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A Vector-valued Trace Formula for Finite Groups

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A VECTOR-VALUED TRACE FORMULA FOR FINITE GROUPS

By

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An Abstract of the Thesis Presented
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We derive a trace formula that can be used to study representations of a finite group G induced from arbitrary representations of a subgroup Γ. We restrict our attention to finite-dimensional representations over the field of complex numbers. We consider some applications and examples of our trace formula, including a proof of the well-known Frobenius reciprocity theorem.

DEDICATION

To Ru, with love.

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PREFACE

It is well-known that the trace of a square matrix is the sum of its diagonal entries and, equivalently, the sum of its eigenvalues. This equality is perhaps the most simple example of a trace formula, which is the equality of a linear operator's trace expressed in two different ways.

The Selberg trace formula, introduced in [Sel56], is commonly used to study linear operators on complex-valued functions of a group G. However, as it was originally presented in [Sel56], the Selberg trace formula considers the more general setting of vector-valued functions. One can find a detailed exposition of the vector-valued case in [Hej76]. While [Hej76] explains in detail the analysis that was omitted in [Sel56], it assumes a comprehensive understanding of linear algebra; consequently, much of the algebra underlying the vector-valued case is implicit. Our goal in this thesis is to explain the linear algebra of the Selberg trace formula, and for this it is enough to focus on the simplest case of a finite group.

We consider the representation π_τ of a finite group G induced from an arbitrary finite-dimensional representation τ of a subgroup Γ. If we denote the space of τ by V, then π_τ defines a group action of G on the space vector-valued functions,

$$C(\tau) = \{\varphi : G \to V \mid \varphi(\gamma g) = \tau(\gamma)\varphi(g) \, \forall \, \gamma \in \Gamma, g \in G\}.$$

The group action is given by right translation. That is, if $\varphi \in C(\tau)$ and $g \in G$, then $[\pi_\tau(g)\varphi](x) = \varphi(xg)$ for all $x \in G$. For a complex-valued function $f : G \to \mathbb{C}$, we define the linear operator $\pi_\tau(f) : C(\tau) \to C(\tau)$ by the linear combination

$$\pi_\tau(f) = \sum_{g \in G} f(g)\pi_\tau(g).$$

Our formula examines the trace of $\pi_\tau(f)$. It is derived using only basic results from linear algebra and representation theory.

In the last few decades, several authors have examined analogs of the Selberg trace formula for finite groups. It is well-known that the Selberg trace formula, when applied to a finite group, yields the Frobenius reciprocity theorem. This is mentioned in [Art89]. The finite group case was examined in detail by [Ter99], deriving the trace formula for $\pi_\tau(f)$ where τ is the trivial representation of Γ. In this case, $\pi_\tau(f)$ is a linear operator on the space of complex-valued functions $\{\varphi : \Gamma\backslash G \to \mathbb{C}\}$. Later, [Yan06] derived a trace formula for $\pi_\tau(f)$ where τ is the trivial representation of Γ on a finite-dimensional complex vector space W. Here, $\pi_\tau(f)$ is a linear operator on the space of vector-valued functions $\{\varphi : \Gamma\backslash G \to W\}$, which is isomorphic to a direct sum of $\dim W$ copies of $\{\varphi : \Gamma\backslash G \to \mathbb{C}\}$. The trace formula we present in this thesis was left as an exercise in [Whi10, p. 8], though to our knowledge, it has not been derived anywhere in the published literature.

The first two chapters of this thesis provide some relevant and well-known background on the theory of group representations. Chapter 1 reviews the most basic results and definitions from representation theory. Along the way, we also prove some simple results that will assist us in deriving our trace formula. In Chapter 2, we introduce the group algebra and examine its properties and applications in representation theory. In Chapter 3, we derive our trace formula and conclude with some examples and applications.

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CHAPTER 1

REPRESENTATIONS OF FINITE GROUPS

In this chapter, we review some basic definitions and results from representation theory. We will restrict our attention to representations of finite groups on finite-dimensional complex vector spaces. In Section 1.1, we provide the definition of a group representation and consider some examples. In Section 1.2, we examine the notion of equivalence between two representations. Section 1.3 examines how representations can be combined to form new representations. Sections 1.4 through 1.6 examine how a representation is decomposed into subrepresentations, and the extent to which this decomposition is unique. We then conclude by examining how a representation of a finite group gives rise to representations of subgroups and ambient groups.

Many of the results and definitions in this chapter are from [FH91] and [Ser77]. Our treatment of intertwining numbers in Section 1.6 and our construction of the induced representation in Section 1.7 are both adapted from unpublished notes by Rogawski.

1.1 Definitions and examples

Let V be a finite-dimensional vector space over the field of complex numbers. The *general linear group of* V is the set $\mathrm{GL}(V) = \{T : V \to V \mid T \text{ is a bijective linear map}\}$ with the operation of function composition. We verify that $\mathrm{GL}(V)$ is indeed a group. Suppose $S, T \in \mathrm{GL}(V)$. Since $S : V \to V$ and $T : V \to V$ are bijective, it follows that $S \circ T : V \to V$ is also bijective. Furthermore, for any $v, w \in V$ and $\alpha \in \mathbb{C}$ we have

$$S(T(v + \alpha w)) = S(T(v) + T(\alpha w)) = S(T(v) + \alpha T(w)) = S(T(v)) + \alpha S(T(w))$$

since S and T are both linear. Therefore $S \circ T : V \to V$ is a bijective linear map and $\mathrm{GL}(V)$ is closed under function composition. The group operation is associative because function composition is associative. The identity element in $\mathrm{GL}(V)$ is simply the identity map $\mathrm{id}_V(v) = v$, and since each $S \in \mathrm{GL}(V)$ is bijective, there exists a bijective map

$S^{-1} : V \to V$ such that $S \circ S^{-1} = S^{-1} \circ S = \mathrm{id}_V$. To show that S^{-1} is linear, note that

$$S^{-1}(v) + \alpha S^{-1}(w) = S^{-1}\left(S\left(S^{-1}(v) + \alpha S^{-1}(w)\right)\right) = S^{-1}(v + \alpha w).$$

Hence $\mathrm{GL}(V)$ is a group under function composition.

Let G be a finite group with the operation $G \times G \to G$ given by $(g, h) \mapsto gh$. Suppose that $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. If $e \in G$ is the identity element in G, then $\rho(e)$ is the identity map $\mathrm{id}_V : V \to V$ and for any $v \in V$ we have $\rho(e)(v) = \mathrm{id}_V(v) = v$. If $g, h \in G$, then $[\rho(g) \circ \rho(h)](v) = \rho(gh)(v)$ for all $v \in V$. Hence, a group homomorphism $\rho : G \to \mathrm{GL}(V)$ defines a group action of G on V. We formalize this notion as follows.

Definition 1.1. Let G be a finite group. A *representation* of G is an ordered pair (ρ, V), where V is a finite-dimensional complex vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homormorphism. We call V the *representation space* of G, and $\dim V$ the *dimension* of the representation.

Given a representation (ρ, V) of a group G, we may refer to ρ as a representation of G if the representation space V is understood from context. Likewise, we may refer to V as a representation of G if the homomorphism $\rho : G \to \mathrm{GL}(V)$ is understood from context. We conclude this section by considering some examples.

Example 1.2. *Let G be a finite group. The map $\rho : G \to \mathrm{GL}(\mathbb{C})$ defined by $\rho(g) = \mathrm{id}_{\mathbb{C}}$ for all $g \in G$ is a group homomorphism, which we call the trivial representation of G.*

Example 1.3. *Let G be a finite group, and let $\mathbb{C}[G]$ denote the space of complex-valued functions on G equipped with pointwise addition and scalar multiplication. We define a map $R : G \to \mathrm{GL}(\mathbb{C}[G])$, where $R(g) : \mathbb{C}[G] \to \mathbb{C}[G]$ is defined by*

$$[R(g)\varphi](x) = \varphi(xg)$$

for all $\varphi \in \mathbb{C}[G]$ and $x \in G$. Then $(R, \mathbb{C}[G])$ is a representation of G, which we call the right regular representation of G.

Proof. Clearly $R(g)$ is a linear map for each $g \in G$: if $\varphi, \psi \in \mathbb{C}[G]$ and $\lambda \in \mathbb{C}$, then we have for all $x \in G$,

$$\begin{aligned} [R(g)(\varphi + \lambda\psi)](x) &= (\varphi + \lambda\psi)(xg) \\ &= \varphi(xg) + \lambda\psi(xg) \\ &= [R(g)\varphi](x) + \lambda[R(g)\psi](x). \end{aligned}$$

Now observe that $R(g)R(g^{-1}) = R(g^{-1})R(g) = \mathrm{id}_{\mathbb{C}[G]}$, so $R(g)$ is invertible for each $g \in G$. To verify that R is a group homomorphism, let $g, h \in G$ and note that

$$\begin{aligned} [R(gh)\varphi](x) &= \varphi(xgh) \\ &= [R(h)\varphi](xg) \\ &= [R(g) \circ R(h)\varphi](x) \end{aligned}$$

for all $x \in G$. Hence $(R, \mathbb{C}[G])$ is a representation of G. □

1.2 Intertwining operators and equivalence of representations

The purpose of this section is to examine the notion of equivalence between two representations of a finite group G. We begin by defining the notion of an intertwining operator between two representations, which is analogous to a homomorphism between two groups.

Definition 1.4. Let (ρ, V) and (σ, W) be representations of a finite group G. A linear map $T : V \to W$ is called an *intertwining operator* if

$$T\rho(g)v = \sigma(g)Tv$$

for all $g \in G$ and all $v \in V$. That is, $T : V \to W$ is a linear map that makes the following diagram commute for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \sigma(g)} \\ V & \xrightarrow[T]{} & W \end{array}$$

The set of all intertwining operators $T : V \to W$ is denoted $\mathrm{Hom}_G(V, W)$, or $\mathrm{Hom}_G(\rho, \sigma)$ if the underlying vector spaces are understood from context.

Suppose (ρ, V) and (σ, W) are representations of a finite group G. If $T : V \to W$ is an invertible intertwining operator, then $T(V) = W$ is a relabeling of the vectors in V, and the group actions defined by ρ and σ are essentially identical. This motivates the following.

Definition 1.5. Let (ρ, V) and (σ, W) be representations of G. We say that (ρ, V) and (σ, W) are *equivalent* and write $(\rho, V) \cong (\sigma, W)$ if there exists a bijective intertwining operator $T : V \to W$.

Lemma 1.6. *Let G be a finite group. Then Definition 1.5 gives rise to an equivalence relation on the set of representations of G.*

Proof. Suppose (τ, U), (ρ, V) and (σ, W) are representations of a finite group G. We must show that

(1) $(\rho, V) \cong (\rho, V)$;

(2) $(\rho, V) \cong (\sigma, W)$ if and only if $(\sigma, W) \cong (\rho, V)$;

(3) $(\tau, U) \cong (\rho, V)$ and $(\rho, V) \cong (\sigma, W)$ implies $(\tau, U) \cong (\sigma, W)$.

For the first claim, it is easy to show that $(\rho, V) \cong (\rho, V)$ via the identity map $\mathrm{id}_V : V \to V$.

For the second claim, suppose $(\rho, V) \cong (\sigma, W)$. Then there exists an invertible linear map $T : V \to W$ such that $T\rho(g)v = \sigma(g)Tv$ for all $g \in G$ and $v \in V$. Because T is a bijective linear map, it has an inverse $T^{-1} : W \to V$ which is also linear. It follows that, for any $g \in G$ and any $Tv = w \in W$,

$$T^{-1}\sigma(g)w = T^{-1}\sigma(g)Tv = T^{-1}T\rho(g)v = \rho(g)T^{-1}w.$$

Thus, $T^{-1} : W \to V$ is a bijective intertwining operator and we have $(\sigma, W) \cong (\rho, V)$.

For the final claim, suppose that $(\tau, U) \cong (\rho, V)$ via $S : U \to V$ and suppose $(\rho, V) \cong (\sigma, W)$ via $T : V \to W$. Note that $T \circ S : U \to W$ is a bijective linear map because S and T are both bijective and linear. Now for any $u \in U$ and $g \in G$, we have

$$TS\tau(\gamma)u = T\rho(\gamma)Su = \sigma(g)TSu.$$

Hence (τ, U) and (σ, W) are equivalent. □

If V and W are complex vector spaces, we let $\mathrm{Hom}(V, W)$ denote the set of $\mathbb{C}$-linear maps $V \to W$. It is easy to show that $\mathrm{Hom}(V, W)$ forms a complex vector space, with addition and scalar multiplication defined pointwise. We conclude this section by proving the following result.

Proposition 1.7. *Let (ρ, V) and (σ, W) be representations of a finite group G. Then $\mathrm{Hom}_G(\rho, \sigma)$ is a subspace of $\mathrm{Hom}(V, W)$.*

Proof. Let $S, T \in \mathrm{Hom}_G(\rho, \sigma)$ and let $\lambda \in \mathbb{C}$. Clearly $\mathrm{Hom}_G(\rho, \sigma) \subseteq \mathrm{Hom}(V, W)$ because each intertwining operator in $\mathrm{Hom}_G(\rho, \sigma)$ is a linear map from V to W. Note also that $S + \lambda T : V \to W$ is a linear map since S and T are both linear. Now for any $v \in V$ and $g \in G$, we have

$$(S + \lambda T)\rho(g)v = S\rho(g)v + \lambda T\rho(g)v = \sigma(g)Sv + \lambda\sigma(g)Tv$$

because S and T are intertwining operators. Now recall that $\sigma(g) \in \mathrm{GL}(W)$ is a linear map, whence

$$\sigma(g)Sv + \lambda\sigma(g)Tv = \sigma(g)Sv + \sigma(g)\lambda Tv = \sigma(g)(S + \lambda T)v.$$

Thus, $S + \lambda T$ is an intertwining operator and $\mathrm{Hom}_G(\rho, \sigma)$ is a subspace of $\mathrm{Hom}(V, W)$. □

1.3 Operations on group representations

Our goal in this section is to explain some common linear algebraic operations defined on representations of finite groups. Much of the information in this section can be found in [FH91]. For definitions and results from linear algebra, we refer to [Rom08].

1.3.1 Direct sums of representations

Suppose (ρ, V) and (σ, W) are representations of a finite group G. Recall that the *direct sum* of V and W is the complex vector space

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}$$

with addition and scalar multiplication defined by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

$$\lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$$

for all $(v_1, w_1), (v_2, w_2) \in V \oplus W$, and $\lambda \in \mathbb{C}$. The representations (ρ, V) and (σ, W) give rise to a representation of G on the space $V \oplus W$ in a natural way.

Proposition 1.8. *Suppose (ρ, V) and (σ, W) are representations of a finite group G. Define the map $(\rho \oplus \sigma) : G \to \mathrm{GL}(V \oplus W)$, where $(\rho \oplus \sigma)(g) : V \oplus W \to V \oplus W$ is given by*

$$[(\rho \oplus \sigma)(g)](v, w) = (\rho(g)v, \sigma(g)w)$$

for all $(v, w) \in V \oplus W$. Then $(\rho \oplus \sigma, V \oplus W)$ is a representation of G.

Proof. Let $g \in G$. Clearly $(\rho \oplus \sigma)(g)$ is a linear map since $\rho(g)$ and $\sigma(g)$ are both linear. Furthermore, we have

$$[(\rho \oplus \sigma)(g)]\,[(\rho \oplus \sigma)(g^{-1})] = \mathrm{id}_{V \oplus W} = [(\rho \oplus \sigma)(g^{-1})]\,[(\rho \oplus \sigma)(g)].$$

Thus $(\rho \oplus \sigma)(g)$ is an invertible linear map for each $g \in G$. Now for any $g, h \in G$ and $(v, w) \in V \oplus W$, observe that

$$\begin{aligned}
[(\rho \oplus \sigma)(gh)](v, w) &= (\rho(gh)v, \sigma(gh)w) \\
&= (\rho(g)\rho(h)v, \sigma(g)\sigma(h)w) \\
&= [(\rho \oplus \sigma)(g)](\rho(h)v, \sigma(h)w) \\
&= [(\rho \oplus \sigma)(g)]\,[(\rho \oplus \sigma)(h)](v, w).
\end{aligned}$$

Hence $(\rho \oplus \sigma) : G \to \mathrm{GL}(V \oplus W)$ is a group homomorphism, as needed. □

Remark. By a simple inductive argument, if $(\rho_1, V_1), \ldots, (\rho_n, V_n)$ is a finite collection of representations of G, we obtain a representation $\bigoplus_{i=1}^n \rho_i : G \to GL(\bigoplus_{i=1}^n V_i)$ by defining

$$[(\bigoplus_{i=1}^n \rho_i)(g)](v_i)_{i=1}^n = (\rho_i(g)v_i)_{i=1}^n$$

for all $g \in G$ and $(v_i)_{i=1}^n \in \bigoplus_{i=1}^n V_i$.

1.3.2 Dual representations

Before we define the dual representation, we recall some relevant definitions and results from linear algebra. Let V denote a finite-dimensional complex vector space and suppose $X = \{x_1, \ldots, x_n\}$ is a basis for V. The *dual space* of V is the set V^* of linear maps $f : V \to \mathbb{C}$ under pointwise addition and scalar multiplication. For each $x_i \in X$, we define the linear map $x_i^* : V \to \mathbb{C}$ by $x_i^*(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_i$. The set $X^* = \{x_1^*, \ldots, x_n^*\}$ forms a basis for V^*, which we call the *dual basis* corresponding to X. In particular, $\dim V = \dim V^*$.

Proposition 1.9. *Let (ρ, V) be a representation of G. Define the map $\rho^* : G \to \mathrm{GL}(V^*)$, where $\rho^*(g) : V^* \to V^*$ is given by*

$$(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$$

for all $f \in V^*$ and $v \in V$. Then (ρ^*, V^*) is a representation of G, which we call the dual representation of (ρ, V).

Proof. We first verify that $\rho^*(g) \in GL(V^*)$ for all $g \in G$. Let $f_1, f_2 \in V^*$ and let $\lambda \in \mathbb{C}$. Then for any $v \in V$, we have

$$\begin{aligned}
[\rho^*(g)(f_1 + \lambda f_2)](v) &= [f_1 + \lambda f_2](\rho(g^{-1})v) \\
&= f_1(\rho(g^{-1})v) + \lambda f_2(\rho(g^{-1}v)) \\
&= [\rho^*(g)f_1](v) + \lambda[\rho^*(g)f_2](v).
\end{aligned}$$

Hence $\rho^*(g) : V^* \to V^*$ is a linear map. Now note that,

$$\begin{aligned}
\rho^*(g^{-1})\,[\rho^*(g)f](v) &= [\rho^*(g)f](\rho(g)v) \\
&= f(\rho(g^{-1})\rho(g)v) \\
&= f(\rho(e)v) \\
&= f(v).
\end{aligned}$$

Thus, $\rho^*(g^{-1})\rho^*(g) = \mathrm{id}_{V^*} = \rho^*(g)\rho^*(g^{-1})$ and we have $\rho^*(g) \in \mathrm{GL}(V^*)$. To show ρ^* is a homomorphism, let $g, h \in G$. Then for any $v \in V$,

$$\begin{aligned}
[\rho^*(gh)f](v) &= f(\rho(h^{-1}g^{-1})v) \\
&= f(\rho(h^{-1})\rho(g^{-1})v) \\
&= [\rho^*(h)f](\rho(g^{-1})v) \\
&= \rho^*(g)[\rho^*(h)f](v).
\end{aligned}$$

Hence, (ρ^*, V^*) is a representation of G. □

1.3.3 Tensor products of representations

Two representations (ρ, V) and (σ, W) of a finite group G give rise to a representation of $G \times G$ on the tensor product $V \otimes W$ in a natural way. We will begin by defining the tensor product and recalling some of its basic properties.

Definition 1.10. Let U and V be complex vector spaces and let $U \times V$ denote the cartesian product of U and V. A function Φ with domain $U \times V$ is called *bilinear* if

$$\Phi(u + \lambda u', v) = \Phi(u, v) + \lambda \Phi(u', v)$$

$$\Phi(u, v + \lambda v') = \Phi(u, v) + \lambda \Phi(u, v')$$

for all $(u, v), (u', v') \in U \times V$ and $\lambda \in \mathbb{C}$.

Definition 1.11. Let U and V be complex vector spaces. A *tensor product* of U and V is an ordered pair (T, t) where T is a complex vector space and $t : U \times V \to T$ is a bilinear map such that, for any complex vector space W and any bilinear map $\Phi : U \times V \to W$, there exists a unique linear map $\tilde{\Phi} : T \to W$ satisfying $\Phi = \tilde{\Phi} \circ t$.

Remark. A tensor product of U and V exists, and it is unique up to linear isomorphism. We will frequently refer to the vector space T as the tensor product of U and V. In the following proposition, we present one common construction of the tensor product. For a more thorough discussion, we refer to [Rom08].

Proposition 1.12. *Suppose U and V are complex vector spaces with respective bases $\mathcal{B}_U$ and $\mathcal{B}_V$. Let $U \otimes V$ denote the complex vector space defined by the basis $\{x \otimes y \mid (x, y) \in \mathcal{B}_U \times \mathcal{B}_V\}$. Let $\otimes : U \times V \to U \otimes V$ denote the bilinear map defined by*

$$\otimes \left(\sum_{x \in \mathcal{B}_U} \alpha_x \, x, \sum_{y \in \mathcal{B}_V} \beta_y \, y \right) = \sum_{x \in \mathcal{B}_U} \sum_{y \in \mathcal{B}_V} \alpha_x \beta_y \, (x \otimes y)$$

for arbitrary $u = \sum_{x \in \mathcal{B}_U} \alpha_x \, x$ in U and $v = \sum_{y \in \mathcal{B}_V} \beta_y \, y$ in V. Then $U \otimes V$ and $\otimes : U \times V \to U \otimes V$ form a tensor product of U and V.

Proof. Let W be a complex vector space and let $\Phi : U \times V \to W$ be a bilinear map. Since $\{x \otimes y \mid (x, y) \in \mathcal{B}_U \times \mathcal{B}_V\}$ is a basis for $U \otimes V$, we obtain a unique linear map $\tilde{\Phi} : U \otimes V \to W$ by letting

$$\tilde{\Phi}(x \otimes y) = \Phi(x, y)$$

for all $x \in \mathcal{B}_U$ and $y \in \mathcal{B}_W$. Now for any $u = \sum_{x \in \mathcal{B}_U} \alpha_x \, x$ in U and $v = \sum_{y \in \mathcal{B}_W} \beta_y \, y$ in V, since Φ is bilinear we have

$$\begin{aligned} \Phi\,(u, v) &= \Phi \left(\sum_{x \in \mathcal{B}_U} \alpha_x \, x, \sum_{y \in \mathcal{B}_W} \beta_y \, y \right) \\ &= \sum_{x \in \mathcal{B}_U} \sum_{y \in \mathcal{B}_W} \alpha_x \beta_y \, \Phi(x, y). \end{aligned}$$

On the other hand,

$$\begin{aligned} \otimes\,(u, v) &= \otimes \left(\sum_{x \in \mathcal{B}_U} \alpha_x \, x, \sum_{y \in \mathcal{B}_V} \beta_y \, y \right) \\ &= \sum_{x \in \mathcal{B}_U} \sum_{y \in \mathcal{B}_V} \alpha_x \beta_y \, (x \otimes y). \end{aligned}$$

Hence, $\Phi(u, v) = \tilde{\Phi} \circ \otimes (u, v)$. □

Remark. For $(u, v) \in U \times V$ and $\otimes : U \times V \to U \otimes V$ as defined above, we call $\otimes(u, v)$ an *elementary tensor* and write $\otimes(u, v) = u \otimes v$. The following result tells us that every element of $U \otimes V$ can be written as a sum of elementary tensors.

Corollary 1.13. *Suppose U and V are complex vector spaces and let $U \otimes V$ and $\otimes : U \times V \to U \otimes V$ be as defined in Proposition 1.12. Then for any $u, u' \in U$, $v, v' \in V$, and $\lambda \in \mathbb{C}$, we have*

(1) $(u \otimes v) + (u' \otimes v) = (u + u') \otimes v$;

(2) $(u \otimes v) + (u \otimes v') = u \otimes (v + v')$;

(3) $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$;

(4) $u \otimes 0 = 0 \otimes 0 = 0 \otimes v$.

Proof. Parts (1) through (3) all follow from the observation that $\otimes : U \times V \to U \otimes V$ is a bilinear map. Part (4) follows from the definition of $\otimes : U \times V \to U \otimes V$. □

Corollary 1.14. *If U and V are finite-dimensional complex vector spaces and $U \otimes V$ is as defined in Proposition 1.12, then* $\dim(U \otimes V) = \dim U \, \dim V$.

Proposition 1.15. *If $(S, s : U \times V \to S)$ and $(T, t : U \times V \to T)$ are tensor products of U and V, then $S \cong T$ as vector spaces.*

Proof. Since the map s is bilinear and (T, t) is a tensor product of U and V, there exists a unique linear map $\tilde{s} : T \to S$ such that

$$s(u, v) = \tilde{s} \circ t(u, v)$$

for all $(u, v) \in U \times V$. By a similar argument, there exists a unique linear map $\tilde{t} : S \to T$ such that

$$t(u, v) = \tilde{t} \circ s(u, v)$$

for all $(u, v) \in U \times V$. We now have

$$t = \tilde{t} \circ (\tilde{s} \circ t) = (\tilde{t} \circ \tilde{s}) \circ t.$$

We claim that $\tilde{t} \circ \tilde{s} = \mathrm{id}_T$. Indeed, since (T, t) is a tensor product of U and V, there exists a unique linear map $t' : T \to T$ such that such that $t' \circ t = t$. Since we have both $(\tilde{t} \circ \tilde{s}) \circ t = t$ and $\mathrm{id}_T \circ t = t$, it follows that $(\tilde{t} \circ \tilde{s}) = t' = \mathrm{id}_T$. By a similar argument, we have $\tilde{s} \circ \tilde{t} = \mathrm{id}_S$. Therefore, $\tilde{t} : S \to T$ is an invertible linear map and it follows that $S \cong T$ as vector spaces. □

Proposition 1.16. *Suppose (ρ, V) and (σ, W) are representations of G. For each $(g, h) \in G \times G$, there exists a unique linear map $(\rho \otimes \sigma)(g, h) \in \mathrm{GL}(V \otimes W)$ that is defined on the elementary tensors by*

$$[(\rho \otimes \sigma)(g, h)]\,(v \otimes w) = \rho(g)v \otimes \sigma(h)w, \tag{1.1}$$

and the map $(\rho \otimes \sigma) : G \times G \to \mathrm{GL}(V \otimes W)$ that sends $(g, h) \mapsto (\rho \otimes \sigma)(g, h)$ is a representation of $G \times G$.

Proof. Let $(g, h) \in G \times G$ and consider the map

$$\begin{aligned} U \times V &\longrightarrow U \otimes V \\ (u, v) &\longmapsto \rho(g)u \otimes \sigma(h)v. \end{aligned}$$

The map is bilinear due to Corollary 1.13 and the observation that $\rho(g)$ and $\sigma(h)$ are both linear maps. Thus, there exists a unique linear map $U \otimes V \to U \otimes V$ that is defined on the elementary tensors by $u \otimes v \mapsto \rho(g)u \otimes \sigma(h)v$; we will denote this map by $(\rho \otimes \sigma)(g, h)$. It is easy to show that $(\rho \otimes \sigma)(g^{-1}, h^{-1})$ is the inverse of $(\rho \otimes \sigma)(g, h)$. Hence $(\rho \otimes \sigma)(g, h) \in \mathrm{GL}(V \otimes W)$.

By the above, the map $(\rho \otimes \sigma) : G \times G \to \mathrm{GL}(V \otimes W)$ that sends $(g, h) \mapsto (\rho \otimes \sigma)(g, h)$ is well-defined. To see that $(\rho \otimes \sigma)$ is a group homomorphism, let $(g_1, h_1), (g_2, h_2) \in G \times G$.

Then for any $v \otimes w \in V \otimes W$,

$$
\begin{aligned}
[(\rho \otimes \sigma)(g_1 g_2, h_1 h_2)]\, v \otimes w &= \rho(g_1 g_2) v \otimes \sigma(h_1 h_2) w \\
&= \rho(g_1)\rho(g_2) v \otimes \sigma(h_1)\sigma(h_2) w \\
&= [(\rho \otimes \sigma)(g_1, h_1)]\, [(\rho \otimes \sigma)(g_2, h_2)]\, (v \otimes w).
\end{aligned}
$$

Hence $(\rho \otimes \sigma, V \otimes W)$ is a representation of $G \times G$. □

1.3.4 Representations on the space of $\mathbb{C}$-linear maps

Suppose (ρ, V) and (σ, W) are representations of G. We will use (ρ, V) and (σ, W) to define a representation of $G \times G$ on the space $\mathrm{Hom}(V, W)$. The representation arises naturally from the following result.

Proposition 1.17. *Let U and V be finite-dimensional complex vector spaces. Then $V \otimes U^* \cong \mathrm{Hom}(U, V)$ as vector spaces.*

Proof. Suppose $\mathcal{B}_U = \{x_1, \ldots, x_n\}$ and $\mathcal{B}_V = \{y_1, \ldots, y_m\}$ are bases for U and V, respectively. We let $V \otimes U^*$ denote the vector space defined by the basis $\{y_j \otimes x_i^* \mid x_i \in \mathcal{B}_U, y_j \in \mathcal{B}_V\}$, as in Proposition 1.12. Define a map $\Phi : V \times U^* \to \mathrm{Hom}(U, V)$ where $\Phi(v, f) : U \to V$ is defined by

$$\Phi(v, f)u = f(u)v$$

for all $u \in U$. Clearly the map Φ is bilinear. Thus, there exists a linear map $\tilde{\Phi} : V \otimes U^* \to \mathrm{Hom}(U, V)$ such that $\tilde{\Phi}(v \otimes f) = \Phi(v, f)$ for all $(v, f) \in V \times U^*$.

We show that $\tilde{\Phi}$ is injective. Suppose

$$\sum_{x_i \in \mathcal{B}_U} \sum_{y_j \in \mathcal{B}_V} \alpha_{ij}\, (y_j \otimes x_i^*) \in \ker \tilde{\Phi}.$$

where $\alpha_{ij} \in \mathbb{C}$ for each $x_i \in \mathcal{B}_U$ and $y_j \in \mathcal{B}_V$. For fixed $x_k \in \mathcal{B}_U$, we have

$$\sum_{x_i \in \mathcal{B}_U} \sum_{y_j \in \mathcal{B}_V} \alpha_{ij} x_i^*(x_k)\, y_j = \sum_{y_j \in \mathcal{B}_V} \alpha_{kj} x_k^*(x_k)\, y_j = \sum_{y_j \in \mathcal{B}_V} \alpha_{kj}\, y_j = 0.$$

Now since $\mathcal{B}_V$ is a basis for V, we must have $\alpha_{kj} = 0$ for all $y_j \in \mathcal{B}_V$. Since $x_k \in \mathcal{B}_U$ was arbitrary, we conclude $\alpha_{ij} = 0$ for all $x_i \in \mathcal{B}_U$ and $y_j \in \mathcal{B}_V$. Hence $\tilde{\Phi}$ is injective.

To see that $\tilde{\Phi}$ is surjective, suppose $T \in \mathrm{Hom}(U, V)$. For each $x_i \in \mathcal{B}_U$, we write $T(x_i)$ as a linear combination of the basis elements in $\mathcal{B}_V$,

$$T(x_i) = \sum_{y_j \in \mathcal{B}_V} \beta_{ij}\, y_j,$$

where $\beta_{ij} \in \mathbb{C}$ for each $y_j \in \mathcal{B}_V$. It now follows that

$$T(u) = \sum_{x_i \in \mathcal{B}_U} \sum_{y_j \in \mathcal{B}_V} \beta_{ij}\, x_i^*(u)\, y_j$$

for all $u \in U$. Thus $\tilde{\Phi}$ is surjective, and we conclude that $\tilde{\Phi}$ is a linear isomorphism. □

Let (ρ^*, V^*) denote the dual representation of (ρ, V). By Proposition 1.16 the map $(\sigma \otimes \rho^*) : G \times G \to \mathrm{GL}(W \otimes V^*)$ where $(\sigma \otimes \rho^*)(g, h) : W \otimes V^* \to W \otimes V^*$ is defined on elementary tensors by

$$[(\sigma \otimes \rho^*)(g, h)](w \otimes f) = \sigma(g)w \otimes \rho^*(h)f$$

defines a representation of $G \times G$. Now taking $\Phi : W \otimes V^* \to \mathrm{Hom}(V, W)$ as defined in the last proposition, we have for any $v \in V$

$$\begin{aligned}
\Phi(\sigma(g)w \otimes \rho^*(h)f)v &= (\rho^*(h)f(v))\,(\sigma(g)w) \\
&= f(\rho(h^{-1})v)\sigma(g)w \\
&= \sigma(g)f(\rho(h^{-1})v)w \qquad \text{(since } \sigma(g) \in \mathrm{GL}(W) \text{ and } f(\rho(h^{-1})v) \in \mathbb{C}\text{)} \\
&= \sigma(g)\,\Phi(w \otimes f)\,\rho(h)^{-1}v.
\end{aligned}$$

The linear isomorphism $\Phi : W \otimes V^* \to \mathrm{Hom}(V, W)$ is in fact an intertwining operator of $(\sigma \otimes \rho^*, W \otimes V^*)$ and the representation of $G \times G$ defined below.

Proposition 1.18. *Suppose (ρ, V) and (σ, W) are representations of G. Define the map $\Psi_{\sigma,\rho} : G \times G \to \mathrm{GL}(\mathrm{Hom}(V, W))$ where $\Psi_{\sigma,\rho}(g, h) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ is defined by*

$$[\Psi_{\sigma,\rho}(g, h)](A) = \sigma(g)\, A\, \rho(h)^{-1}$$

for all $A \in \mathrm{Hom}(V, W)$. Then $(\Psi_{\sigma,\rho}, \mathrm{Hom}(V, W))$ is a representation of $G \times G$.

Proof. It is straightforward to show that $\Psi_{\sigma,\rho}(g,h) : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$ is a linear map for each $(g,h) \in G \times G$. Now if $(g_1,h_1),(g_2,h_2) \in G \times G$, then for any $A \in \mathrm{Hom}(V,W)$

$$\begin{aligned}[\Psi_{\sigma,\rho}(g_1g_2,h_1h_2)]\,(A) &= \sigma(g_1g_2)\,A\,\rho(h_1h_2)^{-1} \\ &= \sigma(g_1)\sigma(g_2)\,A\,\rho(h_2)^{-1}\rho(h_1)^{-1} \\ &= [\Psi_{\sigma,\rho}(g_1,h_1)]\,(\sigma(g_2)\,A\,\rho(h_2)^{-1}) \\ &= [\Psi_{\sigma,\rho}(g_1,h_1)]\,[\Psi_{\sigma,\rho}(g_2,h_2)]\,(A).\end{aligned}$$

Thus $(\Psi_{\sigma,\rho}, \mathrm{Hom}(V,W))$ is a representation of $G \times G$. □

1.4 Subrepresentations

Let (ρ, V) be a representation of a finite group G. We know that $\rho : G \to \mathrm{GL}(V)$ defines a group action of G on the vector space V. It is natural to ask whether G acts on the subspaces of V. That is, if (ρ, V) is a representation of G and W is a subspace of V, is (ρ, W) also a representation of G? In order for this to be true, the group homomorphism $\rho : G \to GL(V)$ must also define a group homomorphism $\rho : G \to GL(W)$. In other words, for each $g \in G$ we must have $\rho(g)|_W \in GL(W)$. When this is true, we say that (ρ, W) is a *subrepresentation* of (ρ, V). We formalize this notion as follows:

Definition 1.19. Let (ρ, V) be a representation of a finite group G. A subspace $W \subseteq V$ is said to be *G-stable* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$.

Proposition 1.20. *Let V be a finite-dimensional complex vector space and let W be a subspace of V. Then (ρ, W) is a subrepresentation of (ρ, V) if and only if W is G-stable.*

Proof. If (ρ, W) is a subrepresentation of (ρ, V), then $\rho : G \to \mathrm{GL}(W)$ is a group homomorphism and $\rho(g) : W \to W$ is a bijective linear map for all $g \in G$. Therefore $\rho(g)w \in W$ for all $g \in G$ and all $w \in W$, whence W is G-stable.

On the other hand, suppose W is G-stable. For each $g \in G$, we let $\rho(g)|_W$ denote the restriction of $\rho(g)$ to W. We will show that $\rho(g)|_W \in GL(W)$. Since $\rho(g)$ is a linear map so too is $\rho(g)|_W$. Likewise, $\rho(g)|_W$ is injective because $\rho(g)$ is injective. Now since W is G-stable, it follows that $\rho(g)|_W$ maps W into W. Surjectivity now follows from the observation that W is finite-dimensional and the rank-nullity theorem. Therefore, $\rho : G \to \mathrm{GL}(W)$ is a group homomorphism. Hence (ρ, W) is a representation of G. □

If (ρ, V) is a representation of a group G, then the subspaces $\{0\}, V \subseteq V$ are both G-stable. Therefore, (ρ, V) and $(\rho, \{0\})$ are both subrepresentations of (ρ, V). We regard these as the trivial subrepresentations of (ρ, V).

Let U and W be subspaces of a vector space V. We say that U is a *complement* of W in V if $\{u + w \mid u \in U, w \in W\} = V$ and $U \cap W = \{0\}$. If U and W are complements in V, then V is isomorphic to the direct sum $U \oplus W = \{(u, w) \mid u \in U, w \in W\}$, and each $v \in V$ can be written uniquely as a sum $v = u + w$ for $u \in U$ and $w \in W$. Consequently, the projection maps $\pi_U : V \to U$ and $\pi_W : V \to W$ that send each $v \in V$ to its component in U and W, respectively, are well-defined linear maps. It is straightforward to show that every subspace $W \subseteq V$ has a complement in V. Below, we will show that every G-stable subspace of a representation (ρ, V) has a G-stable complement.

Proposition 1.21. *Let (ρ, V) be a representation of a finite group G. If $W \subseteq V$ is a G-stable subspace, then there exists a G-stable complement of W in V.*

Proof. Let X be a complement of W in V. Then each $v \in V$ can be written as $v = w_v + x_v$ for unique $w_v \in W$ and $x_v \in X$. Define $T : V \to W$ by

$$Tv = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_W \rho(g)^{-1} v$$

for all $v \in V$. Note that if $w \in W$, then $\rho(g)^{-1} w \in W$ since W is G-stable. Therefore, $\pi_W \rho(g)^{-1} w = \rho(g)^{-1} w$ and we have

$$Tw = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_W \rho(g)^{-1} w = \frac{1}{|G|} \sum_{g \in G} \rho(g) \rho(g)^{-1} w = w.$$

Since T maps V into W, the above implies that $T(Tv) = Tv$ for all $v \in V$.

We claim that $\ker T$ is a G-stable complement of W in V. Note that any $v \in V$ may be written $v = Tv + (v - Tv)$, where $Tv \in W$ and $(v - Tv) \in \ker T$ because

$$\begin{aligned} T(v - Tv) &= Tv - TTv \\ &= Tv - Tv \\ &= 0. \end{aligned}$$

Further, we have $w \in U \cap W$ if and only if $Tw = w = 0$. Thus, U is a complement of W in V. To show that U is G-stable, first note that for any $x \in G$

$$\begin{aligned} \rho(x)T\rho(x)^{-1} &= \rho(x)\left(\frac{1}{|G|}\sum_{g\in G}\rho(g)\pi_W\rho(g)^{-1}\right)\rho(x)^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(x)\rho(g)\pi_W\rho(g)^{-1}\rho(x)^{-1} && \text{(since } \rho(x) \text{ is a linear map)} \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(xg)\pi_W\rho(xg)^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(g)\pi_W\rho(g)^{-1} && \text{(change of variables } g \mapsto x^{-1}g) \end{aligned}$$

Therefore, $\rho(x)T\rho(x)^{-1} = T$ and it follows that $\rho(x)T = T\rho(x)$ for all $x \in G$. Now given $u \in U$, we have $T\rho(x)u = \rho(x)Tu = 0$ for all $x \in G$. Thus $\rho(x)u \in U$ for all $x \in G$ and $u \in U$ and we conclude that U is G-stable. □

1.5 Reducibility of representations

In the previous section, we saw that if W is a G-stable subspace of a representation (ρ, V), then there exists a complementary subspace U of W which is also G-stable. When this is true, the representation V is equivalent to the direct sum of its subrepresentations, $U \oplus W$. If either U or W is a proper, nonzero subspace of V, the representation (ρ, V) is reducible in the sense that it is understood by the subrepresentations (ρ, U) and (ρ, W) with $\dim U$, $\dim W < \dim V$. This motivates the following.

Definition 1.22. Let (ρ, V) be a representation of a finite group G. We say that (ρ, V) is *reducible* if there is a proper, nonzero, G-stable subspace of V. Otherwise, we say that (ρ, V) is *irreducible.*

Given a reducible representation $V \cong U \oplus W$ with U and W both G-stable, if either U or W is reducible, then we may use Proposition 1.21 to write V as a direct sum of three subrepresentations. Indeed, since V is finite dimensional, we should be able to repeat this process finitely many times to write V as a direct sum of irreducible representations.

Proposition 1.23. *Let G be a finite group. Each representation (ρ, V) of G is equivalent to a direct sum of irreducible representations.*

Proof. We induct on $\dim V$. If $\dim V = 1$, then the only subspaces of V are $\{0\}$ and V itself. Hence, V is irreducible. Now let k be a positive integer and suppose inductively that each representation V of G with $\dim V \leq k$ is equivalent to a direct sum of irreducible representations. Let V' denote a representation of G with $\dim V' = k + 1$. If V' is irreducible then we are done. Otherwise V' is reducible, so by Proposition 1.21, we may write $V' \cong U \oplus W$ where $\dim U, \dim W < \dim V'$. Using the induction hypothesis, we may write both U and W as a direct sum of irreducible representations, whence $V' \cong U \oplus W$ is a direct sum of irreducible representations. □

1.6 Intertwining numbers and Schur's lemma

For representations (ρ, V) and (σ, W) of a finite group G, we call the dimension of $\mathrm{Hom}_G(\rho, \sigma)$ the *intertwining number of ρ and σ*, and write $\dim \mathrm{Hom}_G(\rho, \sigma) = n_G(\rho, \sigma)$.

Proposition 1.24. *If $(\rho_1, V_1) \cong (\rho_2, V_2)$ and $(\sigma_1, W_1) \cong (\sigma_2, W_2)$ are representations of a finite group G, then $n_G(\rho_1, \sigma_1) = n_G(\rho_2, \sigma_2)$.*

Proof. Since (ρ_1, V_1) and (ρ_2, V_2) are equivalent, there exists a bijective intertwining operator $R : V_1 \to V_2$. Likewise, because (σ_1, W_1) and (σ_2, W_2) are equivalent, there exists a bijective intertwining operator $S : W_1 \to W_2$. Consider the map $\Phi : \mathrm{Hom}_G(\rho_1, \sigma_1) \to \mathrm{Hom}_G(\rho_2, \sigma_2)$ defined by $\Phi(T) = STR^{-1}$. We show that Φ is a linear isomorphism. First note that $STR^{-1} : V_2 \to W_2$. Furthermore, for any $g \in G$ we have

$$\sigma_2(g)STR^{-1} = S\sigma_1(g)TR^{-1} = ST\rho_1(g)R^{-1} = STR^{-1}\rho_2(g),$$

so $STR^{-1} \in \mathrm{Hom}_G(\rho_2, \sigma_2)$. To verify that Φ is linear, let $T_1, T_2 \in \mathrm{Hom}_G(\rho_1, \sigma_1)$ and let $\lambda \in \mathbb{C}$. Then we have

$$\Phi(T_1 + \lambda T_2) = S(T_1 + \lambda T_2)R^{-1} = ST_1R^{-1} + \lambda ST_2R^{-1} = \Phi(T_1) + \lambda\Phi(T_2)$$

by linearity of S.

To show that Φ is bijective, note that for any $B \in \mathrm{Hom}_G(\rho_2, \sigma_2)$, we have $S^{-1}BR \in \mathrm{Hom}_G(\rho_1, \sigma_1)$, whence $\Phi(S^{-1}BR) = B$ and Φ is surjective. For injectivity, suppose that $A \in \ker\Phi$. Then $\Phi(A)x = SAR^{-1}x = 0$ for all $x \in V_2$. Because S is bijective, it follows that $AR^{-1}x = 0$ for all $x \in V_2$. Likewise, because R^{-1} is bijective, we have $Ay = 0$ for all $y \in V_1$. Thus $A = 0$ and Φ is injective. □

Proposition 1.25. *If (ρ_1, V_1), (ρ_2, V_2), and (σ, W) are representations of a finite group G, then $n_G(\rho_1 \oplus \rho_2, \sigma) = n_G(\rho_1, \sigma) + n_G(\rho_2, \sigma)$ and $n_G(\sigma, \rho_1 \oplus \rho_2) = n_G(\sigma, \rho_1) + n_G(\sigma, \rho_2)$*

Proof. Recall that, for any finite-dimensional vector spaces U and W, we have $\dim U \oplus W = \dim U + \dim W$. So it will suffice to show that there is a linear isomorphism

$$\Phi : \mathrm{Hom}_G(\rho_1 \oplus \rho_2, \sigma) \longrightarrow \mathrm{Hom}_G(\rho_1, \sigma) \oplus \mathrm{Hom}_G(\rho_2, \sigma).$$

Let $T : V_1 \oplus V_2 \to W$ be an element in $\mathrm{Hom}_G(\rho_1 \oplus \rho_2, \sigma)$, so that

$$T[(\rho_1 \oplus \rho_2)(g)](x, y) = T(\rho_1(g)x, \rho_2(g)y) = \sigma(g)T(x, y)$$

for all $g \in G$, $x \in V_1$, and $y \in V_2$. Define $\Phi(T) = (T_1, T_2)$, where $T_1x = T(x, 0)$ and $T_2y = T(0, y)$ for all $x \in V_1$, $y \in V_2$. We first verify that T_1 and T_2 are both intertwining operators. Note that T_1 and T_2 are both linear because T is linear. Now for any $g \in G$ and $x \in V_1$, we have

$$\sigma(g)T_1x = \sigma(g)T(x,0) = T(\rho_1(g)x, \rho_2(g)0) = T(\rho_1(g)x, 0) = T_1\rho_1(g)x,$$

hence $T_1 \in \mathrm{Hom}_G(\rho_1, \sigma)$. Likewise, for any $g \in G$ and $y \in V_2$, we have

$$\sigma(g)T_2y = \sigma(g)T(0,y) = T(\rho_1(g)0, \rho_2(g)y) = T(0, \rho_2(g)y) = T_2\rho_2(g)y,$$

so $T_2 \in \mathrm{Hom}_G(\rho_2, \sigma)$.

Next, we show that Φ is a linear map. Let $S, T \in \mathrm{Hom}_G(\rho_1 \oplus \rho_2, \sigma)$ and let $\lambda \in \mathbb{C}$. We have $\Phi(S + \lambda T) = (S_1 + \lambda T_1, S_2 + \lambda T_2)$. Now for any $(x, y) \in V_1 \oplus V_2$,

$$(S_1 + \lambda T_1, S_2 + \lambda T_2)(x,y) = (S_1x + \lambda T_1x, S_2y + \lambda T_2y) = (S_1x, S_2y) + \lambda(T_1x, T_2y).$$

Noting that $(S_1x, S_2y) = \Phi(S)(x,y)$ and $\lambda(T_1x, T_2y) = \lambda\Phi(T)(x,y)$, it follows that Φ is linear.

Finally, we show that Φ is bijective. Assume that $T \in \ker\Phi$. Then $\Phi(T)(x,y) = (0,0)$ for all $x \in V_1$ and all $y \in V_2$. Therefore, $T_1x = T(x,0) = 0$ for all $x \in V_1$, and similarly $T_2y = T(0,y) = 0$ for all $y \in V_2$. So for any $(x,y) \in V_1 \oplus V_2$, we have

$$T(x,y) = T(x,0) + T(0,y) = T_1x + T_2y = 0,$$

hence $T = 0$ and Φ is injective.

For surjectivity, let $S_1 \in \mathrm{Hom}_G(\rho_1, \sigma)$ and let $S_2 \in \mathrm{Hom}_G(\rho_2, \sigma)$. Define the map $S : V_1 \oplus V_2 \to W$ by $S(x,y) = S_1x + S_2y$. Clearly S is linear because S_1 and S_2 are both linear. For any $g \in G$, we have

$$\sigma(g)S(x,y) = \sigma(g)(S_1x + S_2y) = \sigma(g)S_1x + \sigma(g)S_2y = S_1\rho_1(g)x + S_2\rho_2(g)y.$$

On the other hand, $S(\rho_1(g)x, \rho_2(g)y) = S_1\rho_1(g)x + S_2\rho_2(g)y$. Thus S is an intertwining operator and $\Phi(S) = (S_1, S_2)$. It now follows that Φ is bijective, so $n_G(\rho_1 \oplus \rho_2, \sigma) = n_G(\rho_1, \sigma) + n_G(\rho_2, \sigma)$, as needed. The proof of the second equality is similar. □

Corollary 1.26. *Let m be a positive integer and suppose that $\rho \cong \bigoplus_{i=1}^{m} \rho_i$ and σ are representations of a finite group G. Then $n_G(\rho, \sigma) = \sum_{i=1}^{m} n_G(\rho_i, \sigma)$ and $n_G(\sigma, \rho) = \sum_{i=1}^{m} n_G(\sigma, \rho_i)$.*

Proof. We established the basis for induction in Proposition 1.25. So let m be a positive integer and suppose that $n_G(\bigoplus_{i=1}^{m} \rho_i, \sigma) = \sum_{i=1}^{m} n_G(\rho_i, \sigma)$ for any finite collection of representations of G, $\{(\rho_1, V_1), \cdots, (\rho_m, V_m)\}$. Then for any representation (ρ_{m+1}, V_{m+1}) of G we have

$$n_G\left(\bigoplus_{i=1}^{m+1} \rho_i, \sigma\right) = n_G\left(\left(\bigoplus_{i=1}^{m} \rho_i\right) \oplus \rho_{m+1}, \sigma\right),$$

and by Proposition 1.25,

$$n_G\left(\left(\bigoplus_{i=1}^{m} \rho_i\right) \oplus \rho_{m+1}, \sigma\right) = n_G\left(\bigoplus_{i=1}^{m} \rho_i, \sigma\right) + n_G\left(\rho_{m+1}, \sigma\right) = \sum_{i=1}^{m+1} n_G(\rho_i, \sigma),$$

which closes the induction. The proof of the second equality is similar. □

Theorem 1.27 (Schur's lemma)**.** *Suppose (ρ, V) and (σ, W) are irreducible representations of a finite group G.*

(1) If $(\rho, V) \ncong (\sigma, W)$, then $\mathrm{Hom}_G(\rho, \sigma) = \{0\}$.

(2) If $(\rho, V) = (\sigma, W)$, then $\mathrm{Hom}_G(\rho, \sigma) = \mathrm{span}\{\mathrm{id}_V\}$.

Proof. For the first claim, suppose that $(\rho, V) \ncong (\sigma, W)$. Since $\mathrm{Hom}_G(\rho, \sigma)$ is a subspace of $\mathrm{Hom}(V, W)$, we must have $0 \in \mathrm{Hom}_G(\rho, \sigma)$. To show that $\mathrm{Hom}_G(\rho, \sigma)$ contains only the zero map, suppose for the sake of contradiction that $T : V \to W$ is a nonzero intertwining operator in $\mathrm{Hom}_G(\rho, \sigma)$. Note that $\ker T$ is a subspace of V, and for any $g \in G$, $x \in \ker T$, we have

$$T\rho(g)x = \sigma(g)Tx = \sigma(g)0 = 0 \in \ker T,$$

so $\ker T$ is a G-stable subspace of V. Now note that $T(V)$ is a subspace of W, and for any $g \in G$, $v \in V$, we have

$$\sigma(g)Tv = T\rho(g)v \in T(V),$$

hence $T(V)$ is a G-stable subspace of W. Since V is irreducible, the only G-stable subspaces of V are $\{0\}$ and V itself. Because we assumed that T is nonzero, we must have $\ker T = \{0\}$. Likewise, the only G-stable subspaces of W are $\{0\}$ and W itself, so we must have $T(V) = W$. It now follows that $T : V \to W$ is bijective. Hence $(\rho, V) \cong (\sigma, W)$, a contradiction.

For the second claim, let $(\rho, V) = (\sigma, W)$. Note that $\alpha \operatorname{id}_V \in \operatorname{Hom}_G(\rho, \rho)$ for all $\alpha \in \mathbb{C}$, hence $\operatorname{span}\{\operatorname{id}_V\} \subseteq \operatorname{Hom}_G(\rho, \rho)$. Now let $T \in \operatorname{Hom}_G(\rho, \rho)$, suppose $\mathcal{B}$ is an ordered basis for V, and let $[T]_{\mathcal{B}}$ denote the matrix of T relative to $\mathcal{B}$. Recall that the eigenvalues of T are the roots of the characteristic polynomial,

$$c_T(x) = \det(xI - [T]_{\mathcal{B}}),$$

where I is the $\dim V \times \dim V$ identity matrix. Since $c_T(x)$ is a complex polynomial, it has a root $\lambda \in \mathbb{C}$, whence T has an eigenvalue. Let $S = T - \lambda \operatorname{id}_V$. For any $g \in G$ and $v \in V$, we have

$$\begin{aligned}
\rho(g)Sv &= \rho(g)(Tv - \lambda \operatorname{id}_V v) \\
&= \rho(g)Tv - \rho(g)\lambda \operatorname{id}_V v \\
&= T\rho(g)v - \lambda \operatorname{id}_V \rho(g)v \\
&= S\rho(g)v,
\end{aligned}$$

so $S \in \operatorname{Hom}_G(\rho, \rho)$. Since λ is an eigenvalue of T, there exists an eigenvector $v \in V$ such that $Tv = \lambda v$. It follows that $Sv = Tv - \lambda v = 0$, which implies $\ker S = V$ by the proof of the first claim. Therefore, $Tv = \lambda v$ for all $v \in V$ and we have $\operatorname{Hom}_G(\rho, \rho) = \operatorname{span}\{\operatorname{id}_V\}$, as needed. $\square$

Corollary 1.28. *If (ρ, V) and (σ, W) are irreducible representations of a finite group G, then*

$$n_G(\rho, \sigma) = \begin{cases} 1 & \text{if } (\rho, V) \cong (\sigma, W); \\ 0 & \text{if } (\rho, V) \ncong (\sigma, W). \end{cases}$$

Proof. If $(\rho, V) \not\cong (\sigma, W)$, then $\mathrm{Hom}_G(\rho, \sigma) = \{0\}$ by Schur's lemma and therefore $n_G(\rho, \sigma) = 0$. On the other hand, if $(\rho, V) \cong (\sigma, W)$ then $n_G(\rho, \sigma) = n_G(\sigma, \sigma)$, by Proposition 1.24. Now due to Schur's lemma, $\mathrm{Hom}_G(\sigma, \sigma) = \mathrm{span}\{\mathrm{id}_W\}$, so $n_G(\rho, \sigma) = n_G(\sigma, \sigma) = 1$. □

Definition 1.29. Let G be a finite group and let $\hat{G}$ denote a set of representatives for the irreducible representations of G modulo equivalence. If (ρ, V) is a representation of G, we denote the decomposition of (ρ, V) as a direct sum of irreducible representations in $\hat{G}$ by

$$\rho \cong \bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi \tag{1.2}$$

where m_π is a non-negative integer for each $\pi \in \hat{G}$ and

$$m_\pi \cdot \pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{m_\pi\text{-times}}.$$

For each $\pi \in \hat{G}$, we call m_π the *multiplicity of* π *in* ρ.

Proposition 1.30. *Suppose* $\rho \cong \bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi$ *is a representation of a finite group* G*. Then* $n_G(\rho, \tau) = m_\tau$ *for each* $\tau \in \hat{G}$*.*

Proof. Let $\tau \in \hat{G}$. Since intertwining numbers are additive, we have

$$\begin{aligned} n_G(\rho, \tau) &= n_G\left(\bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi, \tau\right) \\ &= \sum_{\pi \in \hat{G}} n_G(m_\pi \cdot \pi, \tau) \\ &= \sum_{\pi \in \hat{G}} m_\pi \, n_G(\pi, \tau). \end{aligned}$$

By the previous corollary, $n_G(\pi, \tau) = 1$ if $\pi \cong \tau$ and $n_G(\pi, \tau) = 0$ otherwise. Hence, $n_G(\rho, \tau) = m_\tau$. □

Remark. By Proposition 1.30 and Schur's lemma, the multiplicities m_π for $\pi \in \hat{G}$ are uniquely determined by ρ.

Corollary 1.31. *If* $\rho \cong \bigoplus_{\pi\in\hat{G}} m_\pi \cdot \pi$ *and* $\sigma \cong \bigoplus_{\pi\in\hat{G}} n_\pi \cdot \pi$ *are representations of* G*, then* $n_G(\rho,\sigma) = \sum_{\pi\in\hat{G}} m_\pi n_\pi$*. In particular,* $n_G(\rho,\sigma) = n_G(\sigma,\rho)$*.*

Proof. The result follows easily from Propositions 1.25 and 1.30. □

1.7 Restricted and induced representations

Suppose Γ is a subgroup of a finite group G. If (ρ, V) is a representation of G, then $\rho : G \to \mathrm{GL}(V)$ is by definition a homomorphism of groups. The restriction of ρ to Γ, denoted $\rho|_\Gamma : \Gamma \to \mathrm{GL}(V)$, is also a group homomorphism. Therefore $(\rho|_\Gamma, V)$ is a representation of Γ. We call $(\rho|_\Gamma, V)$ the *restriction of* ρ *to* Γ, and we will sometimes write $\rho|_\Gamma = \mathrm{Res}^G_\Gamma(\rho)$.

Now suppose (σ, W) is a representation of Γ. In what follows, we will construct the representation of G *induced* by (σ, W). We begin by defining the representation space of the induced representation.

Proposition 1.32. *Let* Γ *be a subgroup of a finite group* G*, let* (σ, W) *be a representation of* Γ*, and let*

$$C(\sigma) = \{f : G \to W \mid f(\gamma g) = \sigma(\gamma) f(g) \text{ for all } \gamma \in \Gamma,\ g \in G\}.$$

Then $C(\sigma)$ *is a complex vector space under pointwise addition and scalar multiplication and* $\dim C(\sigma) = [G : \Gamma] \dim W$*.*

Proof. Let $\mathcal{B} = \{w_1, \ldots, w_m\}$ be a basis for W and let $[\Gamma\backslash G] = \{g_1, \ldots, g_n\}$ be a set of representatives for $\Gamma\backslash G$. For each g_i and w_j, define the map $\varphi_{g_i,w_j} : G \to W$ by

$$\varphi_{g_i,w_j}(x) = \begin{cases} \sigma(xg_i^{-1})w_j & \text{if } x \in \Gamma g_i; \\ 0 & \text{otherwise.} \end{cases}$$

We verify that $\varphi_{g_i,w_j} \in C(\sigma)$. Let $\gamma \in \Gamma$ and $g \in G$. If $\gamma g \in \Gamma g_i$, then $g = \delta g_i$ for some $\delta \in \Gamma$ and we have

$$\begin{aligned}\varphi_{g_i,w_j}(\gamma g) &= \varphi_{g_i,w_j}(\gamma\delta g_i) \\ &= \sigma(\gamma\delta g_i g_i^{-1})w_j \\ &= \sigma(\gamma)\sigma(\delta)w_j \\ &= \sigma(\gamma)\varphi_{g_i,w_j}(g).\end{aligned}$$

On the other hand, if $\gamma g \notin \Gamma g_i$ then $g \notin \Gamma g_i$ and we have $\varphi_{g_i,w_j}(\gamma g) = \varphi_{g_i,w_j}(g) = 0$. Since $\sigma(\gamma)$ is a linear map, it follows that $\sigma(\gamma)\varphi_{g_i,w_j}(g) = 0$. Thus $\sigma(\gamma)\varphi_{g_i,w_j}(g) = \varphi_{g_i,w_j}(\gamma g)$ and $\varphi_{g_i,w_j} \in C(\sigma)$, as needed.

Now we show that the set $\{\varphi_{g_i,w_j} \mid g_i \in [\Gamma\backslash G], w_j \in \mathcal{B}\}$ forms a basis for $C(\sigma)$. Suppose that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\, \varphi_{g_i,w_j}(x) = 0$$

for some $\alpha_{ij} \in \mathbb{C}$ and for all $x \in G$. Then for fixed $k \in \{1, \ldots, n\}$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\, \varphi_{g_i,w_j}(g_k) = \sum_{j=1}^{m} \alpha_{kj}\, w_j = 0.$$

Because $\{w_1, \ldots, w_m\}$ is a basis for W, it now follows that $\alpha_{kj} = 0$ for all $j \in \{1, \ldots, m\}$. Thus $\alpha_{ij} = 0$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, and linear independence follows.

Now suppose that $\psi \in C(\sigma)$. For each $g_i \in [\Gamma\backslash G]$, we have $\psi(g_i) \in W$ and we may write $\psi(g_i) = \sum_{j=1}^{m} \beta_{ij}\, w_j$ for scalars $\beta_{ij} \in \mathbb{C}$. We claim that $\psi = \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij}\, \varphi_{g_i,w_j}$. Indeed, for any $\gamma \in \Gamma$ and $g_k \in [\Gamma\backslash G]$, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} \beta_{ij}\, \varphi_{g_i,w_j}(\gamma g_k) &= \sum_{j=1}^{m} \beta_{kj}\, \sigma(\gamma)w_j \\ &= \sigma(\gamma)\sum_{j=1}^{m} \beta_{kj}\, w_j \\ &= \sigma(\gamma)\psi(g_k) \\ &= \psi(\gamma g_k).\end{aligned}$$

Therefore, the set $\{\varphi_{g_i,w_j} \mid g_i \in [\Gamma\backslash G], w_j \in \mathcal{B}\}$ forms a basis for $C(\sigma)$ and we have $\dim C(\sigma) = [G : \Gamma]\, \dim W$. □

Proposition 1.33. *Suppose Γ is a subgroup of a finite group G, let (σ, W) be a representation of Γ, and let $C(\sigma)$ denote the vector space defined in Proposition 1.32. Define the map $\pi_\sigma : G \to \mathrm{GL}(C(\sigma))$, where for $g \in G$ and $\varphi \in C(\sigma)$,*

$$[\pi_\sigma(g)\varphi](x) = \varphi(xg)$$

for all $x \in G$. Then $(\pi_\sigma, C(\sigma)) = (\mathrm{Ind}_\Gamma^G(\sigma), C(\sigma))$ is a representation of G, which we call the representation of G induced by (σ, W).

Proof. It is straightforward to show that $\pi_\sigma(g) : C(\sigma) \to C(\sigma)$ is a linear map for each $g \in G$. Clearly the inverse of $\pi_\sigma(g)$ is simply $\pi_\sigma(g^{-1})$, so $\pi_\sigma(g) \in \mathrm{GL}(C(\sigma))$ for each $g \in G$. Now if $g, h \in G$ and $\varphi \in C(\sigma)$, then

$$[\pi_\sigma(gh)\varphi](x) = \varphi(xgh) = [\pi_\sigma(h)\varphi](xg) = [\pi_\sigma(g)\pi_\sigma(h)\varphi](x)$$

for all $x \in G$. Hence $\pi_\sigma : G \to \mathrm{GL}(C(\sigma))$ is a group homomorphism, so $(\pi_\sigma, C(\sigma))$ is indeed a representation of G. □

Theorem 1.34 (Frobenius reciprocity)**.** *Let Γ be a subgroup of a finite group G, let (ρ, V) be a representation of G, and let (σ, W) be a representation of Γ. Then*

$$\mathrm{Hom}_G(\rho, \mathrm{Ind}_\Gamma^G(\sigma)) \cong \mathrm{Hom}_\Gamma(\mathrm{Res}_\Gamma^G(\rho), \sigma).$$

Proof. Let $S : V \to W$ be an element of $\mathrm{Hom}_\Gamma(\mathrm{Res}_\Gamma^G(\rho), \sigma)$. Consider the map $\Phi : \mathrm{Hom}_\Gamma(\mathrm{Res}_\Gamma^G(\rho), \sigma) \to \mathrm{Hom}_G(\rho, \mathrm{Ind}_\Gamma^G(\sigma))$, where $\Phi(S) : V \to C(\sigma)$ is given by $v \mapsto S_v$, and $S_v : G \to \mathbb{C}$ is given by $S_v(g) = S(\rho(g)v)$. We first verify that $S_v \in C(\sigma)$. Note that for any $\gamma \in \Gamma$ and any $g \in G$, we have

$$S_v(\gamma g) = S(\rho(\gamma g)v) = S(\rho(\gamma)\rho(g)v) = \sigma(\gamma)S(\rho(g)v) = \sigma(\gamma)S_v(g),$$

because $S \in \mathrm{Hom}_\Gamma(\mathrm{Res}^G_\Gamma(\rho), \sigma)$ is an intertwining operator. Thus $S_v \in C(\sigma)$. Now we verify that $\Phi(S) \in \mathrm{Hom}_G(\rho, \mathrm{Ind}^G_\Gamma(\sigma))$. For linearity, let $v_1, v_2 \in V$ and let $\lambda \in \mathbb{C}$. Then for any $g \in G$, we have

$$\Phi(S)(v_1 + \lambda v_2)(g) = S_{v_1+\lambda v_2}(g) = S(\rho(g)(v_1 + \lambda v_2)) = S(\rho(g)v_1) + \lambda S(\rho(g)v_2),$$

by linearity of S and $\rho(g)$. Hence $\Phi(S)(v_1 + \lambda v_2) = \Phi(S)(v_1) + \lambda\Phi(S)(v_2)$ and $\Phi(S)$ is linear. For $x \in G$, also note that

$$\pi_\sigma(x)S_v(g) = S_v(gx) = S(\rho(g)\rho(x)v) = S_{\rho(x)v}(g).$$

Ergo, $\pi_\sigma(x)\Phi(S) = \Phi(S)\rho(x)$ and $\Phi(S)$ is an intertwining operator.

We now show that Φ is a linear isomorphism. Let $S, T \in \mathrm{Hom}_\Gamma(\mathrm{Res}^G_\Gamma(\rho), \sigma)$, let $\lambda \in \mathbb{C}$, and let $v \in V$. Then $\Phi(S + \lambda T)v = (S + \lambda T)_v$, and we have for any $g \in G$,

$$(S + \lambda T)_v(g) = (S + \lambda T)(\rho(g)v) = S(\rho(g)v) + \lambda T(\rho(g)v) = S_v(g) + \lambda T_v(g).$$

Therefore $\Phi(S + \lambda T) = \Phi(S) + \lambda\Phi(T)$ and Φ is linear.

To show that Φ is injective, note if $S \in \ker\Phi$, then $S_v(g) = S(\rho(g)v) = 0$ for all $v \in V$ and all $g \in G$. In particular, $S(\rho(e)v) = S(v) = 0$ for all $v \in V$, so $S = 0$ and Φ is injective. For surjectivity, let $T \in \mathrm{Hom}_G(\rho, \mathrm{Ind}^G_\Gamma(\sigma))$ and define $S : V \to W$ by $S(v) = [T(v)](e)$. For any $\gamma \in \Gamma$ and $v \in V$, we have

$$S(\rho(\gamma)v) = [T(\rho(\gamma)v)](e) = \pi_\sigma(\gamma)[T(v)](e)$$

since T is an intertwining operator. Furthermore,

$$\pi_\sigma(\gamma)[T(v)](e) = [T(v)](\gamma) = \sigma(\gamma)[T(v)](e),$$

since $T(v) \in C(\sigma)$. It now follows that $S \in \mathrm{Hom}_\Gamma(\mathrm{Res}^G_\Gamma(\rho), \sigma)$, and for any $g \in G$ and $v \in V$, we have

$$S_v(g) = S(\rho(g)v) = [T(\rho(g)v)](e) = [\pi_\sigma(g)T(v)](e) = [T(v)](g).$$

Thus Φ is an isomorphism and $\mathrm{Hom}_G(\rho, \mathrm{Ind}^G_\Gamma(\sigma)) \cong \mathrm{Hom}_\Gamma(\mathrm{Res}^G_\Gamma(\rho), \sigma)$, as needed. □

Proposition 1.35. *Let Γ be a subgroup of a finite group G. Suppose (ρ, V) and (σ, W) are representations of Γ. If $(\rho \otimes \sigma) : \Gamma \times \Gamma \to \mathrm{GL}(V \otimes W)$ is the representation of $\Gamma \times \Gamma$ defined in Proposition 1.16, then*

$$\mathrm{Ind}_{\Gamma}^{G}(\rho) \otimes \mathrm{Ind}_{\Gamma}^{G}(\sigma) \cong \mathrm{Ind}_{\Gamma\times\Gamma}^{G\times G}(\rho \otimes \sigma)$$

as representations of $G \times G$.

Proof. Consider the map

$$\Phi : C(\rho) \times C(\sigma) \longrightarrow C(\rho \otimes \sigma)$$
$$(\varphi, \psi) \longmapsto \varphi\psi$$

where $\varphi\psi : G \times G \to V \otimes W$ is defined by $\varphi\psi(g, h) = \varphi(g) \otimes \psi(h)$ for all $g, h \in G$. Clearly $\varphi\psi \in C(\rho \otimes \sigma)$ because

$$\begin{aligned}\varphi\psi(\gamma g, \delta h) &= \varphi(\gamma g) \otimes \psi(\delta h)\\ &= \rho(\gamma)\varphi(g) \otimes \sigma(\delta)\psi(h)\\ &= [(\rho \otimes \sigma)(\gamma, \delta)](\varphi(g) \otimes \psi(h))\\ &= [(\rho \otimes \sigma)(\gamma, \delta)]\varphi\psi(g, h)\end{aligned}$$

for all $\gamma, \delta \in \Gamma$ and $g, h \in G$. It is straightforward to show that Φ is bilinear, so there exists a unique linear map $\tilde{\Phi} : C(\rho) \otimes C(\sigma) \to C(\rho \otimes \sigma)$ that is defined on the elementary tensors by $\tilde{\Phi}(\varphi \otimes \psi) = \varphi\psi$.

To see that $\tilde{\Phi}$ is bijective, we will show that it maps a basis for $C(\rho) \otimes C(\sigma)$ to a basis for $C(\rho \otimes \sigma)$. Suppose $\mathcal{B}_V$ and $\mathcal{B}_W$ are bases for V and W, respectively, and suppose $[\Gamma\backslash G]$ is a set of representatives for $\Gamma\backslash G$. For each $g \in [\Gamma\backslash G]$ and $v \in \mathcal{B}_V$, define $\varphi_{g,v} : G \to V$ by

$$\varphi_{g,v}(x) = \begin{cases}\rho(xg^{-1})v & \text{if } x \in \Gamma g;\\ 0 & \text{otherwise.}\end{cases}$$

Likewise, for each $h \in [\Gamma \backslash G]$ and $w \in \mathcal{B}_W$ define, $\psi_{h,w} : G \to W$ by

$$\psi_{h,w}(x) = \begin{cases} \sigma(xh^{-1})w & \text{if } x \in \Gamma h; \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 1.32, the set $\{\varphi_{h,v} \mid h \in [\Gamma \backslash G],\, v \in \mathcal{B}_V\}$ forms a basis for $C(\rho)$ and the set $\{\varphi_{g,w} \mid g \in [\Gamma \backslash G], w \in \mathcal{B}_W\}$ forms a basis for $C(\sigma)$. Hence, by Proposition 1.12, the set

$$\{\varphi_{g,v} \otimes \psi_{h,w} \mid g, h \in [\Gamma \backslash G], v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$$

is a basis for $C(\rho) \otimes C(\sigma)$. Now for any $x, y \in G$, we have

$$\tilde{\Phi}(\varphi_{g,v} \otimes \psi_{h,w})(x, y) = \begin{cases} \rho(xg^{-1})v \otimes \sigma(yh^{-1})w & \text{if } x \in \Gamma g \text{ and } y \in \Gamma h; \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$\tilde{\Phi}(\varphi_{g,v} \otimes \psi_{h,w})(x, y) = \begin{cases} [(\rho \otimes \sigma)(xg^{-1}, yh^{-1})]\,(v \otimes w) & \text{if } (x, y) \in \Gamma g \times \Gamma h; \\ 0 & \text{otherwise.} \end{cases}$$

Now by Proposition 1.32, the set $\{\tilde{\Phi}(\varphi_{g,v} \otimes \psi_{h,w}) \mid g, h \in [\Gamma \backslash G], v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$ forms a basis for $C(\rho \otimes \sigma)$. Thus, $\tilde{\Phi}$ is bijective.

To complete the proof, we need only show that $\tilde{\Phi}$ is an intertwining operator. That is, we need to show that

$$\tilde{\Phi} \circ [(\pi_\rho \otimes \pi_\sigma)(g, h)] = [\pi_{\rho \otimes \sigma}(g, h)] \circ \tilde{\Phi}$$

for all $(g, h) \in G \times G$. Let $\varphi \otimes \psi \in C(\rho) \otimes C(\sigma)$. Then for all $x, y \in G$, we have

$$\begin{aligned} \tilde{\Phi} \circ [(\pi_\rho \otimes \pi_\sigma)(g, h)](\varphi \otimes \psi)(x, y) &= [\tilde{\Phi}(\varphi \otimes \psi)](xg, yh) \\ &= \varphi\psi(xg, yh) \\ &= [\pi_{\rho \otimes \sigma}(g, h)]\varphi\psi(x, y) \\ &= [\pi_{\rho \otimes \sigma}(g, h)] \circ \tilde{\Phi}(\varphi \otimes \psi)(x, y), \end{aligned}$$

which completes the proof. □

CHAPTER 2
THE GROUP ALGEBRA

In this chapter, we examine some well-known results pertaining to the group algebra in representation theory. In Section 2.1, we define the group algebra and its inner product. In Section 2.2, we define the notion of a representation of the group algebra. In Section 2.3 we prove the Schur orthogonality relations for matrix coefficients. Section 2.4 examines the characters of group representations and some of their many useful properties.

Many of the results and definitions in this section pertaining to associative $\mathbb{C}$-algebras can be found in [Rom08]. Much of the information in Sections 2.2 and 2.4 can be found in [FH91]. Our discussion of matrix coefficients and our proof of Schur orthogonality in Section 2.3 can be found in [Bum13], and our proof of Maschke's theorem in Section 2.4 is adapted from the unpublished notes of Rogawski.

2.1 Definitions

Let G be a finite group and let $\mathbb{C}[G]$ denote the set of complex-valued functions on G. The set $\mathbb{C}[G]$ forms a complex vector space under pointwise addition and scalar multiplication. The dimension of $\mathbb{C}[G]$ is simply the order of G. To see this, we define for each $g \in G$ the map $\mathbf{1}_g : G \to \mathbb{C}$ by

$$\mathbf{1}_g(x) = \begin{cases} 1 & \text{if } x = g; \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to show that the set $\{\mathbf{1}_g \mid g \in G\}$ forms a basis for $\mathbb{C}[G]$. For $\varphi, \psi \in \mathbb{C}[G]$, the *convolution* of φ and ψ is the map $(\varphi * \psi) : G \to \mathbb{C}$ defined by

$$(\varphi * \psi)(x) = \sum_{y \in G} \varphi(y)\psi(y^{-1}x).$$

The set $\mathbb{C}[G]$ together with the operations of addition, scalar multiplication, and convolution defined above is called the *group algebra of G over* $\mathbb{C}$. The group algebra is an example of an *associative* $\mathbb{C}$-*algebra*.

Definition 2.1. An *associative* $\mathbb{C}$*-algebra* is a complex vector space V together with a bilinear map $V \times V \to V$, denoted $(x, y) \mapsto xy$, such that

(1) $(xy)z = x(yz)$ for all $x, y, z \in V$;

(2) there exists $I \in V$ satisfying $Ix = xI = x$ for all $x \in V$.

The bilinear map $V \times V \to V$ is usually called *multiplication*, and $I \in V$ is usually called the *multiplicative identity element*.

We will verify that $\mathbb{C}[G]$ is indeed an associative $\mathbb{C}$-algebra. Before the proof, we state some familiar examples of algebras used in later sections. If V is a finite-dimensional complex vector space, then $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ is an associative $\mathbb{C}$-algebra with addition and scalar multiplication defined pointwise, and multiplication defined by function composition. If n is a positive integer, the set $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries is likewise an associative $\mathbb{C}$-algebra under the usual addition, multiplication, and scalar multiplication defined on matrices.

Proposition 2.2. *The group algebra* $\mathbb{C}[G]$ *is an associative* $\mathbb{C}$*-algebra.*

Proof. Clearly $\mathbb{C}[G]$ is a complex vector space. So it will suffice to show that convolution on $\mathbb{C}[G]$ is bilinear and associative, and that there exists $I \in \mathbb{C}[G]$ such that $I * \varphi = \varphi * I$ for all $\varphi \in \mathbb{C}[G]$.

To show that convolution is bilinear, let $\varphi, \psi, \chi \in \mathbb{C}[G]$ and let $\lambda \in \mathbb{C}$. For any $g \in G$, we have

$$(\varphi * (\psi + \lambda\chi))\,(g) = \sum_{x \in G} \varphi(x)(\psi + \lambda\chi)(x^{-1}g) = \sum_{x \in G} \varphi(x)\psi(x^{-1}g) + \lambda \sum_{x \in G} \varphi(x)\chi(x^{-1}g).$$

Therefore, $\varphi * (\psi + \lambda\chi) = (\varphi * \psi) + \lambda(\varphi * \chi)$. Similarly,

$$((\varphi + \lambda\psi) * \chi))\,(g) = \sum_{x \in G} (\varphi + \lambda\psi)(x)\chi(x^{-1}g) = \sum_{x \in G} \varphi(x)\chi(x^{-1}g) + \lambda \sum_{x \in G} \psi(x)\chi(x^{-1}g).$$

Thus $(\varphi + \lambda\psi) * \chi = (\varphi * \chi) + \lambda(\psi * \chi)$ and convolution on $\mathbb{C}[G]$ is bilinear. For associativity, note that

$$(\varphi * (\psi * \chi))(g) = \sum_{x \in G} \varphi(x)(\psi * \chi)(x^{-1}g) = \sum_{x \in G} \sum_{y \in G} \varphi(x)\psi(y)\chi(y^{-1}x^{-1}g).$$

On the other hand,

$$((\varphi * \psi) * \chi)(g) = \sum_{y \in G} (\varphi * \psi)(y)\chi(y^{-1}g) = \sum_{y \in G} \sum_{x \in G} \varphi(x)\psi(x^{-1}y)\chi(y^{-1}g).$$

Now we switch the order of summation and make a change of variables $y \mapsto xy$ to obtain

$$((\varphi * \psi) * \chi)(g) = \sum_{x \in G} \sum_{y \in G} \varphi(x)\psi(x^{-1}y)\chi(y^{-1}g) = \sum_{x \in G} \sum_{y \in G} \varphi(x)\psi(y)\chi(y^{-1}x^{-1}g).$$

Thus $\varphi * (\psi * \chi) = (\varphi * \psi) * \chi$ and convolution on $\mathbb{C}[G]$ is associative.

Let $e \in G$ denote the identity element. Then for any $\varphi \in \mathbb{C}[G]$,

$$(\mathbf{1}_e * \varphi)(g) = \sum_{x \in G} \mathbf{1}_e(x)\varphi(x^{-1}g) = \varphi(g) = \sum_{x \in G} \varphi(x)\mathbf{1}_e(x^{-1}g) = (\varphi * \mathbf{1}_e)(g).$$

Therefore $\mathbf{1}_e$ is the multiplicative identity element in $\mathbb{C}[G]$, and we conclude that $\mathbb{C}[G]$ is an associative $\mathbb{C}$-algebra. □

One can also define the group algebra to be the set of formal linear combinations of group elements. We write $\mathbb{C}G = \{\sum_{x \in G} \lambda_x \cdot x \mid \lambda_x \in \mathbb{C}\}$ to avoid any confusion with our preceding definition of the group algebra. The operations of addition and scalar multiplication on $\mathbb{C}G$ are defined by

$$\sum_{x \in G} \alpha_x \cdot x + \sum_{x \in G} \beta_x \cdot x = \sum_{x \in G} (\alpha_x + \beta_x) \cdot x;$$

$$\lambda \sum_{x \in G} \alpha_x \cdot x = \sum_{x \in G} \lambda \alpha_x \cdot x.$$

Multiplication on $\mathbb{C}G$ is defined by

$$\left(\sum_{x \in G} \alpha_x \cdot x\right)\left(\sum_{y \in G} \beta_y \cdot y\right) = \sum_{x \in G} \sum_{y \in G} \alpha_x \beta_y \cdot xy.$$

We then make a change of variables $y \mapsto x^{-1}y$ and switch the order of summation to obtain

$$\left(\sum_{x\in G} \alpha_x \cdot x\right)\left(\sum_{y\in G} \beta_y \cdot y\right) = \sum_{y\in G}\left(\sum_{x\in G} \alpha_x \beta_{x^{-1}y}\right) \cdot y.$$

If we identify each $\varphi \in \mathbb{C}[G]$ with the formal linear combination $(\sum_{x\in G} \varphi(x) \cdot x) \in \mathbb{C}G$, we see that both definitions of the group algebra are essentially identical. Indeed, $\mathbb{C}[G]$ and $\mathbb{C}G$ are *isomorphic* as algebras.

Definition 2.3. Suppose V and W are associative $\mathbb{C}$-algebras. Let I_V and I_W denote the multiplicative identities in V and W, respectively. A linear map $\Phi : V \to W$ is called an *algebra homomorphism* if

(1) $\Phi(I_V) = I_W$,

(2) $\Phi(vw) = \Phi(v)\Phi(w)$ for all $v \in V$ and $w \in W$.

A bijective algebra homomorphism is called an *algebra isomorphism*.

It is straightforward to show that the map $\mathbb{C}[G] \to \mathbb{C}G$ defined by $\varphi \mapsto \sum_{x\in G} \varphi(x) \cdot x$ is in fact an algebra isomorphism. If V is a complex vector space and $\dim V = n$ is finite, then $\mathrm{End}(V)$ and $\mathrm{Mat}_n(\mathbb{C})$ are likewise isomorphic as algebras. We conclude this section by defining an inner product on $\mathbb{C}[G]$.

Proposition 2.4. *The map* $\langle \cdot, \cdot \rangle : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ *given by*

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{x\in G} \varphi(x)\overline{\psi(x)}.$$

defines an inner product on $\mathbb{C}[G]$.

Proof. Let $\varphi, \psi, \chi \in \mathbb{C}[G]$ and let $\lambda \in \mathbb{C}$. We need to show that

(1) $\overline{\langle \varphi, \psi \rangle} = \langle \psi, \varphi \rangle$;

(2) $\langle \varphi + \lambda\psi, \chi \rangle = \langle \varphi, \chi \rangle + \lambda \langle \psi, \chi \rangle$;

(3) $\langle \varphi, \varphi \rangle \geq 0$;

(4) $\langle \varphi, \varphi \rangle = 0$ if and only if $\varphi = 0$.

For (1), we have

$$\overline{\langle \varphi, \psi \rangle} = \frac{1}{|G|} \sum_{x \in G} \overline{\varphi(x)\overline{\psi(x)}} = \frac{1}{|G|} \sum_{x \in G} \psi(x)\overline{\varphi(x)} = \langle \psi, \varphi \rangle.$$

For (2),

$$\langle \varphi + \lambda\psi, \chi \rangle = \frac{1}{|G|} \sum_{x \in G} (\varphi + \lambda\psi)(x)\overline{\chi(x)} = \frac{1}{|G|} \sum_{x \in G} \varphi(x)\overline{\chi(x)} + \lambda \frac{1}{|G|} \sum_{x \in G} \psi(x)\overline{\chi(x)}.$$

Hence $\langle \varphi + \lambda\psi, \chi \rangle = \langle \varphi, \chi \rangle + \lambda \langle \psi, \chi \rangle$. For (3) and (4), note that $\langle \varphi, \varphi \rangle$ is a finite sum of non-negative numbers since

$$\langle \varphi, \varphi \rangle = \frac{1}{|G|} \sum_{x \in G} \varphi(x)\overline{\varphi(x)} = \frac{1}{|G|} \sum_{x \in G} |\varphi(x)|^2.$$

Thus $\langle \varphi, \varphi \rangle \geq 0$, with equality if and only if $\varphi(x) = 0$ for all $x \in G$. □

Remark. We will sometimes denote the inner product on $\mathbb{C}[G]$ by $\langle \cdot, \cdot \rangle_{\mathbb{C}[G]}$. This notation will be especially useful in Chapter 3, where we consider several distinct inner product spaces all at once.

2.2 Representations of the group algebra

Definition 2.5. A *representation of the group algebra* $\mathbb{C}[G]$ is an ordered pair (Φ, V), where V is a finite-dimensional complex vector space and $\Phi : \mathbb{C}[G] \to \mathrm{End}(V)$ is an algebra homomorphism.

A representation of $\mathbb{C}[G]$ defines a representation of G in a natural way. To see this, suppose that (Φ, V) is a representation of $\mathbb{C}[G]$ and let $\mathcal{G} = \{\mathbf{1}_g \mid g \in G\}$. Note that for any $g, h \in G$,

$$(\mathbf{1}_g * \mathbf{1}_h)(x) = \sum_{y \in G} \mathbf{1}_g(y)\mathbf{1}_h(y^{-1}x) = \mathbf{1}_h(g^{-1}x) = \mathbf{1}_{gh}(x).$$

It is easily verified that the map $\iota : G \to \mathcal{G}$ defined by $g \mapsto \mathbf{1}_g$ is a group isomorphism. Now $\Phi(\mathbf{1}_g) : V \to V$ is invertible for each $g \in G$ since

$$\Phi(\mathbf{1}_g)\Phi(\mathbf{1}_{g^{-1}}) = \Phi(\mathbf{1}_g * \mathbf{1}_{g^{-1}}) = \Phi(\mathbf{1}_e) = \Phi(\mathbf{1}_{g^{-1}} * \mathbf{1}_g) = \Phi(\mathbf{1}_g)\Phi(\mathbf{1}_{g^{-1}}),$$

and $\Phi(\mathbf{1}_e) = \mathrm{id}_V$. It is once again easily verified that $\Phi|_{\mathcal{G}} : \mathcal{G} \to GL(V)$ is a group homomorphism. It now follows that $\Phi \circ \iota : G \to \mathrm{GL}(V)$ is a group homomorphism. Ergo, $(\Phi \circ \iota, V)$ is a representation of G. A representation of G likewise extends to a representation of $\mathbb{C}[G]$ in a natural way.

Proposition 2.6. *Let (ρ, V) be a representation of G and define $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ by*

$$\rho(f) = \sum_{x \in G} f(x)\rho(x)$$

for all $f \in \mathbb{C}[G]$. Then (ρ, V) is a representation of $\mathbb{C}[G]$.

Proof. Clearly $\rho(f) \in \mathrm{End}(V)$ for each $f \in \mathbb{C}[G]$, since $\rho(f)$ is a linear combination of elements in $\mathrm{GL}(V) \subseteq \mathrm{End}(V)$. To see that $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ is a linear map, let $f_1, f_2 \in \mathbb{C}[G]$ and let $\lambda \in \mathbb{C}$. Then

$$\rho(f_1 + \lambda f_2) = \sum_{x \in G} [f_1 + \lambda f_2](x)\rho(x) = \sum_{x \in G} f_1(x)\rho(x) + \lambda \sum_{x \in G} f_2(x)\rho(x) = \rho(f_1) + \lambda\rho(f_2).$$

Thus ρ is a linear map.

Let $e \in G$ denote the identity element. To complete the proof, we need only show that $\rho(\mathbf{1}_e) = \mathrm{id}_V$ and that $\rho(f_1 * f_2) = \rho(f_1) \circ \rho(f_2)$ for all $f_1, f_2 \in \mathbb{C}[G]$. For the first claim, we have

$$\rho(\mathbf{1}_e) = \sum_{x \in G} \mathbf{1}_e(x)\rho(x) = \rho(e) = \mathrm{id}_V\,.$$

For the second claim, observe that

$$\begin{aligned}
\rho(f_1 * f_2) &= \sum_{x\in G} [f_1 * f_2](x)\rho(x) \\
&= \sum_{x\in G}\sum_{y\in G} f_1(y)f_2(y^{-1}x)\rho(x) \\
&= \sum_{x\in G}\sum_{y\in G} f_1(y)f_2(y^{-1}x)\rho(y)\rho(y^{-1}x) && (\rho(x) = \rho(y)\rho(y^{-1}x)) \\
&= \sum_{x\in G}\sum_{y\in G} f_1(y)\rho(y)f_2(y^{-1}x)\rho(y^{-1}x) && (\text{since } \rho(y) \text{ is a linear map}) \\
&= \sum_{y\in G} f_1(y)\rho(y)\sum_{x\in G} f_2(y^{-1}x)\rho(y^{-1}x) \\
&= \sum_{y\in G} f_1(y)\rho(y)\sum_{x\in G} f_2(x)\rho(x) && (\text{change of variables } x \mapsto yx) \\
&= \rho(f_1)\circ\rho(f_2).
\end{aligned}$$

Thus ρ is an algebra homomorphism and we conclude that (ρ, V) is a representation of $\mathbb{C}[G]$. □

Note that if (ρ, V) is a representation of G, then $\rho(\mathbf{1}_g) = \rho(g)$ for each $g \in G$. Since $\{\mathbf{1}_g \mid g \in G\}$ forms a basis for $\mathbb{C}[G]$ and $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ is a linear map, we conclude that (ρ, V) is the unique representation of $\mathbb{C}[G]$ on V satisfying $\rho(\mathbf{1}_g) = \rho(g)$ for all $g \in G$.

2.3 Schur orthogonality

Let (ρ, V) be a representation of G and let $X = \{x_1, \dots, x_n\}$ be a basis for V. Recall that the dual space of V is the set V^* of linear maps $f : V \to \mathbb{C}$ under pointwise addition and scalar multiplication. For each $x_i \in X$, recall that we define a linear map $x_i^* : V \to \mathbb{C}$ by $x_i^*(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_i$. The set $X^* = \{x_1^*, \dots, x_n^*\}$ forms a basis for V^*, which we call the dual basis corresponding to X.

A *matrix coefficient* of (ρ, V) is a map $\phi : G \to \mathbb{C}$ of the form $\phi(g) = f(\rho(g)v)$ for some $v \in V$ and some $f \in V^*$. Our goal in this section is to prove the Schur orthogonality relations for matrix coefficients. Before we state the theorem, it will be convenient to express the matrix coefficients of (ρ, V) in terms of an inner product on V. We will assume that $\rho : G \to \mathrm{GL}(V)$ is *unitary* with respect to our choice of an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$, meaning that $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$ for all $v, w \in V$ and all $g \in G$.

Proposition 2.7. *There exists an inner product on V relative to which (ρ, V) is unitary.*

Proof. Suppose that $(\cdot, \cdot)$ is an inner product on V. Define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ by

$$\langle v, w \rangle = \sum_{x \in G} (\rho(x)v, \rho(x)w).$$

It is easily verified that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is an inner product on V. To show that (ρ, V) is unitary with respect to this inner product, let $g \in G$ and observe that

$$\langle \rho(g)v, \rho(g)w \rangle = \sum_{x \in G} (\rho(g)\rho(x)v, \rho(g)\rho(x)w) = \sum_{x \in G} (\rho(gx)v, \rho(gx)w).$$

Making a change of variables $x \mapsto g^{-1}x$ gives us $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$, as needed. □

Theorem 2.8 (Riesz representation theorem)**.** *Let V be a finite-dimensional vector space, let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be an inner product on V, and let $f \in V^*$. Then there exists a unique vector $w \in V$ such that $f(v) = \langle v, w \rangle$ for all $v \in V$.*

Proof. Let X be an orthonormal basis for V and let X^* be the corresponding dual basis. Since $f \in V^*$, there exist unique scalars λ_x for $x \in X$ such that $f = \sum_{x \in X} \lambda_x x^*$. Now for any $v \in V$,

$$f(v) = \sum_{x \in X} \lambda_x x^*(v) = \sum_{x \in X} \lambda_x \langle v, x \rangle = \sum_{x \in X} \langle v, \overline{\lambda_x} x \rangle = \left\langle v, \sum_{x \in X} \overline{\lambda_x} x \right\rangle.$$

Hence $w = \sum_{x \in X} \overline{\lambda_x} x$ satisfies $f(v) = \langle v, w \rangle$ for all $v \in V$, and w is unique by construction. □

By the above, each matrix coefficient of a representation (ρ, V) is given by $g \mapsto \langle \rho(g)v, w \rangle$ for some $v, w \in V$ and some inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. So given a fixed inner product on V, each matrix coefficient of (ρ, V) is determined by a pair of vectors $(v, w) \in V \times V$. We let $\phi_{v,w} : G \to \mathbb{C}$ denote the matrix coefficient $g \mapsto \langle \rho(g)v, w \rangle$.

Note that $\phi_{v,w}$ depends not only on our choice of $v, w \in V$, but also on our choice of an inner product on V. We are now ready to state the main result of this section. We give the proof in several steps.

Theorem 2.9 (Schur orthogonality)**.** *Let (ρ, V) and (σ, W) be irreducible unitary representations of G, let $v, v' \in V$, and let $w, w' \in W$. Then*

$$\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \begin{cases} \dfrac{\langle v, w \rangle \overline{\langle v', w' \rangle}}{\dim \rho} & \text{if } (\rho, V) = (\sigma, W); \\ 0 & \text{if } (\rho, V) \ncong (\sigma, W). \end{cases}$$

Lemma 2.10. *Fix $v \in V$ and $w \in W$. Then there exists an intertwining operator $T : W \to V$ such that $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \langle Tw', v' \rangle$ for all $w' \in W$ and $v' \in V$.*

Proof. Let $Y = \{y_1, \ldots, y_n\}$ be an orthonormal basis for W. For $y_k \in Y$, define the map $\ell_k : V \to \mathbb{C}$ by $\ell_k(x) = \overline{\langle \phi_{v,x}, \phi_{w,y_k} \rangle}$. It is easily verified that $\ell_k : V \to \mathbb{C}$ is a linear map for each $y_k \in Y$. Therefore, by the Riesz representation theorem there exists $v_k \in V$ for each $y_k \in Y$ such that $\ell_k(x) = \langle x, v_k \rangle$ for all $x \in V$. Ergo, $\langle \phi_{v,x}, \phi_{w,y_k} \rangle = \langle v_k, x \rangle$ for each $y_k \in Y$ and for all $x \in V$.

Define a linear map $T : W \to V$ by $Ty_i = v_i$ for each $y_i \in Y$. First, we show that $\langle Ty, x \rangle = \langle \phi_{v,x}, \phi_{w,y} \rangle$ for all $x \in V$ and $y \in W$. Let $x \in V$ and let $y = \sum_{i=1}^{n} \lambda_i y_i \in W$ be arbitrary. Since $T : W \to V$ is linear and $\langle \cdot, \cdot \rangle : V \to V$ is linear in the first coordinate, we have

$$\langle Ty, x \rangle = \left\langle T \sum_{i=1}^{n} \beta_i y_i, x \right\rangle = \left\langle \sum_{i=1}^{n} \beta_i T y_i, x \right\rangle = \sum_{i=1}^{n} \beta_i \langle Ty_i, x \rangle .$$

Now recall $\langle Ty_i, x \rangle = \langle v_i, x \rangle = \langle \phi_{v,x}, \phi_{w,y_i} \rangle$ for each $y_i \in Y$. We have

$$\begin{aligned} \sum_{i=1}^{n} \beta_i \langle Ty_i, x \rangle &= \sum_{i=1}^{n} \beta_i \langle \phi_{v,x}, \phi_{w,y_i} \rangle \\ &= \left\langle \phi_{v,x}, \sum_{i=1}^{n} \overline{\beta_i} \, \phi_{w,y_i} \right\rangle \\ &= \langle \phi_{v,x}, \phi_{w,y} \rangle , \end{aligned}$$

where the last equality follows from the observation that matrix coefficients are conjugate-linear in the second coordinate. Therefore, $\langle Ty, x\rangle = \langle \phi_{v,x}, \phi_{w,y}\rangle$ for all $x \in V$ and $y \in W$, as needed.

By the above, we have $\langle T\sigma(g)y, x\rangle = \left\langle \phi_{v,x}, \phi_{w,\sigma(g)y}\right\rangle$ for all $g \in G$. On the other hand, since (ρ, V) is unitary we have $\langle \rho(g)Ty, x\rangle = \langle Ty, \rho(g)^{-1}x\rangle = \left\langle \phi_{v,\rho(g)^{-1}x}, \phi_{w,y}\right\rangle$ for all $g \in G$. If $h \in G$, then

$$\begin{aligned}
\left\langle \phi_{v,x}, \phi_{w,\sigma(h)y}\right\rangle &= \frac{1}{|G|}\sum_{g\in G} \langle \rho(g)v, x\rangle\, \overline{\langle \sigma(g)w, \sigma(h)y\rangle} \\
&= \frac{1}{|G|}\sum_{g\in G} \langle \rho(g)v, x\rangle\, \overline{\langle \sigma(h^{-1}g)w, y\rangle} && \text{(since } (\sigma, W) \text{ is unitary)} \\
&= \frac{1}{|G|}\sum_{g\in G} \langle \rho(hg)v, x\rangle\, \overline{\langle \sigma(g)w, y\rangle} && \text{(change of variables } g \mapsto hg) \\
&= \frac{1}{|G|}\sum_{g\in G} \left\langle \rho(g)v, \rho(h^{-1})x\right\rangle \overline{\langle \sigma(g)w, y\rangle} && \text{(since } (\rho, V) \text{ is unitary)} \\
&= \left\langle \phi_{v,\rho(h^{-1})x}, \phi_{w,y}\right\rangle.
\end{aligned}$$

Therefore, $\langle T\sigma(h)y, x\rangle = \langle \rho(h)Ty, x\rangle$ for all $h \in G$. Since $x \in V$ and $y \in W$ were arbitrary, it follows that $T\sigma(h) = \rho(h)T$ for all $h \in G$. Thus $T : W \to V$ is an intertwining operator. □

Corollary 2.11. *Fix $v' \in V$ and $w' \in W$. Then there exists an intertwining operator $S : W \to V$ such that $\langle \phi_{v,v'}, \phi_{w,w'}\rangle = \overline{\langle Sw, v\rangle}$ for all $v \in V$ and $w \in W$.*

Proof. First note that

$$\begin{aligned}
\overline{\langle \phi_{v,v'}, \phi_{w,w'}\rangle} &= \frac{1}{|G|}\sum_{g\in G} \langle \sigma(g)w, w'\rangle\, \overline{\langle \rho(g)v, v'\rangle} \\
&= \frac{1}{|G|}\sum_{g\in G} \left\langle \sigma(g^{-1})w, w'\right\rangle \overline{\langle \rho(g^{-1})v, v'\rangle} && \text{(change of variables } g \mapsto g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G} \langle w, \sigma(g)w'\rangle\, \overline{\langle v, \rho(g)v'\rangle} && \text{(since } (\rho, V) \text{ and } (\sigma, W) \text{ are unitary)} \\
&= \frac{1}{|G|}\sum_{g\in G} \langle \rho(g)v', v\rangle\, \overline{\langle \sigma(g)w', w\rangle} \\
&= \langle \phi_{v',v}, \phi_{w',w}\rangle.
\end{aligned}$$

Now by Lemma 2.9, there exists an intertwining operator $S : W \to V$ such that $\langle \phi_{v',v}, \phi_{w',w} \rangle = \langle Sw, v \rangle$ for all $v \in V$ and $w \in W$. Thus, $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \overline{\langle Sw, v \rangle}$. □

Corollary 2.12. *If (ρ, V) and (σ, W) are irreducible representations of G and $(\rho, V) \ncong (\sigma, W)$, then $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = 0$.*

Proof. By Lemma 2.10, there exists an intertwining operator $T : W \to V$ such that $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \langle Tw', v' \rangle$. Since (ρ, V) and (σ, W) are irreducible and $(\rho, V) \ncong (\sigma, W)$, Schur's lemma implies that $T = 0$. Thus $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \langle 0, v' \rangle = 0$. □

Lemma 2.13. *If $(\rho, V) = (\sigma, W)$ is an irreducible representation of G, then there exists a constant $\lambda \in \mathbb{C}$ such that $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \lambda \langle v, w \rangle \overline{\langle v', w' \rangle}$ for all $v, v', w, w' \in V$.*

Proof. If $v, w \in V$ are fixed, then by Lemma 2.9 there exists an intertwining operator $T : V \to V$ such that $\langle \phi_{v,x'}, \phi_{w,y'} \rangle = \langle Ty', x' \rangle$ for all $x', y' \in V$. Since (ρ, V) is irreducible, Schur's lemma implies that the map T is just multiplication by some constant $C(v, w) \in \mathbb{C}$, which depends only on $v, w \in V$. Thus, $\langle \phi_{v,x'}, \phi_{w,y'} \rangle = \langle C(v, w)y', x' \rangle = C(v, w)\overline{\langle x', y' \rangle}$ for all $x', y' \in V$. By a similar argument, if $v', w' \in V$ are fixed then there exists a nonzero constant $K(v', w') \in \mathbb{C}$ depending on $v', w' \in V$ such that $\langle \phi_{x,v'}, \phi_{y,w'} \rangle = K(v', w') \langle x, y \rangle$ for all $x, y \in V$.

Assume $\langle v, w \rangle$ is nonzero and write $C(v, w) = \lambda \langle v, w \rangle$ for some $\lambda \in \mathbb{C}$. Then by the above, $\langle \phi_{v,x'}, \phi_{w,y'} \rangle = C(v, w)\overline{\langle x', y' \rangle} = \lambda \langle v, w \rangle \overline{\langle x', y' \rangle}$ for all $x', y' \in V$. On the other hand, $\langle \phi_{v,x'}, \phi_{w,y'} \rangle = K(x', y') \langle v, w \rangle$ for all $x', y' \in V$. Hence $K(x', y') = \lambda\overline{\langle x', y' \rangle}$ for all $x', y' \in V$. By a similar argument, if we assume $\langle v', w' \rangle$ is nonzero and write $K(v', w') = \lambda' \overline{\langle v', w' \rangle}$, we find that $C(x, y) = \lambda' \langle x, y \rangle$ for all $x, y \in V$. In particular, $C(v, w) = \lambda' \langle v, w \rangle$. Hence $\lambda = \lambda'$ and we have

$$\langle \phi_{x,x'}, \phi_{y,y'} \rangle = \lambda \langle x, y \rangle \overline{\langle x', y' \rangle}$$

for all $x, x', y, y' \in V$. □

Lemma 2.14. *If $(\rho, V) = (\sigma, W)$ is an irreducible unitary representation of G, then $\langle \phi_{v,v'}, \phi_{w,w'} \rangle = (\dim \rho)^{-1} \langle v, w \rangle \overline{\langle v', w' \rangle}$ for all $v, v', w, w' \in V$.*

Proof. Let $v, v', w, w' \in V$. We expand the inner product $\langle \phi_{v,v'}, \phi_{w,w'} \rangle$ to obtain

$$\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, v' \rangle \overline{\langle \rho(g)w, w' \rangle}.$$

Because (ρ, V) is unitary, we have $\overline{\langle \rho(g)w, w' \rangle} = \overline{\langle w, \rho(g^{-1})w' \rangle} = \langle \rho(g^{-1})w', w \rangle$. Therefore,

$$\langle \phi_{v,v'}, \phi_{w,w'} \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, v' \rangle \left\langle \rho(g^{-1})w', w \right\rangle.$$

So by the previous lemma,

$$\frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, v' \rangle \left\langle \rho(g^{-1})w', w \right\rangle = \lambda \langle v, w \rangle \overline{\langle v', w' \rangle} \tag{2.1}$$

for some $\lambda \in \mathbb{C}$. We show that $\lambda = (\dim \rho)^{-1}$. Let $\dim \rho = n$ and let $X = \{x_1, \ldots, x_n\}$ be an orthonormal basis for V. For any $g \in G$, the matrix of $\rho(g)$ relative to X is given by

$$[\rho(g)]_{ij} = \langle \rho(g)x_j, x_i \rangle.$$

The matrix of $\rho(g)\rho(g^{-1}) = \rho(e)$ relative to X is simply the $n \times n$ identity matrix. That is,

$$[\rho(g)\rho(g^{-1})]_{ij} = \sum_{k=1}^{n} \langle \rho(g)x_k, x_i \rangle \langle \rho(g^{-1})x_j, x_k \rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Now we combine (2.1) and (2.2), fixing $v = w = x_j$ and summing over $v' = w' = x_k$ for all $x_k \in X$. We have

$$\sum_{k=1}^{n} \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)x_j, x_k \rangle \langle \rho(g^{-1})x_k, x_j \rangle = \sum_{k=1}^{n} \lambda \langle x_j, x_j \rangle \overline{\langle x_k, x_k \rangle}. \tag{2.3}$$

The right-hand side of (2.3) is simply $\lambda \cdot \dim \rho$. For the left-hand side, we switch the order of summation to obtain

$$\frac{1}{|G|} \sum_{g \in G} \sum_{k=1}^{n} \langle \rho(g)x_k, x_j \rangle \langle \rho(g^{-1})x_j, x_k \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho(g)\rho(g^{-1})]_{jj} = 1.$$

Hence $\lambda = (\dim \rho)^{-1}$, as needed. □

2.4 Characters of group representations

Let (ρ, V) be a representation of G. The *character* of (ρ, V) is the map $\chi_\rho : G \to \mathbb{C}$ given by $\chi_\rho(g) = \mathrm{Tr}\,[\rho(g)]$. We begin by proving some basic properties enjoyed by the characters of group representations.

Proposition 2.15. *If (ρ, V) and (σ, W) are representations of G and $(\rho, V) \cong (\sigma, W)$, then $\chi_\rho = \chi_\sigma$.*

Proof. Since (ρ, V) and (σ, W) are equivalent, there exists an invertible intertwining operator $T : V \to W$. Therefore, $\rho(g) = T^{-1}\sigma(g)T$ for all $g \in G$, and it follows that

$$\chi_\rho(g) = \mathrm{Tr}\,[\rho(g)] = \mathrm{Tr}\left[T^{-1}\sigma(g)T\right] = \mathrm{Tr}\,[\sigma(g)] = \chi_\sigma(g). \qquad \square$$

Proposition 2.16. *If (ρ, V) and (σ, W) are representations of G then $\chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$.*

Proof. Let $X = \{x_1, \ldots, x_m\}$ be a basis for V and let $Y = \{y_1, \ldots, y_n\}$ be a basis for W. Then $Z = \{(x_1, 0), \ldots, (x_m, 0), (0, y_1), \ldots, (0, y_n)\}$ is a basis for $V \oplus W$. Now for any $g \in G$, the matrix of $(\rho \oplus \sigma)(g)$ relative to Z is given by the block diagonal matrix

$$[(\rho \oplus \sigma)(g)]_{ij} = \begin{cases} x_i^*(\rho(g)x_j) & \text{if } i, j \le \dim V; \\ y_{i-n}^*(\sigma(g)y_{j-n}) & \text{if } \dim V < i, j \le \dim V + \dim W; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{Tr}\,[(\rho \oplus \sigma)(g)] = \mathrm{Tr}\,[\rho(g)] + \mathrm{Tr}\,[\sigma(g)]$ and so $\chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$. $\square$

Corollary 2.17. *If $\rho \cong \bigoplus_{\pi\in\hat{G}} m_\pi \cdot \pi$ is a finite-dimensional representation of G, then $\chi_\rho = \sum_{\pi\in\hat{G}} m_\pi \chi_\pi$.*

The characters of the irreducible representations of G, which we will refer to as the irreducible characters of G, enjoy the following orthogonality relations.

Theorem 2.18. *If (ρ, V) and (σ, W) are irreducible representations of G, then*

$$\langle \chi_\rho, \chi_\sigma \rangle = \begin{cases} 1 & \textit{if } (\rho, V) \cong (\sigma, W); \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let X be an orthonormal basis for V and let Y be an orthonormal basis for W. We have

$$\begin{aligned}\langle \chi_\rho, \chi_\sigma \rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\overline{\chi_\sigma(g)} \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X}\langle \sigma(g)x, x\rangle \sum_{y\in Y}\overline{\langle \tau(g)y, y\rangle} \\ &= \sum_{x\in X}\sum_{y\in Y}\frac{1}{|G|}\sum_{g\in G}\langle \sigma(g)x, x\rangle\overline{\langle \tau(g)y, y\rangle} \\ &= \sum_{x\in X}\sum_{y\in Y}\langle \phi_{x,x}, \phi_{y,y}\rangle\end{aligned}$$

If $(\rho, V) \ncong (\sigma, W)$, then by the Schur orthogonality relations for matrix coefficients, we have $\langle \phi_{x,x}, \phi_{y,y}\rangle = 0$ for all $x \in X$ and $y \in Y$. Thus, $\langle \chi_\rho, \chi_\sigma\rangle = 0$. On the other hand, if $(\rho, V) \cong (\sigma, W)$ then $\chi_\rho = \chi_\sigma$ and we have

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{x_i, x_j \in X}\left\langle \phi_{x_i,x_i}, \phi_{x_j,x_j}\right\rangle = \sum_{x_i,x_j\in X}\frac{\langle x_i, x_j\rangle\overline{\langle x_i, x_j\rangle}}{\dim V} = 1. \qquad \square$$

Corollary 2.19. *If $\rho \cong \bigoplus_{\pi\in\hat{G}} m_\pi \cdot \pi$ is a representation of G and $\tau \in \hat{G}$, then $\langle \chi_\rho, \chi_\tau\rangle = m_\tau$.*

Proof. Simply note that

$$\langle \chi_\rho, \chi_\tau \rangle = \left\langle \sum_{\pi\in\hat{G}} m_\pi \chi_\pi, \chi_\tau \right\rangle = \sum_{\pi\in\hat{G}} m_\pi \langle \chi_\pi, \chi_\tau\rangle,$$

and $\langle \chi_\pi, \chi_\tau\rangle = 0$ unless $\pi \cong \tau$. $\square$

Our goal now is to show that a representation is determined (up to equivalence) by its character. That is, $\chi_\rho = \chi_\sigma$ if and only if $(\rho, V) \cong (\sigma, W)$. We have already proved the reverse implication. The proof of the forward implication is more involved. Our strategy will be to show that the irreducible characters $\{\chi_\pi \mid \pi \in \hat{G}\}$ form a basis for some subspace of $\mathbb{C}[G]$.

A function $f \in \mathbb{C}[G]$ is said to be a *class function* if it is constant on the conjugacy classes of G, that is, if $f(h^{-1}gh) = f(g)$ for all $g, h \in G$. The complex-valued class functions on G form a subspace of $\mathbb{C}[G]$, which we denote by $\mathbb{C}_{class}[G]$. Note that for any representation (ρ, V) of G and any $g, h \in G$, we have

$$\chi_\rho(h^{-1}gh) = \operatorname{Tr}\left[\rho(h^{-1}gh)\right] = \operatorname{Tr}\left[\rho(h)^{-1}\rho(g)\rho(h)\right] = \operatorname{Tr}\left[\rho(g)\right] = \chi_\rho(g).$$

Hence the character of (ρ, V) is a class function. We first show that $\hat{G}$ is finite. Then we will show that $\dim \mathbb{C}_{class}[G] = |\hat{G}|$.

To show that $|\hat{G}|$ is finite, we will use the right-regular representation $(R, \mathbb{C}[G])$. Recall from Example 1.3 that $R : G \to \mathrm{GL}(\mathbb{C}[G])$ is given by $\left[R(g)f\right](x) = f(xg)$ for all $f \in \mathbb{C}[G]$ and $x \in G$.

Proposition 2.20. *If G is a finite group and $(R, \mathbb{C}[G])$ is the right-regular representation of G, then*

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = e; \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

where $e \in G$ is the identity element.

Proof. Clearly the set $\{\sqrt{|G|}\,\mathbf{1}_g \mid g \in G\}$ forms an orthonormal basis for $\mathbb{C}[G]$. Note that if $e \in G$ is the identity element, then $R(e) : \mathbb{C}[G] \to \mathbb{C}[G]$ is the identity map and we have

$$\begin{aligned} \operatorname{Tr}(R(e)) &= \sum_{x\in G} \left\langle R(e)\sqrt{|G|}\,\mathbf{1}_x, \sqrt{|G|}\,\mathbf{1}_x \right\rangle \\ &= \sum_{x\in G} \left\langle \sqrt{|G|}\,\mathbf{1}_x, \sqrt{|G|}\,\mathbf{1}_x \right\rangle \\ &= |G|. \end{aligned}$$

On the other hand, if $g \in G$ is not the identity then

$$\begin{aligned} \operatorname{Tr}(R(g)) &= \sum_{x\in G} \left\langle R(g)\sqrt{|G|}\,\mathbf{1}_x, \sqrt{|G|}\,\mathbf{1}_x \right\rangle \\ &= \sum_{x\in G}\sum_{y\in G} [R(g)\mathbf{1}_x](y)\mathbf{1}_x(y) \\ &= \sum_{x\in G}\sum_{y\in G} \mathbf{1}_x(yg)\mathbf{1}_x(y) \\ &= 0. \end{aligned}$$

□

Proposition 2.21. *Let $R \cong \bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi$ denote the right-regular representation of G. Then $m_\pi = \dim \pi$ for all $\pi \in \hat{G}$.*

Proof. Fix $\pi \in \hat{G}$ and observe that

$$m_\pi = \langle \chi_R, \chi_\pi \rangle = \frac{1}{|G|} \sum_{x \in G} \chi_R(x) \overline{\chi_\pi(x)} = \chi_\pi(e) = \dim \pi,$$

since $\pi(e)$ is just the $\dim \pi \times \dim \pi$ identity matrix relative to a basis for V_π. □

Corollary 2.22. *If G is finite, then so too is $\hat{G}$.*

Proof. By the last proposition, $R \cong \bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi$ where $m_\pi = \dim \pi$ is a positive integer for each $\pi \in \hat{G}$. On the other hand, $(R, \mathbb{C}[G])$ is finite-dimensional. Thus $\hat{G}$ must be finite. □

The *center* of an associative $\mathbb{C}$-algebra R is the set of elements in R that commute with all elements of R. Explicitly, $Z(R) = \{x \in R \mid xy = yx \text{ for all } y \in R\}$. The center $Z(R)$ forms a sub-algebra of R.

Proposition 2.23. *Let G be a finite group. Then $Z(\mathbb{C}[G]) = \mathbb{C}_{class}[G]$.*

Proof. Let $\{\mathbf{1}_x \mid x \in G\}$ denote the usual basis for $\mathbb{C}[G]$ and let $f \in Z(\mathbb{C}[G])$. By linearity, a function $f : G \to \mathbb{C}$ lies in the center of $\mathbb{C}[G]$ if and only if it commutes with the basis elements $\mathbf{1}_x$ for all $x \in G$. That is, $f \in Z(\mathbb{C}[G])$ if and only if $f * \mathbf{1}_x = \mathbf{1}_x * f$ for all $x \in G$. For any $g, h \in G$, we have

$$(f * \mathbf{1}_h)(g) = \sum_{x \in G} f(x) \mathbf{1}_h(x^{-1}g) = f(gh^{-1});$$

$$(\mathbf{1}_h * f)(g) = \sum_{x \in G} \mathbf{1}_h(x) f(x^{-1}g) = f(h^{-1}g).$$

Thus $f \in Z(\mathbb{C}[G])$ if and only if $f(gh^{-1}) = f(h^{-1}g)$ for all $g, h \in G$. Equivalently, $f \in Z(\mathbb{C}[G])$ if and only if $f(g) = f(h^{-1}gh)$ for all $g, h \in G$. Hence $Z(\mathbb{C}[G]) = \mathbb{C}_{class}[G]$. □

Proposition 2.24. *Let n be a positive integer. Then $Z(\mathrm{Mat}_n(\mathbb{C})) = \mathrm{span}\{I_n\}$, where I_n is the $n \times n$ identity matrix.*

Proof. For any $\lambda \in \mathbb{C}$ and $A \in \mathrm{Mat}_n(\mathbb{C})$, we have $(\lambda I_n)A = \lambda A = A(\lambda I_n)$. Therefore, $\mathrm{span}\{I_n\} \subset Z(\mathrm{Mat}_n(\mathbb{C}))$. Now for $k, l \in \{1, \ldots, n\}$ define $e_{(k,l)} \in \mathrm{Mat}_n(\mathbb{C})$ by

$$[e_{(k,l)}]_{ij} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l; \\ 0 & \text{otherwise.} \end{cases}$$

If $A \in Z(\mathrm{Mat}_n(\mathbb{C}))$, then $Ae_{(k,l)} = e_{(k,l)}A$ for all $k, l \in \{1, \ldots, n\}$. In particular,

$$[Ae_{(k,k)}]_{ij} = \sum_{t=1}^{n} A_{it}[e_{(k,k)}]_{tj} = \begin{cases} A_{ik} & \text{if } j = k; \\ 0 & \text{otherwise;} \end{cases}$$

$$[e_{(k,k)}A]_{ij} = \sum_{t=1}^{n} [e_{(k,k)}]_{it}A_{tj} = \begin{cases} A_{kj} & \text{if } i = k; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $A_{ij} = 0$ if $i \neq j$. Now for any $k \in \{1, \ldots, n\}$,

$$[Ae_{(1,k)}]_{1k} = \sum_{t=1}^{n} A_{1t}[e_{(1,k)}]_{tk} = A_{11};$$

$$[e_{(1,k)}A]_{1k} = \sum_{t=1}^{n} [e_{(1,k)}]_{1t}A_{tk} = A_{kk}.$$

Therefore $A_{11} = A_{kk}$ for all $k \in \{1, \ldots, n\}$ and so $A \in \mathrm{span}\{I_n\}$. □

Remark. By induction, $Z(\mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_k}(\mathbb{C})) \cong \mathrm{span}\{I_{n_1}\} \oplus \cdots \oplus \mathrm{span}\{I_{n_k}\}$ for any finite collection of matrix algebras $\mathrm{Mat}_{n_1}(\mathbb{C}), \ldots, \mathrm{Mat}_{n_k}(\mathbb{C})$. In particular, we have $\dim Z(\mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_k}(\mathbb{C})) = k$.

Now let $\mathcal{V} = \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi)$. For an arbitrary element $S \in \mathcal{V}$, we write $S = \bigoplus_{\pi \in \hat{G}} S_\pi$. For $S, T \in \mathcal{V}$ and $\lambda \in \mathbb{C}$, we define

$$S + T = \bigoplus_{\pi \in \hat{G}} S_\pi + T_\pi;$$

$$\lambda S = \bigoplus_{\pi \in \hat{G}} \lambda S_\pi;$$

$$S \circ T = \bigoplus_{\pi \in \hat{G}} S_\pi \circ T_\pi.$$

It is straightforward to show that $\mathcal{V}$ forms an associative $\mathbb{C}$-algebra under the given operations. Clearly $\mathcal{V} \cong \bigoplus_{\pi \in \hat{G}} \mathrm{Mat}_{\dim V_\pi}(\mathbb{C})$ as algebras. Hence, by Proposition 2.23 we have $Z(\mathcal{V}) \cong \bigoplus_{\pi \in \hat{G}} \mathrm{span}\{I_{\dim V_\pi}\}$. Thus, $\dim Z(\mathcal{V}) = \dim \bigoplus_{\pi \in \hat{G}} \mathrm{span}\{I_{\dim V_\pi}\} = |\hat{G}|$. We have already established that $Z(\mathbb{C}[G]) = \mathbb{C}_{class}[G]$. In the following theorem we will show that $\mathbb{C}[G] \cong \mathcal{V}$, which allows us to conclude $\mathbb{C}_{class}[G] \cong Z(\mathcal{V})$ and therefore $\dim \mathbb{C}_{class}[G] = |\hat{G}|$.

Theorem 2.25 (Maschke's theorem)**.** *Let G be a finite group and let $\mathcal{V}$ be as defined above. Then $\mathbb{C}[G] \cong \mathcal{V}$ as algebras.*

Proof. The isomorphism is the map $\Phi : \mathbb{C}[G] \mapsto \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi)$ where

$$\Phi(f) = \bigoplus_{\pi \in \hat{G}} \pi(f).$$

Clearly Φ is a linear map. For each $(\pi, V_\pi) \in \hat{G}$, recall that $\pi : \mathbb{C}[G] \to \mathrm{End}(V_\pi)$ is a homomorphism of algebras. Thus, if $e \in G$ is the identity then we have

$$\Phi(\mathbf{1}_e) = \bigoplus_{\pi \in \hat{G}} \pi(\mathbf{1}_e) = \bigoplus_{\pi \in \hat{G}} \mathrm{id}_{V_\pi}.$$

Likewise, if $f, f' \in \mathbb{C}[G]$ then

$$\Phi(f * f') = \bigoplus_{\pi \in \hat{G}} \pi(f * f') = \bigoplus_{\pi \in \hat{G}} \pi(f) \circ \pi(f') = \Phi(f) \circ \Phi(f').$$

Hence Φ is an algebra homomorphism.

Next we show that Φ is injective. Suppose that $f \in \ker(\Phi)$. Then $\Phi(f) = 0$ and it follows that $\pi(f) = 0$ for all $\pi \in \hat{G}$. Now recall that $R \cong \bigoplus_{\pi \in \hat{G}} (\dim \pi) \cdot \pi$. Since $\pi(f) = 0$ for all $\pi \in \hat{G}$ we have $\bigoplus_{\pi \in \hat{G}} (\dim \pi) \cdot \pi(f) = 0$. Therefore $R(f) = 0$, and for any $g \in G$ we have

$$[R(f)\mathbf{1}_e](g) = \sum_{x \in G} f(x) R(x) \mathbf{1}_e(g) = \sum_{x \in G} f(x) \mathbf{1}_e(gx) = f(g^{-1}) = 0$$

Thus $f(g) = 0$ for all $g \in G$, so Φ is injective.

For surjectivity, note that $\dim \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi) = \sum_{\pi \in \hat{G}} (\dim \pi)^2$. By Proposition 2.21, we have $\dim \mathbb{C}[G] = \sum_{\pi \in \hat{G}} (\dim \pi)^2$. Thus Φ is surjective by the rank-nullity theorem and we conclude that Φ is an algebra isomorphism. $\square$

Lemma 2.26. *The irreducible characters* $\{\chi_\pi \mid \pi \in \hat{G}\}$ *form a basis for* $\mathbb{C}_{class}[G]$*.*

Proof. By Maschke's theorem, we have $\mathbb{C}[G] \cong \mathcal{V} \cong \bigoplus_{\pi\in\hat{G}} \mathrm{Mat}_{\dim V_\pi}(\mathbb{C})$. The center of $\mathbb{C}[G]$ is therefore isomorphic to the center of $\bigoplus_{\pi\in\hat{G}} \mathrm{Mat}_{\dim V_\pi}(\mathbb{C})$. On the one hand, $Z(\mathbb{C}[G]) = \mathbb{C}_{class}[G]$. On the other, $Z\left(\bigoplus_{\pi\in\hat{G}} \mathrm{Mat}_{\dim V_\pi}(\mathbb{C})\right) = \bigoplus_{\pi\in\hat{G}} \mathrm{span}\{I_{\dim V_\pi}\}$. We have $\dim \bigoplus_{\pi\in\hat{G}} \mathrm{span}\{I_{\dim V_\pi}\} = |\hat{G}|$. Thus $\dim \mathbb{C}_{class}[G] = |\hat{G}|$, and it follows that $|\{\chi_\pi \mid \pi \in \hat{G}\}| = |\hat{G}|$ is finite. Now by Theorem 2.18, $\{\chi_\pi \mid \pi \in \hat{G}\}$ is an orthonormal subset of $\mathbb{C}_{class}[G]$. In particular, $\{\chi_\pi \mid \pi \in \hat{G}\}$ is linearly independent. Thus, $\{\chi_\pi \mid \pi \in \hat{G}\}$ is a basis for $\mathbb{C}_{class}[G]$. □

Theorem 2.27. *Let* (ρ, V) *and* (σ, W) *be representations of* G*. Then* $(\rho, V) \cong (\sigma, W)$ *if and only if* $\chi_\rho = \chi_\sigma$*.*

Proof. We have already shown that $(\rho, V) \cong (\sigma, W)$ implies $\chi_\rho = \chi_\sigma$. So suppose that $\chi_\rho = \chi_\sigma$. Write $\rho \cong \bigoplus_{\pi\in\hat{G}} m_\pi \cdot \pi$ and $\sigma \cong \bigoplus_{\pi\in\hat{G}} n_\pi \cdot \pi$, so that $\chi_\rho = \sum_{\pi\in\hat{G}} m_\pi \chi_\pi$ and $\chi_\sigma = \sum_{\pi\in\hat{G}} n_\pi \chi_\pi$. Since $\{\chi_\pi \mid \pi \in \hat{G}\}$ is a basis for $\mathbb{C}_{class}[G]$ and $\chi_\rho = \chi_\sigma$, we must have $m_\pi = n_\pi$ for all $\pi \in \hat{G}$. Thus, $(\rho, V) \cong (\sigma, W)$. □

CHAPTER 3

THE TRACE FORMULA

Suppose Γ is a subgroup of a finite group G and let (τ, W) be an arbitrary representation of Γ. We consider the induced representation $(\mathrm{Ind}_{\Gamma}^{G}(\tau), C(\tau)) = (\pi_\tau, C(\tau))$ and write

$$\pi_\tau \cong \bigoplus_{\rho \in \hat{G}} m_\rho \cdot \rho, \tag{3.1}$$

where m_ρ is a non-negative interger for each $\rho \in \hat{G}$. The multiplicities m_ρ for $\rho \in \hat{G}$ are fundamental to understanding the induced representation π_τ. Our goal in this chapter is to develop a formula that will allow us to compute these multiplicities.

Our strategy amounts to evaluating $\mathrm{Tr}\left[\pi_\tau(f)\right]$ for arbitrary $f \in \mathbb{C}[G]$ in two different ways. In Section 3.1, we view $\pi_\tau(f)$ as a block-diagonal matrix and evaluate $\mathrm{Tr}\left[\pi_\tau(f)\right]$ to obtain an expression involving the multiplicities m_ρ for $\rho \in \hat{G}$. In Section 3.2, we evaluate $\mathrm{Tr}\left[\pi_\tau(f)\right]$ as a sum over an orthonormal basis for $C(\tau)$. We then equate the expressions obtained in Sections 3.1 and 3.2 to obtain our trace formula. We conclude by presenting some applications and examples of our trace formula in Section 3.3.

3.1 The spectral side of the trace formula

Assume the same notation as above. For fixed $f \in \mathbb{C}[G]$, recall that

$$\pi_\tau(f) = \sum_{x \in G} f(x) \pi_\tau(x). \tag{3.2}$$

We take the trace on both sides of (3.2) to obtain

$$\mathrm{Tr}\left[\pi_\tau(f)\right] = \sum_{x \in G} f(x)\, \mathrm{Tr}\left[\pi_\tau(x)\right]. \tag{3.3}$$

By Corollary 2.17, for each $x \in G$ we may write

$$\mathrm{Tr}\left[\pi_\tau(x)\right] = \sum_{\rho \in \hat{G}} m_\rho \chi_\rho(x). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\operatorname{Tr}\left[\pi_\tau(f)\right] = \sum_{x\in G} f(x) \sum_{\rho\in\hat{G}} m_\rho\, \chi_\rho(x). \tag{3.5}$$

Rearranging (3.5) then yields

$$\operatorname{Tr}\left[\pi_\tau(f)\right] = \sum_{\rho\in\hat{G}} m_\rho \sum_{x\in G} f(x)\, \chi_\rho(x). \tag{3.6}$$

For fixed $\rho \in \hat{G}$, note that the inner sum over $x \in G$ in (3.6) is simply $\operatorname{Tr}\left[\rho(f)\right]$. Hence,

$$\operatorname{Tr}\left[\pi_\tau(f)\right] = \sum_{\rho\in\hat{G}} m_\rho \operatorname{Tr}\left[\rho(f)\right]. \tag{3.7}$$

We refer to (3.7) as the *spectral side* of the trace formula. Note that (3.7) holds when we replace $\pi_\tau \cong \bigoplus_{\rho\in\hat{G}} m_\rho \cdot \rho$ with an arbitrary representation of G. In what follows, we compute $\operatorname{Tr}\left[\pi_\tau(f)\right]$ as a sum over an orthonormal basis for $C(\tau)$ to obtain the *geometric side* of the trace formula.

3.2 The geometric side of the trace formula

Assume the same notation as in the last section and fix $f \in \mathbb{C}[G]$. Our goal now is to compute $\operatorname{Tr}\left[\pi_\tau(f)\right]$ as a sum over an orthonormal basis for $C(\tau)$. The process is considerably more involved than our derivation of the spectral side. We refer to [Whi10] for well-known background on trace formulas. Our derivation of the geometric side is adapted from [Hej76].

We will begin expanding $\pi_\tau(f)\,\varphi$ for arbitrary $\varphi \in C(\tau)$. Ultimately, we will write $\pi_\tau(f)\,\varphi$ as a sum over the right cosets of Γ in G. Care must be taken to ensure that such a sum is well-defined, that is, the sum must take on the same value for any choice of representatives for $\Gamma\backslash G$. We will delay verifying this detail for the moment. If $x \in G$, then

$$\begin{aligned}
[\pi_\tau(f)\varphi](x) &= \sum_{y\in G} f(y)\pi_\tau(y)\varphi(x) \\
&= \sum_{y\in G} f(y)\varphi(xy) \\
&= \sum_{y\in G} f(x^{-1}y)\varphi(y) \\
&= \sum_{\Gamma y\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\varphi(\gamma y) \\
&= \sum_{\Gamma y\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\tau(\gamma)\varphi(y)
\end{aligned}$$

Define the map $K_f : G \times G \to \operatorname{End} W$ by

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\tau(\gamma). \tag{3.8}$$

We call K_f the *kernel* of $\pi_\tau(f)$. By the above, we have for all $x \in G$

$$[\pi_\tau(f)\varphi](x) = \sum_{\Gamma y\in\Gamma\backslash G} K_f(x,y)\varphi(y) \tag{3.9}$$

We will see that (3.9) is indeed well-defined as a corollary of the following result.

Proposition 3.1. *Let $x, y \in G$, let $\delta_1, \delta_2 \in \Gamma$, and let $K_f : G \times G \to \operatorname{End} W$ be as defined in* (3.8)*. Then $K_f(\delta_1 x, \delta_2 y) = \tau(\delta_1)K_f(x,y)\tau(\delta_2^{-1})$.*

Proof. Suppose $x, y \in G$ and $\delta_1, \delta_2 \in \Gamma$. First observe that

$$\begin{aligned}
K_f(\delta_1 x, \delta_2 y) &= \sum_{\gamma\in\Gamma} f\left((\delta_1 x)^{-1}\gamma(\delta_2 y)\right)\tau(\gamma) \\
&= \sum_{\gamma\in\Gamma} f\left(x^{-1}\delta_1^{-1}\gamma\delta_2 y\right)\tau(\gamma) \\
&= \sum_{\gamma\in\Gamma} f\left(x^{-1}\gamma y\right)\tau(\delta_1\gamma\delta_2^{-1}),
\end{aligned}$$

where the last equality is obtained by making a change of variables $\gamma \mapsto \delta_1 \gamma \delta_2^{-1}$. Now because $\tau : \Gamma \to \mathrm{GL}(W)$ is a group homomorphism and $\tau(\delta_1)$ is a linear map, we have

$$\begin{aligned} K_f(\delta_1 x, \delta_2 y) &= \sum_{\gamma \in \Gamma} f\left(x^{-1}\gamma y\right) \tau(\delta_1)\tau(\gamma)\tau(\delta_2^{-1}) \\ &= \tau(\delta_1) \sum_{\gamma \in \Gamma} f\left(x^{-1}\gamma y\right) \tau(\gamma)\tau(\delta_2^{-1}) \\ &= \tau(\delta_1) K_f(x,y)\tau(\delta_2^{-1}). \end{aligned}$$

□

Corollary 3.2. *Let $\varphi \in C(\tau)$ and let $x \in G$. Then $K_f(x, \delta y)\varphi(\delta y) = K_f(x,y)\varphi(y)$ for all $\delta \in \Gamma$ and $y \in G$.*

Proof. Suppose $\delta \in \Gamma$ and $y \in G$. By the last proposition, $K_f(x, \delta y) = K_f(x,y)\tau(\delta^{-1})$. On the other hand, since $\varphi \in C(\tau)$, we have $\varphi(\delta y) = \tau(\delta)\varphi(y)$. Hence,

$$K_f(x, \delta y)\varphi(\delta y) = K_f(x,y)\tau(\delta^{-1})\tau(\delta)\varphi(y) = K_f(x,y)\varphi(y).$$

□

Remark. Suppose $x \in G$ is fixed. By the above, the map $G \to W$ given by

$$y \longmapsto K_f(x,y)\varphi(y)$$

is *left Γ-invariant*. That is, for any $\delta \in \Gamma$ and $y \in G$, we have

$$\delta y \longmapsto K_f(x,y)\varphi(y).$$

Consequently, the map $\Gamma \backslash G \to W$ defined by

$$\Gamma y \longmapsto K_f(x,y)\varphi(y)$$

is well-defined. Ergo, the sum in (3.9) is also well-defined.

3.2.1 The space $C(\tau^*)$

Let $\langle \cdot, \cdot \rangle_W : W \times W \to \mathbb{C}$ denote an inner product on W relative to which τ is unitary. Let (τ^*, W^*) denote the dual representation of (τ, W), as defined in Proposition 1.9. For each $w \in W$, let $w^* \in W^*$ denote the linear map defined by $w^*(x) = \langle x, w \rangle_W$ for all $x \in W$. We consider the induced representation $(\mathrm{Ind}_\Gamma^G(\tau^*), C(\tau^*))$. Recall that

$$C(\tau^*) = \{f : G \to W^* \mid f(\gamma g) = \tau^*(\gamma) f(g) \ \forall\ \gamma \in \Gamma,\ g \in G\}.$$

Proposition 3.3. $\dim C(\tau) = \dim C(\tau^*)$

Proof. Since (τ, W) is finite dimensional, we have $\dim W = \dim W^*$. Now by Proposition 1.32, $\dim C(\tau) = [G : \Gamma](\dim W) = [G : \Gamma](\dim W^*) = \dim C(\tau^*)$. □

Proposition 3.4. *Let $\varphi \in C(\tau)$ and define $\varphi^* : G \to W^*$ by $\varphi^*(g) = \varphi(g)^*$. Then $\varphi^* \in C(\tau^*)$.*

Proof. Let $\gamma \in \Gamma$ and let $g \in G$. For any $w \in W$, we have

$$\begin{aligned}
[\varphi^*(\gamma g)](w) &= [\varphi(\gamma g)^*](w) \\
&= \langle w, \varphi(\gamma g) \rangle_W \\
&= \langle w, \tau(\gamma)\varphi(g) \rangle \\
&= \left\langle \tau(\gamma^{-1}) w, \varphi(g) \right\rangle \\
&= [\varphi^*(g)](\tau(\gamma^{-1}) w) \\
&= [\tau^*(\gamma)\varphi^*(g)](w).
\end{aligned}$$

Hence $\varphi^*(\gamma g) = \tau^*(\gamma)\varphi^*(g)$, as needed. □

Proposition 3.5. *If $\mathcal{B}(\tau)$ is a basis for $C(\tau)$, then $\mathcal{B}(\tau^*) = \{\varphi^* \mid \varphi \in \mathcal{B}(\tau)\}$ is a basis for $C(\tau^*)$.*

Proof. We show that the map $C(\tau) \to C(\tau^*)$ given by $\varphi \mapsto \varphi^*$ is a conjugate-linear group isomorphism. Let $\varphi, \psi \in C(\tau)$ and let $\lambda \in \mathbb{C}$. Then for any $g \in G$ and $w \in W$,

$$\begin{aligned}
[\varphi + \lambda\psi]^*(g) w &= \langle w, [\varphi + \lambda\psi](g) \rangle_W \\
&= \langle w, \varphi(g) \rangle + \langle w, \lambda\psi(g) \rangle \\
&= \langle w, \varphi(g) \rangle + \bar{\lambda} \langle w, \psi(g) \rangle \\
&= [\varphi^* + \bar{\lambda}\psi^*](g) w.
\end{aligned}$$

Hence $[\varphi + \lambda\psi]^* = \varphi^* + \bar{\lambda}\psi^*$.

To see that the map is injective, suppose $\varphi^* = 0$. Then $\langle w, \varphi(g)\rangle = 0$ for all $g \in G$ and $w \in W$. Thus $\varphi(g) = 0$ for all $g \in G$ and injectivity follows. Now recall that $\dim C(\tau) = \dim C(\tau^*)$. Hence, the map is surjective by the rank-nullity theorem.

Let $\sum_{\varphi\in\mathcal{B}(\tau)} \lambda_\varphi \varphi$ denote an arbitrary element of $C(\tau)$, where $\lambda_\varphi \in \mathbb{C}$ for each $\varphi \in \mathcal{B}(\tau)$. By the above, we have a group isomorphism $C(\tau) \to C(\tau^*)$ defined by

$$\sum_{\varphi\in\mathcal{B}(\tau)} \lambda_\varphi \varphi \longmapsto \sum_{\varphi\in\mathcal{B}(\tau)} \overline{\lambda_\varphi} \varphi^*.$$

Since the map is surjective, it follows that $\mathcal{B}(\tau^*)$ spans $C(\tau^*)$. Now recall that $\dim C(\tau) = \dim C(\tau^*)$, and $|\mathcal{B}(\tau^*)| = \dim C(\tau)$, by construction. Thus, $\mathcal{B}(\tau^*)$ is a spanning subset of $C(\tau^*)$ with $|\mathcal{B}(\tau^*)| = \dim C(\tau^*)$, so it follows that $\mathcal{B}(\tau^*)$ is a basis for $C(\tau^*)$. □

We define an inner product on $C(\tau)$, and give some alternate formulations using $C(\tau^*)$.

Proposition 3.6. *Suppose (τ, W) is a representation of Γ and let $\langle\cdot,\cdot\rangle_W$ be an inner product on W relative to which τ is unitary. Then $C(\tau)$ is a complex inner product space with the inner product*

$$\langle\varphi,\psi\rangle_{C(\tau)} = \sum_{\Gamma g\in\Gamma\backslash G} \langle\varphi(g),\psi(g)\rangle_W\,. \tag{3.10}$$

Proof. Since (τ, W) is unitary with respect to the inner product $\langle\cdot,\cdot\rangle_W$, we have

$$\begin{aligned}\langle\varphi(\gamma g),\psi(\gamma g)\rangle_W &= \langle\tau(\gamma)\varphi(g),\tau(\gamma)\psi(g)\rangle_W \\ &= \langle\varphi(g),\psi(g)\rangle_W\end{aligned}$$

for all $\gamma \in \Gamma$ and $g \in G$. Therefore, the map $g \mapsto \langle\varphi(g),\psi(g)\rangle_W$ is left Γ-invariant and so the sum in (3.10) is well-defined. It is now straightforward to show that (3.10) defines an inner product on $C(\tau)$. □

Proposition 3.7. *If $\varphi, \psi \in C(\tau)$, then*

$$\langle\varphi,\psi\rangle_{C(\tau)} = \sum_{\Gamma g\in\Gamma\backslash G} \psi(g)^*\varphi(g). \tag{3.11}$$

Proof. Recall that $[\psi(g)^*]w = \langle w, \psi(g)\rangle_W$ for all $g \in G$ and $w \in W$. Therefore,

$$\begin{aligned}\langle \varphi, \psi\rangle_{C(\tau)} &= \sum_{\Gamma g \in \Gamma\backslash G} \langle \varphi(g), \psi(g)\rangle_W \\ &= \sum_{\Gamma g \in \Gamma\backslash G} \psi(g)^*\varphi(g),\end{aligned}$$

as needed. □

Remark. Fix $\varphi, \psi \in C(\tau)$ and $g \in G$, and suppose $\mathcal{B}_W$ is an orthonormal basis for W. We like to think of $\varphi(g) \in W$ as a $(\dim W) \times 1$ column vector relative to $\mathcal{B}_W$. Likewise, we view $\psi^*(g) \in W^*$ as a $1 \times (\dim W)$ row vector relative to $\mathcal{B}_W$. Note that $[\psi^*(g)]_{\mathcal{B}_W}[\varphi(g)]_{\mathcal{B}_W}$ is a scalar. On the other hand, $[\varphi(g)]_{\mathcal{B}_W}[\psi^*(g)]_{\mathcal{B}_W}$ is a $(\dim W) \times (\dim W)$ matrix, which gives rise to a linear endomorphism on W. This motivates the following.

Definition 3.8. Let $\varphi, \psi \in C(\tau)$ and let $g, h \in G$. We define the linear map $\varphi(g)\psi(h)^* : W \to W$ by

$$[\varphi(g)\psi(h)^*]w = \langle w, \psi(h)\rangle_W \varphi(g). \tag{3.12}$$

Corollary 3.9. *If $\varphi, \psi \in C(\tau)$, then*

$$\langle \varphi, \psi\rangle_{C(\tau)} = \sum_{\Gamma g \in \Gamma\backslash G} \mathrm{Tr}\left[\varphi(g)\psi(g)^*\right]. \tag{3.13}$$

Proof. Suppose $\mathcal{B}_W = \{w_1, \ldots, w_n\}$ is an orthonormal basis for W. Then for $g \in G$, the matrix of $\varphi(g)\psi^*(g)$ relative to $\mathcal{B}_W$ is given by

$$[\varphi(g)\psi^*(g)]_{ij} = \langle w_j, \psi(g)\rangle_W \langle \varphi(g), w_i\rangle_W. \tag{3.14}$$

Therefore,

$$\mathrm{Tr}\left[\varphi(g)\psi^*(g)\right] = \sum_{w_i \in \mathcal{B}_W} \langle w_i, \psi(g)\rangle_W \langle \varphi(g), w_i\rangle_W. \tag{3.15}$$

On the other hand,

$$\psi(g)^*\varphi(g) = \begin{bmatrix} \langle w_1, \psi(g)\rangle_W & \cdots & \langle w_n, \psi(g)\rangle_W \end{bmatrix} \begin{bmatrix} \langle \varphi(g), w_1\rangle_W \\ \vdots \\ \langle \varphi(g), w_n\rangle_W \end{bmatrix}. \tag{3.16}$$

Therefore, $\psi(g)^*\varphi(g) = \mathrm{Tr}\left[\varphi(g)\psi^*(g)\right]$ and the result now follows from Proposition 3.7. □

3.2.2 The space $C(\tau, \tau^{-1})$

Let $\Psi_{\tau,\tau} : \Gamma \times \Gamma \to \mathrm{GL}(\mathrm{End}(W))$ denote the representation of $\Gamma \times \Gamma$ defined in Proposition 1.18. For $(\gamma, \delta) \in \Gamma \times \Gamma$, recall that

$$\Psi_{\tau,\tau}(\gamma, \delta)A = \tau(\gamma)\, A\, \tau(\delta)^{-1}$$

for all $A \in \mathrm{End}(W)$. Since $\Gamma \times \Gamma$ is a subgroup of $G \times G$, we may consider the induced representation,

$$\mathrm{Ind}_{\Gamma\times\Gamma}^{G\times G}(\Psi_{\tau,\tau}) : G \times G \to \mathrm{GL}\left(C(\tau, \tau^{-1})\right),$$

where $C(\tau, \tau^{-1}) = C(\Psi_{\tau,\tau})$ is the space of functions $\Phi : G \times G \to \mathrm{End}(W)$ that satisfy

$$\Phi(\gamma g, \delta h) = \tau(\gamma)\Phi(g, h)\tau(\delta)^{-1}$$

for all $\gamma, \delta \in \Gamma$ and $g, h \in G$.

For fixed $f \in \mathbb{C}[G]$, recall that the kernel of $\pi_\tau(f)$ is the map $K_f : G \times G \to \mathrm{End}(W)$ defined by

$$K_f(g, h) = \sum_{\gamma\in\Gamma} f(g^{-1}\gamma h)\tau(\gamma)$$

for all $g, h \in G$. By Proposition 3.1, we have $K_f \in C(\tau, \tau^{-1})$. Our goal in this section to prove some basic of properties of the space $C(\tau, \tau^{-1})$, which will allow us to better understand the kernel of $\pi_\tau(f)$.

Proposition 3.10. *Suppose $\mathcal{B}(\tau)$ is a basis for $C(\tau)$ and let $\{\varphi^* \mid \varphi \in \mathcal{B}(\tau)\}$ denote the basis for $C(\tau^*)$ defined in Proposition 3.5. For $\varphi, \psi \in \mathcal{B}(\tau)$, define $\varphi\psi^* : G \times G \to \mathrm{End}(W)$ by*

$$[\varphi\psi^*(g, h)]w = [\varphi(g)\psi(h)^*]w = \langle w, \psi(h)\rangle_W\, \varphi(g)$$

for all $(g, h) \in G \times G$ and $w \in W$. Then $\{\varphi\psi^* \mid \varphi, \psi \in \mathcal{B}(\tau)\}$ is a basis for $C(\tau, \tau^{-1})$.

Proof. The result follows from Propositions 1.18 and 1.35, together with the observation that $\mathrm{Ind}_{\Gamma\times\Gamma}^{G\times G}(\tau \otimes \tau^*) \cong \mathrm{Ind}_{\Gamma\times\Gamma}^{G\times G}(\Psi_{\tau,\tau})$. □

For each $A \in \mathrm{End}(W)$, we let $A^* \in \mathrm{End}(W)$ denote the Hermitian adjoint of A. That is, $A^* : W \to W$ is the linear map that satisfies $\langle Av, w \rangle_W = \langle v, A^*w \rangle_W$ for all $v, w \in W$. We recall some basic properties of the Hermitian adjoint before defining an inner product on $C(\tau, \tau^{-1})$. We refer to [Rom08, p. 227] for a more thorough discussion of the Hermitian adjoint.

Proposition 3.11. *Suppose (τ, W) is a finite-dimensional representation of Γ, and let $\langle \cdot, \cdot \rangle$ be an inner product on W relative to which τ is unitary. Then*

(1) $\tau(\gamma)^* = \tau(\gamma)^{-1}$ *for all* $\gamma \in \Gamma$*;*

(2) $(AB)^* = B^*A^*$ *for all* $A, B \in \mathrm{End}(W)$*;*

(3) $(\varphi\psi^*(x, y))^* = \psi\varphi^*(y, x)$ *for all* $\varphi, \psi \in C(\tau)$ *and* $x, y \in G$*;*

(4) If $[A]_{ij}$ *and* $[A^*]_{ij}$ *are the matrices of* A *and* A^* *relative to an orthonormal basis for* W*, then* $[A^*]_{ij} = \overline{[A]_{ji}}$*;*

(5) $\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$.

Proof. Let $\gamma \in \Gamma$, let $A, B \in \mathrm{End}(W)$, and let $\mathcal{B}_W = \{w_1, \ldots, w_n\}$ be an orthonormal basis for W.

(1) By definition, $\tau(\gamma)^* \in \mathrm{End}(W)$ is the unique map that satisfies $\langle \tau(\gamma)v, w \rangle = \langle v, \tau(\gamma)^*w \rangle$ for all $v, w \in W$. Now because because τ is unitary, we have $\langle \tau(\gamma)v, w \rangle = \langle v, \tau(\gamma)^{-1}w \rangle$. Hence $\tau(\gamma)^* = \tau(\gamma)^{-1}$.

(2) By definition, $\langle ABv, w \rangle = \langle v, (AB)^*w \rangle$ for all $v, w \in W$. Now $\langle Av, w \rangle = \langle v, A^*w \rangle$ and $\langle Bv, w \rangle = \langle v, B^*w \rangle$ for all $v, w \in W$. Hence $\langle ABv, w \rangle = \langle Bv, A^*w \rangle = \langle v, B^*A^*w \rangle$.

(3) Let $v \in V$ and let $w \in W$. Then for $x, y \in G$, we have

$$\begin{aligned}\langle \varphi(x)\psi^*(y)v, w\rangle &= \langle \langle v, \psi(y)\rangle \varphi(x), w\rangle \\ &= \langle \varphi(x), w\rangle \langle v, \psi(y)\rangle \\ &= \langle v, \langle w, \varphi(x)\rangle \psi(y)\rangle \\ &= \langle v, \psi(y)\varphi^*(x)w\rangle .\end{aligned}$$

(4) The matrix of A^* relative to $\mathcal{B}_W$ is given by

$$[A^*]_{ij} = \langle A^* w_j, w_i\rangle .$$

Now we have

$$\langle A^* w_j, w_i\rangle = \langle w_j, Aw_i\rangle = \overline{\langle Aw_i, w_j\rangle} = \overline{[A]_{ji}}.$$

Thus $[A^*]_{ij} = \overline{[A]_{ji}}$, as needed.

(5) The result follows immediately from 4. □

Proposition 3.12. *Suppose W is a finite-dimensional vector space. Then* $\operatorname{End} W$ *is a complex inner product space with the Hilbert-Schmidt inner product,*

$$\langle A, B\rangle_{HS} = \operatorname{Tr}[AB^*] .$$

Proof. Let $A, B \in \operatorname{End} W$ and suppose $\dim W = n$. Clearly $\langle \cdot, \cdot\rangle_{HS}$ is linear in the first coordinate because the trace is linear. We have $\overline{\langle A, B\rangle}_{HS} = \langle B, A\rangle_{HS}$ as a consequence of Proposition 3.11. Now observe that

$$\begin{aligned}\langle A, A\rangle_{HS} &= \operatorname{Tr}[AA^*] \\ &= \sum_{i=1}^{n} [AA^*]_{ii} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} [A]_{ij}[A^*]_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} [A]_{ij}\overline{[A]_{ij}} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} |[A]_{ij}|^2 .\end{aligned}$$

Therefore, $\langle A, A\rangle_{HS}$ is a sum of non-negative numbers. So we have $\langle A, A\rangle_{HS} \geq 0$ with equality if and only if $A = 0$. □

Proposition 3.13. *Suppose (τ, W) is a representation of Γ. Let $\langle\cdot,\cdot\rangle_{HS}$ denote the Hilbert-Schmidt inner product on $\operatorname{End} W$. Then $C(\tau, \tau^{-1})$ is a complex inner product space with the inner product*

$$\langle F, H\rangle_{C(\tau,\tau^{-1})} = \sum_{\Gamma x\in\Gamma\backslash G}\ \sum_{\Gamma y\in\Gamma\backslash G} \langle F(x,y), H(x,y)\rangle_{HS}\,. \tag{3.17}$$

Proof. First, note that $\Psi_{\tau,\tau}$ is unitary relative to the Hilbert-Schmidt inner product. Indeed, if $F, H \in C(\tau, \tau^{-1})$ and $(\gamma, \delta) \in \Gamma \times \Gamma$, then

$$\begin{aligned}
\langle \Psi_{\tau,\tau}(\gamma,\delta)F, \Psi_{\tau,\tau}(\gamma,\delta)H\rangle_{HS} &= \left\langle \tau(\gamma)\, F\, \tau(\delta)^{-1},\, \tau(\gamma)\, H\, \tau(\delta)^{-1}\right\rangle_{HS} \\
&= \operatorname{Tr}\left[\tau(\gamma)\, F\, \tau(\delta)^{-1} \left[\tau(\gamma)\, H\, \tau(\delta)^{-1}\right]^*\right] \\
&= \operatorname{Tr}\left[\tau(\gamma)\, F\, \tau(\delta)^{-1}\tau(\delta)\, H^*\, \tau(\gamma)^{-1}\right] \\
&= \operatorname{Tr}\left[F H^*\right] \\
&= \langle F, H\rangle_{HS}\,.
\end{aligned}$$

Now by Proposition 3.6, it follows that $C(\tau, \tau^{-1})$ is a complex inner product space with the inner product

$$\begin{aligned}
\langle F, H\rangle_{C(\tau,\tau^{-1})} &= \sum_{(\Gamma x,\Gamma y)\in\Gamma\backslash G\times\Gamma\backslash G} \langle F(x,y), H(x,y)\rangle_{HS} \\
&= \sum_{\Gamma x\in\Gamma\backslash G}\ \sum_{\Gamma y\in\Gamma\backslash G} \langle F(x,y), H(x,y)\rangle_{HS}\,.
\end{aligned}$$
□

Proposition 3.14. *Suppose $\mathcal{B}(\tau)$ is an orthonormal basis for $C(\tau)$, and define $\mathcal{B}(\tau, \tau^{-1}) = \{\varphi\psi^* \mid \varphi, \psi \in \mathcal{B}(\tau)\}$. Then $\mathcal{B}(\tau, \tau^{-1})$ is an orthonormal basis for $C(\tau, \tau^{-1})$ relative to the inner product defined in Proposition 3.13.*

Proof. Note that $\mathcal{B}(\tau, \tau^{-1})$ is a basis for $C(\tau, \tau^{-1})$, by Proposition 3.10. Now for any $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{B}(\tau)$,

$$\begin{aligned}
\langle \varphi_1\psi_1^*, \varphi_2\psi_2^* \rangle_{C(\tau,\tau^{-1})} &= \sum_{\Gamma x \in \Gamma\backslash G} \sum_{\Gamma y \in \Gamma\backslash G} \mathrm{Tr}\left[\varphi_1\psi_1^*(x,y)\left[\varphi_2\psi_2^*(x,y)\right]^*\right] \\
&= \sum_{\Gamma x \in \Gamma\backslash G} \sum_{\Gamma y \in \Gamma\backslash G} \mathrm{Tr}\left[\varphi_1\psi_1^*(x,y)\,\psi_2\varphi_2^*(y,x)\right] \\
&= \sum_{\Gamma x \in \Gamma\backslash G} \sum_{\Gamma y \in \Gamma\backslash G} \mathrm{Tr}\left[\varphi_1(x)\psi_1(y)^*\,\psi_2(y)\varphi_2(x)^*\right].
\end{aligned}$$

Now suppose $\mathcal{B}_W$ is an orthonormal basis for W. For fixed $x, y \in G$, $\varphi_1(x)$ and $\psi_2(y)$ are both $(\dim W) \times 1$ column vectors relative to $\mathcal{B}_W$. On the other hand, $\psi_1(y)^*$ and $\varphi_2(x)^*$ are both $1 \times (\dim W)$ row vectors relative to $\mathcal{B}_W$. It follows that $\psi_1(y)^*\,\psi_2(y)$ is a scalar, so we have

$$\begin{aligned}
\langle \varphi_1\psi_1^*, \varphi_2\psi_2^* \rangle_{C(\tau,\tau^{-1})} &= \sum_{\Gamma x \in \Gamma\backslash G} \sum_{\Gamma y \in \Gamma\backslash G} \psi_1(y)^*\,\psi_2(y)\ \mathrm{Tr}\left[\varphi_1(x)\varphi_2(x)^*\right] \\
&= \sum_{\Gamma x \in \Gamma\backslash G} \mathrm{Tr}\left[\varphi_1(x)\varphi_2(x)^*\right] \sum_{\Gamma y \in \Gamma\backslash G} \psi_1(y)^*\,\psi_2(y) \\
&= \langle \varphi_1, \varphi_2 \rangle_{C(\tau)} \langle \psi_1, \psi_2 \rangle_{C(\tau)},
\end{aligned}$$

where the last equality follows from Propositions 3.7 and 3.9. Now because $\mathcal{B}(\tau)$ is an orthonormal basis for $C(\tau)$, it follows that

$$\langle \varphi_1\psi_1^*, \varphi_2\psi_2^* \rangle_{C(\tau,\tau^{-1})} = \langle \varphi_1, \varphi_2 \rangle_{C(\tau)} \langle \psi_1, \psi_2 \rangle_{C(\tau)} = \begin{cases} 1 & \text{if } \varphi_1 = \varphi_2 \text{ and } \psi_1 = \psi_2; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathcal{B}(\tau, \tau^{-1})$ is an orthonormal basis for $C(\tau, \tau^{-1})$. □

3.2.3 The kernel of $\pi_\tau(f)$

In this section, we will complete our expansion of the geometric side of the trace formula. Let Γ be a subgroup of a finite group G, let (τ, W) be a unitary representation of Γ, and let $(\mathrm{Ind}_\Gamma^G(\tau), C(\tau)) = (\pi_\tau, C(\tau))$. For $f \in \mathbb{C}[G]$ and $\varphi \in C(\tau)$, recall that

$$[\pi_\tau(f)\varphi](x) = \sum_{\Gamma y \in \Gamma\backslash G} K_f(x,y)\varphi(y),$$

where $K_f : G \times G \to \operatorname{End} W$ is defined by

$$K_f(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\tau(\gamma)$$

for all $x, y \in G$.

Proposition 3.15. *Let $\mathcal{B}(\tau)$ be an orthonormal basis for $C(\tau)$, let $f \in \mathbb{C}[G]$, and let $x, y \in G$. Then*

$$K_f(x, y) = \sum_{\varphi \in \mathcal{B}(\tau)} [\pi_\tau(f)\varphi](x)\varphi(y)^*. \tag{3.18}$$

We call (3.18) *the spectral form of the kernel K_f.*

Proof. Let $\mathcal{B}(\tau, \tau^{-1})$ denote the orthonormal basis for $C(\tau, \tau^{-1})$ defined in Proposition 3.14, and recall that $K_f \in C(\tau, \tau^{-1})$. We write K_f as a linear combination over the basis elements in $\mathcal{B}(\tau, \tau^{-1})$,

$$K_f(x, y) = \sum_{\varphi \in \mathcal{B}(\tau)} \sum_{\psi \in \mathcal{B}(\tau)} c(\varphi, \psi)\, \varphi\psi^*(x, y) \tag{3.19}$$

where $c(\varphi, \psi) = \langle K_f, \varphi\psi^* \rangle_{C(\tau, \tau^{-1})}$ for each $\varphi\psi^* \in \mathcal{B}(\tau, \tau^{-1})$. First observe that

$$\begin{aligned}
c(\varphi, \psi) &= \langle K_f, \varphi\psi^* \rangle_{C(\tau, \tau^{-1})} \\
&= \sum_{\Gamma x \in \Gamma \backslash G} \sum_{\Gamma y \in \Gamma \backslash G} \operatorname{Tr}\left[K_f(x, y)[\varphi\psi^*(x, y)]^*\right] \\
&= \sum_{\Gamma x \in \Gamma \backslash G} \sum_{\Gamma y \in \Gamma \backslash G} \operatorname{Tr}\left[K_f(x, y)\psi(y)\varphi(x)^*\right] \\
&= \sum_{\Gamma x \in \Gamma \backslash G} \operatorname{Tr}\left[\sum_{\Gamma y \in \Gamma \backslash G} K_f(x, y)\psi(y)\varphi(x)^*\right] \\
&= \sum_{\Gamma x \in \Gamma \backslash G} \operatorname{Tr}\left[(\pi_\tau(f)\psi)(x)\varphi(x)^*\right] \\
&= \langle \pi_\tau(f)\psi, \varphi \rangle_{C(\tau)},
\end{aligned}$$

where the last equality follows from Proposition 3.9. Substituting $c(\varphi,\psi) = \langle \pi_\tau(f)\psi, \varphi\rangle_{C(\tau)}$ in (3.19), we obtain

$$\begin{aligned}
K_f(x,y) &= \sum_{\varphi\in\mathcal{B}(\tau)} \sum_{\psi\in\mathcal{B}(\tau)} \langle \pi_\tau(f)\psi, \varphi\rangle_{C(\tau)}\ \varphi\psi^*(x,y) \\
&= \sum_{\psi\in\mathcal{B}(\tau)} \sum_{\varphi\in\mathcal{B}(\tau)} \langle \pi_\tau(f)\psi, \varphi\rangle_{C(\tau)}\ \varphi(x)\psi(y)^* \\
&= \sum_{\psi\in\mathcal{B}(\tau)} [\pi_\tau(f)\psi](x)\psi(y)^*. \qquad \square
\end{aligned}$$

Proposition 3.16. *Let $f \in \mathbb{C}[G]$. Then*

$$\mathrm{Tr}\,[\pi_\tau(f)] = \sum_{\Gamma g\in\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\,\chi_\tau(\gamma). \tag{3.20}$$

Proof. Let $\mathcal{B}(\tau)$ be an orthonormal basis for $C(\tau)$. We write $\mathrm{Tr}\,[\pi_\tau(f)]$ as a sum over $\mathcal{B}(\tau)$:

$$\begin{aligned}
\mathrm{Tr}\,[\pi_\tau(f)] &= \sum_{\varphi\in\mathcal{B}(\tau)} \langle \pi_\tau(f)\varphi, \varphi\rangle_{C(\tau)} \\
&= \sum_{\varphi\in\mathcal{B}(\tau)} \sum_{\Gamma g\in\Gamma\backslash G} \mathrm{Tr}\,[(\pi_\tau(f)\varphi)(g)\varphi(g)^*] && \text{(by Proposition 3.9)} \\
&= \sum_{\Gamma g\in\Gamma\backslash G} \mathrm{Tr}\left[\sum_{\varphi\in\mathcal{B}(\tau)} (\pi_\tau(f)\varphi)(g)\varphi(g)^*\right] \\
&= \sum_{\Gamma g\in\Gamma\backslash G} \mathrm{Tr}\,[K(g,g)] && \text{(by Proposition 3.15)} \\
&= \sum_{\Gamma g\in\Gamma\backslash G} \mathrm{Tr}\left[\sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\tau(\gamma)\right] \\
&= \sum_{\Gamma g\in\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\,\mathrm{Tr}\,[\tau(\gamma)] \\
&= \sum_{\Gamma g\in\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\chi_\tau(\gamma). \qquad \square
\end{aligned}$$

We would like to interchange the order of summation in (3.20). As before, care must be taken to ensure that the sum over $\Gamma g \in \Gamma\backslash G$ is well-defined. For fixed $\gamma \in \Gamma$ the map $G \to \mathbb{C}$ that sends $x \mapsto f(x^{-1}\gamma x)\,\chi_\tau(\gamma)$ is not necessarily left Γ-invariant. Indeed, if $\delta \in \Gamma$ then

$$\delta x \longmapsto f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma),$$

and $f(x^{-1}\delta^{-1}\gamma\delta x)$ need not equal $f(x^{-1}\gamma x)$. To remedy this, we will first rewrite the inner sum over $\gamma \in \Gamma$.

Proposition 3.17. *Let $\gamma \in \Gamma$ and let $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ denote the centralizer of γ in Γ. Then the map $\Gamma \to \mathbb{C}$ defined by $x \mapsto f(x^{-1}\gamma x)\chi_\tau(\gamma)$ is left Γ_γ-invariant.*

Consequently, the sum

$$\sum_{\Gamma_\gamma\delta \in \Gamma_\gamma\backslash\Gamma} f(\delta^{-1}\gamma\delta)\chi_\tau(\gamma)$$

is well-defined.

Proof. If $\delta \in \Gamma_\gamma$, then for any $x \in \Gamma$ we have

$$\begin{aligned} f\Big((\delta x)^{-1}\gamma(\delta x)\Big)\chi_\tau(\gamma) &= f\Big(x^{-1}\delta^{-1}\gamma\delta x\Big)\chi_\tau(\gamma) \\ &= f\Big(x^{-1}\gamma x\Big)\chi_\tau(\gamma). \end{aligned}$$

□

Proposition 3.18. *Let $\gamma \in \Gamma$ and let $\mathrm{Cl}(\gamma) = \{\delta^{-1}\gamma\delta \mid \delta \in \Gamma\}$ denote the conjugacy class of γ in Γ. Then*

$$\sum_{\Gamma_\gamma\delta \in \Gamma_\gamma\backslash\Gamma} f(\delta^{-1}\gamma\delta)\chi_\tau(\gamma) = \sum_{\alpha \in \mathrm{Cl}(\gamma)} f(\alpha)\chi_\tau(\gamma).$$

Proof. Suppose $[\Gamma_\gamma\backslash\Gamma]$ is a set of representatives for $\Gamma_\gamma\backslash\Gamma$. If $\delta \in [\Gamma_\gamma\backslash\Gamma]$, then clearly $\delta^{-1}\gamma\delta \in \mathrm{Cl}(\gamma)$. On the other hand, if $\alpha \in \mathrm{Cl}(\gamma)$ then $\alpha = \beta^{-1}\gamma\beta$ for some $\beta \in \Gamma$. Now there exists $\delta \in [\Gamma_\gamma\backslash\Gamma]$ such that $\delta \in \Gamma_\gamma\beta$, whence $\alpha = \delta^{-1}\gamma\delta$. It now follows that $\{\delta^{-1}\gamma\delta \mid \delta \in [\Gamma_\gamma\backslash\Gamma]\} = \mathrm{Cl}(\gamma)$. Therefore,

$$\begin{aligned} \sum_{\Gamma_\gamma\delta \in \Gamma_\gamma\backslash\Gamma} f(\delta^{-1}\gamma\delta)\chi_\tau(\gamma) &= \sum_{\delta \in [\Gamma_\gamma\backslash\Gamma]} f(\delta^{-1}\gamma\delta)\chi_\tau(\gamma) \\ &= \sum_{\alpha \in \mathrm{Cl}(\gamma)} f(\alpha)\chi_\tau(\gamma). \end{aligned}$$

□

Corollary 3.19. *Let $g \in G$ and let $\gamma \in \Gamma$. Then the sum*

$$\sum_{\Gamma_\gamma\delta \in \Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g)\chi_\tau(\gamma)$$

is well-defined. Furthermore,

$$\sum_{\Gamma_\gamma\delta \in \Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g)\chi_\tau(\gamma) = \sum_{\delta \in \mathrm{Cl}(\gamma)} f(g^{-1}\delta g)\chi_\tau(\gamma).$$

Proof. Let $g \in G$ and let $\gamma \in \Gamma$. For the first result, note that the map $\Gamma \to \mathbb{C}$ defined by $x \mapsto f(g^{-1}x^{-1}\gamma x g)\chi_\tau(\gamma)$ is left Γ_γ-invariant. The proof of the second result is similar to the proof of Proposition 3.18. $\square$

Proposition 3.20. *Let* $[\Gamma]$ *denote a set of representatives for the conjugacy classes in* Γ. *Then for any* $x \in G$,

$$\sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)\chi_\tau(\gamma) = \sum_{\gamma \in [\Gamma]} \sum_{\Gamma_\gamma \delta \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma).$$

Proof. The result follows immediately from Corollary 3.19 and the observation that the conjugacy classes partition Γ. $\square$

Proposition 3.21. *Let* $f \in \mathbb{C}[G]$. *Then*

$$\operatorname{Tr}[\pi_\tau(f)] = \sum_{\gamma \in [\Gamma]} \sum_{\Gamma x \in \Gamma \backslash G} \sum_{\Gamma_\gamma \delta \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma). \tag{3.21}$$

Proof. By Propositions 3.16 and 3.20, we have

$$\begin{aligned} \operatorname{Tr}[\pi_\tau(f)] &= \sum_{\Gamma x \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)\, \chi_\tau(\gamma) \\ &= \sum_{\Gamma x \in \Gamma \backslash G} \sum_{\gamma \in [\Gamma]} \sum_{\Gamma_\gamma \delta \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma). \end{aligned}$$

For fixed $\gamma \in [\Gamma]$, it is straightforward to show that the map $G \to \mathbb{C}$ defined by

$$x \longmapsto \sum_{\Gamma_\gamma \delta \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma)$$

is left Γ-invariant. Therefore, we may switch the order of summation to obtain (3.21). $\square$

Proposition 3.22. *For each* $\gamma \in [\Gamma]$, *let* $G_\gamma = \{x \in G \mid x^{-1}\gamma x = \gamma\}$ *denote the centralizer of* γ *in* G. *Then*

$$\operatorname{Tr}[\pi_\tau(f)] = \sum_{\gamma \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{G_\gamma x \in G_\gamma \backslash G} f(x^{-1}\gamma x)\chi_\tau(\gamma) \tag{3.22}$$

Proof. The result follows from (3.21). For the moment, we will fix $\gamma \in [\Gamma]$ and restrict our attention to the inner sum over $\Gamma x \in \Gamma\backslash G$. First observe that

$$\begin{aligned}\sum_{\Gamma x\in\Gamma\backslash G}\sum_{\Gamma_\gamma\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma) &= \frac{1}{|\Gamma|}\sum_{x\in G}\sum_{\Gamma_\gamma\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma)\\ &= \frac{1}{|\Gamma|}\sum_{x\in G}\frac{1}{|\Gamma_\gamma|}\sum_{\delta\in\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma)\\ &= \frac{1}{|\Gamma|\,|\Gamma_\gamma|}\sum_{x\in G}\sum_{\delta\in\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma).\end{aligned}$$

Now we switch the order of summation and make a change of variables $x \mapsto \delta^{-1}x$ to obtain

$$\begin{aligned}\sum_{\Gamma x\in\Gamma\backslash G}\sum_{\Gamma_\gamma\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma) &= \frac{1}{|\Gamma|\,|\Gamma_\gamma|}\sum_{\delta\in\Gamma}\sum_{x\in G} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma)\\ &= \frac{1}{|\Gamma|\,|\Gamma_\gamma|}\sum_{\delta\in\Gamma}\sum_{x\in G} f(x^{-1}\gamma x)\chi_\tau(\gamma)\\ &= \frac{1}{|\Gamma_\gamma|}\sum_{x\in G} f(x^{-1}\gamma x)\chi_\tau(\gamma).\end{aligned}$$

Clearly the map $x \mapsto f(x^{-1}\gamma x)\chi_\tau(\gamma)$ is left G_γ-invariant. Thus,

$$\sum_{\Gamma x\in\Gamma\backslash G}\sum_{\Gamma_\gamma\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\chi_\tau(\gamma) = \frac{|G_\gamma|}{|\Gamma_\gamma|}\sum_{G_\gamma x\in G_\gamma\backslash G} f(x^{-1}\gamma x)\chi_\tau(\gamma).$$

Now we obtain (3.22) by summing both sides over $\gamma \in [\Gamma]$. □

We will refer to (3.22) as the geometric side of our trace formula, though in some cases, it will be convenient to use (3.20) instead. We are now ready to state our main result.

Theorem 3.23. *Suppose Γ is a subgroup of a finite group G, let (τ, W) be a representation of Γ, and let $\mathrm{Ind}_\Gamma^G(\tau) = \pi_\tau \cong \bigoplus_{\rho\in\hat{G}} m_\rho \cdot \rho$. Then for any $f \in \mathbb{C}[G]$,*

$$\sum_{\rho\in\hat{G}} m_\rho \operatorname{Tr}(\rho(f)) = \sum_{\gamma\in[\Gamma]}\frac{|G_\gamma|}{|\Gamma_\gamma|}\sum_{G_\gamma x\in G_\gamma\backslash G} f(x^{-1}\gamma x)\chi_\tau(\gamma),$$

where $[\Gamma]$ is a set of representatives for conjugacy classes in Γ, G_γ is the centralizer of γ in G, and Γ_γ is the centralizer of γ in Γ.

3.3 Examples and applications

We can apply the trace formula presented in Theorem 3.23 by choosing a test function $f \in \mathbb{C}[G]$, then computing and comparing the spectral and geometric sides. Our first example recovers the well-known formula for the character of an induced representation, which can be found in [Ser77, p. 30].

Example 3.24. *Suppose* Γ *is a subgroup of a finite group* G*, let* τ *be a representation of* Γ*, and let* π_τ *denote the representation of* G *induced by* τ*. Then for each* $g \in G$*,*

$$\chi_{\pi_\tau}(g) = \sum_{\substack{\Gamma x \in \Gamma\backslash G \\ xgx^{-1} \in \Gamma}} \chi_\tau(xgx^{-1}) = \frac{1}{|\Gamma|} \sum_{\substack{y \in G \\ ygy^{-1} \in \Gamma}} \chi_\tau(ygy^{-1}).$$

Proof. Fix $g \in G$. We obtain the first equality by choosing the indicator function $f = \mathbf{1}_g$ as our test function. We write

$$\pi_\tau \cong \bigoplus_{\rho \in \hat{G}} m_\rho \cdot \rho.$$

Note that for any $\rho \in \hat{G}$,

$$\rho(\mathbf{1}_g) = \sum_{x \in G} \mathbf{1}_g(x)\rho(x) = \rho(g).$$

Thus, the spectral side of the trace formula becomes

$$\begin{aligned} \sum_{\rho \in \hat{G}} m_\rho \operatorname{Tr}[\rho(\mathbf{1}_g)] &= \sum_{\rho \in \hat{G}} m_\rho \operatorname{Tr}[\rho(g)] \\ &= \chi_{\pi_\tau}(g), \end{aligned}$$

where the last equality follows from Corollary 2.17. For this example, we will use the geometric side of the trace formula presented in Proposition 3.16. Equating the spectral and geometric sides, we have

$$\chi_{\pi_\tau}(g) = \sum_{\Gamma x \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} \mathbf{1}_g(x^{-1}\gamma x)\chi_\tau(\gamma).$$

For each Γx in the outer sum, note that the inner sum has a nonzero term if and only if $xgx^{-1} \in \Gamma$. When this is true, the only nonzero term is $\chi_\tau(xgx^{-1})$. Thus, we obtain the first equality,

$$\chi_{\pi_\tau}(g) = \sum_{\substack{\Gamma x \in \Gamma \backslash G \\ xgx^{-1} \in \Gamma}} \chi_\tau(xgx^{-1}).$$

To obtain the second equality, we simply write the outer sum over $\Gamma x \in \Gamma \backslash G$ as a sum over $y \in G$ scaled by a factor of $|\Gamma|^{-1}$:

$$\chi_{\pi_\tau}(g) = \frac{1}{|\Gamma|} \sum_{\substack{y \in G \\ ygy^{-1} \in \Gamma}} \chi_\tau(ygy^{-1}). \qquad \square$$

Remark. It should be possible to obtain the geometric side of the trace formula via the formula in Example 3.24. Indeed, if $f \in \mathbb{C}[G]$ is arbitrary then

$$\operatorname{Tr}\left[\pi_\tau(f)\right] = \sum_{g \in G} f(g)\chi_{\pi_\tau}(g).$$

Substituting $\chi_{\pi_\tau}(g)$ with either of the expressions in Example 3.24, we obtain an expression that resembles the geometric side of the trace formula in Theorem 3.23. This approach does not extend to the case where G is infinite, since π_τ may be infinite-dimensional and, in this case, the trace of $\pi_\tau(g)$ does not converge. However, for suitable $f \in L^1(G)$, $\pi_\tau(f)$ does have a meaningful trace which can be found following the methods in [Hej76]. That is, the method outlined in the previous section can be extended to the case where G is infinite.

Our next example is simply an alternative proof of Proposition 2.21. We rely on the observation that the right-regular representation $(R, \mathbb{C}[G])$ is equivalent to the representation of G induced by the trivial representation of $\Gamma = \{e\}$, where $e \in G$ is the identity element.

Example 3.25 (Proposition 2.21)**.** *Let* $R \cong \bigoplus_{\pi \in \hat{G}} m_\pi \cdot \pi$ *denote the right-regular representation of* G*. Then* $m_\pi = \dim \pi$ *for all* $\pi \in \hat{G}$*.*

Proof. Recall that R is the representation of G on the space $\mathbb{C}[G]$, where for each $g \in G$, $R(g) : \mathbb{C}[G] \to \mathbb{C}[G]$ is the invertible linear map defined by

$$[R(g)\varphi](x) = \varphi(xg)$$

for all $\varphi \in \mathbb{C}[G]$ and $x \in G$. Suppose $\Gamma = \{e\}$, and suppose $(\tau, \mathbb{C})$ is the trivial representation of Γ. We consider the induced representation $\pi_\tau : G \to \mathrm{GL}(C(\tau))$, where

$$C(\tau) = \{f : G \to \mathbb{C} \mid f(\gamma g) = f(g)\, \forall\, \gamma \in \Gamma, g \in G\}$$

and $\pi_\tau(g) : C(\tau) \to C(\tau)$ is defined as $[\pi_\tau(g)\varphi](x) = \varphi(xg)$ for all $\varphi \in C(\tau)$ and $x \in G$. Clearly, $(\pi_\tau, C(\tau)) \cong (R, \mathbb{C}[G])$.

Fix an irreducible representation $(\pi, V_\pi) \in \hat{G}$. We want to show that $m_\pi = \dim \pi$. Focusing on the spectral side, we will need to choose a test function $f \in \mathbb{C}[G]$ so that

$$\sum_{\rho \in \hat{G}} m_\rho \operatorname{Tr}[\rho(f)] = m_\pi.$$

Ergo, we need $f \in \mathbb{C}[G]$ such that for all $\rho \in \hat{G}$,

$$\operatorname{Tr}[\rho(f)] = \begin{cases} 1 & \text{if } \rho \cong \pi; \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

By inspection, (3.23) suggests an orthogonality relation. Indeed, if $f = |G|^{-1}\overline{\chi_\pi}$ and $\mathcal{B}_W$ is an orthonormal basis for W, then we have

$$\begin{aligned}
\operatorname{Tr}\Big[\rho(|G|^{-1}\overline{\chi_\pi})\Big] &= \sum_{w \in \mathcal{B}_W} \Big\langle \rho(|G|^{-1}\overline{\chi_\pi})w, w \Big\rangle_W \\
&= \sum_{w \in \mathcal{B}_W} \left\langle \sum_{x \in G} |G|^{-1}\overline{\chi_\pi(x)}\rho(x)w, w \right\rangle \\
&= \frac{1}{|G|} \sum_{x \in G} \overline{\chi_\pi(x)} \sum_{w \in \mathcal{B}_W} \langle \rho(x)w, w \rangle \\
&= \frac{1}{|G|} \sum_{x \in G} \overline{\chi_\pi(x)}\chi_\rho(x) \\
&= \langle \chi_\rho, \chi_\pi \rangle_{\mathbb{C}[G]}.
\end{aligned}$$

Now by Theorem 2.18, we have $\langle \chi_\rho, \chi_\pi \rangle_{\mathbb{C}[G]} = 1$ if $\rho \cong \pi$, and $\langle \chi_\rho, \chi_\pi \rangle_{\mathbb{C}[G]} = 0$ otherwise. Thus

$$\sum_{\rho \in \hat{G}} m_\rho \operatorname{Tr}\left[\rho(|G|^{-1}\overline{\chi_\pi})\right] = \sum_{\rho \in \hat{G}} m_\rho \langle \chi_\rho, \chi_\pi \rangle_{\mathbb{C}[G]} = m_\pi,$$

as needed.

We will compute the value of m_π via the geometric side of the trace formula. Note that since $\Gamma = \{e\}$ is trivial, so too is $[\Gamma]$. Hence the outer sum over $\gamma \in [\Gamma]$ consists of a single term, $\gamma = e$. Further, we have $G_e = G$ and $\Gamma_e = \Gamma$, so the inner sum over $G_\gamma x \in G_\gamma \backslash G$ also consists of a single term. Equating the spectral and geometric sides, we have

$$\begin{aligned} m_\pi &= \sum_{\gamma \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{G_\gamma x \in G_\gamma \backslash G} |G|^{-1}\overline{\chi_\pi(x^{-1}\gamma x)}\chi_\tau(\gamma) \\ &= \frac{|G|}{|\Gamma|}\Big(|G|^{-1}\overline{\chi_\pi(e)}\Big)\chi_\tau(e) \\ &= \chi_\pi(e)\chi_\tau(e) \\ &= \dim \pi, \end{aligned}$$

where the last equality follows from the observation that $\pi(e)$ is the $\dim \pi \times \dim \pi$ identity matrix relative to a basis for V and $\chi_\tau(e) = 1$. □

Frobenius reciprocity, stated in Theorem 1.34, likewise follows from the trace formula in Theorem 3.23.

Example 3.26 (Frobenius reciprocity)**.** *Let Γ be a subgroup of a finite group G, let (σ, V) be an irreducible representation of G, and let (τ, W) be an irreducible representation of Γ. Then*

$$\operatorname{Hom}_G(\sigma, \operatorname{Ind}_\Gamma^G(\tau)) \cong \operatorname{Hom}_\Gamma(\operatorname{Res}_\Gamma^G(\sigma), \tau)$$

as complex vector spaces.

Proof. We write $\operatorname{Ind}_\Gamma^G(\tau) = \pi_\tau$ and $\operatorname{Res}_\Gamma^G(\sigma) = \sigma|_\Gamma$. As in our proof of Theorem 1.34, it will suffice to show that $\dim \operatorname{Hom}_G(\sigma, \pi_\tau) = \dim \operatorname{Hom}_\Gamma(\sigma|_\Gamma, \tau)$. Recall that

$$\dim \operatorname{Hom}_G(\sigma, \pi_\tau) = n_G(\sigma, \pi_\tau) = \langle \chi_\sigma, \chi_{\pi_\tau} \rangle_{\mathbb{C}[G]},$$

and likewise,

$$\dim \mathrm{Hom}_\Gamma(\sigma|_\Gamma, \tau) = n_\Gamma(\sigma, \pi_\tau) = \langle \chi_{\sigma|_\Gamma}, \chi_\tau \rangle_{\mathbb{C}[\Gamma]}.$$

In particular, $\langle \chi_\sigma, \chi_{\pi_\tau} \rangle_{\mathbb{C}[G]}$ is the multiplicity of σ in π_τ, and $\langle \chi_{\sigma|_\Gamma}, \chi_\tau \rangle_{\mathbb{C}[\Gamma]}$ is the multiplicity of τ in $\sigma|_\Gamma$. Suppose that

$$\pi_\tau \cong \bigoplus_{\rho \in \hat{G}} m_\rho \cdot \rho,$$

where $\hat{G}$ is a set of representatives for the irreducible representations of G with $\sigma \in \hat{G}$. We will show that $\langle \chi_\sigma, \chi_{\pi_\tau} \rangle_{\mathbb{C}[G]} = m_\sigma = \langle \chi_{\sigma|_\Gamma}, \chi_\tau \rangle_{\mathbb{C}[\Gamma]}$.

Choosing $f = |G|^{-1}\overline{\chi_\sigma}$ as our test function, by the previous example, the spectral side of the trace formula is given by

$$\sum_{\rho \in \hat{G}} m_\rho \operatorname{Tr}\Big[\rho(|G|^{-1}\overline{\chi_\sigma}\Big] = m_\sigma.$$

Equating the spectral and geometric sides, we obtain

$$\begin{aligned} m_\sigma &= \sum_{\gamma \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{G_\gamma x \in G_\gamma \backslash G} f(x^{-1}\gamma x)\chi_\tau(\gamma) \\ &= \sum_{\gamma \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma||G|} \sum_{G_\gamma x \in G_\gamma \backslash G} \overline{\chi_\sigma(x^{-1}\gamma x)}\chi_\tau(\gamma) \\ &= \sum_{\gamma \in [\Gamma]} \frac{1}{|\Gamma_\gamma|} \overline{\chi_\sigma(\gamma)}\chi_\tau(\gamma) \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in [\Gamma]} \frac{|\Gamma|}{|\Gamma_\gamma|} \overline{\chi_\sigma(\gamma)}\chi_\tau(\gamma) \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_\sigma(\gamma)}\chi_\tau(\gamma) \\ &= \langle \chi_\tau, \chi_{\sigma|_\Gamma} \rangle_{\mathbb{C}[\Gamma]}. \end{aligned}$$

Hence, $\dim \mathrm{Hom}_G(\sigma, \pi_\tau) = m_\sigma = \langle \chi_\tau, \chi_{\sigma|_\Gamma} \rangle_{\mathbb{C}[\Gamma]} = \dim_\Gamma(\sigma|_\Gamma, \tau)$, as needed. □

REFERENCES

[Art89] James Arthur. The trace formula and Hecke operators. In Karl Egil Aubert, Enrico Bombieri, and Dorian Goldfeld, editors, *Number Theory, Trace Formulas and Discrete Groups*, pages 11–27. Academic Press, 1989.

[Bum13] Daniel Bump. *Lie Groups*. Springer New York, NY, 2nd edition, 2013.

[FH91] William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer New York, NY, 1st edition, 1991.

[Hej76] Dennis A. Hejhal. *The Selberg Trace Formula for PSL(2,R)*. Springer Berlin, Heidelberg, 1st edition, 1976.

[Rom08] Steven Roman. *Advanced Linear Algebra*. Springer New York, NY, 3rd edition, 2008.

[Sel56] Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric spaces with applications to dirichlet series. *J. Indian Math. Soc.*, 20:47–87, 1956.

[Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*. Springer New York, NY, 1977.

[Ter99] Audrey Terras. *Fourier Analysis on Finite Groups and Applications*. London Mathematical Society Student Texts. Cambridge University Press, 1999.

[Whi10] David Whitehouse. An introduction to the trace formula. `https://www.ma.imperial.ac.uk/~buzzard/MSRI/Whitehouse_tf.pdf`, 2010. [Online; accessed 2022].

[Yan06] Jae-Hyun Yang. Trace formulas on finite groups. *Communications of the Korean Mathematical Society*, 21(1):17–25, 2006.

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