Z(n,x)^2 - Z(n-1,x)^2) \]
The midsurface strains and curvatures

\[ \text{ABD matrices} \]

\[
\text{ABD(x)} := \begin{pmatrix}
A11(x) & A12(x) & 0 & 0 & A16(x) & B11(x) & B12(x) & B16(x) \\
A21(x) & A22(x) & 0 & 0 & A26(x) & B21(x) & B22(x) & B26(x) \\
A41(x) & A42(x) & A44(x) & 0 & A46(x) & B41(x) & B42(x) & B46(x) \\
A51(x) & A52(x) & 0 & A55(x) & A56(x) & B51(x) & B52(x) & B56(x) \\
A61(x) & A62(x) & 0 & 0 & A66(x) & B61(x) & B62(x) & B66(x) \\
B11(x) & B12(x) & 0 & 0 & B16(x) & D11(x) & D12(x) & D16(x) \\
B21(x) & B22(x) & 0 & 0 & B26(x) & D21(x) & D22(x) & D26(x) \\
B61(x) & B62(x) & 0 & 0 & B66(x) & D61(x) & D62(x) & D66(x)
\end{pmatrix}
\]

\[ \text{ABDinvers(x)} := \text{ABD(x)}^{-1} \]

\[ \text{Solve this equation blocks for } Qy(x) \text{ and } My(x) \]

\[ Qy(x) := \begin{pmatrix}
\text{ABDinvers(x)}_{1,1} & \text{ABDinvers(x)}_{1,6}
\end{pmatrix}
\begin{pmatrix}
\text{ABDinvers(x)}_{6,0}
\end{pmatrix} \text{Mx(x)} \]
\[ \begin{align*}
\sigma_{xx}(x,z) &= \sigma_{xx}(x) + z \sigma_{zz}(x), \\
\sigma_{yy}(x,z) &= \sigma_{yy}(x) + z \sigma_{zz}(x), \\
\tau_{xy}(x,z) &= \tau_{xy}(x) + z \tau_{yx}(x). 
\end{align*} \]

**The strains**

\[ \begin{align*}
\varepsilon_{xx}(x) &= \sigma_{xx}(x)/E, \\
\varepsilon_{yy}(x) &= \sigma_{yy}(x)/E, \\
\gamma_{xy}(x) &= \tau_{xy}(x)/G.
\end{align*} \]

Kirchhoff Hypothesis

**The strains on the top and bottom surfaces**

\[ \begin{align*}
I_{\varepsilon_{xx}(n,x)} &= \varepsilon_{xx}(x,Z(n,x)), \\
I_{\varepsilon_{yy}(n,x)} &= \varepsilon_{yy}(x,Z(n,x)), \\
I_{\gamma_{xy}(n,x)} &= \gamma_{xy}(x,Z(n,x)).
\end{align*} \]

\[ \begin{align*}
I_{\varepsilon_{zz}(n,x)} &= \frac{C13_n \cos(\phi_n)^2 - C33_n \sin(\phi_n)^2}{Db_n} - I_{\varepsilon_{xx}(n,x)} + \frac{C23_n (\cos(\phi_n)^2 - \sin(\phi_n)^2)}{Db_n} \cdot I_{\varepsilon_{yy}(n,x)}, \\
I_{\varepsilon_{zz}(n,x)} &= I_{\varepsilon_{zz}(n,x)} + \frac{C36_n (\cos(\phi_n)^2 - \sin(\phi_n)^2)}{Db_n} \cdot I_{\gamma_{xy}(n,x)}, \\
I_{\gamma_{xz}(n,x)} &= -2 \left( \frac{C13_n + C33_n}{Db_n} \cos(\phi_n) \sin(\phi_n) \right) \cdot I_{\varepsilon_{xx}(n,x)} - 2 \left( \frac{C23_n}{Db_n} \cos(\phi_n) \sin(\phi_n) \right) \cdot I_{\varepsilon_{yy}(n,x)}, \\
I_{\gamma_{yz}(n,x)} &= I_{\gamma_{xz}(n,x)} - \frac{2(C36_n) \sin(\phi_n)}{Db_n} \cdot I_{\gamma_{xy}(n,x)}, \\
I_{\gamma_{yy}(n,x)} &= \tan(\phi_n) \cdot I_{\gamma_{xy}(n,x)}.
\end{align*} \]
The stresses in layer number \( k \)

\[
R(\alpha) := \begin{pmatrix}
\cos(\alpha)^2 & 0 & \sin(\alpha)^2 & 0 & 2\cos(\alpha)\cdot\sin(\alpha) & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin(\alpha)^2 & 0 & \cos(\alpha)^2 & 0 & -2\cos(\alpha)\cdot\sin(\alpha) & 0 \\
0 & 0 & 0 & \cos(\alpha) & 0 & -\sin(\alpha) \\
-\cos(\alpha)\cdot\sin(\alpha) & 0 & \cos(\alpha)\cdot\sin(\alpha) & 0 & \cos(\alpha)^2 - \sin(\alpha)^2 & 0 \\
0 & 0 & 0 & \sin(\alpha) & 0 & \cos(\alpha)
\end{pmatrix}
\]

\[
T(\alpha) := \begin{pmatrix}
\cos(\alpha)^2 & 0 & \sin(\alpha)^2 & 0 & -2\cos(\alpha)\cdot\sin(\alpha) & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin(\alpha)^2 & 0 & \cos(\alpha)^2 & 0 & 2\cos(\alpha)\cdot\sin(\alpha) & 0 \\
0 & 0 & 0 & \cos(\alpha) & 0 & \sin(\alpha) \\
\cos(\alpha)\cdot\sin(\alpha) & 0 & -\cos(\alpha)\cdot\sin(\alpha) & 0 & \cos(\alpha)^2 - \sin(\alpha)^2 & 0 \\
0 & 0 & 0 & -\sin(\alpha) & 0 & \cos(\alpha)
\end{pmatrix}
\]

\[
I_{\text{essyn}}(n, x) := T(\phi_n)^T \cdot (I_{\text{esxx}}(n, x) I_{\text{esyy}}(n, x) I_{\text{esz}z(n, x)} I_{\text{eyyz}(n, x)} I_{\text{eyxz}(n, x)} I_{\text{eyxy}(n, x)})^T
\]

\[
I_{\text{esxy}}(n, x) := (I_{\text{esxx}}(n, x) I_{\text{esyy}}(n, x) I_{\text{esz}z(n, x)} I_{\text{eyyz}(n, x)} I_{\text{eyxz}(n, x)} I_{\text{eyxy}(n, x)})^T
\]

\[
I_{\text{esss}}(n, x) := I_{\text{essyn}}(n, x)_0
\]

\[
I_{\text{esyy}}(n, x) := I_{\text{essyn}}(n, x)_1
\]

\[
I_{\text{esnz}}(n, x) := I_{\text{essyn}}(n, x)_2
\]

**The stresses in layer number \( k \)**

\[
Q_k := \begin{pmatrix}
Q_{11b_k} & Q_{12b_k} & Q_{16b_k} \\
Q_{21b_k} & Q_{22b_k} & Q_{26b_k} \\
Q_{31b_k} & Q_{32b_k} & Q_{36b_k} \\
Q_{41b_k} & Q_{42b_k} & Q_{46b_k} \\
Q_{51b_k} & Q_{52b_k} & Q_{56b_k} \\
Q_{61b_k} & Q_{62b_k} & Q_{66b_k}
\end{pmatrix}
\]

\[
Q_0 := Q_1
\]
Stresses in the global coordinate system

\[
\begin{align*}
I_{\sigma xx}(n,x) & := S_{xyz}(n,x) \\
I_{\sigma yy}(n,x) & := S_{xyz}(n,x) \\
I_{\sigma zz}(n,x) & := S_{xyz}(n,x)
\end{align*}
\]

\[
\begin{align*}
I_{\tau yz}(n,x) & := S_{xyz}(n,x) \\
I_{\tau xz}(n,x) & := S_{xyz}(n,x) \\
I_{\tau xy}(n,x) & := S_{xyz}(n,x)
\end{align*}
\]

Normalized strains and stresses at the specific points

\[
\begin{align*}
I_\varepsilon_{xx} \left( 0, \frac{L}{2} \right) & = 8.7343 \times 10^{-9} \\
I_\varepsilon_{ss} \left( 0, \frac{L}{2} \right) & = 9.1087 \times 10^{-9}
\end{align*}
\]

\[
\begin{align*}
I_\varepsilon_{xx} \left( 0, \frac{L}{2} \right) & = 1.3759 \times 10^3 \\
I_\varepsilon_{ss} \left( 0, \frac{L}{2} \right) & = 1.4187 \times 10^3
\end{align*}
\]

\[
\begin{align*}
I_{\sigma yz} \left( 0, \frac{L}{2} \right) & = 242.6055 \\
I_{\sigma yy} \left( 0, \frac{L}{2} \right) & = 27.4653
\end{align*}
\]

\[
\begin{align*}
\sigma_{yz} \left( \frac{L}{2} \right) & = 15.224
\end{align*}
\]

Energy Method

\[
\begin{align*}
U_{\text{bot}}(x) & := \sum_{k = 1}^{N_b} \int_{Z(k-1,x)}^{Z(k,x)} \left( I_{\sigma xx}(k,x) \cdot I_\varepsilon_{xx}(k,x) + I_{\sigma yy}(k,x) \cdot I_\varepsilon_{yy}(k,x) + I_{\tau xy}(k,x) \cdot I_\gamma xy(k,x) \right) \, dx \\
U_{\text{top}}(x) & := \sum_{k = N_c+1}^{N} \int_{Z(k-1,x)}^{Z(k,x)} \left( I_{\sigma xx}(k,x) \cdot I_\varepsilon_{xx}(k,x) + I_{\sigma yy}(k,x) \cdot I_\varepsilon_{yy}(k,x) + I_{\tau xy}(k,x) \cdot I_\gamma xy(k,x) \right) \, dx \\
U_{\text{core}}(x) & := \int_{Z(N_c,x)}^{Z(N_c-1,x)} \sigma_{yz}(x)^2 + \sigma_{y23} \gamma_{yz}(x)^2 \, dx
\end{align*}
\]

Nubot := \int_{0}^{L} U_{\text{bot}}(x) \, dx \\
Nutop := \int_{0}^{L} U_{\text{top}}(x) \, dx \\
Nucore := \int_{0}^{L} U_{\text{core}}(x) \, dx

\[
U_{\text{total}} := \text{Nubot} + \text{Nutop} + \text{Nucore}
\]

\[
\text{Deflection} := \frac{U_{\text{total}}}{P}
\]

Deflection = 1
Huyue Zhao was born in Dongyang, Zhejiang, in P.R. China, on September 16, 1974. He was raised in Weishang, Zhejiang and graduated from Weishang High School in 1992. He attended the Huazhong University of Science and Technology (HUST) and graduated in 1996 with a Bachelor's degree in Mechanical Engineering. In 1996, he worked for Baoma Micro-Electrical Motor Corp. for a year. Then he returned to HUST and entered the Mechanical Engineering graduate program in the fall of 1997. After receiving his Master's degree in HUST, he entered the Mechanical Engineering graduate program at The University of Maine in the fall of 2000.

After receiving his degree, Huyue will be joining Northwestern University to begin his pursuit of a Ph.D. degree in the field of Mechanical Engineering. Huyue is a candidate for the Master of Science degree in Mechanical Engineering from The University of Maine in December, 2002.