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Local Converse Theorem For 2-Dimensional Representations of Weil Groups

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LOCAL CONVERSE THEOREM FOR 2-DIMENSIONAL REPRESENTATIONS OF WEIL GROUPS

By

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A local converse theorem is a theorem which states that if two representations χ_1, χ_2 have equal γ-factors for all twists by representations σ coming from a certain class, then χ_1 and χ_2 are equivalent in some way. We provide a direct proof of a local converse theorem in two distinct settings. Previous proofs published in the literature for these settings were indirect proofs making use of various correspondences between representations of other groups.

We first prove a Gauss sum local converse theorem for representations of $\mathbb{F}_{p^2}^\times$ twisted by representations of $\mathbb{F}_p^\times$ where equality means that χ_1 and χ_2 are in the same Frobenius orbit. Specifically, we prove that if χ_1, χ_2 are regular representations of $\mathbb{F}_{p^2}^\times$ such that $S(\chi_1 \otimes \sigma \circ N_{\mathbb{F}_{p^2}/\mathbb{F}_p}) = S(\chi_2 \otimes \sigma \circ N_{\mathbb{F}_{p^2}/\mathbb{F}_p})$ for all representations σ of $\mathbb{F}_p^\times$, then $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^p$. We then apply this theorem to tamely ramified, 2-dimensional representations of the Weil group $\mathcal{W}_F$ for a local field F where we show that if ρ_1, ρ_2 are 2-dimensional, tamely ramified representations of $\mathcal{W}_F$ such that for all 1-dimensional representations χ of $\mathcal{W}_F$ it is true that $\gamma(\rho_1 \otimes \chi, s, \psi) = \gamma(\rho_2 \otimes \chi, s, \psi)$, then $\rho_1 \cong \rho_2$.

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CHAPTER 1

MAIN STATEMENTS

1.1 Introduction

A local converse theorem is, broadly speaking, a theorem which says that although we cannot uniquely identify an n-dimensional representation by its γ-factor; we can uniquely identify an n-dimensional representation, up to an appropriate notion of equivalence, using the γ-factors of all its twists by representations of dimension at most k for some $k < n$. The exact details of a local converse theorem depend on the setting in which the theorem is stated, but the case when $k = \lfloor \frac{n}{2} \rfloor$ came to be known as Jacquet's conjecture after [JPSS83]. Here we shall establish an $n = 2$, $k = 1 = \lfloor \frac{n}{2} \rfloor$ local converse theorem for characters of $\mathbb{F}_{p^n}$; then we will use that theorem to prove an $n = 2$, $k = 1$ local converse theorem for tamely ramified representation of $\mathcal{W}_F$, the Weil group of a local field F.

Both theorems have been proven previously: the Gauss sum version of Jacquet's conjecture as a corollary of [Nie14] pushed through the Green correspondence between representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and characters of $\mathbb{F}_{q^n}^\times$; the local field version of Jacquet's conjecture independently as a corollary of [JL18] and [Cha19], both pushed through the local Langlands correspondence. Our contribution here is a direct proof of the $n = 2$ Gauss sum local converse theorem in the vein of the proof of the $n = 4, 5$ cases in [NZ21] avoiding the Green correspondence. The other contribution is a more elementary proof of the $n = 2$ local fields converse theorem for tamely ramified representations of the Weil group which appeals directly to the Gauss sum local converse theorem instead of requiring the local Langlands correspondence. Neither of these proofs appear in the present literature as stated, though as mentioned, the $n = 4, 5$ Gauss sum local converse for $\mathbb{F}_p$ (not for $\mathbb{F}_q$) was proven directly in [NZ21] by Nien. We leave open the possibility of generalizing these proofs for $n > 2$.

1.2 Finite Fields

On the finite field Gauss sum side of things, we use Stickelberger's theorem to prove that if χ_1 and χ_2 are characters of $\mathbb{F}_{p^2}^\times$ and the Gauss sums satisfy

$$S(\chi_1 \otimes \sigma \circ N_{\mathbb{F}_{p^2}/\mathbb{F}_p}) = S(\chi_2 \otimes \sigma \circ N_{\mathbb{F}_{p^2}/\mathbb{F}_p})$$

for all σ, a character of $\mathbb{F}_p^\times$, then $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^p$.

We define the objects of interest in this case in Section 2.1. In Section 2.2 we state and prove Stickelberger's theorem, a main tool in our theorem which allows us to reduce from a statement about characters to a statement about integers. Finally, in Section 2.3, we prove the $n = 2$ local converse theorem for Gauss sums.

At this time, we have only proven the $n = 2$ local converse theorem for the extension $\mathbb{F}_{p^n}/\mathbb{F}_p$. A useful extension would be to use the same or similar methods to prove the $n = 2$ local converse theorem for the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$; this would allow a proof of the $n = 2$ local converse theorem for tamely ramified representations of $\mathcal{W}_F$ that takes advantage of neither the Green Correspondence nor the local Langlands correspondence. As is, we only have a fully direct proof of the $\mathcal{W}_F$ theorem when the residue field of F is $\mathbb{F}_p$. More generally, extending the $k = 1$ Gauss sum local converse theorem to $n = 3, 4, 5$ should be possible, but may require using the Gross-Koblitz theorem, a far reaching generalization of Stickelberger's theorem, as in Nien's paper [NZ21]. These extensions may allow proving the $k = 1$ Weil group local converse theorem for $n = 3$ and either $k = 1$ or $k = 2$ for $n = 4, 5$.

1.3 Weil Groups

On the Weil Group side of things, our main tools are the stability theorem for characters of $F^\times$ and the previously proven local converse theorem for Gauss sums. What we end up proving is that if ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of the Weil group $\mathcal{W}_F$ of a local field F and the γ-factors satisfy

$$\gamma(\rho_1 \otimes \chi, s, \psi) = \gamma(\rho_2 \otimes \chi, s, \psi)$$

for all characters χ of $\mathcal{W}_F$, then $\rho_1 \cong \rho_2$.

We first have to define what the objects of interest look like for $F^\times$ which we do in Section 3.1 and Section 3.2. We then define the Weil group in Section 3.3. Local class field theory is stated in brief in Section 3.4 which we can use define to the objects of interest for $\mathcal{W}_F$ in Section 3.5. Finally, in Section 3.6, we state and prove the $n = 2$ local converse theorem for tamely ramified, semisimple representations of the Weil group.

Like for the finite field case, at this time we have only proven the $n = 2$ local converse theorem for $\mathcal{W}_F$, and only for tamely ramified, semisimple representations of $\mathcal{W}_F$. Given a Gauss sum $n = 3$ local converse theorem, it should be fairly simple to prove an $n = 3$ local converse theorem for $\mathcal{W}_F$ using the same methods. For $n = 2$ highly ramified representations, the methods displayed here fail for irreducible representations of dimension ≥ 2, due to only specifying a representation up to its behaviour on ϖ^l for some $l > 1$, but not defining it on ϖ like for tamely ramified representations. It is possible that further elementary methods could allow a full proof of an $n = 2$ local converse theorem for highly ramified representations of $\mathcal{W}_F$. Additionally, since γ-factors are equal for representations with isomorphic semisimplifications, we do not need to consider a local converse theorem for non-semisimple representations. We have also decided to omit the topic of Weil-Deligne representations from this thesis for simplicity of exposition as there is not much significantly different when dealing with them.

CHAPTER 2

LOCAL CONVERSE THEOREM FOR GAUSS SUMS

We will use the following notation in this chapter; mostly following the notation choices in [Lan90].

$\mathbb{F}_p$ will be the finite field with p elements.

$\mathbb{F}_q$ will be the finite field with $q = p^n$ elements for some $n \geq 1$.

For any finite field $\mathbb{F}_q$, $\mathbb{F}_q^\times$ will be the multiplicative unit group of the field.

ε will be a primitive p^{th} root of unity. For $\varepsilon \in \mathbb{C}$, we use $\varepsilon = e^{2\pi i/p}$.

μ_N will be the group of N^{th} roots of unity.

Tr will be the trace of $\mathbb{F}_q/\mathbb{F}_p$.

ξ and λ will be additive characters of $\mathbb{F}_q$, typically $\xi = \varepsilon^{\mathrm{Tr}}$, and $\widehat{\mathbb{F}_q}$ will be the group of additive characters of $\mathbb{F}_q$.

χ will be a multiplicative character of $\mathbb{F}_q$ and $\widehat{\mathbb{F}_q^\times}$ will be the group of multiplicative character of $\mathbb{F}_q^\times$.

ω will be the Teichmüller character, a generator of $\widehat{\mathbb{F}_q^\times}$.

$S(\chi, \xi)$ will be the Gauss sum of χ with respect to ξ with $S(\chi)$ used when ξ is understood.

$\mathfrak{p}$ will be a prime ideal of the ring of integers of $\mathbb{Q}(\mu_{q-1})$ lying over $\langle p \rangle$.

$\mathfrak{P}$ will be a prime ideal of the ring of integers of $\mathbb{Q}(\mu_{q-1}, \mu_p)$ lying over $\mathfrak{p}$.

2.1 Gauss Sums for Finite Fields

We will be dealing only with complex representations of finite fields. Since $\mathbb{F}_q$ is a field, all representations are 1 dimensional characters. An additive character of $\mathbb{F}_q$ is a function $\xi : \mathbb{F}_q \to \mathbb{C}^\times$ with $\xi(a + b) = \xi(a)\xi(b)$. A multiplicative character of $\mathbb{F}$ is a function $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ with $\chi(ab) = \chi(a)\chi(b)$. If we ever need to evaluate a multiplicative character χ at $0 \notin \mathbb{F}_q^\times$ then we use $\chi(0) = 0$.

The two most important characters in this setting are the additive character defined by the trace and the multiplicative character called the Teichmüller character. The primary trace we are interested in is the trace for the extension $\mathbb{F}_q/\mathbb{F}_p$ which is defined in the following way.

Definition 2.1.1 The trace from $\mathbb{F}_q/\mathbb{F}_p$ for $q = p^n$ is defined by $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$

$$\mathrm{Tr} : x \mapsto \sum_{i=0}^{n-1} x^{p^i} = x + x^p + x^{p^2} + \cdots + x^{p^{n-1}}.$$

We can also define a trace for any extension; specifically, for the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ we have a trace defined by

$$\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} : x \mapsto \sum_{i=0}^{n-1} x^{q^i}.$$

We can unify these definitions of trace in the following way.

Definition 2.1.2 Let E/F be a Galois field extension and let $G = \mathrm{Gal}(E/F)$ be the Galois group of automorphisms of E fixing F. Then the trace $\mathrm{Tr} : E \to F$ is defined by

$$\mathrm{Tr} : x \mapsto \sum_{\sigma \in G} \sigma x.$$

Then we can define an additive character $\lambda : \mathbb{F}_q \to \mu_p$ by $\lambda(x) = \varepsilon^{\mathrm{Tr}(x)}$. It takes a bit more to define the Teichmüller character.

Consider the field $\mathbb{Q}(\mu_{q-1})$ and let $\mathfrak{p}$ be a choice of prime ideal lying over p. Then the residue field of $\mathbb{Q}(\mu_{q-1})$ mod $\mathfrak{p}$ is isomorphic to μ_{q-1} which in turn is isomorphic to $\mathbb{F}_q^\times$. Because $\mathbb{F}_q^\times \cong \widehat{\mathbb{F}_q^\times}$ is a cyclic group, there is some generator for it.

Proposition 2.1.3 By the above isomorphisms, there is a generator $\omega : \mathbb{F}_q^\times \to \mu_{q-1}$ satisfying

$$\omega(u) \equiv u \mod \mathfrak{p}.$$

We call such a character of $\mathbb{F}_q^\times$ the **Teichmüller character**.

This character generates the character group of $\mathbb{F}_q^\times$; so for all multiplicative characters χ of $\mathbb{F}_q^\times$ there is some integer k such that $\chi = \omega^k$. We also further have that ω^c generates $\widehat{\mathbb{F}_q^\times}$ for all c coprime to $q-1$.

When we have two multiplicative character χ_1 and χ_2, we say that the twist of χ_1 by χ_2 is $\chi_1 \otimes \chi_2$. When written in terms of the Teichmüller character, we have $\omega^{k_1} \otimes \omega^{k_2} = \omega^{k_1+k_2}$.

Now we can define Gauss sums for finite fields, which depend upon a choice of an additive character and a multiplicative one. For finite fields, we will always use the additive character λ defined above.

Definition 2.1.4 The Gauss sum for a multiplicative character χ is denoted $S(\chi)$ or $S(\chi, \lambda)$ and is defined by

$$S(\chi, \lambda) = \sum_{u \in \mathbb{F}_q^\times} \chi(u)\lambda(u).$$

We will use $S(\chi)$ throughout since λ will be fixed as $\varepsilon^{\mathrm{Tr}}$

There are a few useful properties of Gauss sums that can be easily proven.

Proposition 2.1.5 For a nontrivial multiplicative character χ on a finite field $\mathbb{F}_q$, we have that $|S(\chi)| = q^{1/2}$.

Proposition 2.1.6 For a multiplicative character χ on a finite field $\mathbb{F}_q$ with $q = p^n$, we have that $S(\chi^p) = S(\chi)$.

This second proposition is because raising to the p^{th} power only permutes the elements of the sum. For this reason we collect together the characters $\{\chi, \chi^p, \ldots, \chi^{p^{n-1}}\}$ together into a set we call the Frobenius orbit. The converse theorem for Gauss sums concerns separating the multiplicative characters of a finite field $\mathbb{F}_q$ into their Frobenius orbits.

However, first we separate out the characters that live in degenerate Frobenius orbits. These are the non-regular characters of $\mathbb{F}_q$ and are characterized by having $\chi = \chi^{p^k}$ for some $k \mid n-1$, or equivalently, factoring through the norm of a subextension. If we write $\chi = \omega^k$, then χ is non-regular when there is some $k' \mid \frac{q-1}{p-1}$ with $k' \neq 1$ such that $k' \mid k$. It suffices to check $k' = \frac{p^m-1}{p-1}$ for $m \mid n$ and $m \neq 1$. If a character is not non-regular, then it is regular. The regular characters are those with Frobenius orbits of size n when $q = p^n$.

We can now state the conjectured converse theorem for Gauss sums of finite fields.

Conjecture 2.1.7 (Nien) Let χ_1 and χ_2 be two regular multiplicative characters of $\mathbb{F}_q$ with $q = p^n$ and n prime. If

$$S(\chi_1 \otimes \sigma) = S(\chi_2 \otimes \sigma), \text{ for all } \sigma \in \widehat{\mathbb{F}_p^\times}$$

then $\chi_1 = \chi_2^{p^i}$ for some integer i.

For $n = 2$; this becomes our theorem

Theorem 2.1.8 Let χ_1 and χ_2 be two regular multiplicative characters of $\mathbb{F}_{p^2}$. If

$$S(\chi_1 \otimes \sigma) = S(\chi_2 \otimes \sigma), \text{ for all } \sigma \in \widehat{\mathbb{F}_p^\times}$$

then $\chi_1 = \chi_2^p$ or $\chi_1 = \chi_2$.

The version of this that we will actually end up proving for $n = 2$ appears at first to have weaker conditions, but in fact is equivalent in this case.

Theorem 2.1.9 Let χ_1 and χ_2 be two regular multiplicative characters of $\mathbb{F}_{p^2}$. If

$$\langle S(\chi_1 \otimes \sigma) \rangle = \langle S(\chi_2 \otimes \sigma) \rangle, \text{ for all } \sigma \in \widehat{\mathbb{F}_p^\times}$$

then $\chi_1 = \chi_2^p$ or $\chi_1 = \chi_2$.

The difference here is that we only assume the Gauss sums of the twisted characters generate the same ideals, not that they are actually equal.

2.2 Stickelberger's Theorem

Before we set out to prove our converse theorem for finite fields; it is simplest to reframe the problem as a question about the p-adic expansions of integers. The tool that allows us to do this is Stickelberger's theorem which we shall introduce and prove in this section. Working in the field $\mathbb{Q}(\mu_{q-1}, \mu_p)$, Stickelberger's theorem gives us a factorization of the ideal generated by $S(\omega^{-k})$ in terms of the the prime ideals over p and the sum of the p-adic digits of k.

As above; we will consider the case when $\mathbb{F}_q$ is a field of size $q = p^n$. We will let $\mathfrak{p}$ be a prime ideal in $\mathbb{Q}(\mu_{q-1})$ lying above p. Then we shall consider another extension $\mathbb{Q}(\mu_{q-1}, \mu_p)$ with $\mathfrak{P}$ being a prime ideal lying above $\mathfrak{p}$. Additionally; when it is useful for emphasis, we will let $\pi = \varepsilon - 1$

Recall that the Teichmüller character generates $\hat{\mathbb{F}}_q$, so for any $\chi \in \hat{\mathbb{F}}_q$ we have $\chi = \omega^k$ for some $0 \leq k \leq q-1$, or equivalently, $\chi = \omega^{-k}$ for some $0 \leq k \leq q-1$, which will be more convenient when phrasing Stickelberger's theorem. For such a k, we can write the p-adic expansion of k as

$$k = k_0 + k_1 p + \cdots + k_{n-1} p^{n-1}$$

with $0 \leq k_i \leq p-1$. Then we can define the functions $s, \gamma : \mathbb{Z} \to \mathbb{Z}^+$ by

$$s(k) = k_0 + k_1 + \cdots + k_{n-1}$$

$$\gamma(k) = k_0! k_1! \cdots k_{n-1}!$$

for $0 \leq k < q-1$ and requiring that s and γ are $q-1$ periodic for other k.

Our first step towards Stickelberger's theorem will be the following theorem which tells how many times one prime appears in the factorization of a Gauss sum.

Theorem 2.2.1 For any integer k, we have the congruence

$$\frac{\langle S(\omega^{-k}) \rangle}{(\varepsilon - 1)^{s(k)}} \equiv \frac{-1}{\gamma(k)} \mod \mathfrak{P}.$$

In particular, $\mathrm{ord}_{\mathfrak{P}} \langle S(\omega^{-k}) \rangle = s(k)$.

To get the full factorization of $\langle S(\omega^{-k}) \rangle$, we need a few more definitions.

For $t \in \mathbb{R}$, we will let $0 \leq \langle t \rangle < 1$ be the representative of t in $\mathbb{R}/\mathbb{Z}$. Then let $G = \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ and let $\sigma_c \in G$ be defined by $\sigma_c : \zeta \mapsto \zeta^c$ and $\sigma_c|_{\mu_p} = \mathrm{id}$; where $\zeta \in \mu_{q-1}$. Then we make the following definition:

Definition 2.2.2 We define the Stickelberger element in the group ring $\mathbb{Q}[G]$ as

$$\theta(k, \mathfrak{p}) = \sum_{c \in (\mathbb{Z}/m\mathbb{Z})^\times} \left\langle \frac{kc}{q-1} \right\rangle \sigma_c^{-1}$$

Then we can get a full factorization of the ideal generated by $S(\omega^{-k})$. Using $\mathfrak{a} \sim \mathfrak{b}$ to mean that $\mathfrak{a}/\mathfrak{b}$ is the unit ideal, we have the following theorem.

Theorem 2.2.3 The factorization of the ideal generated by $S(\omega^{-k})$ is

$$S(\omega^{-k})\mathcal{O}_{\mathbb{Q}(\mu_p,\mu_{q-1})} \sim \mathfrak{P}^{(p-1)\theta(k,\mathfrak{p})} \sim \left(\mathfrak{p}\mathcal{O}_{\mathbb{Q}(\mu_p,\mu_{q-1})}\right)^{\theta(k,\mathfrak{p})}.$$

2.3 The $n = 2$ Gauss Sum Local Converse Theorem

Recall that we would like to prove Conjecture 2.1.7 for the case of $n = 2$. By using the fact that $\exists k$ such that $\chi = \omega^k$, we can use Stickelberger's theorem rewrite the conjecture in the following way.

Theorem 2.3.1 Suppose that $\alpha, \beta \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ with $p+1 \nmid \alpha, \beta$ and

$$s(\alpha + k(p+1)) = s(\beta + k(p+1))$$

for all $0 \leq k < p-1$. Then we have that $\alpha \equiv \beta \mod p^2-1$ or $\alpha \equiv p\beta \mod p^2-1$.

Proposition 2.3.2 Theorem 2.1.9 and Theorem 2.3.1 are equivalent.

Proof. Suppose the conditions of Theorem 2.1.9 hold. If we write $\chi_1 = \omega^\alpha$ and $\chi_2 = \omega^\beta$ and use the fact that all the twists can be written as $\sigma = \omega^{k(p+1)}$ for $0 \leq k < p-1$; then we are claiming that $S(\omega^{\alpha+k(p+1)}) = S(\omega^{\beta+k(p+1)})$ for all $0 \leq k < p-1$. However, by Theorem 2.2.1 if $S(\omega^{\alpha+k(p+1)}) = S(\omega^{\beta+k(p+1)})$ for all $0 \leq k < p-1$ then $s(-(\alpha+k(p+1))) = s(-(\beta+k(p+1)))$ for all $0 \leq k < p-1$. This implies the conditions of Theorem 2.3.1.

In the other direction, suppose that the conditions of Theorem 2.3.1 hold. Then $p+1 \nmid \alpha, \beta$ ensures that $\chi_1 = \omega^\alpha$ and $\chi_2 = \omega^\beta$ are regular characters of $\mathbb{F}_{p^2}$. Then

$s(\alpha + k(p+1)) = s(\beta + k(p+1))$ for all $0 \le k < p-1$ only provides that $\mathrm{ord}_{\mathfrak{P}} S(\chi_1 \otimes \sigma) = \mathrm{ord}_{\mathfrak{P}} S(\chi_2 \otimes \sigma)$ for all twists σ. However, this is we assume this is true for all α, β so we look at $c\alpha$ and $c\beta$ and because note that ck are in bijection with k mod $p-1$ for c coprime to p^2-1, so we also have $s(c\alpha + ck(p+1) = s(c\beta + ck(p+1))$ which implies

$$\mathrm{ord}_{\sigma_c^{-1}\mathfrak{P}} \langle S(\chi_1 \otimes \sigma) \rangle = \mathrm{ord}_{\sigma_c^{-1}\mathfrak{P}} \langle S(\chi_2 \otimes \sigma) \rangle$$

for all c coprime to p^2-1, which tells us that $\langle S(\chi_1 \otimes \sigma) \rangle = \langle S(\chi_2 \otimes \sigma) \rangle$ as desired.

□

Now that we only need to prove Theorem 2.3.1, we shall prove the $n = 2$ Gauss sum local converse theorem. Our main tool for doing this will be a lemma describing how the digit sum changes with twists.

Lemma 2.3.3 Let $0 \le i, t-i \le p-1$ and $2 \le \ell \le p$, then

$$s((t-i) + ip + (p-\ell)(p+1)) = \begin{cases} p + t - 2\ell + 1 & i \le \ell - 2 \\ t - 2\ell + 2 & i > \ell - 2 \end{cases}.$$

Proof. This proof is mainly a simple computation in which we have to keep track of the possible for cases when we have carry in the 1's place or in the p's place. Specifically, we will have a carry from the 1's into the p's place when $t - i \ge \ell$ which causes $p + t - i - \ell \ge p$. Similarly, we get a carry from the p's place in to the 1's place when either $i \ge \ell$ or $1 + i \ge \ell$, depending on if the previous carry happens. Since $0 \le p - \ell \le p - 2$ and $0 \le i, t - i \le p - 1$, these are the only carry's that can happen because the max value for each digit after summing before carrying is $2p - 3$. This gives us the following calculation that proves the lemma.

$$s((t-i)+(i)p+(p-\ell)(p+1)) = s((p+t-i-\ell)+(p+i-\ell)p)$$

$$= \begin{cases} s((t-i-\ell)+(p+1+i-\ell)p) & t-i \geq \ell \\ s((p+t-i-\ell)+(p+i-\ell)p) & t-i < \ell \end{cases}$$

$$= \begin{cases} s((t+1-i-\ell)+(1+i-\ell)p) & t-i \geq \ell \text{ and } 1+i \geq \ell \\ s((t-i-\ell)+(p+1+i-\ell)p) & t-i \geq \ell \text{ and } 1+i < \ell \\ s((p+t-i-\ell)+(p+i-\ell)p) & t-i < \ell \text{ and } i < \ell \\ s((p+t+1-i-\ell)+(1+i-\ell)p) & t-i < \ell \text{ and } i \geq \ell \end{cases}$$

$$s((t-i)+(i)p+(p-\ell)(p+1)) = \begin{cases} p+t-2\ell+1 & i \leq \ell-2 \\ t-2\ell+2 & i > \ell-2 \end{cases}$$

Note that the only way to have $p+t-2\ell+1 = t-2\ell+2$ is to have $p=1$, which is not a prime, so we will always have that $s((t-i)+ip+(p-\ell)(p+1))$ takes on different values for $i \leq \ell-2$ and $i > \ell-2$.

□

With Lemma 2.3.3 in hand, we can now prove Theorem 2.3.1.

Proof. Note that we only need to consider the case of fixed α and can vary β to find those that satisfy the equality of Gauss sums.

We would like to show that the condition $s(\alpha+k(p+1)) = s(\beta+k(p+1))$ for all $0 \leq k < p-1$ implies that $\alpha \equiv p\beta \mod p^2-1$ or $\alpha \equiv \beta \mod p^2-1$. Another way of stating this is that if $s(\alpha+k(p+1)) = s(\beta+k(p+1))$ for all $0 \leq k < p-1$ then ω^{α} and $\omega^{-\beta}$ are in the same Frobenius orbit. Let $s(\alpha) = t = s(\beta)$ be our fixed digit sum before twists, and let $r = \lceil \frac{t}{2} \rceil - 1$. Then the set of possible Frobenius orbits with a fixed digit sum t is

$$\{\{t, tp\}, \{(t-1)+p, 1+(t-1)p\}, \{t-2)+2p, 2+(t-2)p\}, \ldots, \{(t-r)+rp, r+(t-r)p\}\}.$$

This takes the $t+1$ integers with digit sum t and splits them into orbits of size 2 in a way that corresponds to (cyclically) permuting the digits. When t is even, there is a degenerate orbit $\{(r+1)+(r+1)p\}$ where permuting the digits doesn't change the number or the underlying character; these correspond to non-regular characters, or characters of $\mathbb{F}_p$ lifted to $\mathbb{F}_{p^2}$. We write a generic element of an orbit as $(t-i)+ip$ since it doesn't matter which element of the orbit we pick.

Our goal is then to show that with an appropriate set of twists, we can distinguish between these orbits. I claim that it is sufficient to consider the twists by $k=p-\ell$ for $2 \leq \ell \leq r+1 < t$; note that there are only ℓ satisfying these inequalities for $t \geq 4$, so we will deal with the $t=1,2,3$ cases by hand.

First, the $t=1$ case is trivial because there is only one orbit, $\{1,p\}$. Similarly, the $t=2$ case is trivial because there is only one non-degenerate orbit $\{2,2p\}$ and the orbit $\{1+p\}$ is degenerate. Finally, we have that $t=3$ case. Then there are two orbits $\{3,3p\}$ and $\{2+p,1+2p\}$. For $p \leq 3$, the case of $t=3$ is impossible, and for $p>3$, consider twisting by $k=p-3$, from the lemma we find that $s(3+(p-3)(p+1)) = p-2$ and $s(2+p+(p-3)(p+1)) = 2p-3$ which are different for all primes, so the single twist by $p-3$ serves to distinguish between the orbits.

Now, let us first consider what happens for general t when we twist by $k=p-2$. Lemma 2.3.3 tells us that $s((t-i)+ip)$ is either $p+t-3$ for $i \leq 0$ or $t-2$ for $i>0$. These are different values and so twisting by $k=p-2$ allows us to distinguish $\{t,tp\}$ from $\{(t-i)+ip, i+(t-i)p\}$ for all $i>0$.

Similarly, twisiting by $p-\ell$ allows us to distinguish between $\{(t-\ell+2)+(\ell-2)p, (\ell-2)+(t-\ell+2)p\}$ and $\{(t-\ell+1)+(\ell-1)p, (\ell-1)+(t-\ell+1)p\}$. By letting ℓ vary from 2 up to $r+1$, we are thus able to distinguish between all the non-degenerate Frobenius orbits. □

As a simple corollary of Theorem 2.3.1, we have a proof of Theorem 2.1.8, the Gauss sum converse theorem that we were hoping to prove.

CHAPTER 3

LOCAL CONVERSE THEOREM FOR SMOOTH REPRESENTATIONS OF THE WEIL GROUP

Most of the introductory material on local fields will echo that in Jean-Pierre Serre's *Local Fields* [Ser79], with the material on local class field theory coming from from Bushnell and Henniart's *The Local Langlands Conjecture for* $GL(2)$ Chapter 29 [BH06]. Adopting the notation in the second of these references, we will have the following notation for local fields.

F and E will refer to a non-Archimedean local fields.

$\mathfrak{o}$ is the discrete valuation ring in F.

$\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}$.

ϖ is the uniformizer and a generator of $\mathfrak{p}$.

v_F will be that valuation defined on F by $\mathfrak{o}$.

$\boldsymbol{k} = \mathfrak{o}/\mathfrak{p}$ is the residue field of F.

$q = p^n = |k|$ is the size of the residue field where p is the characteristic.

U_F is the group of units of $\mathfrak{o}$.

$U_F^n = 1 + \mathfrak{p}^n$ for $n \geq 1$ are subgroups of the unit group forming a filtration.

ψ will be an additive character of F and $\widehat{F}$ will be the group of additive characters of F.

χ and ξ will be characters of $F^\times$ or of $\mathcal{W}_F$, frequently viewed as equivalent through Local Class Field Theory, and $\widehat{F^\times}$ will be the group of multiplicative characters of F.

1_F will be the trivial character on $F^\times$ or $\mathcal{W}_F$, which takes the value 1 everywhere. Ω_F will be the absolute Galois group of F and $\mathcal{W}_F$ will be the Weil group of F.

ρ and σ will be semisimple representations of $\mathcal{W}_F$ and ρ will be tamely ramified.

3.1 Local Fields

We are interested in a local converse theorem for smooth, tamely ramified, semisimple, 2-dimensional representations of the Weil group of a local field. First, we need to be able to define all the terms in stating the theorem. Even though we are proving a theorem about representations of the Weil group $\mathcal{W}_F$ of a local field F, the terms in the theorem must first be defined for the mulitplicative group $F^\times$ of the field.

Definition 3.1.1 A **discrete valuation ring** is a principal ideal domain $\mathfrak{o}$ with a unique non-zero prime ideal $\mathfrak{p}$.

We also call such a ring $\mathfrak{o}$ a DVR. An alternative characterization of a DVR is that it is a domain $\mathfrak{o}$ such that its field of fractions K has a non-trivial valuation $v : K \to \mathbb{Z} \cup \{\infty\}$; that is, a function v such that $v(xy) = v(x) + v(y)$, $v(x + y) \geq \min\{v(x), v(y)\}$, $v(x) = 0$ if and only if $x = 0$, and v takes more values than just 0 and ∞.

Definition 3.1.2 A (non-Archimedean) **local field** is a field F with a valuation v that is locally compact with respect to the topology provided by the valuation and has finite residue field k with $|k| = q = p^n$. The valuation defines a basis of open sets as the additive cosets of the $v^{-1}(\{x \in F \mid q^{-v(x)} \leq r\})$ for positive real numbers r.

Given a local field F, it is not difficult to show that $\mathfrak{o} = \{x \in F \mid q^{-v(x)} \leq 1\}$ is the ring of integers of F and is a DVR. Further, $\mathfrak{p} = \{x \in F \mid q^{-v(x)} < 1\}$ is unique maximal (prime) ideal of the DVR $\mathfrak{o}$. Finally, $\mathfrak{o}^\times = U_F = \{x \in F \mid q^{-v(x)} = 1\} = \{x \in F \mid v(x) = 0\}$ is the unit group of $\mathfrak{o}$. A local field is isomorphic as a topological field to a finite extension of the p-adic numbers $\mathbb{Q}_p$ or the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field.

For a DVR $\mathfrak{o}$ or for a local field F, we define a special element called the **uniformizer**, denoted by ϖ, which is a prime in $\mathfrak{o}$, so that $\varpi\mathfrak{o} = \mathfrak{p}$ is the unique prime ideal. This choise is unique up to units.

3.2 L-Functions and ϵ-Factors of Local Fields

Though we are interested in the representation theory of the Weil group, and how we can tell apart the representations of the Weil group, in order to define the objects we will use to do so, we must first define them for local fields. Generally, a local converse theorem for local fields would deal with representations of $GL_n(F)$; however, all of the tools we need are developed while studying $GL_1(F) = F^\times$.

An additive character of a local field F is a continuous homomorphism $\psi : F \to \mathbb{C}^\times$; these are the 1-dimensional representations of the additive group of F. Equivalently to saying ψ is continuous, we can say $\ker \psi$ is open in the topology defined by the valuation. We let $\hat{F}$ be the group of additive characters of F, which are a group under pointwise multiplication.

Definition 3.2.1 Let $\psi \in \hat{F}$ with $\psi \neq 1$. The **level** of ψ is the smallest integer d such that $\mathfrak{p}^d \subseteq \ker \psi$.

We then have the following proposition mostly characterizing the additive characters of F.

Proposition 3.2.2 Let $\psi \in \hat{F}$ with $\psi \neq 1$ be a level d character.

1. Let $a \in F$. The map $a\psi : x \mapsto \psi(ax)$ is a character of F. If $a \neq 0$, then the character $a\psi$ has level $d - v_F(a)$, where v_F is the valuation on F.

2. The map $a \mapsto a\psi$ is a group isomorphism $F \cong \hat{F}$.

We also define multiplicative characters of F which are the 1-dimensional representations of the multiplicative group of F. In this setting, a multiplicative character of a local field F is a continuous homomorphism $\chi : F^\times \to \mathbb{C}^\times$. We also define level for multiplicative characters with a slight modification.

Definition 3.2.3 Let χ be a non-trivial character of $F^\times$. The **level** of χ is defined to be the smallest integer $n \geq 0$ such that $U_F^{n+1} \subseteq \ker \chi$. We further say that χ is **unramified** if $U_F \subseteq \ker \chi$.

The final thing we need to define the functions of interest is the concept of duality for representations of $F^\times$. Because we are working with topological spaces, we need a definition of duality that respects the topological structure which will be called the **smooth dual** of a character χ and will be denoted by $\check{\chi}$. If (χ, V) is a smooth representation of $F^\times$, then let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and let $(v^*, v) \mapsto \langle v^*, v\rangle$ be the canonical evaluation map. Then we can define a representation χ^* of $F^\times$ on the space V^* by

$$\langle \chi^*(g)v^*, v\rangle = \langle v^*, \chi(g^{-1})v\rangle.$$

This is not necessarily a smooth representation, but we can define $\check{V} = \cup_K (V^*)^K$ where K ranges over compact open subgroups of $F^\times$ and the $(V^*)^K$ are the subspaces of K fixed vectors in the representation. Then we define $\check{\chi}$ as the restriction to $\mathrm{Aut}_{\mathbb{C}}(\check{V})$. This $(\check{\chi}, \check{V})$ is the smooth dual of the character (χ, V).

Now we can define the L-function and the ϵ-factor and γ-factor for characters of $F^\times$. We define the L-function for a character of $F^\times$ as a variable of a complex variable s.

Definition 3.2.4 Let χ be a characters of $F^\times$ and q be the size of the residue field of F. Then we define the ***L*-function of** χ as $L(\chi, s) : \mathbb{C} \to \mathbb{C}$ by

$$L(\chi, s) = \begin{cases} (1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise} \end{cases}.$$

This will be independent of the choice of uniformizer because unramified characters χ are trivial on units.

Next, we define the ϵ-factor as another function of a complex variable which is a sum of over the units of the ring of integers of F.

Definition 3.2.5 For characters χ of $F^\times$ with level $n \geq 0$ and not unramified, and $\psi \in \hat{F}$ with level one, we define the **ϵ-factor of χ (relative to ψ)** as

$$\epsilon(\chi, s, \psi) = q^{n(\frac{1}{2}-s)} \sum_{x \in U_F/U_F^{n+1}} \chi(\alpha x)^{-1}\psi(\alpha x)/q^{(n+1)/2}$$

for any $\alpha \in F^\times$ such that $v(\alpha) = -n$. For characters χ of $F^\times$ that are unramified; we define the ϵ-factor by

$$\epsilon(\chi, s, \psi) = q^{s-\frac{1}{2}}\chi(\varpi)^{-1}.$$

As a quick justifcation for only dealing with ψ of level one, we have the following lemma which describes how $\epsilon(\chi, s, \psi)$ changes as ψ changes.

Lemma 3.2.6 Let $a \in F^\times$; then for fixed χ, we have

$$\epsilon(\chi, s, a\psi) = \chi(a)\|a\|^{s-\frac{1}{2}}\epsilon(\chi, s, \psi)$$

Finally, we will define the γ factor as a useful combination of the L-functions and ϵ-factors.

Definition 3.2.7 For characters χ of $F^\times$ and ψ of F, then we define the **γ-factor of χ (relative to ψ)** by

$$\gamma(\chi, s, \psi) = \epsilon(\chi, s, \psi)\frac{L(\check{\chi}, 1-s)}{L(\chi, s)}.$$

The γ-factor encodes information about how χ acts on the uniformizer ϖ and how χ acts on the units U_F in such a way that we hope the γ-factor can help distinguish characters of $F^\times$.

As well as the definitions of L, γ, and ϵ, we will need a theorem that describes how ϵ factors change as we twist them by high level characters; this is the stability theorem.

Theorem 3.2.8 (Stability Theorem) Let θ, χ be characters of $F^\times$ of level $l \geq 0$ and $n \geq 1$. Suppose that $2l < n$. Let $\psi \in \hat{F}$ with $\psi \neq 1$ and let $c \in F$ satisfy $\chi(1 + x) = \psi(cx)$ for $x \in \mathfrak{p}^{\lfloor n/2 \rfloor + 1}$. Then

$$\epsilon(\theta\chi, s, \psi) = \theta(c)^{-1}\epsilon(\chi, s, \psi).$$

To prove this; we first convert the ϵ-factor into a Gauss sum and then prove a lemma about this Gauss sum. If we define the **Gauss sum of χ (relative to ψ)** as

$$\tau(\chi,\psi)=\sum_{x\in U_F/U_F^{n+1}}\check{\chi}(cx)\psi(cx),$$

then this allows us to take the s dependence out of the ϵ-factor. Specifically, we have from our definition above that for ramified χ of level $n\geq 0$

$$\epsilon(\chi,s,\psi)=q^{n(\frac{1}{2}-s)}\tau(\chi,\psi)/q^{(n+1)/2}.$$

Then we have the following lemma

Lemma 3.2.9 Suppose that χ has level $n\geq 1$. Let $c\in F$ satisfy

$$\chi(1+x)=\psi(cx),\text{for all } x\in\mathfrak{p}^{\lfloor n/2\rfloor+1}$$

Then

$$\tau(\chi,\psi)=q^{\lfloor (n+1)/2\rfloor}\sum_{y}\check{\chi}(cy)\psi(cy)$$

where $y\in U_F^{\lfloor (n+1)/2\rfloor}/U_F^{\lfloor n/2\rfloor+1}$.

Proof. Recall that for a level n character χ

$$\tau(\chi,\psi)=\sum_{x\in U_F/U_F^{n+1}}\check{\chi}(cx)\psi(cx).$$

We will make the change of variable $x=y(1+z)$ with $y\in U_F/U_F^{\lfloor n/2\rfloor+1}$ and $z\in\mathfrak{p}^{\lfloor n/2\rfloor+1}/\mathfrak{p}^{n+1}$. Then we can rewrite the $\check{\chi}(cx)\psi(cx)$ in the sum as

$$\check{\chi}(cy(1+z))\psi(cy(1+z))=\check{\chi}(cy)\psi(cy)\psi(c(y-1)z).$$

This follows because $\check{\chi}(cy(1+z))=\check{\chi}(cy)\check{\chi}(1+z)=\check{\chi}(cy)\chi(1+z)^{-1}$ and by the assumptions of the lemma, $\chi(1+z)^{-1}=\psi(cz)^{-1}=\psi(-cz)$. Now, we can rewrite the sum as

$$\tau(\chi,\psi)=\sum_{x}\check{\chi}(cx)\psi(cx)=\sum_{y}\check{\chi}(cy)\psi(cy)\left(\sum_{z}\psi(c(y-1)z)\right).$$

The sum over $y \in U_F/U_F^{\lfloor n/2 \rfloor+1}$ looks superficially like what we want; but note that the desired sum is actually over $y \in U_F^{\lfloor (n+1)/2 \rfloor}/U_F^{\lfloor n/2 \rfloor+1}$. So we need to show that the sum over $z \in \mathfrak{p}^{\lfloor n/2 \rfloor+1}/\mathfrak{p}^{n+1}$ works out as necessary. Since $c(y-1) \in \mathfrak{p}^{-n}$, we find that $z \mapsto \psi(c(y-1)z)$ is a character of $\mathfrak{p}^{\lfloor n/2 \rfloor+1}/\mathfrak{p}^{n+1}$. In fact, $z \mapsto \psi(c(y-1)z)$ will be the trivial character if and only if $y \equiv 1 \mod \mathfrak{p}^{n-\lfloor n/2 \rfloor}$; i.e. $y \in U_F^{\lfloor (n+1)/2 \rfloor}$. Since the sum of a character over its group is 0 except for the trivial character; we find that this sum vanishes except when $y \in U_F^{\lfloor (n+1)/2 \rfloor}$ when the sum is $(\mathfrak{p}^{\lfloor n/2 \rfloor+1} : \mathfrak{p}^{n+1}) = q^{\lfloor (n+1)/2 \rfloor}$. Making this substitution, we find as desired that

$$\tau(\chi, \psi) = q^{\lfloor (n+1)/2 \rfloor} \sum_y \check{\chi}(cy)\psi(cy)$$

where the sum is now over $y \in U_F^{\lfloor (n+1)/2 \rfloor}/U_F^{\lfloor n/2 \rfloor+1}$. $\square$

With this lemma in hand, we can now provide a proof of the stability theorem.

Proof. We only need to consider ψ level one because Lemma 3.2.6 tells us how to convert to ψ of other levels. Since θ is level l and χ is level n with $2l < n$, we have that $\theta\chi$ will be a level n character because it is trivial on $U_F^{n+1} \subset U_F^{l+1}$. Similarly, we have that $\theta\chi$ agrees with θ on U_F^{l+1}. More usefully, because $2l < n$, we have that $\theta\chi$ agrees with θ on $U_F^{\lfloor (n+1)/2 \rfloor}$. Applying Lemma 3.2.9 to the Gauss sum portion of $\epsilon(\theta\chi, s, \psi)$, we get

$$\tau(\theta\chi, \psi) = q^{\lfloor (n+1)/2 \rfloor} \sum_y \check{\theta}\check{\psi}$$

with the sum over $y \in U_F^{\lfloor (n+1)/2 \rfloor}/U_F^{\lfloor n/2 \rfloor+1}$. Since θ is trivial on $U_F^{\lfloor (n+1)/2 \rfloor}$, this becomes $\theta(c)^{-1}\tau(\chi, \psi)$ as we expect. Substituting this into the expression for the ϵ-factor, we find that

$$\epsilon(\theta\chi, s, \psi) = \theta(c)^{-1}\epsilon(\chi, s, \psi)$$

as desired. $\square$

3.3 Weil Groups of Local Fields

Let F be a non-Archimedean local field and pick a separable algebraic closure $\overline{F}$ of F. Then we can define the absolute Galois group $\Omega_F = \mathrm{Gal}(\overline{F}/F)$ which gets a natural topology as

$$\Omega_F = \lim_{\leftarrow} \mathrm{Gal}(E/F)$$

where E/F ranges over finite Galois extensions with $E \subseteq \overline{F}$.

For each $m \geq 1$, F will have a unique unramified extensions F_m/F of degree m with $F_m \subseteq \overline{F}$. Let F_∞ be the composite of all these fields. Then F_∞/F will be the unique maximal unramified extensions of F in $\overline{F}$. Each of the unique subextensions F_m/F has a cyclic Galois group $\mathrm{Gal}(F_m/F)$. Each automorphism in $\mathrm{Gal}(F_m/F)$ is determined by its action on the residue field $\boldsymbol{k}_{F_m}$, which is isomorphic to $\mathbb{F}_{q^m}$ because F_m/F is an unramified extension. So, there will be a unique element $\phi_m \in \mathrm{Gal}(F_m/F)$ that acts on $\boldsymbol{k}_{F_m}$ by $x \mapsto x^q$. We then let $\Phi_m = \phi_m^{-1}$. We get a canonical isomorphism $\mathrm{Gal}(F_m/F) \to \mathbb{Z}/m\mathbb{Z}$ by the map $\Phi_m \mapsto 1$; and by taking the limit over m; we get an isomorphism

$$\mathrm{Gal}(F_\infty/F) \cong \lim_{\leftarrow m \geq 1} \mathbb{Z}/m\mathbb{Z} \cong \hat{\mathbb{Z}}$$

and a unique $\Phi_F \in \mathrm{Gal}(F_\infty/F)$ that acts like Φ_m on each F_m. We call this Φ_F the **geometric Frobenius substitution** on F_∞. Similarly, we define $\phi_F = \Phi_F^{-1}$ to be the **arithmetic Frobenius substitution** on F_∞. If an element of Ω_F has Φ_F as its image in $\mathrm{Gal}(F_\infty/F)$, we call it a **geometric Frobenius element**. We then define $\mathcal{I}_F = \mathrm{Gal}(\overline{F}/F_\infty)$ to be the **inertia group** of F. This subgroup $\mathcal{I}_F \subseteq \Omega_F$ roughly corresponds to the units U_F of F. There is another subgroup called the wild inertia group, denoted $\mathcal{P}_F$, which roughly corresponds to the units U_F^1, and is the unique pro p-Sylow subgroup of $\mathcal{I}_F$. These correspondences are made precise in Theorem 3.4.1

We can then define the Weil group $\mathcal{W}_F$ as a subgroup of Ω_F. We first define ${}_a\mathcal{W}_F$ as the inverse image in Ω_F of the subgroup $\langle \Phi_F \rangle$ of $\mathrm{Gal}(F_\infty/F)$. This is a dense normal subgroup of Ω_F generated by the Frobenius elements. We then define the **Weil group of F** as a topological group with ${}_a\mathcal{W}_F$ as the underlying abstract group, where $\mathcal{I}_F$ is an open subgroup of $\mathcal{W}_F$ and the topology on $\mathcal{I}_F$ as a subspace coincides with the natural topology on $\mathcal{I}_F$ as a subspace of Ω_F.

We then have a proposition defining a few properties of the Weil group relating to field extensions E/F.

Proposition 3.3.1 1. Let E/F be a finite extension with $E \subseteq \overline{F}$.

(a) The group $\mathcal{W}_F$ has a unique subgroup $\mathcal{W}_F^E$ such that

$$\boldsymbol{\iota}_F(\mathcal{W}_F^E) = {}_a\mathcal{W}_F \cap \Omega_E$$

where $\boldsymbol{\iota}$ is the identity map $\mathcal{W}_F \to {}_a\mathcal{W}_F \subseteq \Omega_F$.

(b) The subgroup $\mathcal{W}_F^E$ is open and of finite index in $\mathcal{W}_F$; it is normal in $\mathcal{W}_F$ if and only if E/F is Galois.

(c) The canonical map $\mathcal{W}_F^E \setminus \mathcal{W}_F \to \Omega_E \setminus \Omega_F$ is a bijection.

(d) The canonical map $\boldsymbol{\iota}_E : \mathcal{W}_E \to \Omega_E$ induces a topological isomorphism $\mathcal{W}_E \cong \mathcal{W}_F^E$.

2. The map $E/F \mapsto \mathcal{W}_F^E$ is a bijection between the set of finite extensions E of F inside $\overline{F}$ and the set of open subgroups of $\mathcal{W}_F$ of finite index.

Due to this proposition, we indentify $\mathcal{W}_E$ with the subgroup $\mathcal{W}_F^E$ of $\mathcal{W}_F$ going forward.

The representations of $\mathcal{W}_F$ form a particularly nice subcollection of the representations of Ω_F, and it is these representations that we will be studying.

3.4 Local Class Field Theory

Theorem 3.4.1 There is a canonical continuous group homomorphism

$$\boldsymbol{a}_F : \mathcal{W}_F \to F^\times$$

with the following properties.

1. The map $\boldsymbol{a}_F$ induces a topological isomorphism $\mathcal{W}_F^{\mathrm{ab}} \cong F^\times$.

2. An element $x \in \mathcal{W}_F$ is a geometric Frobenius if and only if $\boldsymbol{a}_F(x)$ is a prime element of F.

3. We have $\boldsymbol{a}_F(\mathcal{I}_F) = U_F$ and $\boldsymbol{a}_F(\mathcal{P}_F) = U_F^1$.

4. If E/F is a finite separable extensions, the diagram

$$\begin{array}{ccc} \mathcal{W}_E & \xrightarrow{\boldsymbol{a}_E} & E^\times \\ \downarrow & & \downarrow N_{E/F} \\ \mathcal{W}_F & \xrightarrow[\boldsymbol{a}_F]{} & F^\times \end{array}$$

commutes.

5. Let $\alpha : F \to F'$ be an isomorphism of fields. The map α induces an isomorphism $\alpha : \mathcal{W}_F^{\mathrm{ab}} \to \mathcal{W}_{F'}^{\mathrm{ab}}$, and the diagram

$$\begin{array}{ccc} \mathcal{W}_F^{\mathrm{ab}} & \xrightarrow{\alpha} & \mathcal{W}_{F'}^{\mathrm{ab}} \\ \boldsymbol{a}_F \downarrow & & \downarrow \boldsymbol{a}_{F'} \\ F^\times & \xrightarrow[\alpha]{} & F'^\times \end{array}$$

commutes.

One consequence of Theorem 3.4.1 is that the map $\boldsymbol{a}_F$, which we call the Artin reciprocity map, gives an isomorphism $\chi \mapsto \chi \circ \boldsymbol{a}_F$ between the group of characters of $F^\times$ and the group of characters of $\mathcal{W}_F$. More precisely, we have that unramified characters of $F^\times$ (trivial on U_F) correspond to unramified characters of $\mathcal{W}_F$ (trivial on $\mathcal{I}_F$); and tamely ramified characters of $F^\times$ (trivial on U_F^1, level $n = 0$) correspond to characters of $\mathcal{W}_F$ trivial on $\mathcal{P}_F$. We can use these correspondences to define the L-functions, ϵ-factors, and γ-factors for representations of the Weil group.

3.5 L-Functions and ϵ-Factors of Weil Groups

Definition 3.5.1 If χ is a character of $\mathcal{W}_F$, then we define

$$L(\chi, s) = L(\chi \circ \boldsymbol{a}_F, s)$$

$$\epsilon(\chi, s, \psi) = \epsilon(\chi \circ \boldsymbol{a}_F, s, \psi)$$

Where the functions on the right are those defined in Section 3.2.

Going forward, we will use χ instead of $\chi \circ \boldsymbol{a}_F$ when it is clear we mean the character of $F^\times$ corresponding to the character of $\mathcal{W}_F$. Now that we have defined L and ϵ for 1 dimensional characters, we can extend their definitions to n-dimensional representations of $\mathcal{W}_F$.

The L function is then easy to extend to semisimple representations of $\mathcal{W}_F$. We simply say that $L(\sigma, s) = 1$ for irreducible representations σ with dimensions ≥ 2. Then we make L multiplicative by requiring that

$$L(\sigma_1 \oplus \sigma_2, s) = L(\sigma_1, s)L(\sigma_2, s).$$

It takes more work to define the ϵ-factor for all semisimple representations of $\mathcal{W}_F$. The main properties we need are that if ρ_1, ρ_2 are semisimple representations of $\mathcal{W}_f$,then

$$\epsilon(\rho_1 \oplus \rho_2, s, \psi_F) = \epsilon(\rho_1, s, \psi_F)\epsilon(\rho_2, s, \psi_F),$$

which is the multipicativity that we expect. We also have an additional property that allows for induction of the local constant ϵ for characters. This induction is the following property:

Proposition 3.5.2 If ρ is a semisimple n-dimensional representation of $\mathcal{W}_E$ and $E \supset F$, then

$$\frac{\epsilon(\mathrm{Ind}_{E/F}\rho, s, \psi_F)}{\epsilon(\rho, s, \psi_E)} = \frac{\epsilon(R_{E/F}, s, \psi_F)^n}{\epsilon(1_E, s, \psi_E)^n}.$$

Where 1_E is the trivial character on $\mathcal{W}_E$ and $R_{E/F} = \mathrm{Ind}_F^E 1_E$.

Multiplicativity and Proposition 3.5.2 allows us to define the ϵ-factor for any semi-simple representation because any irreducible representation of $\mathcal{W}_F$ is the induced representation of a character from an appropriate finite extension. Finally, as was the case for $F^\times$, we still define the γ-factor for semi-simple representation ρ of $\mathcal{W}_F$ as

$$\gamma(\rho, s, \psi) = \epsilon(\rho, s, \psi) \frac{L(\check{\rho}, 1-s)}{L(\rho, s)}.$$

We also will find useful a few results from [BH06] on which representations are induced from a representation of an extension. First, we define an admissible pair.

Definition 3.5.3 Let E/F be a tamely ramified quadratic extension and χ a character of $E^\times$, we call $(E/F, \chi)$ an **admissible pair** if

1. χ does not factor through the norm map $N_{E/F} : E^\times \to F^\times$ and,
2. if $\chi|_{U_E^1}$ does factor through $N_{E/F}$, then E/F is unramified.

We will let $\mathbb{P}_2(F)$ be the set of isomorphism classes of admissible pairs. Further, we will let $\mathcal{G}_2^0$ be the set of isomorphism classes of irreducible 2-dimensional representations of $\mathcal{W}_F$ and $\mathcal{G}_2^{\mathrm{nr}}$ be the set of isomorphism classes $\rho \in \mathcal{G}_2^0$ such that there is a non-trivial unramified character χ of $\mathcal{W}_F$ such taht $\chi \otimes \rho \cong \rho$. With these definition, we have the following theorem, which will allow us to work only with characters for $\mathcal{W}_E$ for appropriate extensions E/F.

Theorem 3.5.4 If $(E/F, \xi)$ is an admissible pair, the representation $\mathrm{Ind}_{E/F}\xi$ of $\mathcal{W}_F$ is irreducible. The map $(E/F, \xi) \mapsto \mathrm{Ind}_{E/F}\xi$ induces a bijection

$$\mathbb{P}_2(F) \to \mathcal{G}_2^0(F) \;\; \text{if } p \neq 2, \text{ or}$$

$$\mathbb{P}_2(F) \to \mathcal{G}_2^{\mathrm{nr}}(F) \;\; \text{if } p = 2.$$

Finally, we have another theorem from Bushnell and Henniart, Theorem A.3 in [BH05], which refines the bijection presented above. For $\sigma \in \mathcal{G}_2^0(F)$, they let $t(\sigma)$ be the number of (isomorphism classes) of unramified characters χ of $\mathcal{W}_F$ such that $\sigma \otimes \chi \cong \sigma$. They say that σ is **essentially tame** if p does not divide $n/t(\sigma)$, or equivalently, that $\sigma_{\mathcal{P}_F}$ is a sum of characters. Then the map $\mathbb{P}_2(F) \to \mathcal{G}_2^{\text{et}}(F)$ defined by $(E/F, \xi) \mapsto \text{Ind}_{E/F}\xi$ is a bijection for all primes p and all $n \geq 1$. Since all tamely ramified ρ are essentially tame; this tells use that all tamely ramified irreducible ρ are induced from a tamely ramified character of an unramified extension E/F.

3.6 Local Converse Theorem for 2 Dimensional Representations of the Weil Group

Finally, we have all the pieces necessary to state the local converse theorem for representations of the Weil Group. What has been proven in the most general case using the Langlands correspondence is the following theorem:

Theorem 3.6.1 (Local Converse Theorem for Weil Groups) Let ρ_1 and ρ_2 be n-dimensional semisimple representations of $\mathcal{W}_F$, with $n \geq 2$, such that for all semi-simple representations σ of $\mathcal{W}_F$ with dimension $k \leq \lfloor \frac{n}{2} \rfloor$ we have

$$\gamma(\rho_1 \otimes \sigma, s, \psi) = \gamma(\rho_2 \otimes \sigma, s, \psi).$$

Then $\rho_1 \cong \rho_2$.

What we would like to prove here is a local converse theorem specifically for the case $n = 2$, which can be stated slightly more simply as follows:

Theorem 3.6.2 (Local Converse Theorem on $\mathcal{W}_F$ with $n = 2$) Let ρ_1 and ρ_2 be 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$, we have

$$\gamma(\rho_1 \otimes \chi, s, \psi) = \gamma(\rho_2 \otimes \chi, s, \psi);$$

then $\rho_1 \cong \rho_2$.

Note that we also restrict to tamely ramified representations in this theorem, though this only matters when the ρ_i are irreducible. Overall, this theorem can be split into three cases depending on the number of poles of γ, then two of those cases can be further split into two more cases each depending on the nature of the representations involved. We shall prove each of those cases first as lemmas, then we shall prove that those are the only cases and so the local converse theorem on $\mathcal{W}_F$ holds for $n = 2$.

For the most part, we are not interested in how γ behaves as a function of $s \in \mathbb{C}$, so we instead write $\gamma(\rho, X, \psi) = \gamma(\rho, s, \psi)$ where $X = q^{-s}$. Then we also have that $\gamma(\rho, 1-s, \psi) = \gamma(\rho, \frac{1}{qX}, \psi)$ and the same for ϵ and L. This simplifies the discussion around the number of poles of γ, so we can say that γ has either 2, 1, or 0 poles without worrying about the periodicity of q^{-s}. In this notation, we will have the following definitions for ϵ, L, and γ restated here for simplicity.

As expected, the γ-factor modification is easy and we just get that

$$\gamma(\chi, X, \psi) = \epsilon(\chi, X, \psi)\frac{L(\check{\chi}, \frac{1}{qX})}{L(\chi, X)}.$$

L is simple as well and we find that

$$L(\chi, X) = \begin{cases} (1-\chi(\varpi)X)^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Finally, the ϵ-factor takes a bit more effort to convert to an expression in terms of X, but we find that

$$\epsilon(\chi, X, \psi) = \begin{cases} \frac{X^n}{\sqrt{q}} \sum_{x \in U_F/U_F^{n+1}} \chi(\alpha x)^{-1}\psi(\alpha x) & \text{for } \chi \text{ of level } n \text{ and } v_F(\alpha) = -n \\ \frac{1}{\sqrt{q}X\chi(\varpi)} & \text{for } \chi \text{ unramified.} \end{cases}$$

With these results, we can now begin proving the individual cases of Theorem 3.6.2.

Lemma 3.6.3 (γ has 2 poles) Suppose ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that

$$\gamma(\rho_1, X, \psi) = \gamma(\rho_2, X, \psi),$$

and $\gamma(\rho_i, X, \psi)$ has 2 poles. Then $\rho_1 \cong \rho_2$.

Proof. Suppose that $\gamma(\rho_1, X, \psi) = \gamma(\rho_2, X, \psi)$ has two poles. Recall that

$$\gamma(\rho_i, X, \psi) = \epsilon(\rho_i, X, \psi)\frac{L(\check{\rho}_i, \frac{1}{qX})}{L(\rho_i, X)}.$$

So $\gamma(\rho_i, X, \psi)$ has poles only when $\epsilon(\rho_i, X, \psi)$ has poles, when $L(\check{\rho}_i, \frac{1}{qX})$ has poles, and when $L(\rho_i, X)$ has zeroes. However, $\epsilon(\rho_i, X, \psi)$ never has poles because it's only X dependence comes from the X^n factor for not unramified level n characters, or the $\frac{1}{\sqrt{q}X}$ factor for unramified characters. Note that as an actual complex function, $X \neq 0$ because $X = q^{-s}$, so we discount this possibility for a pole. Similarly, $L(\rho_i, X)$ never has zeroes because it is defined multiplicatively as 1 for irreducible with dimension > 1 and ramified characters and $(1 - \chi(\varpi)X)^{-1}$ for unramified characters. Specifically, we see that $L(\check{\rho}_i, \frac{1}{qX})$ has one pole for each unramified character in $\check{\rho}_i$. Since $L(\check{\rho}_i, \frac{1}{qX})$ is the only source of poles in $\gamma(\rho_i, X, \psi)$, we must have that $L(\check{\rho}_i, \frac{1}{qX})$ has two poles and therefore ρ_i each contain two unramified characters. Since the ρ_i are two dimensional characters of $\mathcal{W}_F$, we therefore have that each ρ_i is in fact a direct sum of two unramified characters of $\mathcal{W}_F$.

Write $\rho_i = \theta_i \oplus \theta_i'$, with θ_i, θ_i' both unramified characters of $\mathcal{W}_F$. Since $\gamma(\rho_1, X, \psi) = \gamma(\rho_2, X, \psi)$ have the same two poles, we must have that $L(\check{\rho}_1, \frac{1}{qX})$ and $L(\check{\rho}_2, \frac{1}{qX})$ have the same poles which is equivalent to $L(\rho_1, X)$ and $L(\rho_2, X)$ having the same poles. Because the L-function is multiplicative we have that

$$L(\theta_1, X)L(\theta_1', X) = L(\theta_2, X)L(\theta_2', X).$$

Similarly, because we know that all the characters involved are unramified, we find that by treating the θ_i, θ_i' as characters of $F^\times$ we have

$$(1 - \theta_1(\varpi)X)^{-1}(1 - \theta_1'(\varpi)X)^{-1} = (1 - \theta_2(\varpi)X)^{-1}(1 - \theta_2'(\varpi)X)^{-1}.$$

The left side has poles at $X = \frac{1}{\theta_1(\varpi)}, \frac{1}{\theta_1'(\varpi)}$ and the right side has poles at $X = \frac{1}{\theta_2(\varpi)}, \frac{1}{\theta_2'(\varpi)}$. These must be the same, so without loss of generality, we can say $\theta_1(\varpi) = \theta_2(\varpi)$ and

$\theta'_1(\varpi) = \theta'_2(\varpi)$. However, unramified characters of $\mathcal{W}_F$ correspond to unramified characters of $F^\times$ which are trivial on U_F. Since every element of $x \in F^\times$ can be written as $u\varpi^m$ for some $u \in U_F$ and $m \in \mathbb{Z}$, we have that

$$\theta(x) = \theta(u\varpi^m) = \theta(u)\theta(\varpi)^m = \theta(\varpi)^m.$$

So every unramified character of $F^\times$ is fully determined by its value on ϖ, and similarly for characters of $\mathcal{W}_F$. This tells us that $\theta_1 = \theta_2$ and $\theta'_1 = \theta'_2$, so we clearly find that

$$\rho_1 \cong \rho_2$$

which is what we wanted to show. □

Next is the case when $\gamma(\rho_i, X, \psi)$ only has one pole, which can occur in two different ways.

Lemma 3.6.4 (γ has 1 poles) Suppose ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$, we have

$$\gamma(\rho_1 \otimes \chi, X, \psi) = \gamma(\rho_2 \otimes \chi, X, \psi),$$

and $\gamma(\rho_i, X, \psi)$ has 1 poles. Then $\rho_1 \cong \rho_2$.

Proof. Suppose ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$, we have

$$\gamma(\rho_1 \otimes \chi, X, \psi) = \gamma(\rho_2 \otimes \chi, X, \psi),$$

and $\gamma(\rho_i, X, \psi)$ has 1 poles. As was mentioned in the proof of Lemma 3.6.3, the only source of poles in $\gamma(\rho_i, X, \psi)$ is the poles of $L(\check{\rho}_i, \frac{1}{qX}, \psi)$. So in order to have only one pole we must either have that the ρ_i are the sum of an unramified character and a ramified tamely ramified character; or the ρ_i are a sum of two unramified characters and that somehow a pole of $L(\check{\rho}_i, \frac{1}{qX})$ cancels out with a zero of $L(\rho_i, X)$. We will need to show that those two cases cannot coexist, and then prove the converse theorem in each case. We can separate these cases by looking at the zeroes of $\gamma(\rho_i, X, \psi)$ as well as the poles. First, suppose $\rho_i = \theta_i \oplus \xi_i$ with θ_i unramified and ξ_i ramified tamely ramified. Then we find that $\gamma(\theta_i \oplus \xi_i, X, \psi)$ has a pole at

$$X = \frac{1}{q\theta_i(\varpi)}$$

coming from $L(\check{\theta}_i, \frac{1}{qX})$ and a zero at

$$X = \frac{1}{\theta_i(\varpi)}$$

coming from $L(\theta_i, X)$. On the other hand, suppose that $\rho_i = \theta_i \oplus \theta_i'$ with θ_i and θ_i' both unramified. Then as before, we have that (without cancellation of zeroes and poles) $\gamma(\rho_i, X, \psi)$ has poles at

$$X = \frac{1}{q\theta_i(\varpi)}, \frac{1}{q\theta_i'(\varpi)}$$

and has zeroes at

$$X = \frac{1}{\theta_i(\varpi)}, \frac{1}{\theta_i'(\varpi)}.$$

In order to only have a single pole in this case, we must have that one of the zeroes cancels out one of the poles. Without loss of generality, we say that

$$\theta_i'(\varpi) = q\theta_i(\varpi).$$

Since an unramified character θ_i' is fully defined by the value it takes on ϖ, we have that there is only one choice for θ_i' for any given θ. The left over zeroes and poles are then required to be a pole coming from $L(\check{\theta}_i', \frac{1}{qX})$ at

$$X = \frac{1}{q^2\theta_i(\varpi)}$$

and a zero coming from $L(\theta_i, X)$ at

$$X = \frac{1}{\theta_i(\varpi)}$$

What is of note here is that in the case when $\rho_i = \theta_i \oplus \xi_i$ is the sum of an unramified character and a ramified tamely ramified character, then the ratio between the pole and the zero of $\gamma(\rho_i, X, \psi)$ is q. On the other hand, when $\rho_i = \theta_i \oplus \theta_i'$ is the sum of two unramified characters with poles and zeroes that cancel out, then the ratio between the pole and the zero of $\gamma(\rho_i, X, \psi)$ is q^2. So we can tell these cases apart by looking at the zeroes and poles of $\gamma(\rho_i, X, \psi)$.

Now, let us show that both of these types of representations satisfy the converse theorem. We will start with the case where $\rho_i = \theta_i \oplus \theta_i'$ is the sum of two unramified characters with $\theta_i'(\varpi) = q\theta_i(\varpi)$. Since we have

$$\gamma(\theta_1 \oplus \theta_1', X, \psi) = \gamma(\theta_2 \oplus \theta_2', X, \psi)$$

we again must have that the poles are equal on each side. As mentioned above, $\gamma(\theta_i \oplus \theta_i', X, \psi)$ must have a pole at $X = \frac{1}{q^2\theta(\varpi)}$ if θ_i and θ_i' are both unramified and their poles and zeroes cancel. But

$$\frac{1}{q^2\theta_1(\varpi)} = \frac{1}{q^2\theta_2(\varpi)}$$

clearly implies $\theta_1(\varpi) = \theta_2(\varpi)$ and so $\theta_1 = \theta_2$ and $\theta_1' = \theta_2'$ because they are unramified and so determined by their value at ϖ. As desired, this gives us $\rho_1 = \rho_2$ for the unramified case.

Next, consider the case of $\rho_i = \theta_i \oplus \xi_i$ with θ_i unramified and ξ_i ramified tamely ramified (level $n = 0$). We have that $L(\rho_i, X) = L(\theta_i \oplus \xi_i, X) = L(\theta_i, X)L(\xi_i, X) = L(\theta_i, X)$ because ramified characters have trivial L functions. So we can still identify that the poles of $\gamma(\rho_1, X, \psi)$ and $\gamma(\rho_2, X, \psi)$ are the same so $\theta_1(\varpi) = \theta_2(\varpi)$ which like the previous cases tells us that $\theta_1 = \theta_2$, so just call it θ.

The conditions of the converse theorem tell us that

$$\gamma((\theta \oplus \xi_1) \otimes \chi, X, \psi) = \gamma((\theta \oplus \xi_2) \otimes \chi, X, \psi)$$

for all characters χ of $\mathcal{W}_F$. However, because $(\theta \oplus \xi_1) \otimes \chi = (\theta \otimes \chi) \oplus (\xi_1 \otimes \chi)$ and γ is multiplicative, we have that

$$\gamma(\xi_1 \otimes \chi, X, \psi) = \gamma(\xi_2 \otimes \chi, X, \psi)$$

Now, consider twisting by $\chi = \xi_1^{-1}$. Then we have that

$$\gamma(\xi_1 \otimes \xi_1^{-1}, X, \psi) = \gamma(\xi_2 \otimes \xi_1^{-1}, X, \psi).$$

$\xi_1 \otimes \xi_1^{-1}$ is the trivial character 1_F and is clearly unramified. Since $\gamma(\xi_2 \otimes \xi_1^{-1}, X, \psi)$ must have the same poles and zeros as $\gamma(1_F, X, \psi)$, we find that $\xi_2 \otimes \xi_1^{-1}$ must also be unramified, and agree with 1_F on ϖ, making $1_F = \xi_2 \otimes \xi_1^{-1}$. This is equivalent to $\xi_1 = \xi_2$. Combined with $\theta_1 = \theta_2$, we find that $\rho_1 \cong \rho_2$ as desired. □

Now we turn our attention to the final case when γ has no poles, which is when there are no unramified characters present. In this case we have the following lemma.

Lemma 3.6.5 (γ has 0 poles) Let F be a local field with residue field $\boldsymbol{k}_F = \mathbb{F}_q$ where $q = p^n$ and $p \neq 2$. Suppose ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$, we have

$$\gamma(\rho_1 \otimes \chi, X, \psi) = \gamma(\rho_2 \otimes \chi, X, \psi),$$

and $\gamma(\rho_i, X, \psi)$ has 0 poles. Then $\rho_1 \cong \rho_2$.

Proof. Let F be a local field with a residue field not of characteristic 2, and suppose ρ_1 and ρ_2 are 2-dimensional, tamely ramified, semisimple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$, we have

$$\gamma(\rho_1 \otimes \chi, X, \psi) = \gamma(\rho_2 \otimes \chi, X, \psi),$$

and $\gamma(\rho_i, X, \psi)$ has 0 poles. We will need to show that there is no way for all the poles to cancel out if one of the ρ_i has an unramified character as a subrepresentation. Then the two ways in which we can have no poles are either that the ρ_i are the sum of two ramified tamely ramified characters or that the ρ_i are two dimensional ramified tamely ramified irreducibles. Next we will need a way to distinguish between these two cases; then we will show that the converse theorem holds in each case.

The proof of Lemma 3.6.4 shows us why it is not possible for γ to have no poles if there is an unramified portion of the ρ_i. Now, suppose first that $\rho_i = \xi_i \oplus \xi_i'$ with ξ_i, ξ_i' both tamely ramified (level n=0). Consider twisting by $\chi = \xi_i^{-1}$, then we get $\rho_i \otimes \chi = 1_F \oplus (\xi_i^{-1} \otimes \xi_i')$. Note that $\gamma(\rho_i \otimes \chi, X, \psi)$ will have at least one pole in this case.

On the other hand, if ρ_i is a two dimensional ramified tamely ramified irreducible representation of $\mathcal{W}_F$, then there is some unramified quadratic extension E_i/F and ramified tamely ramified character ξ_i of $\mathcal{W}_{E_i}$ such that $\rho_i = \mathrm{Ind}_{E_i/F}\xi_i$. Since there is a unique unramified quadratic extension of F inside a given separable closure $\overline{F}$, we just call it E and let $\rho_i = \mathrm{Ind}_{E/F}$. Recall that by Proposition 3.5.2 we have

$$\frac{\epsilon(\mathrm{Ind}_{E/K}\rho, s, \psi_K)}{\epsilon(\rho, s, \psi_E)} = \frac{\epsilon(R_{E/K}, s, \psi_K)^n}{\epsilon(1_E, s, \psi_E)^n}.$$

Rearranging this and rewriting in terms of X and the representations we are interested in, we have that

$$\epsilon(\rho_i, X_F, \psi_F) = \epsilon(\mathrm{Ind}_{E/F}\xi_i, X_F, \psi_F) = \epsilon(\xi_i, X_E, \psi_E)\frac{\epsilon(\mathrm{Ind}_{E/F}1_E, X_F, \psi_F)^2}{\epsilon(1_E, X_E, \psi_E)^2},$$

where $X_F = q^{-s}$ and $X_E = (q^2)^{-s}$. Now, note that $(\mathrm{Ind}_{E/F}\xi_i) \otimes \chi = \mathrm{Ind}_{E/F}(\xi_i \otimes \chi \circ N_{E/F})$. Since $L(\rho_i) = 1$ for ρ_i irreducible dimension > 1, we have that $\gamma(\rho_i, X, \psi)$ has no poles as expected for this case. However, we further have that $\gamma(\rho_i \otimes \chi, X, \psi)$ will never have a pole because $\mathrm{Ind}_{E/F}(\xi_i)$ is irreducible if and only if ξ is different than its conjugate, i.e. $\xi_i \neq \xi_i^q$. This is also equivalent to saying that ξ_i doesn't factor through the norm. So twisting a ξ_i that doesn't factor through the norm by $\chi \circ N_{E/F}$ will never produce something that does factor through the norm, meaning that $(\mathrm{Ind}_{E/F}\xi_i) \otimes \chi$ is irreducible and so $\gamma(\rho_i \otimes \chi, s, \psi)$ will never have a pole. This allows us to tell apart $\rho_i = \xi_i \oplus \xi_i'$ and $\rho_i = \mathrm{Ind}_{E/F}\xi_i$; the first case will have some twist χ that gives $\gamma(\rho_i, X, \psi)$ at least one pole and the second case will never have such a χ.

Now, let us deal with the $\rho_i = \xi_i \oplus \xi_i'$ case with ξ_i, ξ_i' ramified tamely ramified. This case works out much like the case when γ has one pole and ρ is a sum of an unramified and a ramified tamely ramified character. In fact, consider the twist by $\chi = \xi_1^{-1}$. We get

$$\gamma(1_F \oplus (\xi_1' \otimes \xi_1^{-1}), X, \psi) = \gamma((\xi_2 \otimes \xi_1^{-1}) \oplus (\xi_2' \otimes \xi_1^{-1}), X, \psi)$$

Because 1_F is unramified, we must have at least one pole, possibly two poles. That is either we are in the case of Lemma 3.6.4 or Lemma 3.6.3; either way, we find that, without loss of generality, $\xi_1 = \xi_2$ and $\xi_1' = \xi_2'$ and so $\rho_1 = \rho_2$.

Finally, let us deal with the case when $\rho_i = \mathrm{Ind}_{E/F}\xi_i$ with ξ_i a tamely ramified character of $\mathcal{W}_E$. We will first use stability to find the value of ξ_i on ϖ, then we will use the Gauss sum local converse to show that ξ_1 and ξ_2 agree sufficiently on U_F to have $\xi_1 \cong \xi_2$.

Note that the parts in the fraction for the induction constant depend only on the extension E which is fixed, we in fact have that if $\epsilon(\rho_1, X, \psi_F) = \epsilon(\rho_2, X, \psi_F)$, then $\epsilon(\xi_1, X, \psi_E) = \epsilon(\xi_2, X, \psi_E)$. Now, applying the Stability Theorem, Theorem 3.2.8, for E with a twist by the level 1 character χ of E defined by $\chi(1+x) = \psi(\varpi^{-1}x)$, we find that

$$\xi_1(\varpi^{-1})^{-1}\epsilon(\chi, X, \psi_E) = \xi_2(\varpi^{-1})^{-1}\epsilon(\chi, X, \psi_E).$$

This tells us that $\xi_1(\varpi) = \xi_2(\varpi)$ and so we only need to understand how they act on units. Since the ξ_i are tamely ramified, we already know that they are trivial on $U_E^1 = U_F^1$, we only need to study the character's restriction of $U_E/U_E^1 \cong \mathbb{F}_{q^2}$. Similarly, when we restrict tamely ramified characters χ of $\mathcal{W}_F$ to the units, we can in fact study how they act on $U_F/U_F^1 \cong \mathbb{F}_q$. This puts us in the case of the Gauss sum converse theorem where we have characters of $\mathbb{F}_{q^2}$ being twisted by characters of $\mathbb{F}_q$; and Lemma 3.2.9 provides this reduction for us. Applying that reduction, and then the Gauss sum local converse theorem proves that $\xi_1 \cong \xi_2$ or $\xi_1 \cong \xi_2^q$ on U_E. However, $\mathrm{Ind}_{E/F}\xi_1$ and $\mathrm{Ind}_{E/F}\xi_1^q$ are isomorphic, so we do indeed have that $\rho_1 \cong \rho_2$ as desired.

□

Now that we have the necessary lemmas, we can assemble them into a proof of Theorem 3.6.2.

Proof. Suppose that ρ_1 and ρ_2 are 2-dimensional tamely ramifed semi-simple representations of $\mathcal{W}_F$, such that for all characters χ of $\mathcal{W}_F$ we have

$$\gamma(\rho_1 \otimes \chi, X, \psi) = \gamma(\rho_2 \otimes \chi, X, \psi)$$

As mentioned before, the only place poles can come from is the factor of $L(\check{\rho}_i \otimes \check{\chi}, \frac{1}{qX})$, so there can be at most two poles of $\gamma(\rho_i \otimes \chi, X, \psi)$. In the case when there are two poles, Lemma 3.6.3 proves the converse theorem without using any twists. Alternatively, this can be viewed as twisiting only by the trivial character which could be thought of as a zero dimensional thing. In the case where this is one pole, Lemma 3.6.4 proves the converse theorem using at most 1 twist. Finally, in the case where there are zero poles, Lemma 3.6.5 proves the converse theorem using all of the twists by characters since we must use them to distinguish irreducibles. Since we can have at most two poles and we can't have fewer than zero; this covers all posibilities and concludes the proof of Theorem 3.6.2. □

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