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INVESTIGATION OF STUDENT UNDERSTANDING OF REPRESENTATIONS OF PROBABILITY CONCEPTS IN QUANTUM MECHANICS

By

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B.A. Gustavus Adolphus College, 2017

A DISSERTATION

Submitted in Partial Fulfillment of the

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August 2023

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INVESTIGATION OF STUDENT UNDERSTANDING OF REPRESENTATIONS OF PROBABILITY CONCEPTS IN QUANTUM MECHANICS

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Dissertation Advisor: Dr. John Thompson

An Abstract of the Dissertation Presented
in Partial Fulfillment of the Requirements for the
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The ability to relate physical concepts and phenomena to multiple mathematical representations—and to move fluidly between these representations—is a critical outcome expected of physics instruction. In upper-division quantum mechanics, students must work with multiple symbolic notations, including some that they have not previously encountered. Thus, developing the ability to generate and translate expressions in these notations is of great importance, and the extent to which students can relate these expressions to physical quantities and phenomena is crucial to understand.

To investigate student understanding of the expressions used in these notations and the ways they relate, clinical think-aloud interviews were conducted and an online survey was administered, all to students enrolled in upper-division quantum mechanics courses. The interviews were conducted at a single institution with a “spins-first” instructional approach, while the surveys were administered at ten institutions, including both “spins-first” and “wave functions-first” courses. The interviews and surveys focused on expressions for probabilities and their constituent terms. Analysis of student interviews used the symbolic forms framework to determine the ways that participants interpret and reason about these expressions. Survey

responses were analyzed using network analysis techniques to determine the ways that students conceptualized these expressions and both whether and how they related analogous expressions between notations. Survey responses were also used to compare students' understanding of these expressions and their relations between the two curricula studied.

Multiple symbolic forms—internalized connections between symbolic templates and their conceptual interpretations—were identified in both Dirac and wave function notation, suggesting that students develop an understanding of expressions for probability both in terms of their constituent pieces and as larger composite expressions. Network analysis techniques determined the relative strength of conceptual connections between expressions in different notations and were also used to understand the relative weight of different conceptualizations of expressions that share multiple possible interpretations. Comparative analysis between instructional approaches showed similarity in their conceptions of common expressions within quantum mechanics but also highlighted differences, such as a general preference for and better understanding of expressions in the notation taught first in their respective courses.

DEDICATION

I'll dedicate this to the people who love seemingly incongruous things. Mine are philology and physics, and if I can somehow wrangle a PhD merging those two together, anything is possible.

ACKNOWLEDGEMENTS

No person is an island, and no feat is ever achieved alone. This is certainly true in my case, and there are so many amazing people that have impacted and informed the person I am today and the work I have done up to this point as captured by this dissertation. As such, I am unfortunately only able to highlight a small fraction of these wonderful souls in this section due to length and time constraints.

I will begin, as many such sections should, by expressing my gratitude for the support of my family. My father was always encouraging of my curious spirit, displaying a saint's patience for my incessant questions about the way the world works. His presence and support nurtured my inner scientist's mind, and without him I am quite sure I would never have pursued an advanced degree, and perhaps never discovered my passion for physics. If my father helped me realize the path I wanted to take, it was my mother who helped assure I succeeded while walking it. Graduate school is a long road (and academia longer still), and though my love of physics is deep, my love of singing and the arts was integral to sustaining my spirit through long years of school. Without the (sometimes forceful but always loving) pushes from my mother I likely would have never discovered my love of singing or theatre, and my life would be all the duller for it. Of course, my parents have shaped my personage much more than in simply these aspects (my appreciation for the art of terrible puns and enjoyment of video games being just a few examples), but as I said: I'm on a schedule and this dissertation isn't writing itself. Of course, I need to thank the unwavering support of my siblings for helping to shape the person I am as well. Thank you, Em, Ang, and Dan. Our shared enthusiasm for reading has broadened my horizons and indelibly impacted both my mental health and academic success. Having people

who will always have your back is not something that should be taken for granted, and it truly can't be stated enough how much I appreciate you all.

Then of course I must also state my appreciation for those in Maine that made my time in graduate school as joyful as it was edifying. First and foremost, I thank John Thompson for his support both professionally and personally. John was kind enough to humor several years of my barging into his office and demanding a discussion on the most esoteric features of math in physics—typically with a quantum spin (pun intended). His willingness to support my academic curiosity and to allow me to largely define my own research goals—all while deftly guiding me toward more fruitful questions and away from academic pitfalls—showcases his unbelievably generous nature and his excellence as an advisor. This is all aside from the fact that he was chair of the department for five of my six years in Maine, and thus was giving me precious hours he probably should have been spending drafting another email to the Dean, planning a faculty retreat, or putting out the thousand daily fires roiling the department. John was also the reason I heard about (and joined) the Black Bear Men's Chorus—the most fun ensemble I've had the pleasure of singing with in my life (and as a tenor, no less!). While John has been such a great help professionally, it is truly in the times spent off the clock—in choir, at a conference, or at a PERL potluck—that I've most enjoyed our time together. Overall, just a great guy. 10/10, would recommend. And then there's the *rest* of the members of the PERL group too! Kevin, Ben, Caleb, Abolaji, MJ, Tija, Anthony, Thomas, Miki, Trevor, Em, and Allison, plus our postdocs Drew and Zeynep as well as Saima, Mac, and Michael… I truly couldn't have asked for a better group of mentors, peers, friends, or recipients of fun facts (etymological or otherwise). You've all helped make graduate school a fun and welcoming place for me over the years, and every single

one of you have inspired and helped me in ways too numerous to count. PERL meetings are always a highlight of my week, and I have grown so much both as a presenter and as an engaged audience member by spending those 45 minutes a week discussing research together (the other 15 being awkward business, of course). I'd also like to give special props to my research sister Abolaji; though you may have attended a less-than-excellent undergraduate institution, you've been such a rock of support throughout this whole process and I know you'll forgive me for my sass (you're just beneficent like that). I would also like to thank Suriya, Ram, Muntasir, Ben, Komala, and Brandon for being excellent roommates and for expanding my culinary repertoire (and Minnesotan spice tolerance). I would also like to give thanks to the members of my cohort: Ben, John, MJ, and Katee. My first several years in Maine were made so much richer thanks to each of you, and our time playing board games helped keep me sane during all of our coursework and grading. Similarly, the legendary 10-month spree of Twilight Imperium games with Anthony, Alyce, and Allison (and sometimes Trevor) was, well, legendary. As were the (many) games of Clue with Em, Ethan, and the rest. Basically, Maine is replete with excellent people who know how to have a good time. I would also like to thank Eric Brewe for agreeing to be my external reader, on account of his (excellent) work using network analysis in PER, and Rob and Michael for serving on my committee for the years they did—their feedback in the early days of my project was always insightful. Finally, I want to emphasize John (again) and Zeynep for their huge support in making this dissertation even half as good as it is. They spent way too much time reading and re-reading these chapters and providing critical feedback—though this document is certainly not perfect, it is a hell of a lot better than it would have been without their help. Thank you both.

I would also be remiss if I didn't mention my friends from back home, as they certainly served a vital role in keeping me sane. Will (I'm definitely the Other Will now), Anna, Ian, Blair, and Grant, playing D&D with all of you has been such an excellent experience, both as a creative outlet for me but also as a way to maintain friendships that have only grown in importance to me since moving away. Ian, I swear we'll get back to podcasting once I've defended this thing. We must appease our raucous audience, after all! Will and Anna, abusing your hospitality and staying with you has been a highlight of every trip home, and while the Korean BBQs and Ghibli movies are always impeccable, the real highlight has been the opportunity to just hang out together and to see the many ways we've all grown and yet stayed the same since college.

You can see why I wouldn't be able to thank everyone that has contributed to this work—I'm already on the fourth page and there are so many more to thank (and I haven't said half of what I would like to say about the people I *have* touched on)! I'll instead leave this section with this sentiment, as a way of closing: Reader, think about all that you've personally achieved, be it in the last year, the last five years, the last decade… Those achievements are yours, full stop. Do not let imposter syndrome tell you otherwise—you did those things, and you deserve recognition for them. Yes, those achievements are *yours*, and you *should* take pride in them—and you should also recognize all those that helped you to achieve those things. It's worth ruminating on. Writing this section has been an excellent opportunity for me to reflect on the people that have impacted me and my academic career thus far, and as such I am filled with an immense sense of gratitude. I hope you will do the same, despite (perhaps) not having an Acknowledgements section to write at the moment. It's a wonderful feeling.

…And I unfortunately can’t actually close on that heartfelt point, as I am also obligated to thank the National Science Foundation for supporting this work under grant no. PHY-1912087 as well as the Chase Distinguished Research Assistantship for giving me more time to apply for jobs, attend conferences, and write this dissertation during this final year at UMaine.

Kinda ruins the denouement of it all, doesn’t it?

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CHAPTER 1

INTRODUCTION

Mathematics is the language of physics, and thus any student seeking to model the physical world will by extension be required to develop mathematical skills and reasoning to be successful in the field of physics. This only becomes more true as students begin to study more advanced systems and interactions, where mastering the mathematics required can quickly become one of the most challenging aspects of learning these advanced topics in physics.

There has been extensive work both within the physics education research (PER) and the research in undergraduate mathematics education (RUME) communities to better understand students' physical and mathematical reasoning skills, both individually and where the reasoning between these two domains overlap. Some of this work has been conducted at the introductory level, such as in studies of student understanding of integrals as the area under curves (Nguyen & Rebello, 2011), their use of differentials in electrostatic integrals (Hu & Rebello, 2013), or in attempts to assess students' ability to reason about quantities (White Brahmia et al., 2021). There has also recently been a focus on these math-physics skills in the upper division, including investigations of students working with vector calculus concepts such as divergence and curl in electromagnetic contexts (Bollen, Van Kampen, Baily, & De Cock, 2016) and representations of vector fields (Bollen, Van Kampen, Baily, Kelly, & De Cock, 2017). Some of these studies have made use of epistemological framing, whereby students tend to exist within certain modes—such as one focused on formal mathematical manipulations or focused solely on physical intuition—and transition between them as the need arises while reasoning through physics problems (Bing & Redish, 2012; Modir, Thompson, & Sayre, 2017). Another theoretical framework for analyzing the ways that students reason about the mathematics required in physics that will see much discussion in this dissertation was developed by Sherin (2001). Sherin studied students' understanding of mathematical relationships within symbolic expressions and found that students had developed the ability to separate different symbolic organizations into distinct interpretable elements that he called symbolic forms. This symbolic forms framework was adapted from diSessa's knowledge-in-pieces (KiP) framework (and Hammer's resources framework, another extension of KiP), whereby the complex nature of student conceptual understanding could be modeled by collections of smaller knowledge elements (diSessa, 1993; Hammer, 2000). Where these theoretical frameworks modelled student conceptual understanding of physics, symbolic forms sought to model student understanding and interpretations of symbolic mathematical expressions as they are used to express mathematical and physical relationships. This framework has since been extended to study mathematical sensemaking in contexts such as electrostatics (Hu & Rebello, 2013; Schermerhorn & Thompson, 2019) and quantum mechanics (Dreyfus, Elby, Gupta, & Sohr, 2017). Redish and Kuo (2015) discussed symbolic forms as an example of an extension of embodied cognition—the theory that meaning is grounded in physical experience—into mathematical reasoning. In general, much has been made of attempting to better understand students' mathematical reasoning and the ways in which it relates to their physical understanding.

This research aims to extend this work further into the realm of quantum mechanics. While Dreyfus et al. (2017) did previously investigate mathematical reasoning in quantum mechanics, their work was primarily grounded in an expert perspective and focused primarily on interpretations of eigenvalue equations within a quantum mechanical context (Dreyfus et al., 2017). This quantum mechanical context is particularly well-suited for research into students' physical-mathematical reasoning due to its relative abundance of symbolic notations that are regularly used, and the abstract nature of the mathematics on offer. Because of the non-classical, probabilistic nature of quantum physics, students need to rely more than ever on their mathematical reasoning to succeed, thus making an understanding of their mathematical/physical reasoning even more pertinent within this context. In upper-division quantum mechanics, students are expected to generate, interpret, and work with symbolic expressions written in three different notations: Dirac notation, wave function notation, and matrix-vector notation. While studies have been conducted both on these notations' particular affordances and limitations (Gire & Price, 2015) and students' preferences for selecting a notation for calculations (Schermerhorn, Passante, Sadaghiani, & Pollock, 2019), only a few studies have investigated students' ability to translate between them and the factors that may permit students to do so successfully (Wan, Emigh, & Shaffer, 2019). Similarly, only a small amount of prior work has been done to study how students interpret expressions in these notations and how they reason while generating these expressions (successfully or unsuccessfully) (Wan et al., 2019).

Another reason that this quantum mechanical context is of particular interest for this research topic is due to the curricular differences between quantum mechanics courses at different institutions. There are primarily two different instructional approaches used for upper-division quantum mechanics courses—those that begin by studying the Schrödinger equation with continuous systems expressed in wave function notation (wave functions-first curricula), and those that begin with spin-1/2 systems and the Stern-Gerlach experiment, studying discrete systems with Dirac and matrix-vector notations (spins-first curricula). Each approach eventually transitions to study the alternative systems and use the alternate notations later in the course. Because these notations have such distinct appearances, govern seemingly disparate physical systems, and are taught with very different focuses and in a different order, it is reasonable to suspect that the choice of curriculum by an instructor would affect their students' reasoning about the symbolic expressions they work with.

In pursuit of our goal to extend prior work on student understanding of the notations used in quantum mechanics courses, we aim to answer the following questions:

1. In what ways do students conceptualize and interpret symbolic expressions in quantum mechanics?
2. How strongly and in what ways do students conceptually relate expressions, both within and across notations?
3. To what extent, and in what ways, do students in spins-first and wave functions-first curricula differ in how they conceptualize symbolic expressions in quantum mechanics?

To address these questions, we used a combination of in-person think-aloud interviews and online surveys. Due to the broad nature of our questions, we constrained our purview to study only expressions representing probability concepts within quantum mechanics and only expressions within the Dirac and wave function notations. Because these expressions contain constituent sub-expressions such as Dirac bras and kets and wave functions, we necessarily studied these sub-expressions as well—specifically as they are used in these probability contexts. We defined new symbolic forms identified within student interviews as the elemental building blocks of symbolic expressions in quantum mechanics. The investigation comparing the two approaches in question relied on data from the online surveys, which were distributed at multiple institutions over two years.

The core of this dissertation consists of three chapters (Chapters 3, 4, and 5), the structure of which bears some explanation. Chapter 3 contains a symbolic forms analysis of data from interviews conducted both virtually and in-person with students in a spins-first course. These interviews primarily concerned students generating, translating, working with, and reasoning about expressions for probability (and their constituent sub-expressions), and the findings of this chapter serve as a useful means of helping to explain some of the findings seen in later chapters. Chapter 4 is a manuscript (in preparation for journal submission) describing a study of the survey data collected from students enrolled in spins-first quantum mechanics courses around the US. This survey largely consisted of students categorizing various expressions by their conceptual interpretations. Much of this manuscript concerns a novel use of network analysis techniques to glean meaningful understanding of students' conceptual understanding of expressions from this survey's responses with minimal reliance on free-response questions. Due to this chapter's status as a stand-alone manuscript, it is a self-contained work and thus there will be some repetition of the relevant literature discussed within Chapter 2. These network analysis techniques (among others) are then used in Chapter 5 to analyze survey responses from students in wave functions-first and spins-first courses; the results are then used to compare the conceptual understanding and interpretations of expressions across the two different curricula. Due to the nature of Chapter 4 as a stand-alone paper, there is not a single unified methodology chapter; explanations for the study designs and analysis methods used are instead contained within their respective chapters. In Chapter 6, conclusions from across the study are summarized, larger themes of our results are explored, potential avenues for future research are discussed, and some instructional implications are offered based on our findings.

CHAPTER 2

BACKGROUND AND REVIEW OF RELEVANT LITERATURE

A substantial amount of research has been conducted concerning student use and understanding of mathematical representations, both in physics at large and in quantum mechanics specifically. Much of this work has been conducted to investigate how students interpret meaning from equations and graphs. However, these studies are often either very broad in intended application—and therefore as general as possible—or are limited in scope to a specific relation or representational translation. We seek to extend the current literature base by investigating the ways students engage with two representations that are ubiquitous in upper-division quantum mechanics classrooms: wave function notation and Dirac notation. To do so, we will be drawing from existing literature from both physics education research (PER) and research in undergraduate mathematics education (RUME) regarding student conceptual understanding and the ways in which students interpret meaning from mathematical representations. Section 2.1 will provide a broad overview of the ways that typical spins-first and wave functions-first courses discuss and introduce the two notations we are studying. This context will prove helpful while discussing the results of our analyses in Chapters 3, 4, and especially 5. Sections 2.2 and 2.3 will review prior work in these two areas, respectively, while sections 2.4 and 2.5 will discuss prior studies in these areas within the context of quantum mechanics, specifically. As much of the analysis conducted in Chapters 4 and 5 will be conducted with network analysis techniques, Section 2.6 will discuss prior discipline-based education research using these techniques.

2.1 Instructional Context

Because much of this work will discuss notation, and particularly notation as understood by upper-division quantum mechanics students, it will prove useful to briefly discuss the ways that the texts these students use introduce and discuss the notations in question. While instructors are certainly not expected to (and rarely do) teach in exactly the way the text introduces and discusses topics, it is nonetheless the case that they are likely to hew fairly close if for no other reason than it is the book they selected and that students will be expected to use as reference. Thus, our expectation is that the ways that the chosen text discusses a topic will be roughly indicative of the instruction that the students in these courses will receive. This section will discuss the broad differences between the two primary instructional approaches used within upper-division quantum mechanics: spins-first and wave functions-first approaches. The tables of contents for the most commonly-used textbooks for both of these types of courses are perusable in Appendix A, providing an overview of the topics covered in these courses as well as the order in which they are commonly taught. Appendix B provides a more in-depth examination of the ways that Dirac notation and wave function notation are introduced in the texts used for these two different types of courses.

Spins-first courses generally begin by motivating both the existence of discretized quantum states and the use of Dirac state vectors to describe them via the Stern-Gerlach experiment. The textbooks used in these courses introduce Dirac formalism and connect them to matrix-vector descriptions of superposition states. They also explicitly analogize Dirac eigenstates/eigenvectors with cartesian unit vectors, including connecting dot product ideas seen in earlier physics courses to inner products as used in quantum mechanics. Wave function notation is introduced later in the course, and expressions are directly connected to their Dirac notation analogs (e.g., with "$|\psi\rangle \doteq \psi(x)$," where "$\doteq$" means "is represented by" rather than "is equivalent to"). Because these courses begin by studying systems with discrete observables and are necessarily transitioning to study continuous observables with the introduction of wave functions, explicit parallels are drawn between discrete probability amplitudes (as expressed by Dirac brackets) and continuous probability density amplitudes (as expressed by wave functions). Generally, textbooks used in these courses make use of Dirac expressions and their interpretations as seen earlier in the course to introduce wave function expressions via their analogs in Dirac. More details on the ways that these notations are introduced and the ways that analogs are used to relate them to each other within a spins-first text can be found in Appendix A.

As may be expected, textbooks used in wave functions-first courses begin by introducing wave functions and the Schrödinger equation as a necessary step for describing time evolution of quantum systems. This is motivated by drawing a parallel between the Schrödinger equation and Newton's second law for quantum and classical systems, respectively. Similar to the way that Dirac notation is introduced in spins-first courses, much of the mathematical formalism is simply posited—though without the analogies to geometric spatial vectors used to ground an understanding of state vectors seen in spins-first courses. When these texts then introduce Dirac notation, they do so by drawing parallels between the wave function solutions to various potential wells discussed earlier in the course and general mathematical properties of quantum mechanics—namely, linear algebra. Expressions with the appearance of Dirac kets and brackets are briefly introduced, but discussed in terms of wave function notation operations such as integration. Kets are eventually discussed as being a representation of quantum states and as "vector[s] […] 'living out there in Hilbert space'" (Griffiths, 1995, p. 119), and bras as first "an instruction to integrate" and later as a row vector within a dual space (Griffiths, 1995, p. 122). Once Dirac notation is introduced, it is not used for much of the rest of a one-semester course (a two-semester course will typically include advanced topics later in the text, such as perturbation theory, for which Dirac notation is more appropriate). More details on the ways that these notations are introduced and the ways that analogs are used to relate them to each other within a wave functions-first text can be found in Appendix A.

2.2 Theoretical Frameworks for Analysis of Student Understanding

Multiple theoretical frameworks have been developed for use in analyzing student understanding of concepts and representations. The following frameworks allow for insight into student thought processes regarding these topics, providing a lens with which to interpret student data as well as a foundation from which to construct models of conceptual understanding. This discussion will prove useful as a means of framing our investigation of students' representational understanding within a quantum mechanical context. As we are investigating students' mathematical and physical interpretations of expressions within this context, these theoretical frameworks come from both RUME and PER communities, and address different grain sizes and breadth, ranging from concepts to individual knowledge elements (both mathematical and physical) and from representations to individual symbols in expressions.

2.2.1 Concept Images

Originally used by researchers in the RUME community in the study of functional continuity and limits, concept images are cumulative cognitive structures that are associated with a given concept, including “all mental pictures, associated properties, and processes” associated with the concept (Tall & Vinner, 1981). This idea has been used in PER as well, largely in the context of upper-level electromagnetism and the associated mathematics (Bollen, Van Kampen, Baily, & De Cock, 2016; Roundy et al., 2015; Schermerhorn & Thompson, 2019), but also to study more general mathematical difficulties such as concepts of integration as an area under a curve (Nguyen & Rebello, 2011). The essential idea behind this framework is that students develop a structure of these properties and processes through their own experiences, which will change as they mature and meet new stimuli. Most of the associated mental pictures remain dormant, with a given context causing certain associated meanings to come to the fore; these “excited” aspects of the whole concept image create what is known as an *evoked concept image*, which is what we observe in student behavior. This context-dependent nature of what we can observe is key to this framework, as many different contexts may be necessary to gain a better look at a student’s entire concept image, and thus see what their true understanding of a given concept may be. The concept images developed by students may or may not be in agreement with concept images used by experts, but the goal of instruction is to better align the two. Bollen et al. (2016) studied these concept images of divergence and curl within electromagnetic contexts, while Roundy et al. (2015) studied experts’ concept images related to partial derivatives. Schermerhorn and Thompson (2019), meanwhile, studied students’ concept images for differential length vectors.

2.2.2 Knowledge-In-Pieces and Resources

There are somewhat analogous frameworks to that of concept images that have arisen within the PER community as well, e.g. that of knowledge-in-pieces (diSessa, 1993) and resources (Hammer, 2000). They are similar in that they all view student conceptual understanding as both context-dependent and as an amalgam of multiple smaller concepts that are combined in a given context to form a cumulative whole idea. According to the knowledge-in-pieces framework, students have simple, fundamental ideas related to their embodied experience such as *bigger means slower* and *springiness*. These primitive ideas and relations were (rather appropriately) dubbed phenomenological primitives (p-prims) and were viewed as fundamental building blocks of physics reasoning. The resources framework is a knowledge-in-pieces framework and thus draws upon the same ideas while postulating that students have conceptual and epistemological resources—ideas of physical relations or types of information—that may be in conflict and that may or may not be activated in a given context to generate plausible physical theories. In this way, the resources framework seeks to describe student concept creation, where in a given context certain conceptual resources may be activated in tandem and will inform a student's understanding or belief of how a physical phenomenon may work. While it postulates that p-prims may act as conceptual resources, the resources framework also leaves room for larger conceptual resources that may have their own structure of smaller ideas as well (diSessa & Sherin, 1998).

2.2.3 Symbolic Forms

Similar to how physics education researchers worked to find basic building blocks of student conceptual understanding with p-prims and conceptual resources, so too has other work sought to extend the resources framework in order to describe students' symbolic representational understanding with the symbolic forms framework (Sherin, 2001). Where p-prims and conceptual resources were found to represent building blocks of conceptual knowledge and understanding, symbolic forms serve as building blocks for meaning encoded in symbolic representations. Symbolic forms thus represent a marriage of form (e.g., the shapes and squiggles on paper and their orientation relative to each other) and meaning (i.e., the relationships ascribed to the given arrangements of shapes and squiggles). These are referred to within the framework as symbol templates and conceptual schemata, respectively, and their combinations are dubbed symbolic forms. These are intended to be viewed as the simplest possible relationships; a given equation in physics may be constructed of multiple symbolic forms. An example symbolic form is "opposition," represented by the symbol template "$\square - \square$" and the conceptual schema "two influences working against each other." An example application of the "opposition" symbolic form would be in the sum of vertical forces acting on a block resting on a surface: $N - mg$. This framework allows for theory building where new building blocks of symbolic reasoning are discovered, and has been used as such both for studying mathematical sense-making in electrostatics contexts with vector calculus (Hu & Rebello, 2013; Schermerhorn & Thompson, 2019) and linear algebra concepts in quantum mechanics (Dreyfus et al., 2017; Pina, Topdemir, & Thompson, 2023). Dreyfus et al. (2017) studied students' reasoning about quantum mechanical eigenvalue equations and posited a number of symbolic forms from an expert perspective, while Pina et al. (2023) adapted and extended this work to study symbolic forms for these expressions that were developed by students. As these symbolic representations and their implied meaning are often not taught explicitly (unlike much of the conceptual knowledge built from conceptual resources, which is largely the curricular focus), a greater understanding of these forms and more explicit curricular goals related to symbolic understanding is largely the goal of research within this framework.

2.2.4 Symbol Sense

The RUME community has used the framework of symbol sense (Arcavi, 1994) for studying student understanding of symbolic representations. Designed primarily for analyzing student sense-making in algebra, symbol sense concerns the abilities both to detach meaning from symbols in an expression and to retain a global, gestalt view of the expression as a whole in order to more efficiently conduct algebraic steps. In a way, an aspect of symbol sense would be the ability to work constructively with symbolic forms, as it includes “sensing the different roles symbols can play in different contexts” (Arcavi, 1994). These abilities to aid in symbolic manipulation as well as the reading of symbolic expressions for meaning are all viewed as aspects of symbol sense. Symbol sense goes beyond reading and manipulation of equations, however, as it also includes the ability to choose an appropriate representation as well as switch between options when a choice becomes unsatisfactory. The ability to construct symbolic representations to correctly model desired mathematical relationships is also a sign of symbol sense—perhaps one of the more difficult aspects to master. Related to our work, symbol sense also encompasses “the ability to select a possible symbolic representation of a problem, and, if necessary … [to search] for a better one as a replacement” (Arcavi, 1994). When looking at how to foster and improve symbol sense in students, researchers have found that the development of symbol sense may be affected more by an attitude toward knowledge and learning rather than simply cognitive ability alone (Arcavi, 2005).

2.2.5 Representational Competence

Representational competence is a theoretical framework that was originally developed in chemistry education research to specifically target students' understanding of and ability to work with multiple representations—including symbolic and graphical representations (Kozma & Russell, 1997). Researchers using this framework found that students possess an impressive ability to generate, judge the quality of, and refine representations of given phenomena. This aspect of representational competence was dubbed meta-representational competence (MRC) (diSessa, Hammer, Sherin, & Kolpakowski, 1991), as it was not only their competence with generating or understanding representations that proved valuable, but their ability to reason about the representations as well—critiquing and refining them *as deemed necessary by the students*. They found that even young children possess a "deep, rich, and generative (if intuitive and sometimes limited) understanding of representation" (diSessa & Sherin, 2000), and that this inherent MRC may be key to deepening student understanding of the power and limitations of representations both in physics and more generally (diSessa, 2004). Specifically, MRC focuses on the benefits of representations beyond the "sanctioned" representations such as graphs and tables, and more on student-generated representations that may prove more useful for a given context. This framework has been used by both mathematics and physics education researchers to study student understanding of linear algebra representations in quantum mechanics (Wawro, Watson, & Christensen, 2020).

Related work within PER also describes two skills that are needed to benefit from using multiple representations in physics: representational fluency and representational flexibility (De Cock, 2012). Representational fluency refers to "the ability to construct or interpret certain representations like equations, diagrams, or graphs, but also to what extent someone can switch between different representations on demand," and representational flexibility involves "making appropriate representational choices when solving problems" (Bollen et al., 2017). The idea of representational fluency has been used to investigate the challenges students face when working with symbolic and graphical representations of vector fields (Bollen et al., 2017).

2.3 Meaning from Mathematical Representations

Research in physics education has shown that the ways in which conceptual meaning is tied to representations are multifaceted, and authors often use different combinations of the various frameworks previously mentioned to analyze their student data and to inform their claims. Redish and Kuo (2015) used some aspects of cognitive semantics to compare how meaning is made in language and to "show how those same mechanisms can be used to understand how meaning is made with mathematical expressions in both science and math" (Redish & Kuo, 2015). They discuss how embodied cognition—the theory that conceptualization and meaning are grounded in physical experience and actions—can be extended to mathematical reasoning, using the symbolic forms framework as an example (Sherin, 2001). They argued that the conceptual schema—the understanding of the relationships between the different objects within a symbolic form—is obtained through embodied experience (Redish & Kuo, 2015). They cite the *parts-of-a-whole* symbolic form, where the concept of pieces of a larger whole are inherently connected to physical experiences with real-life objects that are made up of smaller objects.

Other researchers have conducted studies on students' ability to translate a concept represented graphically into an equation and vice versa (Van den Eynde, van Kampen, Van

Dooren, & De Cock, 2019), and how their success in doing so depends on the context given. Particularly, they found that students do significantly better at translating between the two in a "pure" mathematical context than they do in contexts grounded in physics. They also showed that students were more successful at constructing a symbolic equation from a graph than they were at selecting the appropriate graphical representation for a given equation.

In the realm of a more "pure" math context, work has been done in the RUME community on student understanding of linear algebra concepts such as eigenvalue equations (Henderson, Rasmussen, Zandieh, Wawro, & Sweeney, 2010). This work has explored common student notational confusion in this subject, where an operator and its corresponding eigenvalue can be interpreted to be similar despite one being a matrix and the other a scalar. Other work in this field has pursued the actual type of equation that eigenvalue equations are, and have considered how this equivalence is viewed by students (Thomas & Stewart, 2011). Their findings include that the equality signified in the eigenvalue equation seems to represent a different kind of relationship to students than other equalities they are more accustomed to. This type of investigation has recently been somewhat extended into the realm of quantum mechanics eigenvalue problems as well. Wawro and colleagues (2017) used the theory of meta-representational competence (diSessa et al., 1991) alongside Gire and Price's structural features framework (a framework for understanding the affordances and limitations of the various algebraic notations used in quantum mechanics) (Gire & Price, 2015) to analyze students' critical use of and selection between different notational forms. Gire and Price's structural features framework is discussed in greater detail in Section 2.5.2.

2.4 General Quantum Mechanics Meaning and Understanding

Research into student understanding of quantum mechanics is nothing new to the field of PER. There has been a large amount of work conducted in this topic for many reasons, perhaps foremost because it is widely considered a difficult subject for students due to its unintuitive nature and the relatively under-used mathematics common in its implementation. Studies of common student difficulties in quantum mechanics go as far back as the 1990s (Styer, 1996), and other such compilations of scientifically-validated areas of difficulty in quantum have been published in the years since (Singh, 2001, 2008; Singh & Marshman, 2015). These studies typically delve slightly into how to improve instruction based on their findings, but such interventions are not typically the focus of the paper. Other work has been done to try and improve student outcomes by drawing parallels between subjects in physics that are more well-understood by students—like introductory mechanics—and quantum mechanics (Bao & Redish, 2002). This and similar work focus on an aspect of quantum physics that is often seen as a barrier to students' understanding: the inherently probabilistic nature of quantum particles. This work attempts to ground this probabilistic nature of quantum mechanics by showing that quantities such as probability densities of position can be used in classical mechanical systems, and thus attempts to demystify some aspects of quantum physics. More research has been conducted in this field more recently as well, implementing interactive simulations to better show the mathematical connections between quantum and non-quantum systems with regard to classical probability densities (Kohnle, Jackson, & Paetkau, 2019).

Other recent studies focused on the different ontologies students assign to quantum entities such as wave packets, and the way in which these wave-particle duality-based ontologies are constructed (Hoehn & Finkelstein, 2018). Somewhat in contrast to the broader investigations of student difficulties in quantum physics, recent work has been focused on deep dives into specific difficulties to better understand them (Emigh, Passante, & Shaffer, 2015; Passante, Emigh, & Shaffer, 2015), breaking down difficulties in time dependence and measurements in quantum physics, respectively, looking closer at the contributing factors to these challenging topics. Tutorials have also been designed to improve outcomes for these and other specific difficulties in quantum mechanics as well (Emigh, Passante, & Shaffer, 2018; Passante, Emigh, & Shaffer, 2014). These recent studies have greatly expanded our understanding of student thinking about these particular topics, their associated difficulties, and effective ways to combat these difficulties, while leaving much still to be explored further.

2.5 Meaning from Quantum Mechanical Representations

Recent work has also investigated representational understanding in quantum mechanics specifically. This includes the development of simulation-tutorials specifically designed to target students' graphical understanding of wave functions after measurement and time evolution, as well as work conducted to study conceptual interpretations and understanding of symbolic representations within quantum mechanics.

2.5.1 Simulations to Help Students Interpret Quantum Mechanics Graphical Representations

Like with the previously-mentioned simulations to connect classical sensibilities with quantum mechanical concepts (Kohnle et al., 2019), work has also gone into combination simulation-tutorial activities to help students better understand aspects of measurement in quantum mechanics (Zhu & Singh, 2012). These simulations largely focus on building graphical understanding of wave functions and their changes through time and after measurements (if any). The researchers were able to show substantial growth in student understanding of several common sticking points in student understanding of wave functions, including stationary states, shapes of eigenstates, and wave function shape post-measurement. Other work has since built on the idea of graphically-focused simulations, tackling other common areas of difficulty such as time evolution of quantum states (Passante & Kohnle, 2019). These simulation-tutorial activities aim to develop students' representational competence by displaying four different representations of the same quantum state as it evolves through time (see Figure 2.1). The simulation gives students the option of viewing either the ground or first excited states within a one-dimensional infinite square well potential or a linear superposition of the two. The representations shown in the simulation consist of a symbolic algebraic representation, a three dimensional view of the chosen wave function(s) (along the length of the well as well as the complex plane), a complex plane cross-section of this three-dimensional view at a single position value, and the plot of the probability density of the total wave function along the well. Three of the representations used are graphical in design, and all serve to underscore the complicated nature of the time evolution of quantum states, as well as the connections between the graphical and symbolic representations. Students transitioned from classical visual reasoning (such as a traveling wave on a string or a standing wave) toward quantum mechanical visual reasoning (requiring understanding of complex geometry and concepts such as arbitrary/non-arbitrary phase and relative magnitudes) by working through this highly visual simulation-based tutorial.

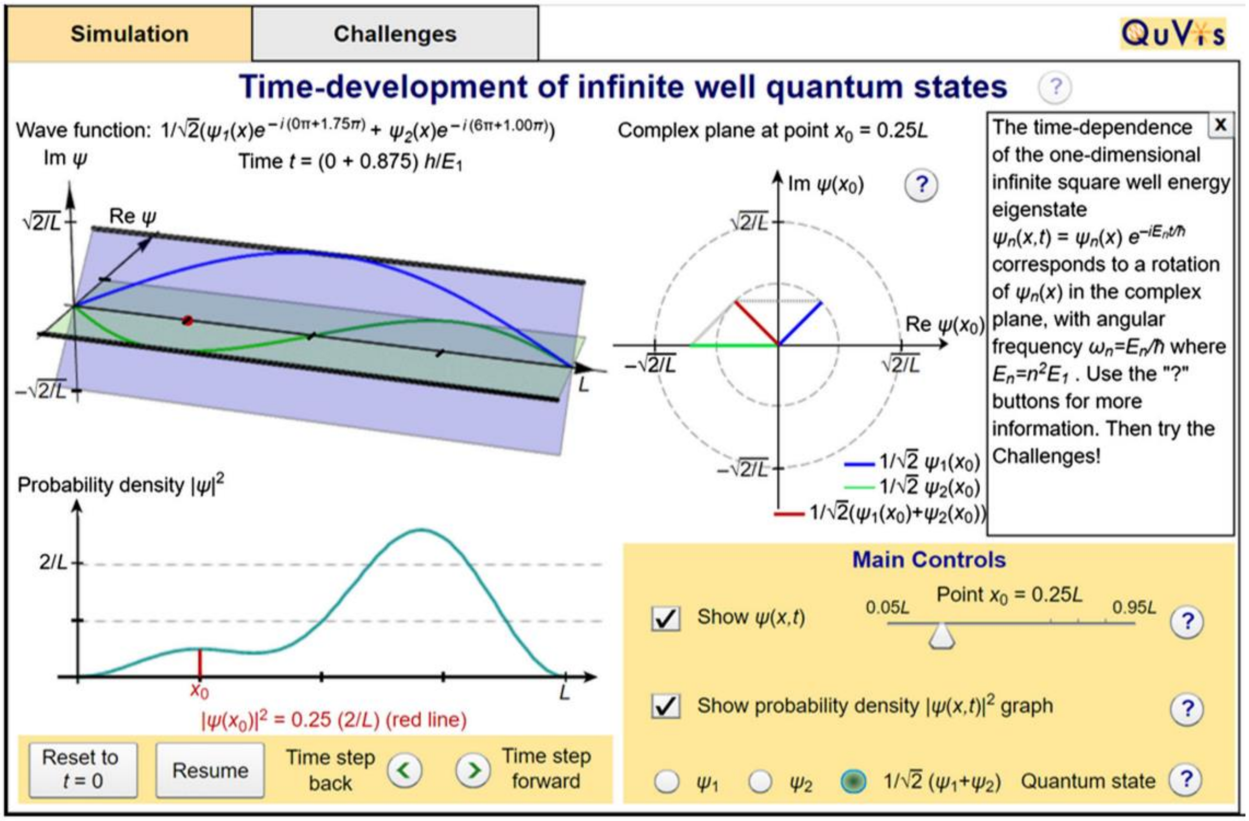


Figure 2.1: Screenshot of the QuVis simulation-tutorial focused on representations of time evolution for superpositions of energy states for an infinite square well potential (Passante & Kohnle, 2019). This simulation is available at https://www.st-andrews.ac.uk/physics/quvis/.

2.5.2 Symbolic Representations in Quantum Mechanics

Other research on mathematical sense-making in quantum mechanics has been used in an attempt to expand Sherin's symbolic forms framework into quantum mechanics by defining new symbolic forms from an expert perspective (Dreyfus et al., 2017). They claimed that many of the "classical" symbolic forms identified by Sherin still hold for quantum mechanical systems, but define two new forms that are inherently necessary for mathematical sense-making in quantum mechanics: the *transformation* symbolic form—represented by the symbol template $\hat{\square}|\ \rangle$—and the *eigenvector-eigenvalue* symbolic form—represented by the symbol template $\hat{\square}|\ \rangle = C|\ \rangle$. They claimed that these symbolic forms are more abstract and require more indirect reasoning, which reflect the complexity and indirectness of physical interpretations in quantum physics. This research, while useful in its extension of the symbolic forms framework into quantum reasoning, is ultimately somewhat limited due to its emphasis on proposing symbolic forms from an expert point of view, and thus leaves much room still to be investigated in this realm in terms of symbolic forms used by students.

Gire and Price introduced a structural features framework to directly investigate the three most prevalent symbolic representations in undergraduate quantum mechanics: Dirac notation, matrix notation, and algebraic wave function notation (Gire & Price, 2015). They provided evidence of four aspects of the symbolic notations, and investigated how these aspects manifest in students' physical and mathematical interpretations of symbolic statements across the different notations (Gire & Price, 2015). The first of these aspects is *individuation*, which is a measure of how easily separable and/or elemental important features are represented in a given notation. The second aspect is a notation's degree of *externalization*, which is a measure of how explicitly relevant elements and features are externalized via markings included within the notation. The third, *compactness*, is simply a measure of how much space and writing is required to express quantities and expressions in a given notation. Finally, *symbolic support for computation* measures how well the symbols themselves support computation due entirely to their visual properties, such as shape. The relative strength of the three notations Gire and Price studied, as well as examples of the reasoning for the notations' scores in each category, can be seen in Figure 2.2. They used this framework to help explain students' work with the notations, as well as their success (or lack thereof) in "using an expression in one notation to guide the development of the analogous expression in another notation." They also discussed their work's implications for instruction, including "support for the value and appropriateness of Dirac notation for undergraduates … and the need for practice using and coordinating multiple notational systems."

Notation	Individuation	Externalization	Compactness	Symbolic support
Dirac	High. Kets provide an elemental representation of basis states	Moderately low. Label for quantum state is ambiguous	High for individual kets (only a few simple symbols). Low for superposition states with many components; kets cannot be combined	High. The asymmetry of the bra-ket notation makes an inner or outer product easily distinguishable
Wave function	Low. Coefficients, normalization constants, and basis states may be combined algebraically	High. Aspects such as functional form, periodicity, etc., are explicit	Low for individual eigenstates, but may be high for infinite superpositions. Basis states require many symbols, but superposition states may be expressed compactly, e.g., via Fourier synthesis	Low. The symbols themselves provide little clue on how to perform operations
Matrix	High. Individual entries in a vector correspond to a basis state	Low. Nothing indicates the basis elements beyond the implicit designation of a space for an entry in the vector	High for low dimension spaces such as spin-$1/2$ systems, but low for infinite-dimensional Hilbert spaces	Moderately low. Differently shaped row and column vectors facilitate remembering to include one of each in inner and outer products, but provide no clue as to the ordering

Figure 2.2: Gire & Price's four "structural features" of notations in quantum mechanics. Relative scores are given for each notation, as well as examples explaining their relative scores (Gire & Price, 2015).

Researchers have since used this structural features framework to investigate student relations between inner products and probabilities in quantum mechanics (Wan et al., 2019). They found that Dirac notation's more directly obvious geometric interpretations aided students in thinking about inner products as analogous to dot products, which helped when thinking of the meaning of the coefficients. This benefit aside, they found Dirac notation's low externalization to hinder student interpretations of position eigenstates, and its high compactness may have led to more student difficulties differentiating between $\langle\psi|\psi\rangle$ and $\psi^*(x)\psi(x)$. In summary, this research looks both at how the structural features framework can be applied to student sense-making, and analyzes symbolic difficulties prevalent in translating between algebraic wave functions and Dirac notation.

The structural features framework has also been adapted to study student preferences for using a given notation when computing expectation values in quantum mechanics within a spins-first curriculum context (Schermerhorn, Passante, Sadaghiani, & Pollock, 2019). In adapting the structural features framework for the purposes of studying students' use of different methods for computation, they identified four slightly different features that were similar to those identified by Gire and Price (2015). Instead of *individuation*, they used *identification* to refer to whether elements within an expression are identifiable for a given student. *Externalization* remained, but was recontextualized as a measure of a student's ability to express the previously identified components successfully. *Compactness* returned as well, though as *computational compactness*, referring to the amount of space/writing required to compute a given quantity, thus allowing for documentation of both how compact a students' computation is as well as whether they value a given computation technique's compactness. Finally, they removed the *symbolic support for computation* feature from the structural features framework and replaced it with a student's level of *computational confidence*, which is a measure of a student's level of comfort with various mathematical operations. In the end, they found that students were more likely to use matrix or integral methods for calculating expectation values, even where summation methods would be much simpler (i.e., when the state is expressed in the basis of the operator in question in Dirac notation). They found that the primary drivers of student preference for a given notation were their confidence conducting mathematics with the notation and the amount of space required to conduct the computation.

In other words, students were more likely to use a notation with which they were confident performing calculations and that required the least amount of space and/or writing during the computation.

2.6 Network Analysis in Discipline-Based Education Research

Network analysis is the collective name of a body of work that has been developed to study anything where actors and connections between them are worth studying. This includes study of topics as diverse as physical infrastructure networks (e.g., airports and the flights between them), neural networks (e.g., neurons or areas of the brain and the synapses connecting them), information networks (e.g., websites and hyperlinks between them), and social networks (e.g., students at a university and social interactions between them) among others (Newman, 2010). In these analysis techniques, the actors (e.g., airports, websites, and students) are called nodes or vertices, and the connections between them (e.g., flights, hyperlinks, and social interactions) are called links or edges. Network analysis consists of a large number of techniques that have been developed to study and interpret the relationships between these nodes and the edges connecting them. These techniques have recently also seen extensive use in both education research in general and physics education research in particular.

Community detection and cluster analysis techniques have been used to study response groupings for various conceptual inventories in physics education research (Brewe, Bruun, & Bearden, 2016; Wells et al., 2019; Wells, Henderson, Traxler, Miller, & Stewart, 2020; Wells, Sadaghiani, Schermerhorn, Pollock, & Passante, 2021; Wheatley, Wells, Henderson, & Stewart, 2021; Yang et al., 2020). As these inventories were designed to study student understanding in several conceptual categories using several questions apiece, network analysis techniques such as community detection were used as another means of validation of these inventories by determining which non-normative responses are often used by the same students, and determining whether they represent shared conceptual difficulties. These techniques have also seen recent use in interpreting results of Likert-style surveys, where they were showed to be a viable alternative or complement to a principal component analysis (PCA) of this format of surveys by categorizing survey questions into similar categories as a previously-conducted PCA on the aspects of student experience scale (ASES) survey (Dalka, Sachmpazidi, Henderson, & Zwolak, 2022).

There has also been extensive work done with these techniques to study social communities and their various impacts in physics, both among communities of educators (Hopkins, Ozimek, & Sweet, 2017; Smith, Hayes, & Lyons, 2017) and students (Brewe et al., 2012; Hopkins et al., 2017; Thomas, 2000). These studies investigated the effects of peer coaching on improving teachers' abilities (Hopkins et al., 2017) and their sense of self-efficacy (Smith et al., 2017), as well as the effects of peer interactions on students' sense of belonging within a learning environment (Brewe et al., 2012) and their levels of persistence in higher education (Hopkins et al., 2017; Thomas, 2000). Recent work has also been conducted with network analysis techniques to characterize how these social communities are affected by different active-learning pedagogies (Commeford, Brewe, & Traxler, 2021), as well as by remote physics courses (Sundstrom, Schang, Heim, & Holmes, 2022). No work has yet been conducted within PER to study student conceptual interpretations of mathematical expressions in physics using network analysis techniques, and so in Chapter 4 we contribute to this research base by showcasing a means to do so within the context of quantum mechanics.

2.7 Summary

A great deal of research has been conducted in attempting to understand how students reason about physical phenomena and the mathematical representations that describe them. Some of this was geared toward simply describing students' conceptual understanding such as the development of the concept images and knowledge-in-pieces frameworks (including resources and symbolic forms). Other work developed frameworks for understanding the ways students reason about mathematical representations, such as symbol sense and representational competence/fluency/flexibility. In recent years, some of this work in conceptual and representational understanding has been conducted in quantum mechanics contexts due to the conceptual difficulty of quantum mechanics and the multitude of representations that are regularly used to describe quantum systems. This work has included cataloguing the challenges students face while taking courses in this topic, determining the ways students reason about certain quantum mechanical phenomena such as wave-particle duality, and assisting in their conceptual understanding of multiple challenging topics within this context. Some work has even been conducted on the mathematical representations used in quantum mechanics, including the development of targeted interventions to assist students in the development of their representational competence within quantum-specific mathematics such as time evolution within complex function spaces. Other research with the goal of understanding the ways that students may conceptualize symbolic representations used in quantum mechanics such as Dirac eigenvalue equations through utilization of the symbolic forms framework was conducted. This work was valuable in that it codified potential expert-like symbolic forms present within representations specific to quantum mechanics such as Dirac notation, but is similarly limited due to not focusing on students' own developed symbolic forms. Similarly, while all of this research has been conducted within either spins-first or wave functions-first contexts, there has been no prior research comparing the two populations with regard to their understanding of the different notations used within quantum mechanics. As notational focus is one of the primary distinguishing factors between these two curricula, this is potentially fertile ground for further research. It is with the goal of helping to fill in these perceived gaps within the existing literature that we pose our research questions (as discussed in Chapter 1):

1. In what ways do students conceptualize and interpret symbolic expressions for probability concepts in quantum mechanics?
2. How strongly and in what ways do students conceptually relate expressions for probability concepts in quantum mechanics, both within and across notations?
3. To what extent, and in what ways, do students in spins-first and wave functions-first curricula differ in how they conceptualize symbolic expressions for probability concepts in quantum mechanics?

It is our hope that in answering these questions, we can contribute to the field's understanding of student representational understanding within this context. We also hope that future work both by us and others will both continue to investigate these topics as well as extend our initial work on comparing the spins-first and wave functions-first curricula.

CHAPTER 3

SYMBOLIC FORMS ANALYSIS OF EXPRESSIONS FOR PROBABILITY IN DIRAC AND WAVE FUNCTION NOTATIONS FOR SPINS-FIRST STUDENTS

3.1 Introduction

Physics students enrolled in upper-division quantum mechanics courses are expected to learn and use Dirac formalism to represent quantum systems and compute relevant values such as probabilities and expectation values for specific measurements and physical observables, respectively. Students in these courses are also expected to learn and use other, more familiar mathematical notations, such as matrix-vector and wave function notation, that they may have used previously in linear algebra or modern physics contexts. Perhaps most crucially, they are expected to learn how these different notational styles interrelate, as calculating properties related to different physical observables will often require a student to work partially in one notation before needing to finish a calculation using another. If this translation is not required, it is nonetheless often preferred, as certain calculations are less computationally demanding with a given notation for a given context.

As discussed in Section 2.2.3, the symbolic forms framework was proposed by Sherin in an attempt to capture the ways in which students reason about formal mathematical expressions in physics (Sherin, 2001). In particular, it functions on the premise that students learn to interpret expressions via a vocabulary of smaller elements arranged via some syntactical rules. One goal of instruction is to assist students in developing and refining an understanding of these elements, such that the students are eventually able both to make sense of new expressions they encounter as well as to generate mathematical expressions to describe physical phenomena.

Viewing upper-division quantum mechanics courses through a symbolic forms lens thus provides a means of studying the mathematical and physical interpretations of quantum mechanical quantities that students develop in these courses. That students in this context are typically learning an entirely new mathematical representation (in the form of Dirac formalism) makes the application of this lens even more interesting, as it allows for an investigation into the mathematical and physical interpretations that students develop for expressions that are entirely new to them, and that will be of great relevance should they continue on to graduate study in physics. This study investigates the ways that students reason about expressions commonly used in upper-division quantum mechanics courses, particularly those used to represent probabilities in Dirac and wave function notations. To this end, we seek to determine the various symbolic forms students develop relating to probability concepts throughout a spins-first, upper-division quantum mechanics course.

3.2 Mathematical Background and Symbolic Forms' Suitability

Before discussing the experimental design and digging into our analyses, it will be helpful to first discuss the normative expressions for these probability concepts in these notations and to discuss the suitability of the symbolic forms framework for this analysis.

3.2.1 Dirac Notation Expressions for Probability

Given that the focus of this work is on the interpretation of expressions for probability that commonly occur within upper-division quantum mechanics courses, there are two normative expressions within Dirac notation that are of explicit interest for this study. First is inner products, which represent probability *amplitudes*:

$$\langle a_n|\psi\rangle.$$

These are also expressed as c_n, the coefficient of an eigenstate in that operator's basis expansion. The associated probabilities are found by taking the complex square of these inner products:

$$\mathcal{P}_{a_n} = |\langle a_n|\psi\rangle|^2 = |c_n|^2.$$

For both of these expressions, $|\psi\rangle$ is a state vector (ket) and is presumed to represent the initial state of a system prior to a measurement and $|a_n\rangle$ is also a state vector (ket) and is presumed to represent an eigenstate of a quantum mechanical observable (i.e., an eigenvector of the associated physical observable's mathematical operator). $\langle a_n|\psi\rangle$ is the mathematical inner product between the initial state vector $|\psi\rangle$ and the dual of the eigenstate vector $|a_n\rangle$, which is represented as $\langle a_n|$ (and called a bra). The bras and kets get their titles from the name ascribed to the inner product expressions formed from the coupling of a bra with a ket: that of a Dirac bracket. The complex square of this inner product (bracket), $|\langle a_n|\psi\rangle|^2$, is the probability of measuring the $|a_n\rangle$ eigenstate's associated eigenvalue from a system described by the initial state vector $|\psi\rangle$. Thus, the principal (normative) expression in Dirac notation that is expected to show up within students' responses as representing probability is the complex square of an inner product, $|\langle a_n|\psi\rangle|^2$. It would also be expected for students to have developed conceptual knowledge about the constituent parts of this expression, such as the inner product, the ket, and/or the bra, separately. It should be noted that bras and kets are formally distinct mathematical objects: a bra is the covector of its associated ket and the set of bras forms a dual vector space to the vector space within which the set of ket vectors reside. These nuances of dual spaces are not discussed in depth in most upper-division quantum mechanics courses, and thus students are expected to view these distinctions as relatively "fuzzy." Primarily, the instructional goal for this distinction is for students to treat $\langle\psi|$ and $|\psi\rangle$ as representing the "same" vector, with the main distinction being that $\langle\psi|$ and $|\psi\rangle$ are representable as a row vector and column vector, respectively, with the elements within one row/column vector being the complex conjugates of the elements in the other.

3.2.2 Wave Function Notation Expressions for Probability

When studying physical observables that have continua of possible measurable values (such as position or momentum/energy for unbound states), it is no longer fruitful to consider probabilities for measuring single values. Instead, one must consider the probabilities for the continuous observable to lie within given ranges of values. These probabilities are expressed as

$$\mathcal{P}_{a\to b} = \int_a^b \psi^*(m)\psi(m)dm\,,$$

where m represents the continuous physical observable in question, and the probability in question is being calculated for the values of m lying between a and b. The function $\psi(m)$ is generally called the wave function, and represents the probability density amplitude for a given system with respect to the observable m. In practice, this means that $\psi^*(m)\psi(m) = |\psi(m)|^2$ is the probability density function such that $|\psi(m_0)|^2 dm$ is the probability for the system in question to have a value of m measured between m_0 and $m_0 + dm$.

A probability for a specific value of a discretized observable (such as intrinsic angular momentum or energy for a bound state) can be represented in wave function notation as well.

While in Dirac notation this probability is represented with the complex square of a Dirac bracket (e.g., $\mathcal{P}_n = |\langle a_n|\psi\rangle|^2$), in wave function notation this is expressed instead as

$$\mathcal{P}_n = \left| \int_{-\infty}^{\infty} \xi_n^*(m)\psi(m)dm \right|^2 ,$$

where $\xi_n(m)$ is the probability density amplitude (i.e., the wave function) for the system following a measurement of the n^{th} eigenvalue of the discretized observable. These $\xi_n(m)$s are thus eigenfunctions of the operators corresponding to these observables, just as the $|a_n\rangle$ kets were the eigenvectors of these observables as discussed in Section 3.2.1. Essentially, where $\psi(m)$ is the wave function form of the initial state (analogous to $|\psi\rangle$ in Dirac notation), $\xi_n(m)$ is the wave function form of an eigenstate of the discretized observable (analogous to $|a_n\rangle$ in Dirac notation). Just as the bras represented the dual vectors of the kets in Dirac notation, complex conjugates of wave functions represent the dual function space counterparts of the wave functions.

The principal (normative) wave function expression expected of students as representative of probabilities in quantum mechanics would therefore be twofold. For situations where a probability for a single value of a discrete observable is being represented in wave function notation, a complex square of an integral with two different functions, similar to $\mathcal{P}_n = \left|\int_{-\infty}^{\infty} \xi_n^*(m)\psi(m)dm\right|^2$, would be expected. For scenarios where the students discuss probabilities for a system to have a measured value of a continuous observable within a given region, an integral over the range of values of the complex square of a single function should be expected, either expressed as exactly that ($\int_a^b |\psi(m)|^2 dm$) or as the product of the wave function and its complex conjugate ($\int_a^b \psi^*(m)\psi(m)dm$). As was the case with the expressions in Dirac notation, it is reasonable to suspect that students would also learn to view the components of these expressions—the wave functions $\psi(m)$ and $\xi_n(m)$ as well as their complex conjugates—as individual objects with their own interpretations as well.

3.2.3 Symbolic Forms' Suitability for This Analysis

Because these expressions for probability concepts in principle follow a simple formula as inner product expressions and complex squares of inner product expressions, the symbolic forms framework is particularly apt for this analysis. The symbolic forms framework describes consistent symbolic templates to which students learn to ascribe specific meaning, and the types of expressions discussed earlier in this section very much fit the description as fitting certain symbolic templates. In particular, Dirac brackets and inner product integrals, as well as their complex squares and their constituent pieces (the bras, kets, and functions), are seen and used extensively enough within these courses that it is reasonable to expect that students would learn to recognize them quickly and treat them as representative of physical/mathematical objects/processes. In short, we would expect students to develop symbolic forms for many of the expression types discussed in this section by connecting recognizable symbol templates with distinct conceptual schemata.

3.3 Research Design and Methodology

To determine the ways that students interpret expressions in these two different notations, clinical interviews were conducted over several years with students enrolled in the upper-division quantum mechanics course at the University of Maine. This course is offered every fall semester, and follows the spins-first curricular structure. All students enrolled within these classes were offered financial compensation for agreeing to participate in the interviews, and all students who expressed interest were interviewed. The COVID-19 pandemic greatly affected both the distribution of and participation rates for the interviews conducted—interviews planned for the spring of 2020 were cancelled due to safety concerns, and interviews were required to be virtual the following spring. In-person interviews were conducted in the fall of 2021 and 2022. The differences inherent in virtual and in-person interview settings required the structure of these interviews to differ significantly, and participation rates were lower than expected for all three instances of interview data collection. In total, two individual virtual interviews, a single in-person interview of a pair of students, and two individual in-person interviews were conducted. All interviews were planned to take place over an hour.

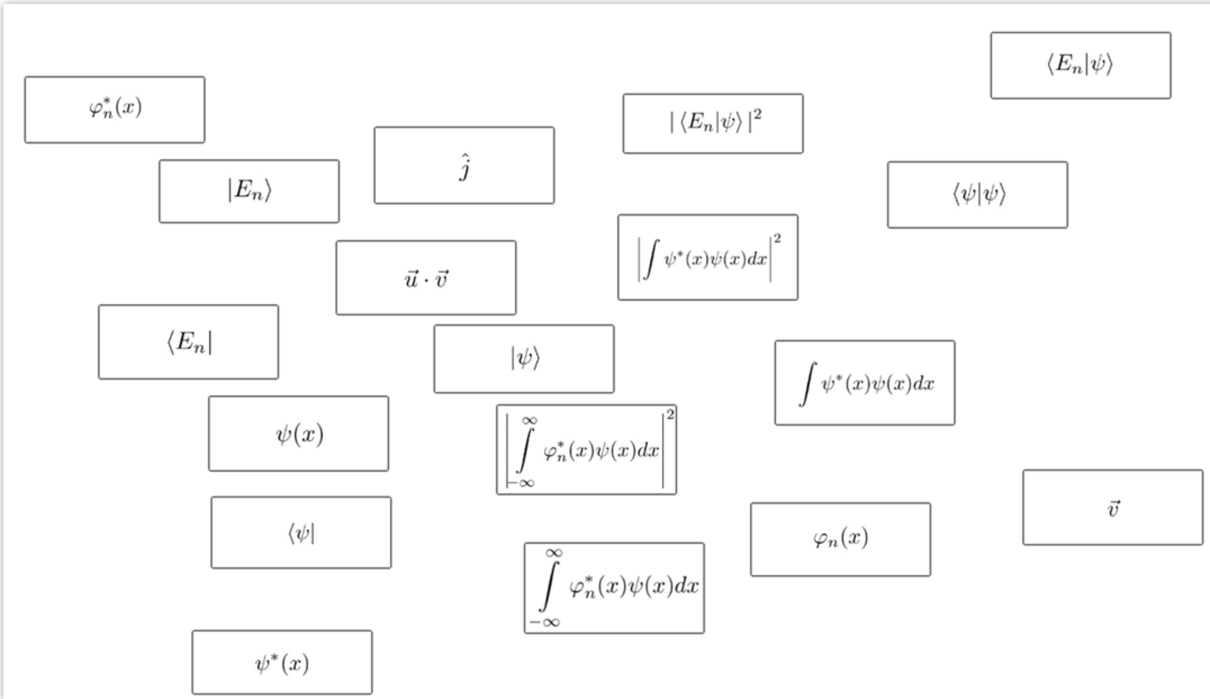
Pseudonyms have been selected for all six students, with Aaliyah and Bilbo as the two virtual interviewees, Castor and Delilah as the participants in the pair interview, and Enoch and Frodo as the participants in the individual interviews. The perceived gender of participants' pseudonyms do not necessarily correspond to the participants' own gender identities.

3.3.1 Virtual Interviews

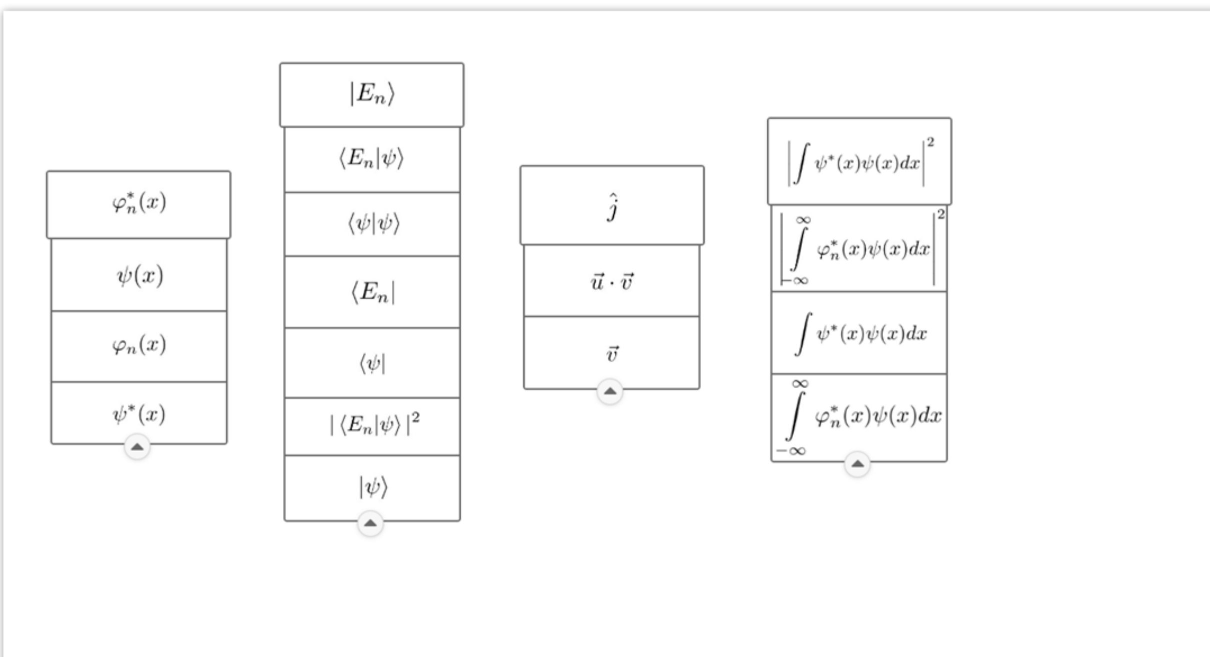
Due to concerns about the spread of COVID-19, the two individual interviews conducted in the spring of 2021 were conducted virtually over Zoom conference video calls. Because the participants could not be expected to have equipment at hand such as tablets and styluses or other means of writing that would be convenient and visible in a virtual environment, the interview tasks were necessarily designed to be conducted solely with a computer mouse. Accordingly, two tasks were administered to the participants: a card-sorting task and an expression-construction task.

3.3.1.1 Card-Sorting Interview Task

The card-sorting task made use of the card-sorting functionality in Desmos, and effectively tasked participants with categorizing and recategorizing a number of expressions by whatever means they deemed appropriate. An example of the expressions and a potential sorting done by a participant is shown in Figure 3.1. The expressions selected for inclusion in this task were selected for a number of reasons. First, they were all expressions with which the students should have been familiar, either from the quantum mechanics course they were enrolled in or from earlier physics courses (e.g., $\vec{v}$, $\vec{u} \cdot \vec{v}$, and $\hat{\jmath}$). Second, they were all normatively "correct" statements, in that each expression has some legitimate physical interpretation. Third, they largely covered every normative expression for probability and their constituent parts within Dirac and wave function notation, such that groups could be constructed containing analogous or equivalent expressions from both notations. Participants were expected to determine categories that "made sense" for these expressions, and to group them accordingly. They were asked to reason aloud about their thought processes, and were encouraged to sort the expressions multiple times to capture as many different categorizations that they thought made sense. The goals of this interview task were to allow for insights into (a) the ways that students think of expressions conceptually, by seeing the categories into which they would sort the expressions and (b) the ways that various expressions interrelate for students, both within and between the given representations. This task generally took the first 20 minutes of the interview; upon the student being satisfied with their categorizations, the second task was initiated.



(a)

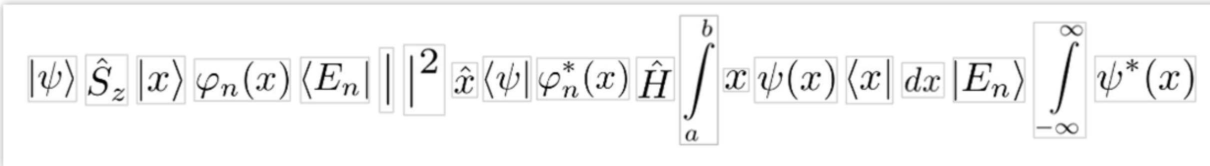


(b)

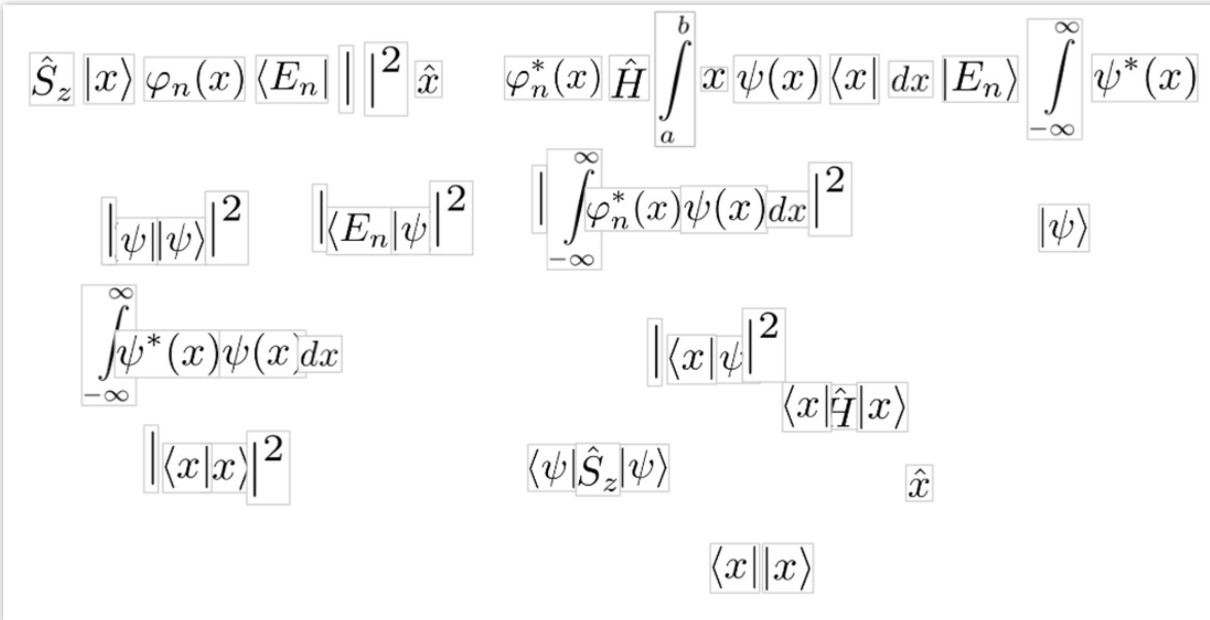
Figure 3.1: (a) The original view of the expression cards in the card-sorting task as seen by the participants. (b) One way that a participant chose to sort the given expression cards.

3.3.1.2 Expression-Construction Interview Task

The second task was an expression-construction task, where students were provided with an assortment of expressions and parts of expressions that were commonly used in their quantum mechanics course. They were then tasked with constructing as many expressions as possible that they deemed as representative of a quantum mechanical probability. This was conducted via a shared Google Slides file, with the student clicking and dragging the parts of expressions around to form the desired probability expressions. The expressions used, as well as an example of some expressions formed by participants, are shown in Figure 3.2. Similar to the card-sorting task, participants were asked to think aloud and explain their reasoning for each expression they constructed. Due to the nature of this task, follow-up questions posed to participants depended greatly on the particular expressions they constructed and the expression elements they indicated, selected, and manipulated. The goals of this interview task were to elicit student thinking on (a) what the individual components of expressions mean and (b) how these expression components interact to form larger expressions with their own meaning. Due to the structure of the questions, the meaning ascribed to these expressions would be expected to relate to probability concepts. By observing the language students used to describe both the components and larger expressions as they were in the process of constructing them, these goals could be addressed. The remaining 40 minutes of interview time were generally dedicated to this task.



(a)



(b)

Figure 3.2: (a) The list of expression components provided to the students for the expression-construction task. (b) An example of a participant's constructed expressions. Some expression components are free-floating from the process of constructing other expressions.

3.3.2 In-Person Interviews

By the fall of 2021, concerns over the spread of COVID-19 were lessened enough that in-person student interviews could be conducted. The questions and tasks posed to these participants differed from the virtual interviews but were identical between the two years these in-person interviews were conducted, with the exception of one additional question asked during the two interviews in the fall of 2022. These interviews took place in a room with a whiteboard and markers, with both the participants and their writing captured by a video camera. The initial prompts given to the students are shown in Table 3.1, and generally required students to either generate or translate expressions in Dirac and/or wave function notations. Participants were asked to voice their thought processes throughout the interview, and further follow-up questions were asked to get a clearer idea of their thinking based on their responses. This methodology is known as a "think-aloud" interview structure, and has been used extensively within physics education research (Lewis & Rieman, 1993; Otero, Harlow, & Harlowe, 2009). These interviews are semi-structured, and the interviewer and interviewees are encouraged to follow trains of thought until they are ultimately satisfied with their responses and move on to the next structured prompt in the sequence.

Table 3.1: The structured prompts given in the in-person interviews. Prompt 2 (in blue) was not asked during the Fall 2021 interview, but was added for the Fall 2022 interviews.

Prompt 1	How would you express the probability for an electron within a potential well to be measured as having the ground state energy of that well?
Prompt 2	Let's say we have an electron in a potential well—perhaps an infinite square well. If we know that it has an even 33% chance of having any of the three lowest possible measurable energy values for that well, how could you express its current quantum state mathematically?
Prompt 3a	Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. It is currently in the superposition state described by: $$\lvert\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\lvert E_1\rangle + \lvert E_2\rangle + 2\lvert E_3\rangle\right)$$ How would you go about finding the probability of measuring that particle to be in the left half of the square well?
Prompt 3b	How would you go about finding the probability of measuring that particle to be in the lowest energy state?
Prompt 4a	Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. The particle is described by the following wave function: $$\psi(x) = \frac{4}{\sqrt{5L}}\sin^3\left(\frac{4\pi x}{L}\right)$$ How would you go about finding the probability of measuring that particle to be in the lowest energy state?
Prompt 4b	How would you go about finding the probability of measuring that particle to be on the left half of the square well?

The primary goal of these interviews was to ascertain students' functional understanding of the expressions used to represent probability concepts in quantum mechanics, by which we mean the understanding these students exhibit in an authentic setting such as in the classroom, on homework problem sets, or on an exam (McDermott, 2001). This idea of a functional understanding is critical, as we care about what students think while they are learning and working with these ideas. One could imagine simply asking "how do you reason about probability expressions in quantum mechanics?" or "what are the requisite components of a complete expression for probability?", but these hypothetical students' responses would be representative of how students respond when prompted with vague, high-level questions about the meaning of mathematical expressions, and not of situations in the classroom or on assessments. Most physics instructors can relate to the difference in the response a student will give when asked what Newton's 2nd Law *is* versus when they are asked to *use* Newton's 2nd Law in a given context. Prompts 3 and 4 were thus designed to be similar to homework problems the participants would have seen throughout the course. Prompts 1 and 2 were chiefly focused on how students generate an expression with only verbal prompting, as well as seeing what participants' first choice for notational style would be. These prompts and their respective follow-up questions were thus intended to gain an understanding of the ways students reason about these expressions, including what they mean as a whole, what their constituent parts mean, and how they are related to their analogous expressions in other notational styles. These responses were analyzed through a symbolic forms lens.

3.4 Symbolic Forms Analysis Results and Discussion

Analysis of students' responses in both virtual and in-person interviews was conducted to determine any conceptual interpretations that were consistently applied to expressions and components of expressions when working with or discussing them. The results of this analysis will be broken down primarily by notational representation and secondarily by the types of expressions identified as separable by students. There will then be discussion about the instances of students identifying conceptual overlap between the two notations, as often occurred in situations where they translated between the different representations.

As discussed in Chapter 2, symbolic forms are the means by which students learn to interpret—and eventually generate—mathematical expressions to represent physical processes or quantities (Sherin, 2001). Symbolic forms are made of two constituent parts: symbol templates and conceptual schemata. A symbol template represents the visual aspects to an expression or part of an expression that cues a student to apply a certain symbolic form. The conceptual schema is the meaning that a student has learned to ascribe to that symbol template—and thus, the meaning that is applied to an expression once a symbolic form has been cued. An example of this is the idea that students could learn to interpret a sum of separate terms as describing a combination of multiple separate parts of a larger quantity being combined to form that larger whole. The symbol template used to represent this "parts-of-a-whole" symbolic form is $[\square + \square + \square + \cdots]$, and a common example of this symbolic form being used in physics is the total energy of a system being described by the sum of several different types of energy such as kinetic, gravitational potential, electrostatic potential, etc.: $E_{tot} = KE + PE_g + PE_e + \cdots$. As simplified by the creator of the symbolic forms framework, "the schema is the idea to be expressed in the equation, and the symbol template specifies how that idea is written in symbols" (Sherin, 2001, p. 13). The symbolic forms we have identified are presented in Table 3.2, along with their associated symbol templates. It is notable that multiple forms seen in this table share identical symbol templates; this is not unprecedented, as Sherin himself noted cases where templates can appear similar or identical, depending on the context. This includes the "base $\pm$ change" symbolic form having the symbol template $[\square \pm \Delta]$, which can appear identical to the templates used for "parts-of-a-whole" or "whole – part" symbolic forms depending on context ($[\square + \square + \square + \cdots]$ and $[\square - \square]$, respectively). The thing that distinguishes two symbolic forms with identical symbol templates is the interpretations that students apply to them—that is, their conceptual schemata. The conceptual schemata applied to each symbolic form is summarized somewhat by their titles and is made explicit within the following sections where they are discussed in detail. To assist the reader, the end of each section will include a smaller table containing the specific symbolic forms and symbol templates identified in that section.

Table 3.2: The symbolic forms identified through our analysis, as well as their associated symbol templates. Symbolic forms are divided into two categories, based on whether their associated conceptual schemata imply more physical or mathematical interpretations of their associated symbol templates.

(More) mathematical symbolic forms	**Symbol templates**	**(More) physical symbolic forms**
Ket as vector	$\|\ \rangle$	Ket as quantum state
Bra as vector	$\langle\ \|$	Bra as quantum state
Bracker as projection	$\langle\ \|\ \rangle$	Bracket as probability amplitude
	$\|\langle\ \|\ \rangle\|^2$	Square bracket as probability

Function as vector	$f(\square)$	Function as superposition state

Table 3.2 Continued.

Conjugate function as vector	$f^*(\square)$	Conjugate function as superposition state
	$f_n(\square)$	Function as specific state
	$f_n^*(\square)$	Conjugate function as specific state
Inner product integral of two identical functions as projection	$\int f^*(\square)f(\square)\,d\square$	Inner product integral of two identical functions as probability amplitude
		Inner product integral of two identical functions as probability
Inner product integral of two different functions as projection	$\int f^*(\square)g(\square)\,d\square$	Inner product integral of two different functions as probability amplitude
		Inner product integral of two different functions as probability
	$\left\lvert\int f^*(\square)f(\square)\,d\square\right\rvert^2$	Complex square of inner product integral of two identical functions as probability
	$\left\lvert\int f^*(\square)g(\square)\,d\square\right\rvert^2$	Complex square of inner product integral of two different functions as probability
Coefficient as component	c_n	
	$\lvert c_n\rvert^2$	Squared coefficient as probability

3.4.1 Symbolic Forms Identified Within Dirac Notation Expressions

Many symbolic forms were identified within Dirac notation expressions for probability concepts. We begin by discussing the symbolic forms identified for the smallest constituent pieces of these expressions: Dirac bras and kets.

3.4.1.1 Dirac Bras and Kets as Quantum States

One consistent observation is that students appeared to consider bras and kets as representative of quantum states. Aaliyah was seen doing this in the virtual card-sorting task, when discussing the elements within a category containing several Dirac bras and kets and a Dirac inner product:

> This [$|E_n\rangle$] represents a ket energy eigenstate, and this [$\langle E_n|$] represents a bra energy eigenstate. So these [$|\psi\rangle$ and $\langle\psi|$] are general ones, these [$|E_n\rangle$] and $\langle E_n|$] are specific energy eigenstates, and this thing [$\langle E_n|\psi\rangle$]- this inner product represents the amplitude of the energy eigenstate E_n if I […] map out all the […] energy eigenstates that make up the ψ.

Here Aaliyah called out the E_n bras and kets as being different from the ψ bras and kets, as she drew a distinction between "specific" eigenstates and "general" states. This distinction shows up again later in the interview, when Aaliyah discusses an expression for probability that she constructed ($|\langle E_n|\psi\rangle|^2$): "The E_n φ_n [gestures at $|\langle E_n|\psi\rangle|^2$] represents, like the probability of finding ψ, which is a general state, in a particular energy eigenstate E_n." This may suggest that, for Aaliyah at least, there may be a distinction between the symbolic forms for a "general state" and for a "specific eigenstate." The exact nature of the meaning of a "general state" may be hinted at by Aaliyah's discussion of ψ being "[made] up" of the energy eigenstates. This may be indicative of an interpretation of ψ and "general" states as relating to superposition states made of a combination of "specific" eigenstates.

Bilbo, in his interview, discussed the ket $|x\rangle$ in the following terms: "you could make x an eigenstate, you could make it a spin state […] put anything in there […] I just need it to be a ket." This implies that Bilbo was very much treating the ket symbol as a marker for a quantum state.

Interestingly, he appeared to treat x here as a mathematical variable, with it being a possibility to swap it with some other symbol to signify a given quantum state; so long as it is a ket, it represents some kind of quantum state, with the marker inside the ket determining the exact type of state.

Castor and Delilah quite frequently discussed E_n bras and kets as representing quantum states, such as early in their interview when explaining an expression they wrote to represent an electron in a potential well. They had written

$$|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots,$$

and Delilah described it with: "if psi is written in terms of the energy states, in Dirac notation like this [points to kets in the expression] then $\mathcal{P}$ [probability] is, you know, as [Castor] said, is just [writes $\mathcal{P}_{E_0} = {c_0}^2$]." Much later in the interview, Delilah reflected on their earlier response to Prompt 3a, where they were given the expression

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$$

as representing a particle in an infinite square well potential. She discussed her interpretation of the expression as a whole and why it was written that way:

> I think it's just, by design. The point of this is to give information. And so the coefficients are designed to give us the probability. And well, […] these [points to the different terms in $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$] are all our possible values. […] And so we just represent it as […] [points to $\sqrt{3}$] the square root of the probability times the first state [points to $|E_1\rangle$] plus the square root of probability [points in front of $|E_2\rangle$] times the second state [points to $|E_2\rangle$] plus the square root of the probability [points to the 2] times the third state [points to

$|E_3\rangle$] and then do that. We can do that infinitely. […] So yeah, […] we essentially just have square root of probability times each state.

As can be seen from this excerpt, Delilah directly referred to the ket symbols as states, with the E_n's representing "all our possible values" (presumably the possible measurable energy values for each state). She also notably described the coefficients in front of each eigenstate as "the square root of the probability" for measuring that state's energy, which is a normatively correct interpretation, albeit without explicitly referring to the possibility of complex coefficients and the necessity of a complex square to attain the probabilities.

Delilah also wrote an expression equating $\langle E_n|$ and $\hat{x}$, $\hat{y}$, and $\hat{z}$ with a large and exaggerated "≈" sign (see Figure 3.3). When asked to explain her expression, she replied with "I'm just trying to say that's how I reconciled the complete orthonormal basis of the E_n energy states," calling out a conceptual similarity to them between a bra representing an "energy state" and Cartesian unit vectors. This vector interpretation of bras and kets will be discussed in more depth in Section 3.4.1.2.

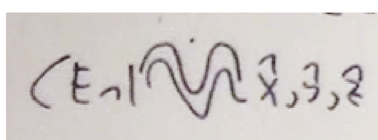


Figure 3.3: Delilah's expression relating an $\langle E_n|$ bra to Cartesian unit vectors.

While Aaliyah, Bilbo, Castor, and Delilah all claimed bras and kets represented quantum states, there was some ambiguity as to what they meant when they said "state." Did they have a clear conceptual interpretation for that phrase, or was that simply a learned name from lecture? Evidence that they did in fact have a clear conceptual understanding of what a quantum state "is" or describes can be found in the interviews with Enoch and Frodo. When

Enoch interpreted the expression given in Prompt 3a ($|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$), he identified the kets as energy states: "What this equation is saying is that [...] the particle could be in any of the three energy states [gestures at the three E_n kets]." Later, Enoch discussed his error in writing $\psi(x)$ on the LHS of the expression he initially wrote for Prompt 2 ($\psi(x) = \frac{1}{\sqrt{3}}|E_1\rangle + \frac{1}{\sqrt{3}}|E_2\rangle + \frac{1}{\sqrt{3}}|E_3\rangle$): "[$\psi(x)$] is a function of x, and then I wrote it as a sum of vectors of the distinct energy levels." Frodo likewise discussed kets as representing explicit energy levels, discussing the terms in his expression for prompt 2 (written as $\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$) with "so each of these [gestures to the three kets]… these are the three lowest energy levels," later describing the state of the particle as "a mixture of the three different energy states." Frodo did explain that he was being imprecise with calling it a "mixture," and was conscientious of the difference between superposition and mixed states. In these examples, Enoch and Frodo were clear about what a (quantum) state means to them—that it was related to a property associated with a particle, be it describing a particle's (possible) energy value, or a superposition of possible energy values.

Based on the responses discussed in this section, two symbol templates are present here—$|\quad\rangle$ and $\langle\quad|$—both of which appear to share a conceptual schema of representing a particle or system in a specific quantum state (either a general state or an eigenstate of an observable), with their associated physical properties/eigenvalues. These symbol template/conceptual schema pairs then define the "*ket as quantum state*" and "*bra as quantum state*" symbolic forms (Table 3.3). Looking at the responses from Aaliyah, it is also possible that there are distinctions to be made between a "general" state—typically with a ψ inside the ket or bra—and a "specific" eigenstate—with some other label within the bras/kets. This would suggest that perhaps there should be a differentiation between two types of kets/bras as symbol templates based on the symbol within them, and that there are distinct conceptual schemata for the two "flavors" of kets and bras. While this is representative of a normative understanding as there is indeed a distinction between a general ket that can represent any state (typically expressed as $|\psi\rangle$) and kets such as $|+\rangle_z$ or $|E_2\rangle$ that describe specific eigenstates of physical observables, without more evidence of this from the other five interviewees there does not appear to be enough evidence to entirely support that claim. Thus, we will only lay out the two symbolic forms discussed above (ket and bra as quantum state).

Table 3.3: Symbolic forms identified for Dirac bras and kets as describing quantum states and their associated symbol templates.

Symbolic form	**Symbol template**
Ket as quantum state	$| \quad \rangle$
Bra as quantum state	$\langle \quad |$

3.4.1.2 Dirac Bras and Kets as Vectors, and Dirac Brackets as Dot Products

Another cluster of symbolic forms that appeared in student responses relates to treating bras and kets as vectors, and Dirac brackets as vector dot products. As students often discussed brackets in terms of bras and kets—particularly in the context of bras/kets as vectors—and brackets as dot products between bra and ket vectors, these three symbol templates and their associated vector-like conceptual schemata will all be discussed together.

In the virtual card-sorting task, Aaliyah discussed both the $\langle E_n|\psi\rangle$ and $\langle\psi|\psi\rangle$ brackets in terms relating to a geometric interpretation of dot products, where she calculated a projection of one vector onto another. When discussing the $\langle E_n|\psi\rangle$ bracket, she said: "I will get the amplitude from this inner product […] like in vector form, like how one vector along the other, like projection of one thing along the other, so that is like the projection of ψ function along the E_n-basis." When discussing the $\langle\psi|\psi\rangle$ bracket, she said "why would I do ψ of ψ? Because physically like I'm thinking in terms of vectors, it represents ψ along ψ." Upon being asked what she meant by "_____ along _____," she explicitly connected their interpretation of the Dirac bracket to that of a dot product's projection idea: "it's a traditional way to think about vectors, like because our dot product represents– like $\vec{a} \cdot \vec{b}$ represents, basically, the projection of $\vec{a}$ along $\vec{b}$ or projection of $\vec{b}$ along $\vec{a}$." Later, Aaliyah was thinking aloud about the meaning of $|\langle x|x\rangle|^2$, and asked herself "what does a vector dot-product-ed with itself represent? The magnitude? Squared…" She eventually settled on the convention that a dot product of a vector with itself would in fact produce the magnitude of that vector squared (e.g., $\vec{v} \cdot \vec{v} = |v|^2$), and thus the complex square on the outside of the bracket was redundant. In all of these cases, the connection between Dirac brackets and the projection ideas she associated with dot products is clear. She also appeared to relate the Dirac bracket as a combination of a bra and a ket, and associated them as her $\vec{a}$ and $\vec{b}$ vectors (e.g., treating the bracket as an analog: $\langle \quad | \quad \rangle \leftrightarrow \vec{a} \cdot \vec{b}$), where the bracket is explicitly the projection of a ket vector onto a bra vector with a geometric spatial interpretation to both the vectors and the inner product.

During the virtual card-sorting task, Bilbo provided explicit categories for "vector" and "vector inner product," wherein he placed the bras/kets and Dirac brackets, respectively. He initially grouped the $\vec{v}$ and $|\psi\rangle$ together, saying "I'm going to be grouping these things as vectors," then added $\hat{\jmath}$ and $|E_n\rangle$ sequentially, explicitly calling out that they go into that group because they are all vectors. Bilbo briefly grouped $\langle E_n|$ and $\langle\psi|$ together separately, saying “bras are effectively vectors as well, they’re just conjugate vectors,” before then combining the two groups together, stating “I could combine the bras with the kets, surely, because to me, those are just— they’re all— they’re vectors.” He crystallized this point for himself by declaring “If you’re gonna take an inner product between a bra and a ket, you can only have an inner product of two vectors.” This connected in with his earlier grouping of Dirac brackets and dot product expressions, which began by grouping $\langle\psi|\psi\rangle$ together with $\vec{u}\cdot\vec{v}$, with him stating “yeah so, I mean, got some dot products here,” and continued with adding $\langle E_n|\psi\rangle$, saying, “this is also going to be a dot product.” Bilbo applied other rules of dot products and vectors to this category, when he stated “$\langle E_n|\psi\rangle$, $\langle\psi|\psi\rangle$, $\vec{u}\cdot\vec{v}$] should be scalars because they're inner products”—upon being asked why that meant they were necessarily scalars, he expounded with “a dot product produces a scalar. Due to mathematics […] that’s the way it goes—there are two types of vector multiplication. You’re doing an inner product that is the […] scalar multiplication.” He later clarified that “[he’d] been taught to think of [a dot/inner product] as like a projection, so then you know how much does one vector project onto the other.” Bilbo also treated kets and bras as geometric vectors in contexts beyond inner products. For example, during the expression construction task, he discussed what operating an $\hat{S}_z$ operator on $|\psi\rangle$ or $\langle\psi|$ would do.

> Certainly changes the state […] well actually […] does it have to change the state? I mean […] if the state is purely in *z*, I believe it’ll still change it, but I think by only lengthwise stretching […] rather than rotating.
> [Asked why it would only stretch and not rotate] Because it would be an eigenstate of that matrix […] we just know if you have a vector […] that is an eigenvector of the operator, then when you operate you just get you know ‘λ

your eigenvalue times your vector,' which is thus the same vector and not rotated at all, but its magnitude may have changed.

Here it is clear that Bilbo was treating these bras and kets very geometrically, with "stretching" and "rotating" as viable operations that could occur to them. It is also of note that Bilbo discussed this action as occurring not only to a vector, but to a state as well. This is further evidence of the *ket as quantum state* symbolic form from Section 3.4.1.1, as well as evidence of Bilbo thinking fluidly about these symbolic forms.

Castor and Delilah likewise treated brackets as dot products, and even had a discussion on the distinctions between an inner product and a dot product, starting when the interviewer asked them about their calling $\langle E_1|\psi(x)\rangle$, an expression they wrote prior, a dot product.

> Interviewer: Okay, so you called the– this thing [$\langle E_1|\psi(x)\rangle$] a dot product.
>
> Delilah: Uh, inner product, yes.
>
> Castor: They're basically the same.
>
> Delilah: I think a dot product is a form- one of them is a form of the other
>
> Castor: Like, one is more broad than the other, but they're basically the same thing.
>
> Delilah: Which one is which though […] I think dot product is a form of an inner product.

Here they can be seen determining (correctly) that an inner product is a generalization of a dot product, but they nonetheless referred to Dirac brackets as dot products. Later in their interview, they made the assertion that for $n \neq m$, $\langle E_n|E_m\rangle = 0$. When asked to explain why that was the case, Castor stated "because of like orthonormality, the eigenstates are perpendicular in a space," ascribing spatial geometric properties such as orthogonality/perpendicularity to a Dirac bracket. As was discussed in Section 3.4.1.1, Delilah wrote an exaggerated version of $\langle E_n| \approx \hat{x}, \hat{y}, \hat{z}$ on the board. Later in the interview, the two discussed it in the following way:

> Delilah: But like that [$\langle E_n| \approx \hat{x}, \hat{y}, \hat{z}$], that helps me realize, why it's orthonormal- why it's orthogonal. And complete.
>
> Castor: And like the dot product, or inner product is like the, if you do it with just with vectors, it's like, how much is a projection onto the other... thing.
>
> Delilah: Yeah, how much of them are in the same direction.
>
> Castor: So if they're 90 degrees from each other, then their components are just in their directions. They're not, like, a superposition or like a vector that has multiple pieces. [sketches an arrow lying between two perpendicular dotted arrows (presumably two axes)]

Here Castor and Delilah were discussing the similarities between the bra $\langle E_n|$ and the Cartesian unit vectors $\hat{x}, \hat{y}$, and $\hat{z}$. They explicitly connected $\langle E_n|$ to ideas of dot products, projection, and directionality. Later on, when working through some calculations for Prompt 3b, they determined that $\langle E_2|E_2\rangle = 1$. When asked to explain that step, Castor responded with "because, like 100% of E2 [points to the $\langle E_2|$ in $\langle E_2|E_2\rangle$] is in the direction of E2 [points to the $|E_2\rangle$ in $\langle E_2|E_2\rangle$]."

Enoch and Frodo also described Dirac bras/kets and brackets in terms of a vector- and dot product-like interpretation, respectively. While explaining his answer to prompt 3b, Enoch described $|\langle E_1|\psi\rangle|^2$ as "giving the component of this [$|\psi\rangle$] in a particular direction or in this case of the particular energy, and then norm squaring it." Enoch was referring to the Dirac bracket within the complex square as a process of determining a component along a direction, a clearly geometric dot product interpretation. Enoch also discussed the ket $|\psi\rangle$ alone, referring to it as "a vector sum of each of the probabilities of the different energy states or observables that you can do," where he explicitly connected a ket for a superposition state to a vector sum. Very early in Enoch's interview, he wrote $\psi(x) = \frac{1}{\sqrt{3}}|E_1\rangle + \frac{1}{\sqrt{3}}|E_2\rangle + \frac{1}{\sqrt{3}}|E_3\rangle$ for prompt 2. Upon reflection near the end of the interview, he corrected himself: "Yeah, that's wrong [...] [$\psi(x)$] is a function of x, and then I wrote it as a sum of vectors of the distinct energy levels." Here, Enoch again referred to a sum of kets as a vector sum. It is notable that there are two ways that Enoch could have corrected this expression: he could have changed $\psi(x)$ into $|\psi\rangle$, thus matching it to the "vector" ontology he identifies the E_n kets as sharing; alternatively, he could have changed the E_n kets into their corresponding eigenfunctions written as functions of position to match the $\psi(x)$'s "function of x" identity. It is also worth noting the somewhat "sloppy" language Enoch uses, referring earlier to a vector sum of probabilities (which are scalar quantities) and "vectors of [...] energy levels." Based on his other responses, we believe this sloppiness is not indicative of a low level of understanding and view his first statement as referring to the probability amplitudes being the coefficients in front of the basis vectors and his reference to "energy levels" referring to energy eigenstates (which describe states at certain energy levels). Frodo, meanwhile, used perpendicularity to explain the Dirac bracket $\langle E_1|E_2\rangle$: "so, these two [gestures at $\langle E_1|E_2\rangle$], when they're not the same state, they're perpendicular to each other. Like these [gestures at $\langle E_1|$ and $|E_2\rangle$ in $\langle E_1|E_2\rangle$] are each orthogonal to each other." Similarly to Castor and Delilah, Frodo conceptually connected inner products and dot products together, saying “we were calling these inner products in class. […] But I mean, it's the same as a dot product.”

As can be seen from these interview excerpts, these students all made very strong conceptual connections between bras and kets and vector ideas, and between projection/dot product ideas and Dirac brackets. In this case, there are three symbol templates: | 〉, 〈 |, and 〈 | 〉. The conceptual schemata that these students appear to have connected to the | 〉 and 〈 | symbol templates include ideas related to vectors in a geometric sense, such as length/magnitude and directionality. They appear to have identical conceptual schemata tied to both: although Bilbo does potentially draw a distinction between | 〉 as a “vector” and 〈 | as a “conjugate vector,” he quickly sorted the two together into one overarching “vector” category so a strong conceptual distinction does not seem likely. We name the symbolic forms formed from these symbol template-conceptual schema pairs “*ket as vector*” and “*bra as vector*” (Table 3.4). The 〈 | 〉 symbol template, meanwhile, appears to elicit a very strong conceptual response as representing a dot product, complete with a geometric projection interpretation. We call the combination of this symbol template with these ideas of two vectors being projected together via a dot product the “*bracket as projection*” symbolic form (Table 3.4).

Table 3.4: Symbolic forms identified for Dirac bras, kets, and brackets in the context of vector-like conceptualizations.

Symbolic form	**Symbol template**
Ket as vector	| 〉
Bra as vector	〈 |
Bracket as projection	〈 | 〉

3.4.1.3 Dirac Brackets – and Squared Brackets – as Probability Concepts

Another common interpretation of Dirac brackets was that of probabilities or probability amplitudes (meaning a quantity that will represent a probability upon being multiplied with its complex conjugate). Recall from Section 3.2.1 that the complex square of a Dirac bracket between a state vector (often represented as $|\psi\rangle$) and an eigenstate of an operator is generally representative of a probability (e.g., $|\langle a_n|\psi\rangle|^2$ is the probability of measuring the eigenvalue associated with $|a_n\rangle$, the n^{th} eigenstate of the operator $\hat{A}$). The bracket alone generally represents the probability amplitude. The only deviation from this rule is when the bracket includes two identical vectors (e.g., $\langle\psi|\psi\rangle$ or $\langle a_n|a_n\rangle$). These expressions have two interpretations: as a step in the process of normalizing the vectors within the inner product, or as a sum of probabilities of all states included within a superposition expansion of the vector within the bracket in a basis (which, not coincidentally, need to sum to one—hence the normalization condition). In the card-sorting task, Aaliyah discussed the bracket $\langle E_n|\psi\rangle$ as representing a probability amplitude in this way, saying "this inner product represents the amplitude of the energy eigenstate E_n if I, you know, map out all the […] energy eigenstates that make up the ψ […] and this will give me- squaring it will give me the probability." Aaliyah also discussed a similar interpretation of the complex square of the bracket $\langle\psi|\psi\rangle$, constructing $|\langle\psi|\psi\rangle|^2$ and saying "I'm trying to represent probability in bra-ket notation […] and this will be just one, if psi is normalized […] that should represent probability of one." In these excerpts, Aaliyah was connecting the square of a Dirac bracket with the means of calculating a probability. Similarly, Aaliyah later described $|\langle E_n|\psi\rangle|^2$ as "the probability of finding ψ, which is a general state, in a particular energy eigenstate E_n." Aaliyah also explicitly connected $|\langle\psi|\psi\rangle|^2$ and $|\langle E_n|\psi\rangle|^2$ later in the interview:

> for me, this [$|\langle\psi|\psi\rangle|^2$] also represents probability in the sense like if I go all– to like all space […] This [$|\langle\psi|\psi\rangle|^2$] is like a summation of all the possible variations of this [$|\langle E_n|\psi\rangle|^2$], okay. If I add all of these guys [$|\langle E_n|\psi\rangle|^2$] together I'll get– end up getting a one […] since every ψ is made up of all the possible energy eigenstates […] so if I find the individual probabilities of all of these E_n's [$|\langle E_n|\psi\rangle|^2$] and add them together what do I get? I get 1. Because that's how we represent probability. […] One means hundred percent of the time so— so that's what like this [$|\langle\psi|\psi\rangle|^2$] is like more broader [sic] representation of the right thing [$|\langle E_n|\psi\rangle|^2$], but again, essentially […] they would both represent probabilities to me.

Aaliyah was clearly treating both $|\langle\psi|\psi\rangle|^2$ and $|\langle E_n|\psi\rangle|^2$ as representations of probability; she also noted (incorrectly) that $|\langle\psi|\psi\rangle|^2$ is the sum of all possible $|\langle E_n|\psi\rangle|^2$ probabilities (in fact, $\langle\psi|\psi\rangle$ is the sum of all possible $|\langle E_n|\psi\rangle|^2$'s). Earlier in her interview, however, Aaliyah also explained a Dirac bracket she constructed—$\langle x|\psi\rangle$—as representing a probability, despite the fact that it lacked the complex square: "this will also represent the probability of finding x— sorry, the probability of finding the general state ψ in the eigenstate x." It is unclear if this is a mere slip of the tongue for Aaliyah, as she was consistent in requiring the complex square in all other cases, or if the position representation for this expression had a different meaning for her.

Bilbo also treated $|\langle E_n|\psi\rangle|^2$ as a probability in the card-sorting task, saying "I'm looking at [$|\langle E_n|\psi\rangle|^2$] and I'm thinking like, probability. […] In this case, probability of being in that eigenstate. Of this wave function [indicates ψ] being in that [indicates E_n] eigenstate." Later in the card-sorting task, Bilbo grouped $|\langle E_n|\psi\rangle|^2$, $|\int \psi^*(x)\psi(x)dx|^2$, and $|\int \varphi_n^*(x)\psi(x)dx|^2$

together and said "okay, here we got our inner products- our probabilities, excuse me, because they're [...] magnitude squared of inner products." Here he explicitly stated the necessity of taking the complex square of an inner product to effectively represent a probability. Later on, Bilbo expressed some confusion as to whether he should square $\langle\psi|\psi\rangle$, asking "do we want to square [$\langle\psi|\psi\rangle$] here? Do we need to square this again? I'm not so sure here, because it's already a magnitude. It is already a scalar. That is just an inner product, though I had been saying the inner product squared is a probability and that this [$\langle\psi|\psi\rangle$] is just a density." Here Bilbo exhibited a behavior that was observed quite often: mixing up terminology, particularly probability amplitude and probability density. He also appeared to be confused due to the repeated label within both the bra and ket of $\langle\psi|\psi\rangle$, asking if it was necessary to "square [it] again". Taking the statement about it being a magnitude, this is presumably a result of his conceptualizing the result of taking a dot product of a vector with itself as the magnitude of the vector squared, hence the question of whether he was squaring it "again." Regardless, it is clear that Bilbo thought both an inner product and a square of some kind was required (be it implicit in the repeated ψ, or explicit in the complex square). Enoch showcased similar reasoning to Aaliyah and Bilbo, even describing his own pseudo-symbol template when asked how to write a probability in prompt 1: "The probability, which I will call squiggly P, is going to be something like the norm squared of some business with some kets and... [writes $\mathcal{P} = |\langle \quad | \quad \rangle|^2$] something like that."

Frodo likewise discussed the complex squares of Dirac brackets representing probabilities, discussing his answer to prompt 2 ($\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$) by declaring the criteria for its correctness as "when you do the square- when you do this... [writes $|\langle\psi|\psi\rangle|^2 =$]

you need it to spit out the one third for the probability." While the expression he gave for the desired probability was incorrect (we believe he meant to write $|\langle 1|\psi\rangle|^2$), he nonetheless wrote a complex square of a Dirac bracket. Similarly to Enoch, he then went on to (correctly) write for their solution to prompt 3b that the probability could be written $\mathcal{P}_{E_1} = |\langle E_1|\psi\rangle|^2$. When asked why the square was necessary, he explained "because we're looking for the probability, and without it, we just have the probability amplitude."

These students have developed some symbol templates—$\langle \quad | \quad \rangle$ and $|\langle \quad | \quad \rangle|^2$—as well as some consistent conceptual schemata tied to them. The latter template has developed a strong conceptual association as a representation of a probability, while the former, commonly referred to as a probability amplitude but occasionally as a probability density, appears to have a strong association as a quantity that is squared to become a probability. In some cases, as with Aaliyah with $\langle x|\psi\rangle$, the Dirac bracket is declared a probability despite lacking the square. In others, as with Bilbo getting confused about whether $|\langle\psi|\psi\rangle|^2$ is squaring a quantity one too many times, leads him to wonder if $\langle\psi|\psi\rangle$ is in fact a probability and not a probability amplitude. While these fuzzy distinctions do appear to exist, for the majority of cases these students appeared to have robust conceptual schemata ascribed to these symbol templates. We name these symbolic forms the "*square bracket as probability*" symbolic form, and the "*bracket as probability amplitude*" symbolic form, as seen in Table 3.5. While some students referred to the probability amplitude as effectively the square root of the probability, rather than using the technically rigorous term, we have elected to use the normatively correct term for this symbolic form, but note that most students appear to interpret it in terms of an object that needs to be squared.

Table 3.5: Symbolic forms identified for Dirac brackets (and complex squares of Dirac brackets) in the context of probability concepts.

Symbolic form	Symbol template
Square bracket as probability	$\lvert\langle\ \mid\ \rangle\rvert^2$
Bracket as probability amplitude	$\langle\ \mid\ \rangle$

3.4.2 Symbolic Forms Identified Within Wave Function Notation Expressions

Just as the focus of our work on expressions for probability determined the particular expressions of interest within Dirac notation to primarily be Dirac brackets and their constituent parts, this same focus is true for the wave function notation. As in Section 3.4.1, we begin by looking at symbolic forms identified for the smallest constituent components of these expressions: the wave functions themselves.

3.4.2.1 Functions as Quantum State

A common interpretation of wave function expressions was that—similar to Dirac bras and kets—they represented quantum states. During the card-sorting task, Aaliyah sorted $\psi(x)$, $\varphi_n(x)$, $\psi^*(x)$, and $\varphi_n^*(x)$ all into the same category, and said they "would represent a general eigenstate $\psi(x)$ or its conjugate [indicates $\psi^*(x)$], or like a specific energy eigenstate phi of- … Those represent states […] some of them represent general states [$\psi(x)$ and $\psi^*(x)$], some of them represent specific energy states [$\varphi_n(x)$ and $\varphi_n^*(x)$], but they represent states." Here Aaliyah drew a parallel to the distinction she drew between "general" states and "specific" states in Dirac notation, as discussed in Section 3.4.1.1. If taken in conjunction with her discussion in that section of the general states $\langle\psi|$ and $|\psi\rangle$ as being "made up of" the specific states $\langle E_n|$ and $|E_n\rangle$, it is reasonable that this interpretation of general vs. specific applies to the ψ and φ_n wave functions here.

Bilbo also discussed wave functions as representing quantum states, particularly while discussing them in the context of inner product integrals. While discussing the expression $\int \varphi_n^*(x)\psi(x)dx$, he said “I'm thinking okay, you have this state [indicates $\psi(x)$] … and you want to ask the question of, you know, ‘what about that state [indicates the $\psi(x)$] being in this [indicates the $\varphi_n^*(x)$] eigenstate.’” Later in his interview, he discussed the expression $|\int \psi^*(x)\psi(x)dx|^2$ on two different occasions. First, he seemed somewhat puzzled by what it could mean but suggested that “it’s like the probability of a state being… in… its own state? I- yeah I’m not quite sure honestly.” He later came back to it and declared that “[$|\int \psi^*(x)\psi(x)dx|^2$] I believe should be one […] because it's a state with itself- what's the probability of a state being in itself? What's the probability of a heads-up coin being heads? It's one.” Here he drew a parallel between the wave function $\psi(x)$ and a heads-up coin. Both objects have a certain quality that describes them—whatever quantum state a system is in and that the heads side is facing up, respectively. The analogy is not a perfect one, as heads-up would be more analogous to an eigenstate of a coin flip, and thus the quantum state described by $\psi(x)$ would need to be in a post-measurement eigenstate to be perfectly analogous to a coin with a predetermined coin flip result. Regardless, Bilbo did appear to treat the functions inside of the integral as representing a quantum state, and moreover the complex conjugate of the wave function ($\psi^*(x)$) as representing the same state as $\psi(x)$. He also treats the integral as determining the probability of one function “being in” the other.

While responding to prompt 1 in his interview, Frodo described his thinking after writing $\psi(x) = c_1\varphi_1(x)$ as "$\psi(x)$ is […] c_1 times $\varphi_1(x)$ […] but I think these [gestures at the $\varphi_1(x)$] are the […] energy eigenstates written in the position basis." Later, during his response to prompt 2, Frodo wrote an expression in Dirac notation for a superposition state: $\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$. When asked whether he could translate it into another notation, he wrote a φ_1 directly above the $|1\rangle$, and discussed the difference between the ψ and φ_1, saying "so this [gestures at ψ] would be this- like the whole state, and then these [gestures at the φ_1] are the individual energy functions, I believe." Frodo can be seen here describing $\psi(x)$ and $\varphi_1(x)$ as representative of "whole" states and "energy" eigenstates, respectively. This is reminiscent of the distinction between "general" and "specific" states drawn by Aaliyah earlier.

As was seen with the Dirac bras and kets, these wave function expressions appear to be treated as representative of quantum states. There appear to be two symbol templates present: that of a function $f(\square)$ and of a function with a subscript $f_n(\square)$, as well as distinct symbol templates for their complex conjugates (i.e., $f^*(\square)$ and $f_n^*(\square)$). We use generic function notation (the f) for these symbol templates because we do not wish to make a claim about what specific letters the students cue these templates from (if there exist specific letters at all). The conceptual schemata that students have applied to these functions seem to consistently differ, with $f(\square)$ connoting a "whole" or "general" state. This suggests they regard these functions as representative of superposition states, which is distinct from their conception of $f_n(\square)$ as representing a "specific" state associated with a given quantity, i.e., an eigenstate; this manifested as a specific energy in the situations prompted by the interview setting. This is similar to the *quantum state* conceptual schemata observed for Dirac bras and kets in Section

3.3.1.1, though there appears to be a more obvious symbolic distinction between the "general" states and the "specific" states in the context of wave function expressions. We note that while the symbol templates differ between the functions and their respective complex conjugates ($f(\square)/f^*(\square)$ and $f_n(\square)/f_n^*(\square)$), there were no real differences between the conceptual schemata applied to the complex conjugate functions when compared to their respective non-conjugated function—i.e., these students did not appear to draw a physical or mathematical distinction between the wave functions or their dual functions. We name the symbolic forms associated with the $f(\square)$, $f^*(\square)$, $f_n(\square)$, and $f_n^*(\square)$ symbol templates with their associated conceptual schemata the "*function as superposition state*," "*conjugate function as superposition state*," "*function as eigenstate*," and "*conjugate function as eigenstate,*" respectively. These symbolic forms and their symbol templates are summed up within Table 3.6.

Table 3.6: Symbolic forms identified for functions in the context of describing quantum states.

Symbolic form	**Symbol template**
Function as superposition state	$f(\square)$
Conjugate function as superposition state	$f^*(\square)$
Function as eigenstate	$f_n(\square)$
Conjugate function as eigenstate	$f_n^*(\square)$

3.4.2.2 Functions as Vectors, and Integrals as Dot Products

One potentially surprising conceptualization of wave function expressions is the combination of wave functions representing vectors and integrals representing dot products. This was most often exhibited within the context of inner product integrals, such as Aaliyah describing $|\int \psi^*(x)\psi(x)dx|^2$ with "it basically represents an inner product of ψ with itself. So in the Cartesian world it will look like the projection of ψ along itself." Aliyah drew an explicit analogy between the integral and projection in a Cartesian space. During the card-sorting task, after discussing how he interpreted the expression, Bilbo added $\int \varphi_n^*(x)\psi(x)dx$ to the group containing $\langle\psi|\psi\rangle$ and $\langle E_n|\psi\rangle$: "So this integral without being squared [referring to $\int \varphi_n^*(x)\psi(x)dx$], also a dot product, I believe, effectively- or an inner product of those functions over space." He similarly remarked upon subsequently adding $\int \psi^*(x)\psi(x)dx$ to the same group, saying that it is also "just a dot product." Later, he discussed his interpretation of $\int \varphi_n^*(x)\psi(x)dx$:

> "in this case here [...] I'm thinking okay, you have this state [indicates the $\psi(x)$] [...] and you want to ask the question of, you know, 'what about that state [indicates the $\psi(x)$] being in this [indicates the $\varphi_n(x)$] eigenstate.' [...] to me I'm looking at this thing I'm thinking, 'what is the projection of this eigenstate onto this wave function,' or maybe vice versa, but I don't think it should matter— dot products are [...] commutative."

Bilbo seemed to be categorizing these integrals into a group of what he considered to be dot products, which bear the conceptualization of a geometric projection. He also sorted $\psi(x)$ and $\varphi_n(x)$ together with $|E_n\rangle$, $\vec{v}$, $|\psi\rangle$, and $\hat{\jmath}$, saying "these [$\psi(x)$, $\varphi_n(x)$] could also, you know, be kets in functional form, so we could think of all of these [$|E_n\rangle$, $\vec{v}$, $|\psi\rangle$, $\hat{\jmath}$, $\psi(x)$, and $\varphi_n(x)$] as just vectors, kets I guess," explicitly referring to the wave functions as representing vectors. Bilbo also discussed wave functions in terms of representing vectors to him in the context of operating an operator on a function. During the expression construction task, Bilbo constructed $|\int \varphi_n^*(x)x\psi(x)dx|^2$, which is something of a conflation of an expression for a probability of measuring an energy value and that of an expectation value for position. He discussed the effect of placing the x in this expression in the following way: “if I am saying [the x in $|\int \varphi_n^*(x)x\psi(x)dx|^2$] is an operator, an operator on a vector is just […] a vector, so it's the same thing. It's just another probability […] it’s a different state after being operated on.” In this context, Bilbo was expressing that the expression still represented a probability because he treated the x as an operator, which acts on one of the wave functions like it would a vector, which will just generate another vector (albeit one that’s been scaled and/or rotated from the original). This is further evidence that these wave functions represented vector objects to these students.

As has been shown, some students appear to have developed an understanding relating wave functions to vector properties, and inner product integrals to dot products. The symbol templates at play here include $f(\square)$, $f^*(\square)$, $\int f^*(\square)f(\square)\, d\square$, and $\int f^*(\square)g(\square)\, d\square$. While there does not appear to be any conceptual distinction between wave functions with and without subscripts when students are interpreting these functions in this way, students did discuss integrals in this manner both when the two functions within the integrand were the same (e.g., $\int \psi^*(x)\psi(x)dx$) and different (e.g., $\int \varphi_n^*(x)\psi(x)dx$). These students did not appear to focus on the bounds of integration (i.e., distinguishing between an indefinite integral, a definite integral over all space and a definite integral over a finite region), and so these are excised from the symbol template that they have developed. The conceptual schema tied to both of the first two symbol templates ($f(\square)$ and $f^*(\square)$) appears to be that of a vector with an associated directionality, which is the same as was connected to the bras and kets in Section 3.4.1.2. Thus, we call the symbolic forms formed from these symbol template-conceptual schema pairs “*function as vector*” and “*conjugate function as vector*.” The latter symbol templates

($\int f^*(\square)f(\square)\,d\square$ and $\int f^*(\square)g(\square)\,d\square$) appear to also both share a conceptual schema that appeared in Section 3.4.1.2 when discussing the Dirac brackets: that of a dot product's projection along a direction. In this context, the "direction" is that of one of the two functions' vector interpretation. We name these symbolic forms "*inner product integral of two identical/different functions as projection*." These symbolic forms and their symbol templates are again shown in Table 3.7.

Table 3.7: Symbolic forms identified for describing functions and integrals in vector-like terms.

Symbolic form	**Symbol template**
Function as vector	$f(\square)$
Conjugate function as vector	$f^*(\square)$
Inner product integral of two identical functions as projection	$\int f^*(\square)f(\square)\,d\square$
Inner product integral of two different functions as projection	$\int f^*(\square)g(\square)\,d\square$

3.4.2.3 Integrals as Probabilities

Finally, students also viewed complex squares of inner product integral expressions as representative of probabilities. Because the distinctions between expressions for probability, probability density, and probability amplitude for continuous variables are quite similar, a brief refresher of the distinctions is warranted, similar to what was done earlier for discrete quantities. Recall from Section 3.2.2 that there are a number of normatively correct expressions for probabilities in wave function notation. Similar to those in Dirac notation discussed in Sections 3.2.1 and 3.4.1.3, complex squares of inner products between quantum states (often expressed as generic wave functions $\psi(x)$ in wave function notation) and eigenfunctions of operators represent probabilities for measuring the eigenfunctions' associated eigenvalue. In wave function notation, these inner products are represented by integrals: for example, $|\int \varphi_n^*(x)\psi(x)dx|^2$ represents the probability of measuring the n^{th} eigenvalue of the operator associated with the $\varphi_n(x)$ eigenfunctions. The integral without the complex square is thus the probability amplitude, typically written as c_n. Similar as well to Dirac the inner products discussed in Section 3.4.1.3, inner products between the same function have a different normatively correct interpretation than those with a wave function and an eigenfunction. As was the case with Dirac brackets between two identical vectors, an integral over the entire space of two identical wave functions represents both a normalization condition and the sum of all probabilities within the basis in which the functions are written (e.g., $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1$). If the bounds of integration are not over the entire space, however, these integrals with identical wave functions have a different interpretation: that of the probability for the system to be measured with any value of the (continuous) observable that the wave function is a function of within the bounds of integration. Essentially, the product of two identical wave functions (e.g., $\psi^*(x)\psi(x)$ or $|\psi(x)|^2$) is the probability density for a system. These different integrals and their interpretations can look quite similar, and it is thus reasonable that they could prove challenging to students in the development of firm symbolic forms.

In her card-sorting task, Aaliyah grouped two pairs of expressions together, with each pair containing a Dirac expression and a wave function expression. She discussed the distinction between one pair ($|\langle\psi|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$) and the other ($|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$) as follows:

> So this [$|\langle\psi|\psi\rangle|^2$] and this [$\int \psi^*(x)\psi(x)dx$] will represent the same thing, [$|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$] will represent the same thing, which is

probability- […] [$|\langle\psi|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$] will be just [makes finger quotations] "one" if ψ is normalized […] and then [$|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$] will be like some number, less than one, unless it's an eigenstate itself.

Here, Aaliyah was drawing parallels between the Dirac and wave function notation expressions she viewed as representative of probabilities. Interestingly, the squares that are present in the Dirac expressions are missing from their associated integrals. It is not entirely clear why she paired these together despite their visual (and mathematical/physical) dissimilarity, though it is perhaps evidence of some confusion as to whether and/or how a complex square translates between Dirac and wave function notation. She also drew a distinction again between the expressions that contained an eigenstate (and thus have a probability less than one), and expressions that contained two identical states (which have a probability of one).

Bilbo sorted $|\langle E_n|\psi\rangle|^2$, $|\int \psi^*(x)\psi(x)dx|^2$, and $|\int \varphi_n^*(x)\psi(x)dx|^2$ all together, and explained the grouping as "okay, here we got our inner products— our probabilities, excuse me, because they're […] magnitude squared of inner products." The square on the outside of the integrals appeared to be crucial to him for delineating which expressions represented probabilities.

When Enoch was asked the question in prompt 3a about finding the probability for measuring a particle (provided as in a superposition state in Dirac notation) within the left half of an infinite square well, his immediate response was "I remember it being an integral of sorts […] it was something like the probability equals integral of ψ of x complex conjugate $\psi\ x\ dx$ [writes $\mathcal{P} = \int \psi^*(x)\psi(x)dx$]." Later in his interview, while working with the expression given in prompt 3 ($|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$), Enoch explained the difference between the probabilities calculated by $|\langle E_1|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$:

> The particle could be in any of the three energy states [gestures at $\frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$] by a probability given by whatever you calculate this to be [gestures at $|\langle E_1|\psi\rangle|^2$]. Whereas this [gestures at $\int \psi^*(x)\psi(x)dx$] is saying […] what is the probability of the particle to be at a certain position, regardless of which energy state you're looking at.

Enoch later successfully translated the expression given in prompt 3 into generic wave function notation:

$$\psi(x) = \frac{1}{2\sqrt{2}}\left[\sqrt{3}\varphi_1(x) + \varphi_2(x) + 2\varphi_3(x)\right].$$

He was then asked how he would find the probability for measuring the particle to be in the left half of the well, and said that he would "take this whole conflagration [draws brackets around the expression he translated, and squares it all] […] just do that from 0 to L/2." It seems that Enoch had developed a fairly consistent symbol template of an integral of a product of two functions that he drew from when expressing probabilities in wave function notation.

Again, a few symbol templates appear to capture the work of these students when generating or selecting function-based expressions for probability. $\int f^*(\square)f(\square)\, d\square$ shows up here as it did in Section 3.2.2.2, though this time it is called upon to represent either a probability for a measurement or a quantity that must be complex-squared to get a probability (i.e., a probability amplitude). Unlike in Section 3.2.2.2, however, it appears that some of the students treated integrals differently depending on whether they contained a product of the same function (albeit with one being the function's complex conjugate) or a product of two different functions. Because this appears to be a meaningful distinction within this context, we will propose another symbol template here, $\int f^*(\square)g(\square)d\square$, with the same (or at least fundamentally similar) interpretations as a (square root of) probability as was applied to the $\int f^*(\square)f(\square)\, d\square$ symbol template. Relatedly, $|\int f^*(\square)f(\square)\, d\square|^2$ appears to represent probabilities of a measurement as well, and thus shares a conceptual schema. We name these symbolic forms the "*inner product integral of two identical/different functions as probability amplitude*," "*inner product integral of two identical/different functions as probability*," and "*complex square of inner product integral of two identical/different functions as probability*" symbolic forms, as shown in Table 3.8. Similar to what was noted in Section 3.4.1.3, we are naming some of these forms "probability amplitudes" to better reflect convention, though students often appeared to reason about them as a thing that must be squared. Given the discussion of normative interpretations of inner product integrals at the beginning of this section, it is worth pointing out which of these symbolic forms are generally normatively correct. These would be the "*inner product integral of two identical functions as probability*," "*inner product integral of two different functions as probability amplitude*," and "*complex square of inner product integral of two different functions as probability*" symbolic forms.

Table 3.8: Symbolic forms identified for inner product integrals in the context of describing probability concepts.

Symbolic form	**Symbol template**
Inner product integral of two identical functions as probability amplitude	$\int f^*(\square)f(\square)\, d\square$
Inner product integral of two identical functions as probability	$\int f^*(\square)f(\square)\, d\square$
Inner product integral of two different functions as probability amplitude	$\int f^*(\square)g(\square)\, d\square$

Table 3.8 Continued.

Inner product integral of two different functions as probability	$\int f^*(\square)g(\square)\, d\square$
Complex square of inner product integral of two identical functions as probability	$\left\lvert\int f^*(\square)f(\square)\, d\square\right\rvert^2$
Complex square of inner product integral of two different functions as probability	$\left\lvert\int f^*(\square)g(\square)\, d\square\right\rvert^2$

3.4.3 Castor and Delilah's Focus on Coefficients as an Intermediate Step Between Inner Products and Probability Expressions

While Aaliyah, Bilbo, Enoch, and Frodo all shared very similar interpretations of the ways that inner products expressed in both notations related to probability concepts, Castor and Delilah appeared to connect the expressions and concepts differently. They instead seemed to prefer to make an additional symbolic step between inner products and the probabilities they represent: that of first converting the inner product to a coefficient associated with a term in the initial state's representation as a superposition of orthonormal eigenstates. For example, after they had finished with prompt 3a, where they were given $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$ and asked to find the probability for the left half of the well, they were asked to calculate the probability for measuring the lowest energy value for the given particle. The following exchange then occurred:

Castor: I mean, it's written in the energy basis.

Delilah: Yeah. I mean, I'd go back to the energy basis

Castor: And just square the coefficient.

[…]

[Castor writes $\left|\frac{\sqrt{3}}{2\sqrt{2}}\right|^2$]

Interviewer: Okay. Why is that the probability of the first energy state?

Castor: Because it's the coefficient for the first energy state.

In this exchange, Castor has squared the coefficient in front of the lowest energy state's ket and claimed that as the probability of a measurement of E_1. Earlier in their interview, they defined their own initial state for prompt 1 as $|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots$, and, when explaining how they would find the probability for the lowest energy, Delilah said, "if psi is written in terms of the energy states, in Dirac notation like this [gestures at their expression for $|\psi(x)\rangle_E$], then $\mathcal{P}$ is […] just [writes $\mathcal{P}_{E_1} = {c_1}^2$]," defining the probability as an eigenstate's associated component (squared) when expressed as a superposition state. Later, they expanded that expression for the probability and added in complex square bars to make it $\mathcal{P}_{E_1} = |c_1|^2 = |\langle E_1|\psi\rangle|^2$; Castor justified their addition of the complex square thusly: "because this [$\langle E_1|\psi(x)\rangle$] gives you the coefficient. […] Because when you do, like, the inner product of that they end up like, the dot product of something times itself [referring to the $\langle E_1|$ multiplication being distributed through the superposition state, specifically the $|E_1\rangle$ component to form $\langle E_1|E_1\rangle$] it all works out to give you the coefficient." Again, they were very focused on the coefficient as the expression that represents the probability amplitude, and treat the inner product as a means of obtaining the coefficient, rather than the probability itself. However, they then got confused as to whether they needed to square the integrand of a wave function version of an inner product or square the result of the integral (i.e., whether the complex square happens inside or outside the integral). In the end, they made sense of this problem through the following conversation:

> Castor: Isn't it inside the integral?
>
> Delilah: Well, I'm not sure. I have to think…
>
> Castor: I'm pretty sure- I don't think we've ever seen-
>
> Delilah: -No, but I think, I think this equation is [writes $c_1 = \int_1^2 [\text{squiggle}]$]
>
> Castor: Oh, yeah.
>
> […]
>
> Castor: Because that's where you get your coefficient and then you square it. I was like, thinking I'm like, we've never really seen that on the outside.
>
> Delilah: Yeah, no, we've never done that. But I think that's just because we write it as this [gestures to $c_1 = \int_1^2 [\text{squiggle}]$]
>
> Castor: Because typically we write it as the coefficient and square it to be able to get the probability

Here Castor and Delilah are calling back to a recognizable expression in the $c_1 = \int_1^2 [\text{squiggle}]$. Their statements such as "I don't think we've ever seen ____," "we write it as ____," and Castor's "Oh, yeah" upon seeing Delilah's expression template fits very well with the symbolic forms framework.

While Castor and Delilah do occasionally make direct connections between inner products in both notations and probabilities (displaying use of the symbolic forms discussed in Sections 3.4.1.3 and 3.4.2.3), they appear to have two symbol templates that show up with much more importance in their thinking than did in the other students' interviews: c_n and $|c_n|^2$. Given the role these appear to play in their reasoning and interpretation of these expressions, the symbolic forms framework suggests there are different conceptual schemata tied to each of these symbol templates. The first is that of a coefficient in a linear combination of terms, as seen in their superposition state $|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots$, with the interpretation of the relative importance or size of its associated component in the sum. The inner products' symbolic forms relating to projection along axes thus interfaces with this conceptual schema with the interpretation of the inner product "picking out" the coefficient. This manifests in their equality $c_1 = \langle E_1|\psi\rangle$. This conceptual schema pairs with the c_n symbol template in what we call the "*coefficient as component*" symbolic form. The conceptual schema connected to the $|c_n|^2$ symbol template is that of the squared coefficient being a representation for the probability for the coefficient's associated component. This pair forms the symbolic form "*squared coefficient as probability*." These two symbolic forms are ultimately manifested in their equality $\mathcal{P}_{E_1} = |c_1|^2 = |\langle E_1|\psi\rangle|^2$. Given this interpretation, this equality can be read out as "the probability of measuring E_1 is the complex square of the coefficient for the $|E_1\rangle$ term in the expansion of $|\psi\rangle$, which can be obtained by taking the inner product of $\langle E_1|\psi\rangle$," and understood as relating $|\langle E_1|\psi\rangle|^2$ to $\mathcal{P}_{E_1}$ by means of picking out the E_1 component of $|\psi\rangle$. These two symbolic forms and their symbol templates are shown in Table 3.9.

Table 3.9: Symbolic forms identified for coefficients and complex squares of coefficients as describing probability concepts.

Symbolic form	**Symbol template**
Coefficient as component	c_n
Squared coefficient as probability	$\|c_n\|^2$

3.5 Conclusions and Future Work

One conclusion that can be drawn from these excerpts is evident merely by comparing the lengths of Sections 3.4.1 and 3.4.2: these students used and referred to wave function expressions less often than they did those expressed in Dirac notation. This is perhaps explainable by the curricular focus of the course, as in a spins-first course students may be expected to be more comfortable with Dirac notation than with wave function notation. Viewed through a symbolic forms lens, it could be that the increased time spent working with and thinking about expressions in Dirac notation increased the strength of the connections between symbol templates and conceptual schemata for expressions in that notation. This is potentially significant as, due to the timing of these interviews during or following the course, students had likely used more wave functions than Dirac expressions in the weeks before the interview. This relative recency nonetheless does not appear to override the comfort working with Dirac expressions that has been engendered by working within this notation since day one of the course.

It is also apparent from Section 3.4 and Table 3.2 that students in this course developed numerous symbolic forms to aid them in interpreting, generating, and translating expressions in one or both notations explored in this study. Of note is that the vast majority of the symbolic forms explored in this work are normative; most non-normative symbolic forms are explainable by a lack of a complex square. This is potentially a problem when it leads to students failing to take complex squares when calculating probabilities in quantum mechanical contexts. This confusion did appear to manifest more often within wave function contexts (with students not being sure where to put the square, or if a square is necessary at all), though often students would mix up the terms "probability," "probability amplitude," and "probability density" in both contexts—they would simply write the correct expressions more often in Dirac notation, regardless of what they called them. This confusion regarding the different probability concepts has been noted in prior work as well (Marshman & Singh, 2017), and challenges in correctly translating between Dirac and wave function inner product expressions were previously documented by Wan et al. (2019). While one would like to believe that this data suggests that students mostly develop normative symbolic forms in these courses, it is possible that the tasks given within these interviews were not conducive to capturing any other non-normative symbolic forms. It is also possible that there is a selection bias to the participants, as the interviews were entirely opt-in, and thus only the students with well-developed normative symbolic forms (and thus likely well-performing students in the class) were willing to volunteer to answer questions about quantum mechanics. Regardless, these symbolic forms provide a useful means for understanding the interpretations students hold for common expressions for probabilities in quantum mechanics, both of the expressions as a whole and their constituent parts.

Indeed, the symmetry between the symbol templates for the Dirac and wave function expressions—due largely to very similar conceptual schemata cropping up within both notations—is perhaps evidence of the symbolic forms framework's usefulness in explaining how students translate between expressions that they deem "equivalent." Within the symbolic forms framework, shared conceptual schemata—such as that between the "bracket as dot product" and "integral as dot product"—may be the means by which students coordinate the expressions they choose as a direct translation from one to the other. Misapplying these shared schemata is a possible explanation for the findings by Wan et al. (2019).

This work is limited by a number of factors, which provide obvious avenues for future research. First, this work is entirely concerned with expressions for probability. There are many other types of expressions that are commonly used in upper-division quantum mechanics, such as eigenequations and expectation values, that would benefit from further investigations into the symbolic forms students learn to apply when reasoning about these expressions. Second, the subject pool for this study is limited to students from a single institution using a spins-first curriculum. It is very likely that gathering data from other spins-first institutions and/or from wave functions-first institutions would expand the pool of symbolic forms students develop within their curriculum at their institutions and may even show distinctions due to the instructional approach. Third, the COVID-19 pandemic impacted every course studied within this work, and very likely affected individual students' learning (and thus likely affected the number and/or type of symbolic forms that they were able to develop within the course). Fourth, COVID-19 also noticeably lowered interview participation rates, and thus this study is only representative of the symbolic forms developed by six students. It is likely that with a larger pool of participants, there would have been a larger pool of symbolic forms observed as well.

CHAPTER 4

NETWORK ANALYSIS OF STUDENTS' CONCEPTUAL UNDERSTANDING OF MATHEMATICAL EXPRESSIONS FOR PROBABILITY IN UPPER-DIVISION QUANTUM MECHANICS

4.1 Introduction

Physicists use mathematics for far more than simply computation: mathematical expressions and relationships are utilized to help them understand and reason about the world (Uhden, Karam, Pietrocola, & Pospiech, 2012). Due to its often non-intuitive nature, this is certainly the case in quantum mechanics (QM), where one needs to rely on mathematical reasoning to understand and make predictions for systems on the quantum scale. The level of abstraction and mathematical sophistication used in upper-division QM coursework has been shown to present many challenges to students, including when interpreting Dirac formalism (Singh, 2001), reasoning about possible wave functions both symbolically and graphically (Singh & Marshman, 2015), distinguishing between Euclidean and Hilbert spaces (Singh, 2008; Singh & Marshman, 2015), and studying time dependence and time evolution (Emigh et al., 2015; Passante, Schermerhorn, Pollock, & Sadaghiani, 2020; Singh, 2008; Singh & Marshman, 2015). Student understanding of representations of eigenequations has been studied by education researchers both in mathematics (Karakok, 2019; Thomas & Stewart, 2011; Wawro, Watson, & Zandieh, 2019) and physics (Wawro et al., 2017), as has the number of different notations that are frequently used and the varied mathematics that each notation requires (Gire & Price, 2015; Schermerhorn et al., 2019; Wawro et al., 2020). These notations typically include Dirac, vector-matrix, and wave function notations—all of which require varied mathematical operations and understanding for fruitful application to QM systems.

One challenge when studying student understanding at the upper division in general is the smaller sample size when compared to introductory physics courses. Due to attrition and a transition away from a general education audience, the number of students taking upper-division courses is naturally far smaller (Stewart, Hansen, & Burkholder, 2022); this typically manifests in research studies as a focus on more qualitative methodologies (clinical interviews being a classic example). While these methodologies are excellent opportunities for providing a deep view into individual students' conceptual understanding, they often also lead to a loss in the generalizability of claims that can be made. This loss in generalizability applies both within a given course (unless every enrolled student is studied) as well as across equivalent courses at different institutions. One way to improve the generalizability of a study's findings is to expand the data pool to include students at multiple institutions. This invites its own set of challenges, however: the amount of time and effort required to analyze the results of qualitative data collected from a large pool of participants and the difficulty of procuring IRB approval for a study at several institutions being two examples.

A technique that has seen increased use within the physics education research (PER) community in recent years is that of network analysis. Network analysis encompasses any technique that focuses on connections between different actors. Historically, these techniques were developed to study transportation and information networks (Newman, 2010), but have since been used within the PER community to study social communities and interactions among students and instructors (Hopkins, Ozimek, & Sweet, 2017; Smith, Hayes, & Lyons, 2017; Thomas, 2000) and to assess conceptual inventories developed for physics courses (Brewe et al., 2016; Wells et al., 2019, 2020, 2021; Wheatley et al., 2021; Yang et al., 2020). In general, these techniques are useful whenever connections between actors are of interest.

While prior work examining student understanding of the various notations used in quantum mechanics has been conducted, this has commonly been done only at individual institutions. Also, the ability to work and reason across multiple representations is important for success in QM. To better understand the ways in which students reason about expressions in multiple representations and to glean a more generalizable understanding of the same, we sought to answer the following research questions:

1. How successfully can survey design be used in conjunction with network analysis techniques to efficiently collect and analyze data on students' conceptual connections between different QM expressions for many students at multiple institutions?
2. What can network analysis show about students' conceptual connections in spins-first courses at multiple institutions?

We will begin by laying out the work that has already been conducted in this space and providing a brief background on the relevant terms and concepts within network analysis that we will be using later. Then we will discuss the design of our survey and the ways our networks were generated, before discussing the results of our analyses.

4.2 Background

We begin our discussion of relevant prior research by discussing work on student understanding of different mathematical representations that are used within quantum mechanics courses. We then will provide a brief overview of prior work within PER that has made use of network analysis techniques, before laying out an overview of the specific network analysis concepts we will be using by means of a toy model network.

4.2.1 Prior Work with QM Representations

While much education research has been conducted at the boundary of physics and mathematics, there has more recently been a focus on the mathematics found in upper-division QM courses. This is especially so for the three mathematical notations commonly used to describe identical or analogous physical phenomena or concepts in upper-division QM: Dirac, wave function, and vector-matrix notations. Gire and Price examined all three notations from an expert perspective, noting the affordances and limitations of each for the purposes of computation (Gire & Price, 2015). For example, they found that students elected to use Dirac notation as a medium for coordinating expressions in other notations, and attributed that preference to qualities of the notation, such as its compactness and symbolic support for computation. This framework was then modified by Schermerhorn et al. (2019) to capture student preferences when calculating expectation values; they found that both the compactness of a notation and its relative familiarity were the primary drivers for student preference. Wawro et al. (2020) similarly studied student judgements of vector-matrix and Dirac notations' suitability for particular applications, and showed that a comprehensive understanding of both how similar expressions in these various notations interrelate and how to translate between them is crucial for a deep, cohesive understanding of QM (Wawro et al., 2020). Additionally, incorrect translation between wave function and Dirac expressions has been shown to lead to difficulties when developing models for calculating probabilities (Wan et al., 2019). Because of the importance of this skill, instructional materials to assist students in working among and reasoning with multiple representations concurrently have been developed, including simulation-based tutorials utilizing multiple types of graphs, integrals, and algebraic expressions (Kohnle & Passante, 2017). We believe more work is necessary to understand how students reason about the ways in which expressions in these different notations interrelate, as this will allow better characterization of student thinking and ultimately allow for tailored pedagogy and instructional materials to improve this essential skill.

4.2.2 Prior Work with Network Analysis in PER

Network analysis techniques have been developed and used to study topics as diverse as physical real-world infrastructure, neural networks, and social behaviors among groups. These techniques have recently also seen extensive use in both physics education research and education research more generally. Community detection and cluster analysis techniques alone have been used to study response groupings for various conceptual inventories (Brewe et al., 2016; Wells et al., 2019, 2020, 2021; Wheatley et al., 2021; Yang et al., 2020). These techniques have also seen recent use in interpreting results of Likert-style surveys (Dalka et al., 2022). These techniques have also been used extensively to study social communities and their various impacts, both among communities of educators (Hopkins et al., 2017; Smith et al., 2017) and students (Brewe, Kramer, & Sawtelle, 2012; Hopkins et al., 2017; Thomas, 2000), and to characterize how these social communities are affected by different active-learning pedagogies (Commeford et al., 2021). Recent work has even looked at how these social communities have been affected by remote physics courses (Sundstrom et al., 2022).

4.3 Network Analysis Primer

Because network analysis is so broad and contains a multitude of terms with which the reader may be unfamiliar, a brief introduction to the specific terms and analysis methods that will later be applied to our data is merited. A toy model network is shown in Figure 4.1, with which different methods of our analysis will be demonstrated. For the purposes of our discussion, we will be using the terms as defined in Table 4.1.

Table 4.1: Definitions and descriptions of relevant terms that will be used to discuss networks within this paper. For clarity, several of these terms are highlighted in Figure 4.1 for the toy model network.

Term	Definition
Node	Individual objects that may or may not be connected to each other.
Edge	Represents connections between nodes.
Edge weight	Corresponds to the strength of the connection between two nodes. If an edge has a large weight, that means that the connection between the two nodes on either end of that edge is particularly strong.
Community	Clusters of nodes that have been determined by some means to be more closely connected to each other than to the other nodes in the network.
Geodesic	The shortest possible path between two nodes in a network.
Edge Betweenness	Determined by finding the geodesics between every pair of nodes in a network and counting the number that pass through the edge in question. For a weighted network like our toy network, this number is then divided by the edge's weight. Effectively, an edge has low betweenness if there are many alternative geodesics that can sidestep it, such as within a tightly-knit community.

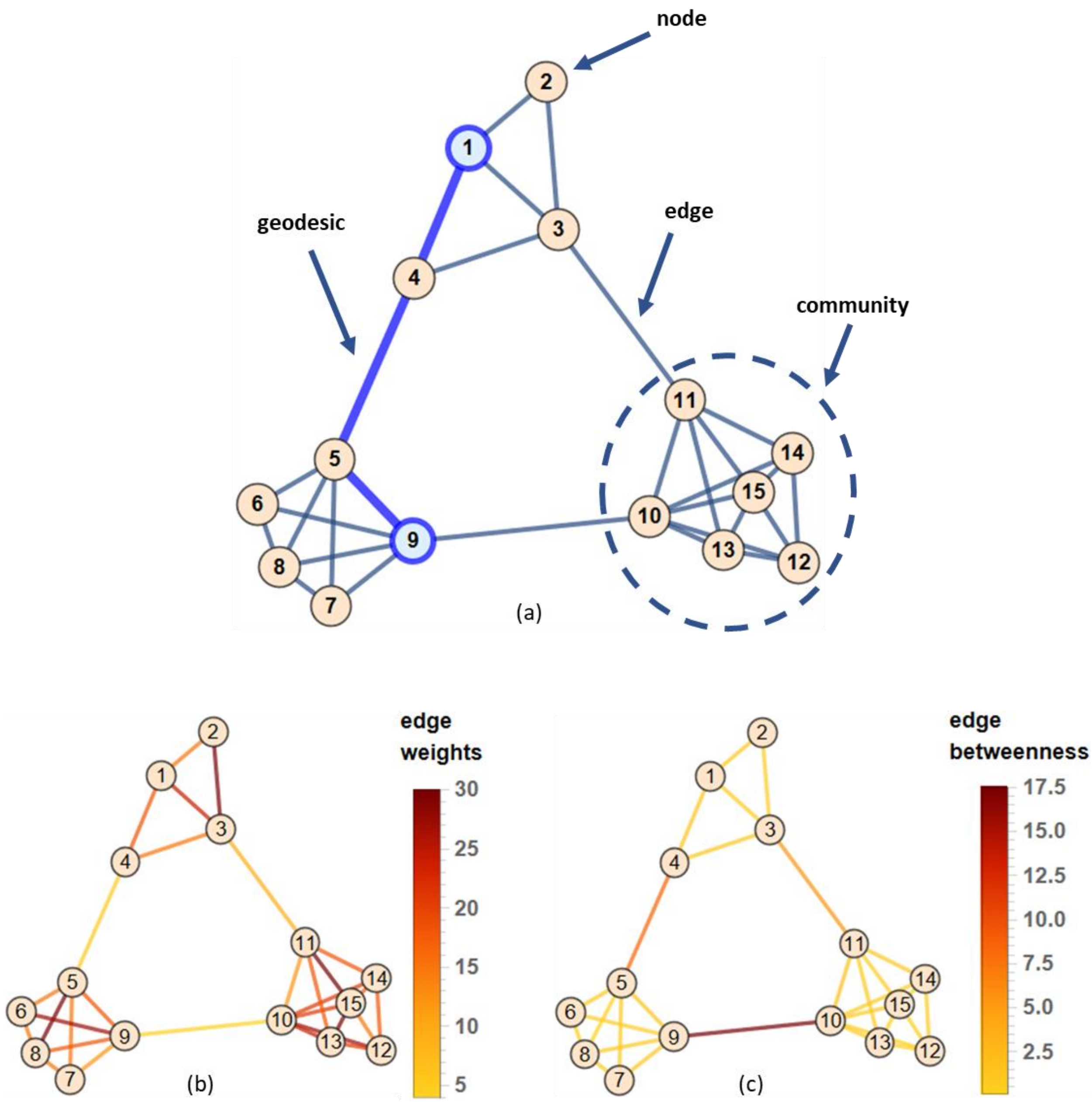


Figure 4.1: A toy model network highlighting terms discussed in Table 4.1. (a) Network highlighting the geodesic between nodes 1 and 9. (b) The same network, with the edges shaded by their edge weights; the edges lying within the three communities were made to have more weight than those that cross between communities. (c) The network with the edges shaded by their edge betweennesses; the edges lying between communities tend to have high betweennesses, while those within communities tend to have lower betweennesses.

4.3.1 Community Detection Methods

There are many ways to detect community structure within networks, including by maximizing a network's modularity (Newman, 2006a, 2006b), agglomerative hierarchical clustering algorithms (Springuel, Wittmann, & Thompson, 2007), and using edge betweennesses to continuously subdivide a network (Newman & Girvan, 2004). These various methods each have their strengths and weaknesses, which often manifest as trade-offs between computational speed, granularity of results, and level of confidence that can be ascribed to the specific communities formed. Modularity maximization is relatively quick to compute but has limitations to its resolution of smaller communities (Fortunato & Barthélemy, 2007a). The latter two algorithms discussed above (hierarchical clustering and edge betweenness-based methods) both provide a much higher resolution of sub-communities by generating a hierarchical community structure, which can be visualized succinctly with a dendrogram (see Figure 4.2 for the dendrogram for our toy model from Figure 4.1). While hierarchical clustering algorithms typically start with every node disconnected and slowly cluster them together via some similarity (distance) measure, betweenness-based methods start with the larger network as a whole and separate it into smaller and smaller subdivisions. In terms of creating a dendrogram, one could view hierarchical clustering as building the dendrogram from the bottom up, and betweenness-based methods doing so from the top down. Both are computationally intensive for networks of even moderate size (for sparse graphs with n edges, completion times are, at best, $O(n^2)$ for hierarchical clustering and $O(n^3)$ for betweenness-based methods, respectively). While hierarchical clustering techniques are afforded some flexibility from their reliance on similarity measures between nodes—from which there are many options to choose—this also means that different metrics can provide different clusters without an obvious way to know if one is more correct than the others (Fortunato, 2010). Additional drawbacks of hierarchical clustering are that even central members of communities can be left out of the communities they “should” belong within, and that it often leaves some peripheral members out as well (Newman, 2004b). Given the drawbacks of modularity maximization and hierarchical clustering, and the fact that the networks we will study are not overly large (at only n=20 nodes), we decided to use edge betweenness to determine how our expressions were clustered.[1]

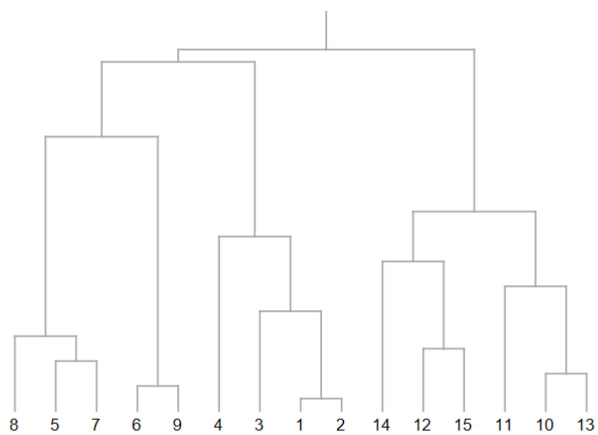


Figure 4.2: Dendrogram showing the community structure of the toy model network found using the edge betweenness method. The three expected communities (1-4, 5-9, and 10-15) are clearly visible, with the potential for the 5-9 community to be made from two sub-communities (5,7,8 and 6,9).

The way that our chosen algorithm works is by sequentially removing the edge within our network that has the highest betweenness. As can be seen from Figure 4.1(c), edges that have the largest betweennesses are most likely to connect communities (Newman & Girvan, 2004). Once the edge with the highest betweenness is found, it is deleted, and the edge

[1] For the curious reader, Fortunato (2010) provides an excellent review of various community detection methods and their respective affordances and limitations.

betweennesses of the resultant network are recalculated; this process is repeated until every edge in the network is removed and all vertices are fully disconnected. This process will tend to separate a network into progressively smaller communities, saving the most tight-knit for last; the result can be visualized with a dendrogram (see Figure 4.2).

In these dendrograms, the network begins with a single community, represented by a vertical line. As the algorithm runs its course, this single community splits into multiple branches containing fewer and fewer vertices. This is visualized in the diagram by moving down the diagram until every edge is removed from the network and every vertex is separated from every other at the bottom of the dendrogram. While originally conceived as a method for community detection in unweighted networks, Newman extended this procedure to include networks with weighted edges, as is the case both for our toy model and for our survey data (Newman, 2004a).

4.3.2 Determining Community Robustness

The relative height of a given vertical segment on these edge betweenness-based dendrograms is indicative of the number of edges that were removed between community divisions. Divisions with very little vertical space between them therefore occurred fairly close together during the community detection process. Before beginning analysis of these communities and drawing conclusions based on the order in which they are formed, some questions are worth asking: How robust is the community structure, in that small perturbations to the network would not produce a meaningfully different community structure? How confident can we be about these communities? Where should we “stop” along the vertical axis, to determine which community division or number of communities is “best”? There are numerous possible methods to bring to bear to answer these questions, including determining which division has the largest modularity (Newman & Girvan, 2004) and various bootstrapping procedures (Fortunato, 2010). Due to the level of granularity and transparency afforded by the latter, we elected to utilize a modified bootstrapping procedure to determine which community configuration represents the overall data most effectively and is robust and stable to perturbations.

In particular, we elected to use a technique based on statistical bootstrapping discussed by Fortunato (2010) and Efron & Tibshirani (1993) and subsequently modified by Speirs (2020). The basic idea is to resample from the pool of student responses to generate multiple hypothetical datasets. For a data set of *N* student respondents, hypothetical datasets are created, each comprised of *N* responses drawn at random from the actual student responses. When creating a hypothetical dataset, some respondents' data may be selected multiple times, and others' not at all—otherwise the process would simply reproduce the original dataset. A network is then created from this hypothetical dataset and the community detection algorithm run. This process is then repeated many times, with the resulting dendrograms then being compared to see where significant deviations occur and what structure is common across all or most hypothetical datasets.

4.4 Methods

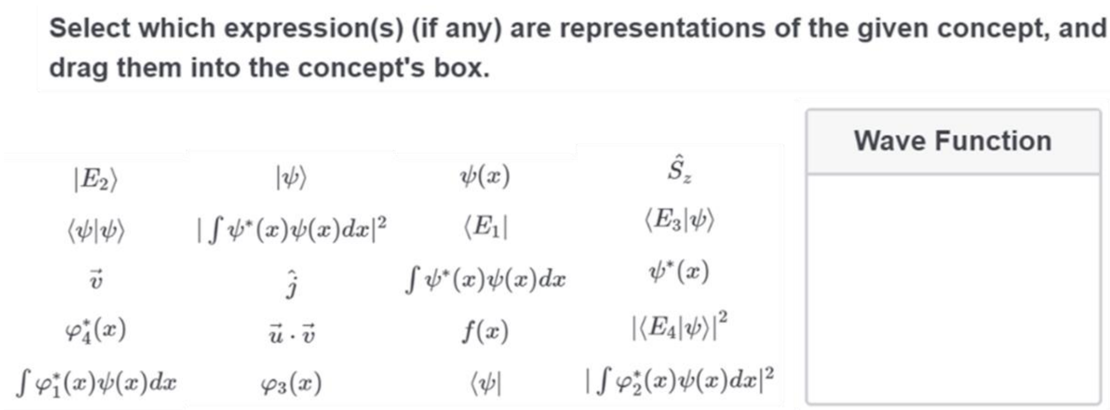
The discussion of our research methods begins with a discussion regarding our survey design and implementation, before then describing the creation of our networks from survey data.

4.4.1 Survey Design

In an effort to be able to make more generalizable claims about students' understanding of the representations used in upper-division QM, we developed and administered an online survey with two primary goals in mind: to collect and analyze responses from many students across multiple institutions in a way that would scale efficiently, and to create a dataset that allows for analysis of students' conceptual understanding of mathematical expressions commonly used in QM—particularly those used to express probability concepts. Because both wave function and Dirac notations are used extensively in upper-division QM courses, and because there are equivalent expressions that look quite different between the two notations (e.g., $\langle E_n|\psi\rangle$ and $\int \varphi_n^*(x)\psi(x)dx$) as well as similar-looking expressions that represent concepts with very subtle distinctions (e.g., $|E_n\rangle$ & $\langle E_n|$), we decided to focus on the conceptual interpretations that various commonly-used expressions shared.

The first goal was tackled by reducing the number of free-response text entry questions as much as possible, both to reduce participant attrition and to help our analysis scale well to a large participant pool. This meant that the second goal—gleaning students' conceptual connections between expressions—would need to be accomplished without explicit student reasoning. To this end, the survey tasks were designed as sorting tasks, where the students were presented with a list of expressions commonly used in upper-division QM courses (Figure 4.3(a)) and a single quantum mechanical concept. Students were tasked with selecting all of the expressions in the list that they felt represented that concept. In all, the survey consisted of 11 different concepts (Figure 4.3(b)). This survey was given to students in upper-division QM courses at six different institutions (N=139); all courses were taught using a spins-first textbook

(McIntyre (2012) or Townsend (2000)). These courses typically begin by using the results of the Stern-Gerlach experiment to motivate the treatment of quantum states as vectors, often represented by Dirac notation bras and kets. After some time studying systems with discrete measurement outcomes in Dirac notation, including time evolution of these systems, these courses eventually transition to studying continuous systems, connecting the Dirac state vectors to their associated wave functions. The survey was distributed near the end of the course, after students had worked extensively with both Dirac and wave function notations. Courses using the spins-first approach were chosen for this study due to the increasing prevalence of this curricular style as compared to more traditional wave functions-first courses, which typically use the text by Griffiths (1995).



(a)

Concepts		
Vector	Wave function	Eigenvector
Quantum state	Unit Vector	Probability amplitude
Inner product	Basis Vector	Probability
Dot product	Eigenstate	

(b)

Figure 4.3: (a) example of a prompt in the online survey administered to students. (b) table showing the different concepts they were tasked with selecting expressions for.

As a result of the survey design, student responses were entirely relational, forming expression-concept pairs and/or expression-expression pairs (for those expressions that were selected simultaneously for the same concept, i.e., dragged into the same box). Given this nature of the survey responses and our interest in how and whether students view these expressions as conceptually connected, we decided to make use of network analysis techniques to analyze our survey responses. In particular, we implemented the edge betweenness community detection method discussed in Section 4.3.

4.4.2 Creating Our Network

To turn our survey results into a network that can be analyzed to gather information about how these expressions are related conceptually, we first collapsed individual students' responses into their own networks. In these networks, the nodes were the 20 different expressions provided on the survey; a connection was placed between two nodes if the student declared those two expressions as representative of the same concept, i.e., placed those two expressions in the same concept box. Also, even if a student selected both expressions for multiple concepts, each pair of expressions was only counted once per student. This "unweighting" of individual student response networks was done to avoid overweighting any connections that may be due to similar concepts given on the survey. For example, if a student selected $\langle E_1|$ and $|E_2\rangle$ simultaneously for the concepts "vector," "basis vector," and "eigenvector," those two expressions would be connected by an edge of weight one for that student. While we wanted the granularity of these different types of vectors for other analyses, we didn't want to overweight the strength of those connections simply because there were more distinct vector concepts on the survey than there were variations of other concepts (e.g., those related to probability or quantum states) while constructing these students' individual networks. These links then served to connect expressions that students believed both can represent the same concept. For N respondents, this process resulted in N unweighted networks of 20 nodes each. These N networks were then superimposed to generate the full weighted network (see Figure 4.4), with a maximum possible edge weight of N if all respondents selected the two expressions connected by that edge simultaneously at least once on the survey. This network is rather complicated, with myriad low-weight edges that make it difficult to interpret. One way to cut through this visual clutter and help meaningful information rise to the surface is to find the clusters or communities within the larger network as a whole. To this end, we implemented the edge betweenness community detection method discussed in Section 4.3.

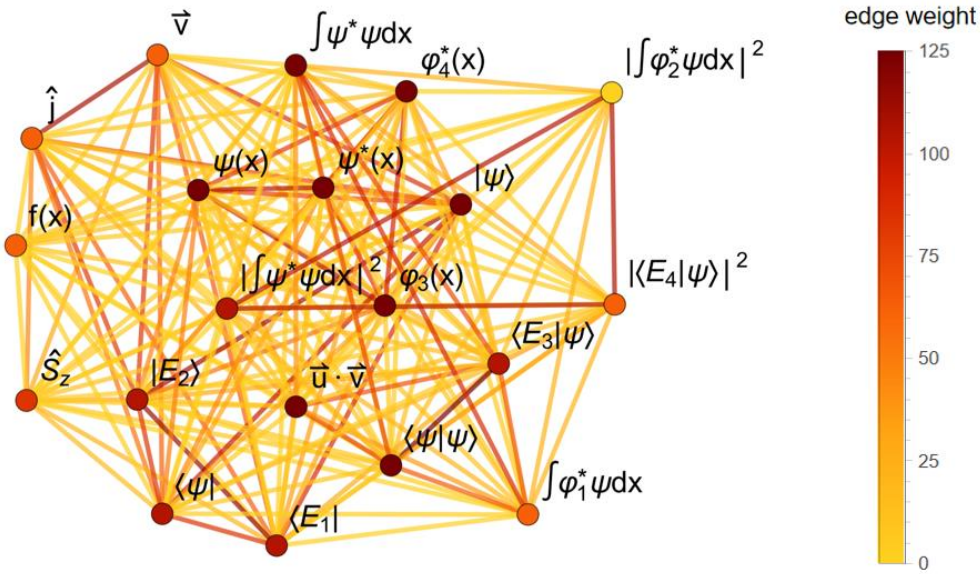


Figure 4.4: The expression-concept network generated from 139 student survey respondents. The nodes represent the different expressions on the survey. Edges between nodes show that students selected those two expressions simultaneously for at least one concept. The edge weights represent the number of students that used the two expressions simultaneously, and is shown by the shading applied to the edges. The shading of the nodes is not a part of our analysis here, but is representative of the nodes' degree (the number and weight of the edges connected to each node).

4.5 Results and Discussion

We begin the discussion of our results by first running a community detection algorithm. We then discuss the reliability of the detected communities as well as the communities' implications for student understanding of mathematical representations within this context.

4.5.1 Running the Betweenness Algorithm and Determining its Limits

The dendrogram generated by the edge betweenness method for our network is shown in Figure 4.5. Cutting horizontally across this dendrogram at any point is representative of a snapshot of the betweenness algorithm—it represents the set of communities that exist at any given point during the procedure.

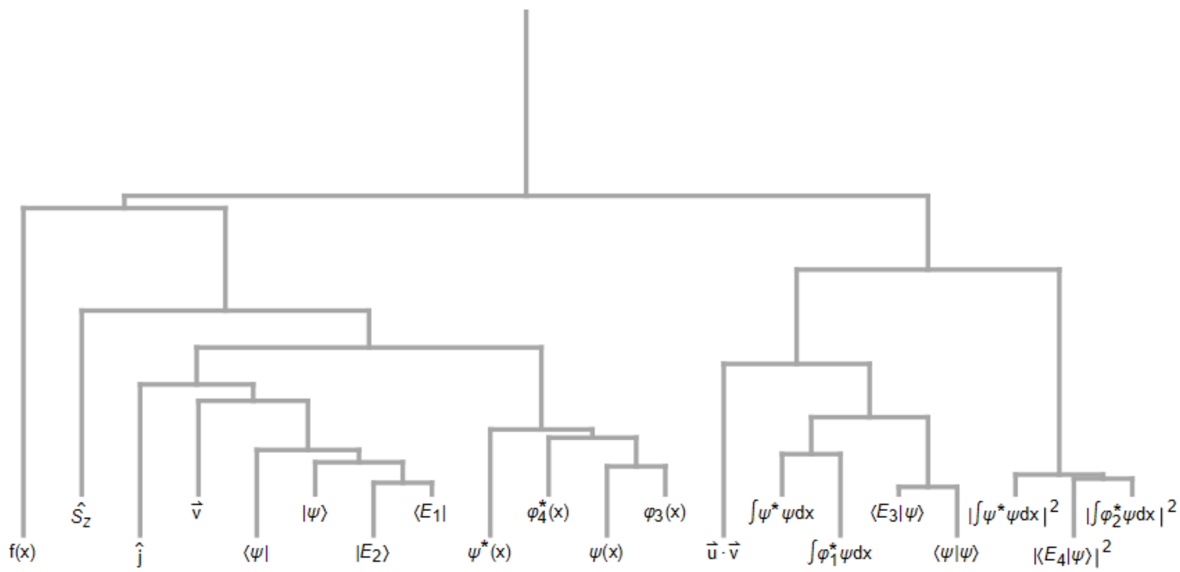


Figure 4.5: Dendrogram showing the community structure of the network in Figure 4.4 after implementing the edge betweenness community detection algorithm.

As discussed in Section 4.3.2, community divisions that are separated by relatively small vertical segments on the dendrogram occurred relatively close together during the betweenness algorithm. Given the importance of the order in which these divisions occur for our analyses, a determination of the community divisions that can be considered robust is of great importance. To this end, we implemented the statistical bootstrapping-based method discussed in Section 4.3.2 to determine the robustness of these community divisions. In particular, we ran the edge betweenness community detection algorithm on 1000 bootstrapped networks. Figure 4.6 shows the number of dendrograms that have identical community structure at a given number of communities (level of the dendrogram). Each bar on this plot represents a specific number of communities—from one community (with every node included) to twenty (with every node separated into their own communities at the end of the betweenness procedure). Each differently-shaded section of a bar represents how many of the 1000 hypothetical (bootstrapped) networks have the exact same set(s) of communities, i.e., these communities have the same sets of nodes. Note that each end of the plot has only one bar, which makes sense: there should only be one possible structure with one community—with all nodes connected—and one with twenty (i.e., all) communities—each node as its own community. One interesting aspect of these types of figures can be seen by observing the second and third bars on this plot. The second bar shows that the 1000 bootstrapped networks showcase three different initial divisions of the networks into two communities: 525 networks result in one common pair of communities, 343 in another distinct pair, and 132 divide a third way. Upon dividing once more and reaching a three-community division of the bootstrapped networks, however, more than 750 of the bootstrapped networks share identical communities. This consolidation effect is due largely to variability in the order of the early community divisions, as the first two divisions visible in Figure 4.5 frequently swap their order among the bootstrapped networks. The number of communities for which there are very few different community structures, evidenced by a small number of stacked bars, are thus indicative of high agreement among bootstrapped networks at that level of their respective dendrograms, and vice versa. With this in mind, Figure 4.6 shows where there is high and low agreement among the bootstrapped dendrograms, and thus we can look for the level of the community detection algorithm for which the community structure is most stable under perturbations.

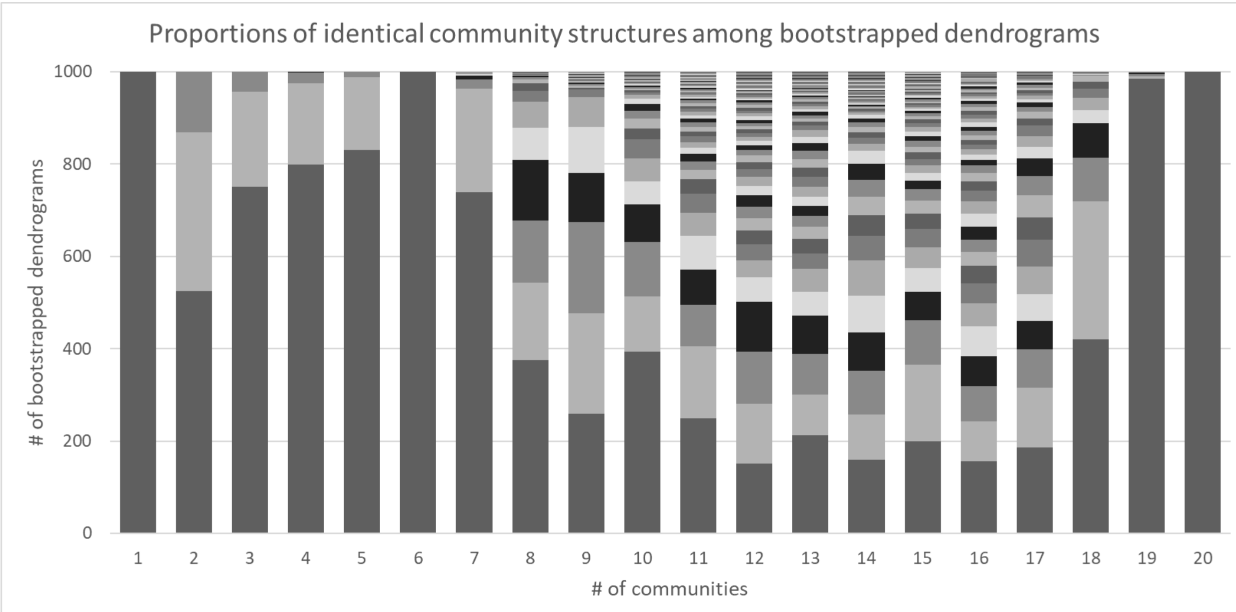


Figure 4.6: Stacked bar chart showing the relative proportions of different community structures at each level across the bootstrapped dendrograms. Each individual bar represents a specific community structure that is shared among some portion of the bootstrapped networks at a given level of the edge betweenness algorithm.

As can be seen in Figure 4.6, some variation occurs within the 2-5 community range, but 100% of the bootstrapped networks have an identical community structure once they are broken into six communities. For more than seven communities, the stability swiftly devolves, before ultimately agreeing strongly again once nearly every vertex is separated for all of the bootstrapped networks (as should be expected—there is only one way for 20 nodes to be separated into 20 communities). One finding to take from this is that while there is minor variability in the relative order of the first four divisions of our network (the four highest splits on the dendrogram in Figure 4.5), we can be very confident that the fifth division of the network (into the six communities seen in Figure 4.7) happens before any of the ones below it on the dendrogram, suggesting that these six communities are highly stable.

4.5.2 Interpreting Community Structure

Separating our initial network into the first six communities determined by the edge betweenness algorithm gives us the network seen in Figure 4.7. Two of the communities consist of $\hat{S}_z$ and $f(x)$ individually (labeled SZ and FX in Figure 4.7, respectively), which shows that students did not think of either of these expressions as especially conceptually similar to the other 18 expressions. The Dirac bras, kets, and generic vector expressions $\vec{v}$ and $\hat{\jmath}$ were found to form their own community (DV), as were the wave function expressions (WF). The expressions for inner products, including a generic dot product $\vec{u} \cdot \vec{v}$, were also separated into a community (IP), as were the expressions for the complex squares of inner products (IS).

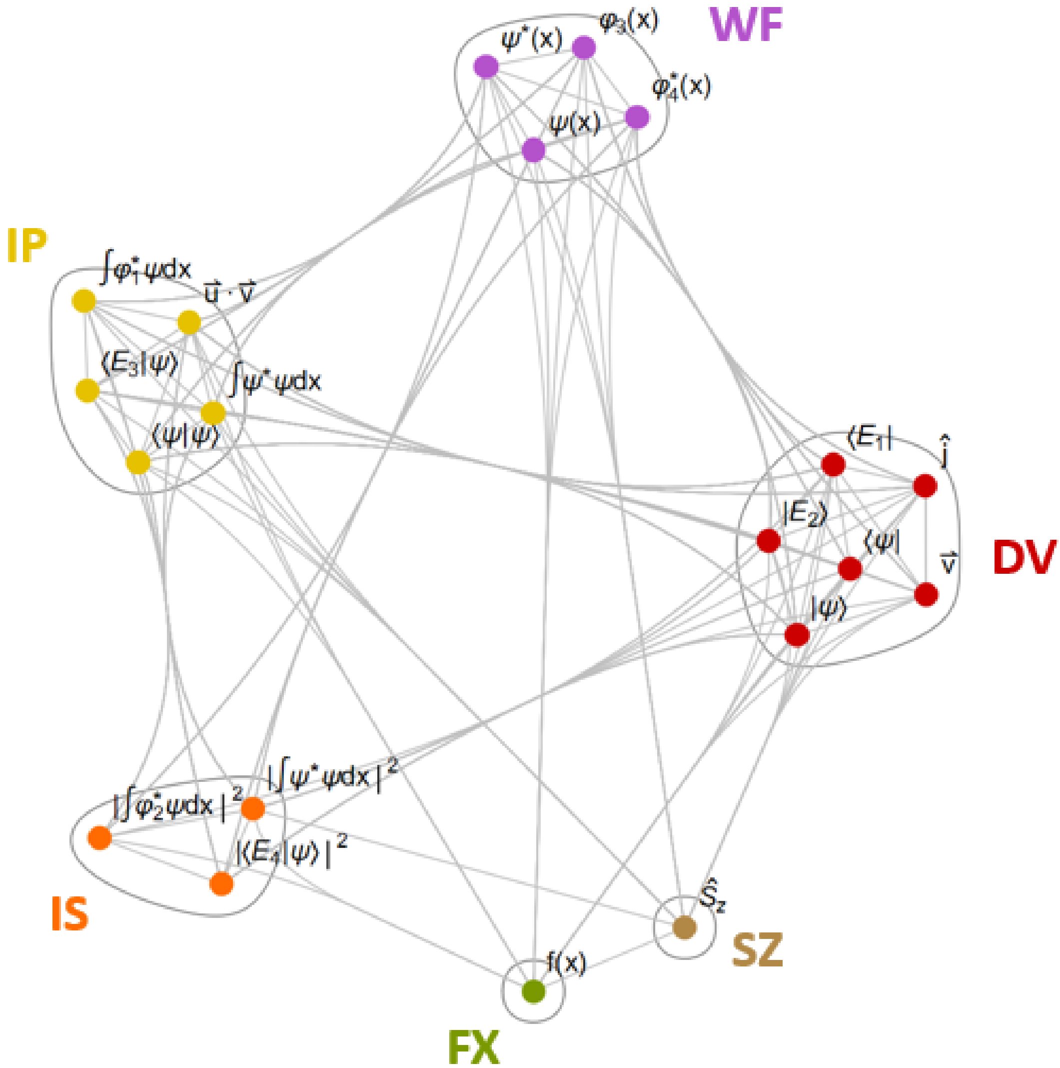


Figure 4.7: The network built from student survey responses as in Figure 4.4, grouped into the six stable communities as determined by the results of the bootstrapping procedure shown in Figure 4.6. FX contains only the generic function $f(x)$; SZ contains only the spin-Z operator $\hat{S}_z$; DV contains Dirac bras, kets, and generic vector expressions $\vec{v}$ and $\hat{j}$; WF contains the wave function and its conjugate ($\psi(x)$ and $\psi^*(x)$), as well as the wave function and conjugate wave function expressions for eigenstates ($\varphi_3(x)$ and $\varphi_4^*(x)$); IP contains inner product expressions, including a generic dot product in both Dirac and wave function notation; IS contains complex squares of inner products in both notations.

IP and IS's separation suggests a conceptual distinction between inner products and the squares of inner products (in QM, this is often a distinction between probability amplitudes and probabilities). Similarly, the separation of communities WF and DV suggests a meaningful distinction between wave functions and Dirac bras and kets and generic vector expressions.

Looking closer at the earlier divisions of the network seen in the dendrogram in Figure 4.5, there are two larger-grain-size communities: that of IP+IS, and that of WF+DV (and potentially SZ as well). A simplified version of this dendrogram is shown in Figure 4.8, where the divisions beyond the resolution limit determined by the bootstrapping procedure are eliminated. The combination of this larger-scale (IP+IS and WF+DV) community structure and our survey design leads us to the conclusion that the expressions in WF and DV are viewed as conceptually more similar to each other than to the remaining expressions; the same is true for the expressions in IP and IS. While the edges connecting these subcommunities together represent conceptual connections for our students, the connections *within* each subcommunity are stronger than those connecting the subcommunities to each other.

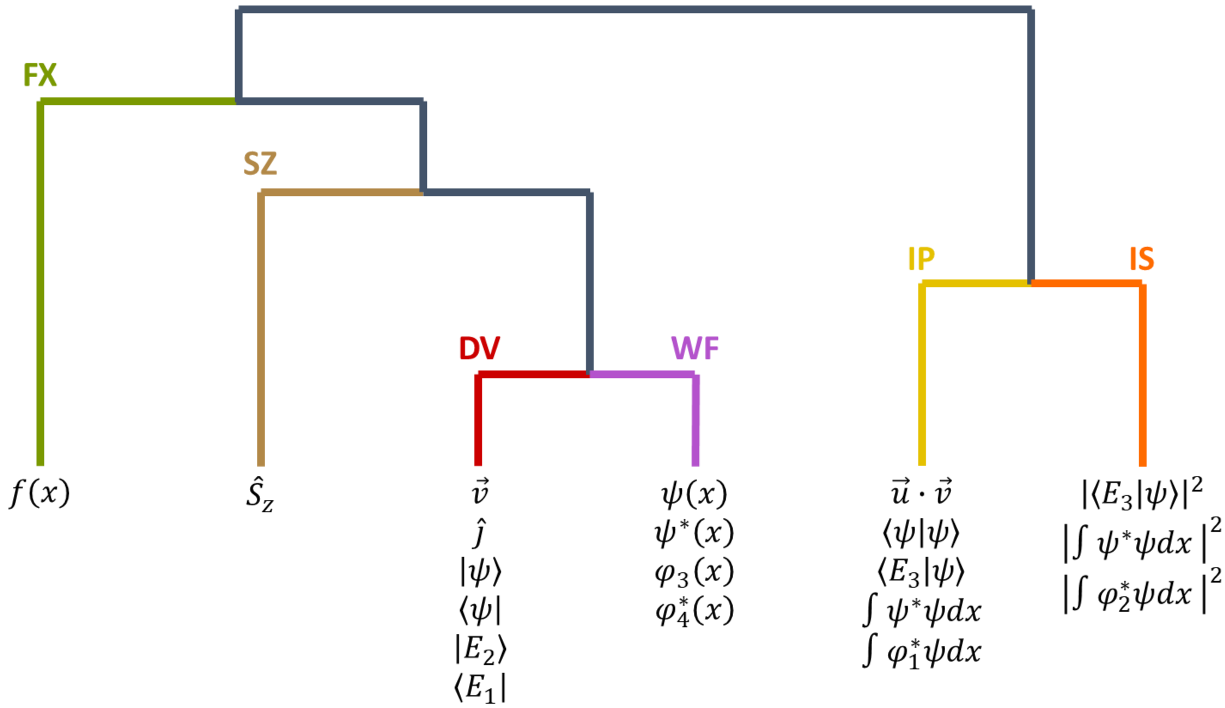


Figure 4.8: Simplified dendrogram showing the stable communities from the bootstrapping procedure. Color coded and labeled to match Figure 4.7.

For the larger WF+DV community, we can also look at the conceptual make-up of these connections to obtain a deeper understanding of which specific concepts connect these expressions. To this end, we can separate the expressions within this larger group into three different groups, by notational style: one for Dirac bras and kets ($|\psi\rangle$, $\langle\psi|$, $|E_2\rangle$, and $\langle E_1|$), one for the generic individual vector expressions ($\vec{v}$ and $\hat{j}$), and one for the wave function expressions ($\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$). We can then look at the concepts that connect these different types of expressions together according to the students. Like we did with the expressions themselves, we can also largely break down the concepts used by students to connect these types of expressions into three camps: vectors ("vector," "eigenvector," "unit vector," and "basis vector"), quantum states ("quantum state" and "eigenstate"), and wave functions (just "wave function"). We can then break down the proportions of each type of concept that connected the various types of expressions, both within a given expression type or between different expression types. We will focus on a subset of these conceptual connections between expression types, shown in Figure 4.9.

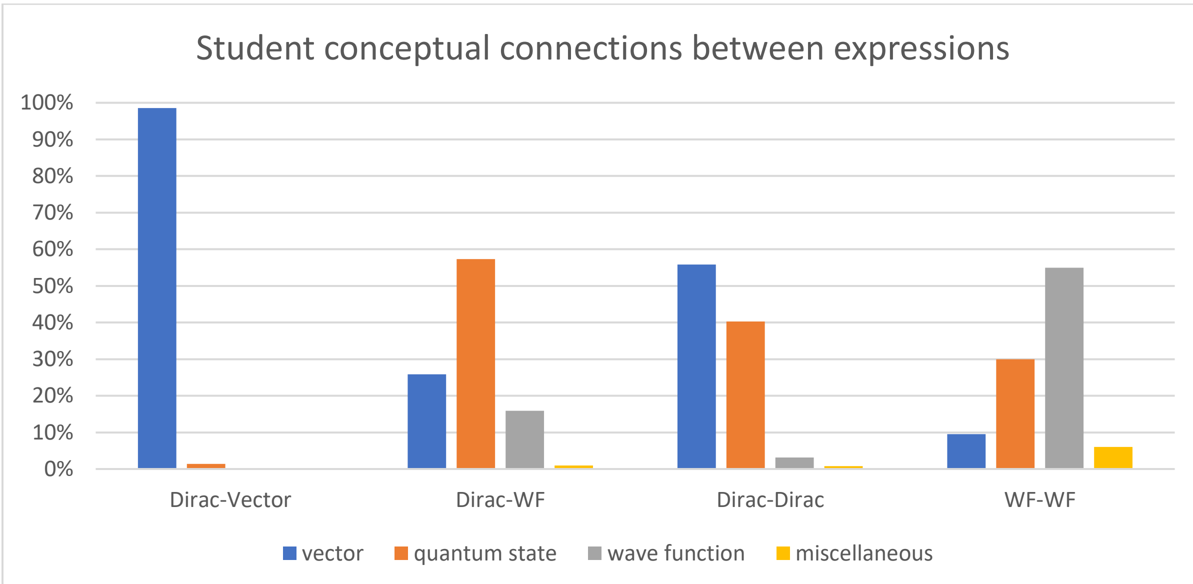


Figure 4.9: Histogram comparing the types of concepts used to link Dirac bras and kets with generic vector expressions and wave function expressions, as well as the types of concepts that connect Dirac bra and ket expressions to each other and wave function expressions to each other.

By examining these conceptual breakdowns, we can see that the concepts for which students used both a Dirac bra or ket ($|\psi\rangle$, $\langle\psi|$, $|E_2\rangle$, or $\langle E_1|$) and a generic vector expression ($\vec{v}$ or $\hat{j}$) (i.e., concepts represented by the connections between those two subsets of nodes) are almost entirely vector concepts. This is noticeably distinct from the distribution of concepts that consistently linked the Dirac bras and kets to the wave function expressions on the survey ($\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$), more than half of which were those related to quantum states. Taken in combination with the stable and distinct WF and DV communities as seen in Figure 4.7 and Figure 4.8 (i.e., that these vector-Dirac connections were stronger than the Dirac-wave function connections), this suggests that the vector-like identity of the Dirac bras and kets was stronger for these students than either their quantum state- or wave function-like conceptualizations. Also, the edges connecting the Dirac bras and kets to each other are split between vector and quantum state concepts, while the connections between wave function expressions are split between wave function and quantum state concepts. This would seem to suggest that while Dirac bras and kets share a strong identity as representing vector-like concepts and wave function expressions share a strong identity as representing wave function concepts, they both appear to represent quantum state concepts. This can be seen more explicitly by looking at a network composed of the expressions students used simultaneously as representing the quantum state concept, shown in Figure 4.10. This figure shows the strong connectivity between bras, kets, functions, and conjugate functions, evidence that these are all treated by these students as representing quantum states, despite the more siloed interpretation of bras and kets representing vectors (and not so much wave functions) and of wave functions representing wave functions (and not so much vectors).

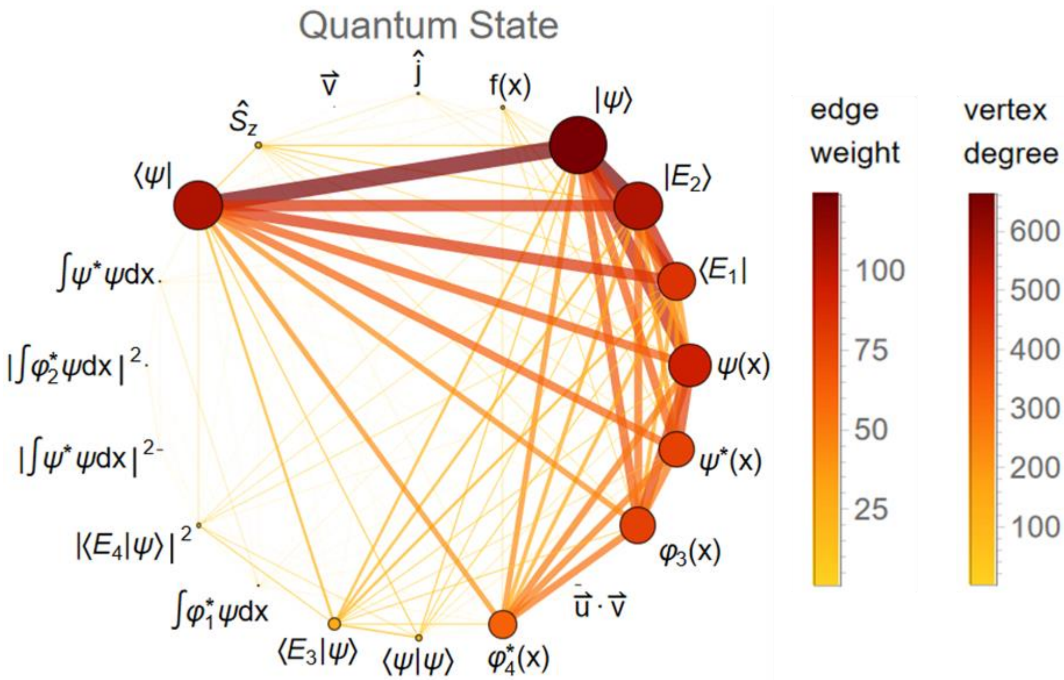


Figure 4.10: Network showing spins-first students' connections between different expressions when prompted to select expressions representative of "quantum state." Edges are sized proportional to and colored according to their weight (i.e., the number of students that selected the pair of expressions for this concept). Nodes are similarly sized proportional to and colored according to their vertex degree (i.e., the sum of the weights of all edges that connect to them).

These findings are to be expected from the curricular focus of the courses in which these students were enrolled. Spins-first quantum mechanics courses begin by introducing Dirac notation in two-dimensional spin-1/2 bases, and lean heavily on familiar vector interpretations to help students understand the mathematical operations at play. Probability amplitude inner products in the beginning of the course are very much treated as geometric dot product projections, with state vectors having components along the different eigenstates' "directions." This analysis suggests that these curricular goals have been successful in getting these students to think about Dirac bras and kets as simultaneously representative of both vectors and quantum states. That $\vec{u}\cdot\vec{v}$ is included within the IP community likewise suggests that these students see this connection between inner products (both as Dirac brackets and wave function integrals) as sharing conceptual backing with dot products—another common instructional goal within these courses.

4.6 Conclusions and Future Work

Prior studies have shown that each notation used in QM has certain aspects that make them more suited for certain tasks (Gire & Price, 2015; Schermerhorn et al., 2019), and that the ability to effectively and efficiently translate and work between notations is a crucial skill for students to develop (Wawro et al., 2020). In our study, network analysis techniques were used to investigate the strengths of students' conceptual connections between common expressions in upper-division quantum mechanics. We found that Dirac bras and kets and their associated wave functions both have distinct shared conceptual identities as vectors and wave functions, respectively, but they also both represent quantum state concepts to the participants in our study. It is possible that this shared quantum state identity aids students by serving as a touchstone when translating from one notation to the other, or that this could serve as a starting point for curriculum or pedagogy aimed at improving this skill.

We also found from our community detection analysis that Dirac bras and kets were considered more conceptually similar to generic vector expressions than they were to their equivalent wave function representations. An implication of this result is that Dirac bras and kets bear a strong association with vector-like concepts—even more than with conceptualizations they share with wave functions, such as both being representations for quantum states. The strength of this association with vector-like concepts is perhaps to be expected due to the curricular structure of these courses, as all of these spins-first courses begin by first drawing attention to the discrete vector-like nature of bras and kets before eventually connecting them to continuous wave function interpretations. While a lack of resolution prevents us from looking closer at the connections between the expressions within the inner product community (IP in Figure 4.7), that the generic dot product remains a part of that community suggests a seemingly dot product-like understanding of even the wave function inner product integrals. This is an encouraging finding, as the conceptual connections between discrete and continuous inner products are important to develop and can often prove elusive.

Overall, we have shown that with novel use of online survey design and network analysis techniques, an investigation of student understanding of mathematical expressions in quantum mechanics and their interrelated conceptual interpretations is feasible for a large number of students at multiple institutions. We believe that the scalability of this methodology can allow for greater generalizability of findings. Also, while much work has been done with network analysis and community detection algorithms within the PER community, there has been relatively little work done in examining the relative stability and robustness of the communities that have been studied. As illustrated by Figure 4.6, taking small grain-size community structure at face value can be fraught with potential errors. As the resolution of any community detection algorithm is limited, the use of bootstrapping or similar techniques to help determine the resolution of a given community structure may be necessary for making community-based claims about network structure.

Our next step is to extend this work to include students enrolled in the more traditional wave functions-first courses. We suspect that while many of the broader findings from community detection for respondents in wave functions-first courses may be similar to those seen here, there may be quite different findings when it comes to the vector-like interpretations of the more quantum mechanical expressions due to the relative lack of focus on linear algebraic interpretations in such courses. Similarly, a comparison of the expressions chosen to represent individual concepts may be of interest, particularly vector-like concepts due to the lessened focus on linear algebra-based reasoning in the beginning of wave functions-first courses.

CHAPTER 5

COMPARATIVE ANALYSIS OF SPINS-FIRST AND WAVE FUNCTIONS-FIRST STUDENTS' UNDERSTANDING OF EXPRESSIONS IN QUANTUM MECHANICS

5.1 Introduction

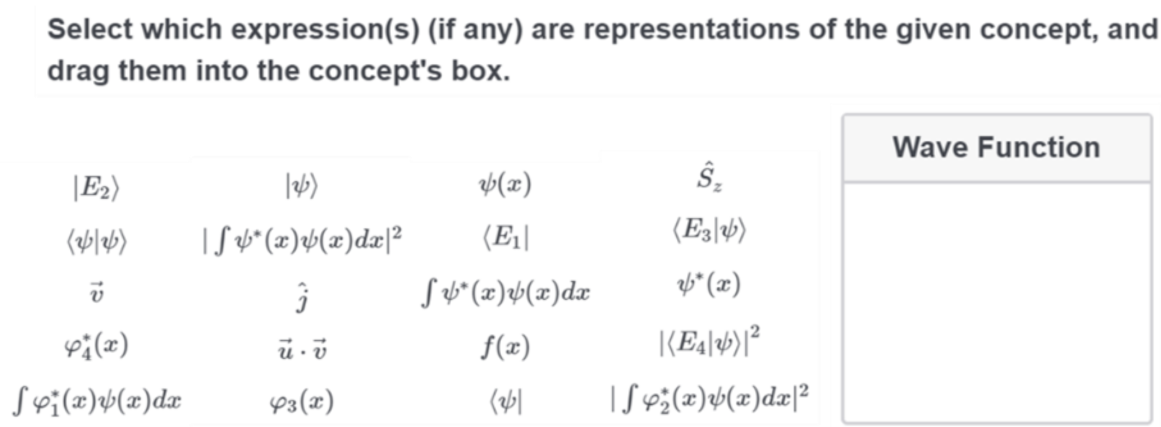
As was shown in Chapter 4, network analysis of survey data can be used to gain an understanding of when and, to some extent, how strongly students understand mathematical expressions to be conceptually related, so long as the survey is designed to elicit such connections. It was shown that students in spins-first courses tend to view Dirac bras and kets as broadly similar to both generic expressions for vectors and expressions for quantum mechanical wave functions. Similarly, these students tended to view Dirac brackets, quantum mechanical inner product integrals, and an expression for a generic vector dot product as conceptually similar. However, the students connected Dirac bras and kets more closely with the expressions for generic vectors than with the expressions for wave functions. Moreover, the concepts that they used to connect the Dirac expressions to each other were more often vector-like concepts than those associated with quantum states. This vector-heavy conceptual understanding could be at least partially explained by the curricular focuses of a spins-first quantum mechanics course. This raises two questions:

1. Given the different instructional emphasis of a wave functions-first quantum mechanics course, what might one expect from a similar analysis of students enrolled in wave functions-first courses?
2. How, if at all, do any differences align with the different emphases of the courses?

This chapter aims to answer these questions.

5.2 Research Design and Methodology

To answer these research questions, the same survey discussed in Section 4.4.1 (reproduced in Figure 5.1 for convenience) was distributed to students enrolled in wave-function first upper-division quantum mechanics courses at four institutions around the US (N=55). The survey was taken near the end of the course, after students had used and become comfortable with both Dirac and wave function notation. Given the somewhat idiosyncratic notational choices of the various texts and instructors used in these courses, the expressions seen in the survey were tweaked to most closely reflect the equivalent expressions used in the text for each course. For example, an energy eigenfunction was expressed as either $\psi_3(x)$ or $\varphi_3(x)$, depending on the textbook used ($\psi_3(x)$ for Griffiths and Townsend, and $\varphi_3(x)$ for McIntyre, (Griffiths, 1995; McIntyre, 2012; Townsend, 2000)).



(a)

Concepts		
Vector	Wave function	Eigenvector
Quantum state	Unit Vector	Probability amplitude
Inner product	Basis Vector	Probability
Dot product	Eigenstate	

(b)

Figure 5.1: Online survey administered to students (McIntyre notation version). (a) Example of a prompt with all expressions and one concept. (b) List of concepts for which participants were asked to select expressions.

The same methodology described in Section 4.4.2 was used for translating these students' survey responses into networks. That is, individual student networks were formed with 20 nodes, each representing one of the expressions used in the survey. A link was created between two expressions if the student ever dragged those two expressions into the same concept box, thus demarcating a shared conceptual understanding for the student. This resulted in N individual unweighted networks, which were summed together to form a single weighted network (e.g., if 45 students had an edge connecting $\psi(x)$ and $|\psi\rangle$, the cumulative network would have a single edge connecting those two nodes, with an edge weight of 45). Both the larger cumulative networks and the individual student networks were used in analysis.

5.3 Data Analysis Methods

Multiple analyses were conducted on the survey data, both from spins-first courses and wave function-first courses. The first analysis of these is that of detecting community structure within the networks generated from the survey results from each type of institution. We used edge betweenness-based methods to do so, as discussed in Section 4.3. In short, an edge's betweenness is representative of its likelihood to lie between two different communities. As such, the edge-betweenness community detection algorithm sequentially removes whichever edge has the highest betweenness within the network, and thus slowly pares out the "connective tissue" between communities within the network. This process terminates once every edge is removed, and thus once all nodes are disconnected into their own individual communities. As the algorithm starts with the network fully intact and ends with the network fully separated, meaningful community separations occur during the procedure and can be determined by observing the relative order of community separation (wherein those communities that separate later on are more closely connected). The question of which number of communities is "best" for a given network (i.e., in our context, when during the edge-removal procedure is it no longer meaningful to keep separating communities) can be answered several ways, as was discussed in Section 4.5.1. A modified bootstrapping procedure has been shown to be effective at answering this exact question (Speirs, 2020), and it is this method that we chose for our analysis. This bootstrapping procedure is described in detail in Section 4.5.1, but in short is a method of generating a large number of slightly different variations on an original network by resampling from the pool of individual student networks. The edge-betweenness community detection algorithm can then be run on each of these slightly different networks, and where there is large agreement between their community structures, claims can be made about those community divisions being robust, while the point at which they start largely disagreeing is the point at which the communities should be divided no further.

Another analytical lens that was applied to these survey results was to compare the two populations' conceptual interpretations of the various expressions present on the survey. To do this, statistical analysis was conducted on the number of students from each curriculum that selected a given expression for each concept. By determining which expression-concept pairs showed statistically significant differences between the curricula, differences in the conceptual interpretations of these expressions were found. These comparisons, as well as simply the fraction of students that paired various concepts and expressions within a given curriculum, offered insight into the conceptual understanding of students in each curriculum as well as the cases where their interpretations differ. These conceptual interpretations can then be discussed in the context of the curricular structure and pedagogical focus of each curriculum.

A principal component analysis (PCA) was also used to see what the primary causes of variance among student survey responses were, using a technique invented for conducting PCAs on network-based data (Wolf et al., 2012). In this way, individual student responses can be compared and primary differentiators between them can be determined *a posteriori*. This process requires the selection of a "distance" metric between student networks—essentially, a way to assign a number that tells how different two networks are. Numerous distance metrics have been invented for various purposes (Tantardini et al., 2019). Wolf et al. (2012) elected to use a very simple distance metric, which was effectively just a count of how many edges would need to be added/removed from one network to make it identical to another. One benefit of this technique is that it is quite efficient to calculate, as many distance metrics can become very inefficient for comparing networks with a large number of nodes. Because the individual student networks generated from their survey responses is limited to 20, we chose a more nuanced distance, known as DeltaCon (Koutra et al., 2016). This distance metric was chosen because it takes into account structural differences between two networks (Koutra et al., 2016), and thus is useful for our purposes, as the community structure (i.e., which expressions are commonly conceptually linked together) is relevant to our analysis. Essentially, the distances between every pair of students can be treated as an $N \times N$ matrix D, where element d_{ij} is the distance between student i and student j. A singular value decomposition can then be applied to this distance matrix, which effectively applies a change of basis to the matrix, where the most variation within the data is captured in the first principal component (PC1), the second-most variation within the second principal component (PC2), and so on. The data can then be projected onto only the first few principal components, to allow for a visualization of only the most influential components. Once this procedure has been conducted, the traits that these abstract primary principal components correspond to within the networks can be determined *a posteriori*.

5.4 Results and Discussion

The discussion of our results is broken down by the analysis methods discussed above. First, the community structure determined by network analysis of students' responses are compared between the students enrolled in courses using the two instructional approaches.

Then, the conceptual interpretations of the expressions selected by students are compared to see where the two instructional approaches differ and agree. Finally, a principal component analysis of each individual student's survey responses is used to determine which factors explain the greatest variance between student responses.

5.4.1 Community Detection Comparison

As was discussed in Section 4.5.1, upon running the edge betweenness algorithm and the bootstrapping procedure, the network generated from the spins-first student responses was best divided into six communities, as shown in Figure 5.2a. The requirement for this division was for at least 70% of the 1000 bootstrapped networks to agree on a specific community division. Upon performing the same bootstrapping procedure on the cumulative network created from the wave functions-first students' survey responses (and requiring the same >70% agreement among bootstrapped networks), the community divisions shown in the dendrogram in Figure 5.2(b) were found to be stable.

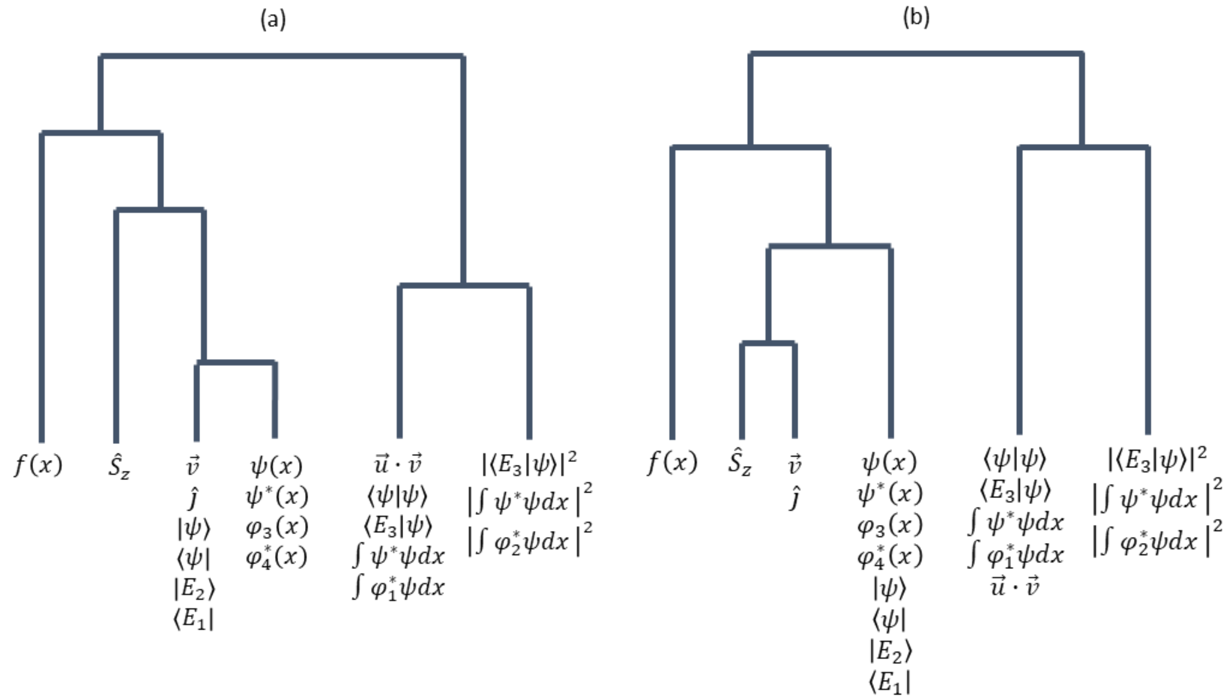


Figure 5.2: Dendrograms showing the relative order of community divisions, determined by the bootstrapping process, of (a) the spins-first network and (b) the wave functions-first network. Community divisions only appear if there is >70% agreement among bootstrapped networks.

Upon inspection, the community structures are largely very similar. There is one major difference between the two dendrograms shown in Figures 5.2(a) and 5.2(b), however. In the dendrogram for the spins-first students (Figure 5.2(a)), the wave function terms ($\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$) split off from the community containing both the Dirac bras and kets ($|\psi\rangle$, $\langle\psi|$, $|E_2\rangle$, and $\langle E_1|$) and the generic individual vector expressions ($\vec{v}$ and $\hat{j}$). The opposite occurs in the dendrogram for the wave functions-first dendrogram (Figure 5.2(b)), where the generic vector expressions split off from the community containing the Dirac bras and kets and the wave function terms. This difference in community structure suggests that the Dirac bras and kets are thought of slightly differently by the students in these different types of courses. In the spins-first courses, while the connections between the Dirac bras and kets and their associated wave functions are certainly present—as these expressions take quite a while for their communities to separate—the bras and kets are more closely connected to expressions the students think of almost exclusively in geometric vector terms. This is counter to the structure present in the wave functions-first students' network, where the conceptual connections between the Dirac bras and kets and their associated wave functions are stronger than the connections between either of these types of expressions and the generic individual vector expressions. This suggests that these different curricula have indeed affected the way that students conceptualize the meanings of both Dirac bras and kets as well as wave functions, or at the very least that they have affected the ways that these two types of expressions are viewed as conceptually connected.

5.4.2 Comparing Expressions' Conceptual Interpretations

In analyzing the conceptual understanding of each expression individually, the survey responses for each concept can be broken down to see which expressions were chosen as representations for a given concept by student. This section goes through the responses for concepts that are exemplary of common trends; discusses which expressions were commonly chosen, where the two populations significantly differed in the expressions they chose; and discusses potential reasons for these differences. Due to the categorical nature of these data, comparisons were conducted on an expression-by-expression basis, and statistical significance was determined using the Fisher's exact test. This was necessary (rather than a simple χ^2 test) due to the low number of students that selected some expressions for a given concept. Discussion will include interesting expression-concept pairs where the two curricula agreed, as well as where they were proven to disagree ($p < 0.05$ is shown, as are the situations where $p < 0.01$). Appendix C and Appendix D contain the charts for each concept, the p-values for every comparison of expressions for each concept, as well as the ϕ-values (effect sizes) for those with statistically significant *p*-values. For the purposes of data visibility and ease of comparison between populations with different N, figures will show the fraction of respondents that selected each expression for each concept rather than the raw numbers.

5.4.2.1 Expressions Chosen for Vector-Like Concepts

The breakdown of student responses for expressions they viewed as representing a *vector* is shown in Figure 5.3. This concept was chosen due to it showcasing common trends among various vector-like concepts, particularly *basis vector* and *eigenvector*. As can be seen, both populations largely agreed that $\vec{v}$ and $\hat{\jmath}$ are representative of vectors. Given these expressions' non-quantum mechanical nature, this is to be expected and is an indication that these students have similar backgrounds. There are a few different categories of expressions with consistently statistically significant differences that suggest a theme here. The first is that $f(x)$, $\psi^*(x)$, and $\varphi_3(x)$ were all statistically more likely to be selected as representing vectors to students enrolled in wave functions-first courses, suggesting that students enrolled in wave functions-first courses were more commonly interpreting functions as potentially representative of vectors than students enrolled in spins-first courses. It is worth noting that while these functions were statistically significantly selected for this concept more by wave functions-first students, this was not the case for $\psi(x)$ or $\varphi_4^*(x)$, suggesting that while this was a consistent pattern, it was nonetheless not a terribly strong one (as can be attested by the effect sizes). The second group of expressions worth discussing is that $|\psi\rangle$ and $\langle E_1|$, both of which were statistically more likely to be selected as representing vectors to students enrolled in spins-first courses, which suggests that students enrolled in spins-first courses were more likely to conceptualize Dirac bras and kets as representative of vectors than students enrolled in wave functions-first courses. As with the prior discussion of the wave function expressions, this should be tempered somewhat by the lack of a statistically different difference between the two curricula regarding $\langle\psi|$ and $|E_2\rangle$. Nonetheless these differences could be explained by the students' respective curricula. It would appear that the fundamental expression unit of whichever notation the students learn first (Dirac bras/kets or wave functions) is more likely to be viewed as representative of vector concepts than that of the notation they learn later. It is possible that students turn to vector interpretations early in their coursework due to their familiarity with vectors from prior coursework, and perhaps this conceptualization aids in interpreting inner products in their respective notation. Once a new notation system is introduced later in the course, perhaps this vector identity simply remains most closely tied to the notation they began using. It is also of interest that students in wave functions-first courses were more likely to consider the spin-Z operator $\hat{S}_z$ as representing a vector, likely due to a combination of a lack of familiarity due to spending significantly less time studying spin-1/2 systems within a wave functions-first course, and a notational association of the "hat" commonly used to denote unit vectors.

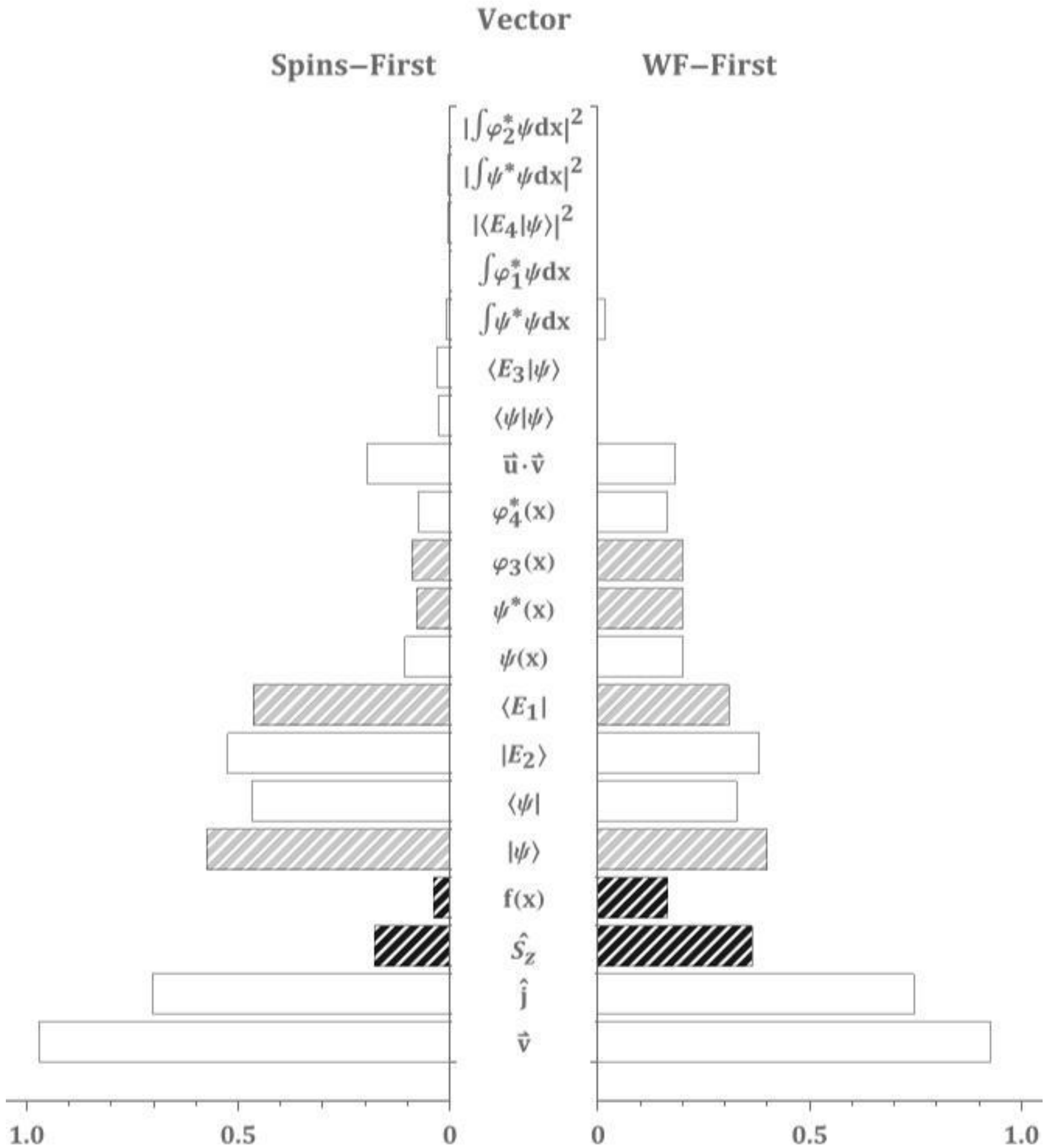


Figure 5.3: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "vector" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

5.4.2.2 Expressions Chosen for the Wave Function Concept

The frequencies of the two populations' choices for expressions as representative of the *wave function* concept is shown in Figure 5.4. Both populations generally agreed that the various wave function expressions ($\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$) were representative of wave functions, which is to be expected. What is particularly interesting is that the wave functions-first students were consistently more likely to consider Dirac bras and kets as representing wave functions, particularly in the case of $\langle E_1|$ and $|E_2\rangle$ (p<0.01, with a medium effect size (0.3<ϕ<0.5)). This can perhaps be explained by the different curricular structures, as wave functions-first courses learn to think of a wave function as the fundamental object in quantum mechanics, and thus perhaps view Dirac bras and kets in those terms, referring to them as such. Conversely, in spins-first courses, wave functions are introduced later on in the course after students have perhaps already developed a vocabulary for Dirac bras and kets as state vectors, eigenvectors, etc., and so perhaps this term is not viewed as strongly as a blanket term by the time they begin to use wave functions later in the semester. That more students in the spins-first course selected the wave function notation versions of inner products as representative of wave functions is less explicable, aside from a potential interpretation of the prompts as asking for any expression involving a wave function that occurred more often with spins-first students.

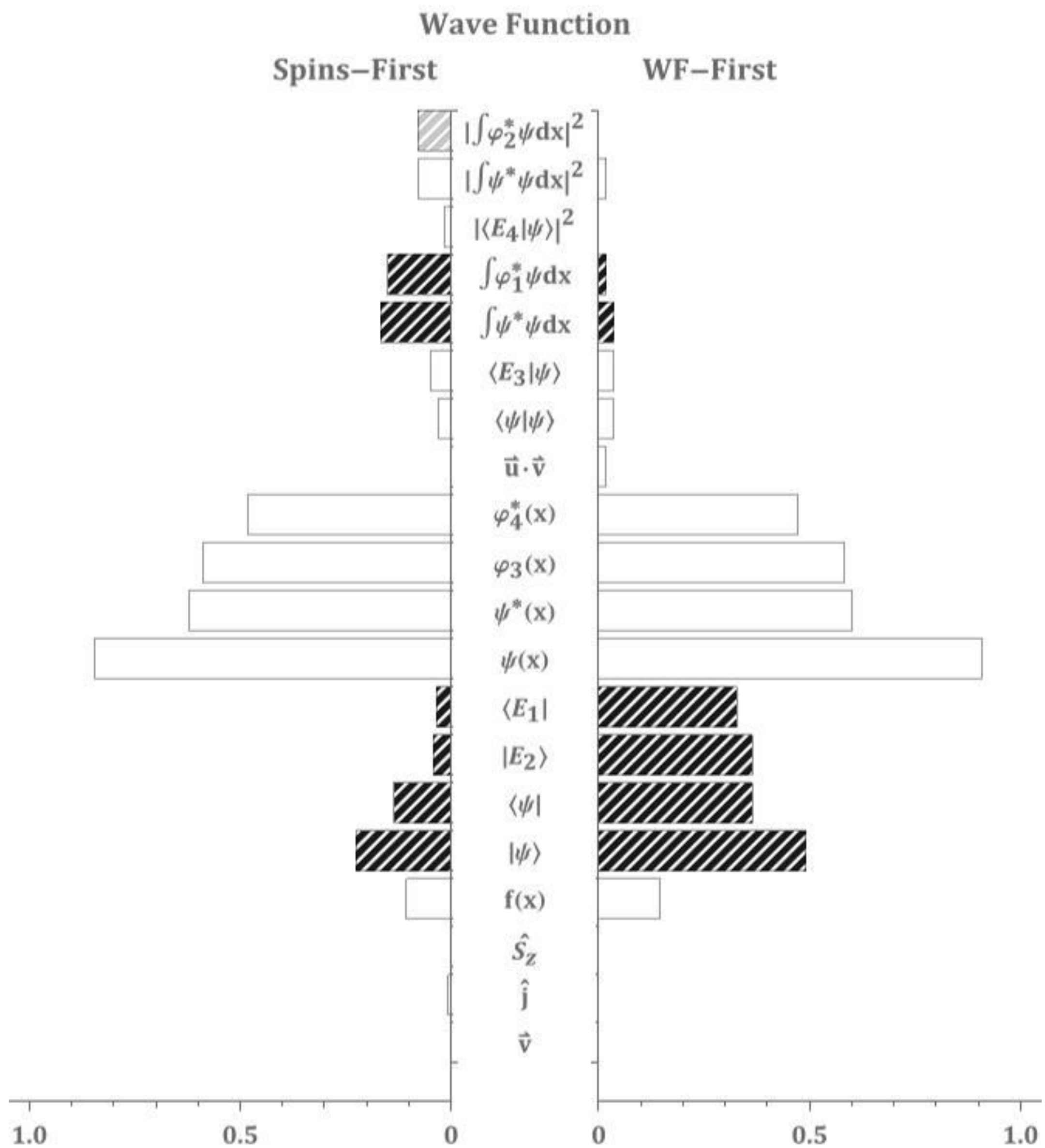


Figure 5.4: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "wave function" concept. The gray-shaded bar represents a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. For $\langle E_1|$ and $|E_2\rangle$, 0.3<ϕ<0.5 (medium effect size), and the other statistically significant differences have 0.1<ϕ<0.3 (small effect size).

5.4.2.3 Expressions Chosen for Quantum State-Like Concepts

As an exemplary case of the trends seen regarding student choices for representing quantum states, see Figure 5.5 for the expression breakdown of *eigenstate*. Here we see a similar effect as we did for *vector* in Section 5.4.2.1, where the different curricula each are consistently more likely to select expressions in the notation they learned first in their course as representative of eigenstates. This suggests that students better understand expressions they are more familiar with, or perhaps that they are more confident in selecting them on a survey. Regardless of curriculum, $\langle E_1|$ and $|E_2\rangle$ were selected more than $\langle\psi|$ and $|\psi\rangle$, and $\varphi_3(x)$ and $\varphi_4^*(x)$ were selected more than $\psi(x)$ and $\psi^*(x)$. This suggests that students were cuing on the difference in letter or presence of a subscript when determining which expressions correspond to eigenstates. These are normatively the markers used to distinguish between an eigenstate (eigenvector/eigenfunction) of an observable (operator) and a general (unspecified) or superposition quantum state's state vector or wave function. The consistent preference for kets and non-complex conjugate functions over bras and complex conjugate functions is also of note, as this suggests that students more think of the kets/wave functions as quantum states more than they do bras/complex conjugate wave functions. This is likely due to a lack of experience with dual vectors and functions, as was consistently observed in our interviews (see Chapter 3).

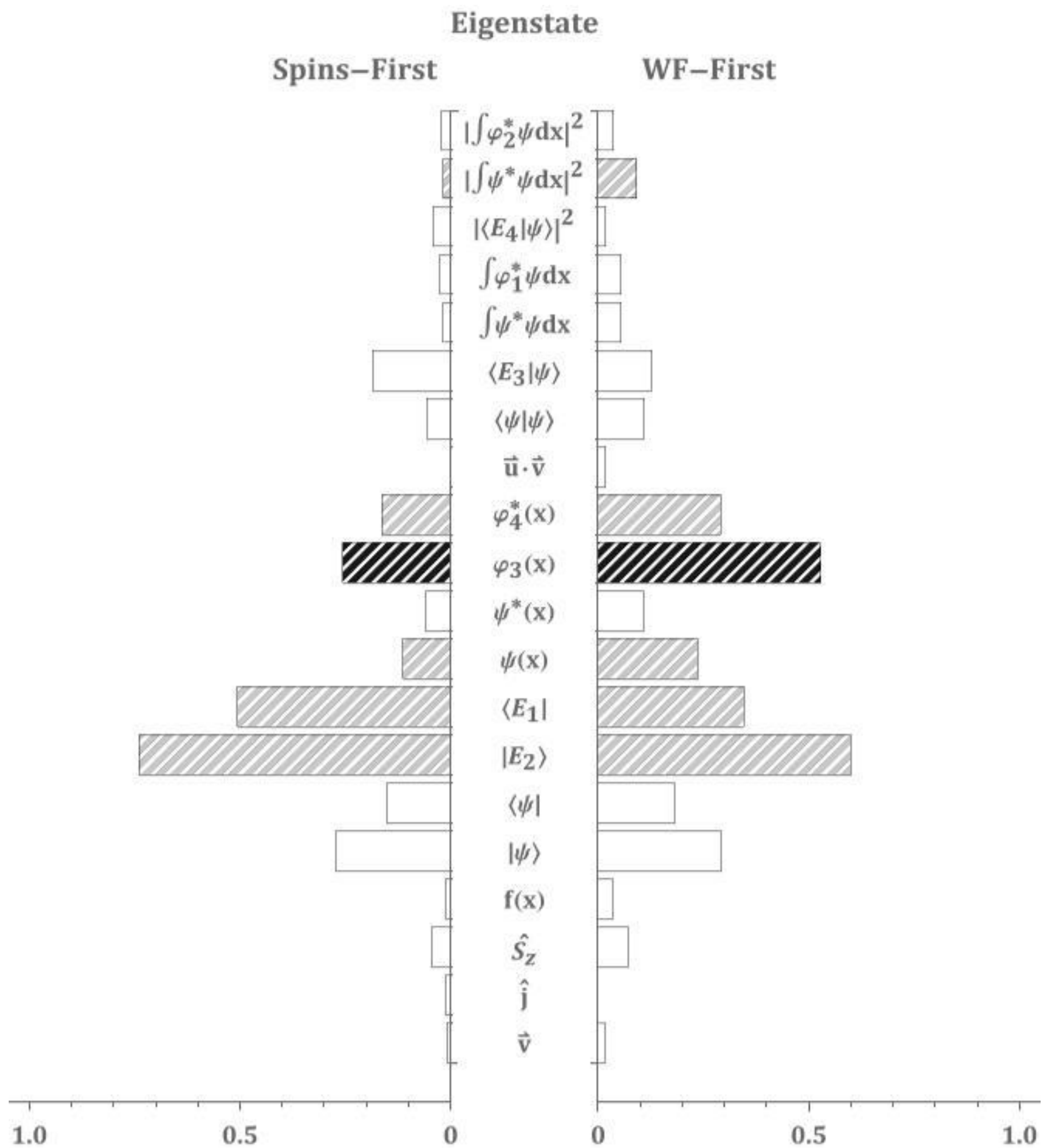


Figure 5.5: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "eigenstate" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

5.4.2.4 Expressions Chosen for Probability-Like Concepts

Finally, we will discuss the differences in the expressions chosen as representative of probability-like concepts, using the results for the *probability amplitude* concept as an example (see Figure 5.6). Interestingly, unlike the divides between the curricula discussed in the previous sections, the divisions here are *not* along notational boundaries. Where before the curricula were seen to generally select expressions that they either learned earlier or later in the course as representative of various concepts, here the spins-first students selected some Dirac expressions more often and some wave function expressions more often, depending on the concept, and vice versa for the wave functions-first students. The factor that instead appears to primarily distinguish the populations is whether the function has a complex square or not. Spins-first students were more likely to select $|\int \varphi_2^* \psi dx|^2$ or $|\langle E_4|\psi\rangle|^2$ than the wave functions-first students, and the wave functions-first students were more likely to select $\int \psi^* \psi dx$ and $\langle\psi|\psi\rangle$ than the spins-first students. It is worth noting that the wave functions-first students selected most expressions including quantum mechanical inner products with fairly similar frequency, and that the primary cause for the differences between the populations is the consistent preference for terms with a complex square among the spins-first students. This suggests that students in spins-first courses may have developed a stronger expectation that probability-like expressions use complex squares than students in wave functions-first courses do. This could be due to the students in spins-first courses developing a symbolic form tied to probability concepts that includes a complex square in the symbol template (see Chapter 3), while perhaps students in wave functions-first courses do not develop a symbolic form with that requisite component, or do not develop it as strongly or consistently.

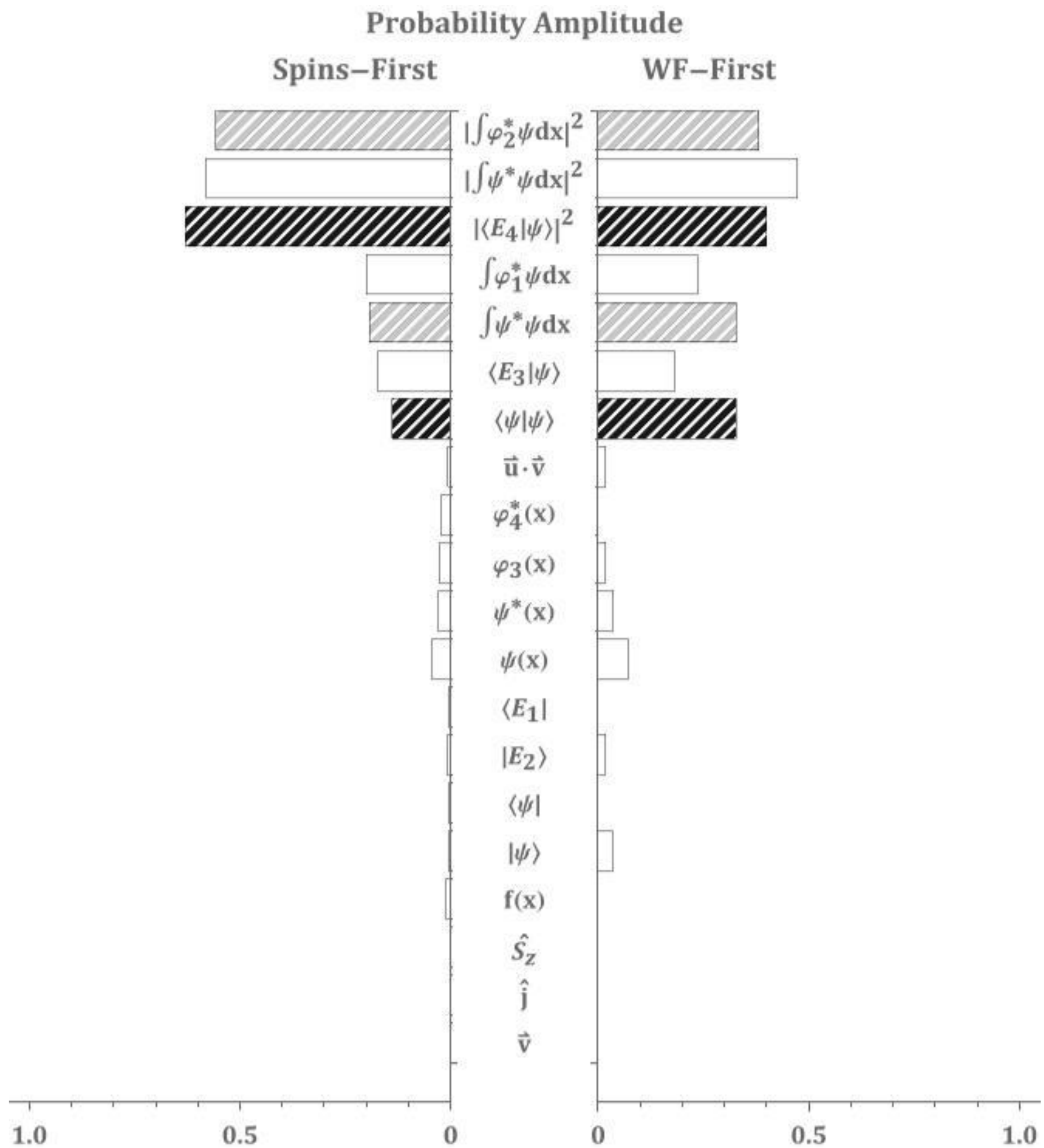


Figure 5.6: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability amplitude" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have $0.1<\phi<0.3$ (small effect size).

There are two potential alternative explanations for these findings: that perhaps the spins-first students were more likely to misinterpret the question as asking for *probability* than *probability amplitude* (hence the focus on complex-square expressions), or that the pattern present is instead that wave functions-first students selected expressions with matching symbols ($\int \psi^* \psi dx$ and $\langle\psi|\psi\rangle$) as representing probability amplitudes and that spins-first students selected expressions with different ones ($|\int \varphi_2^* \psi dx|^2$ and $|\langle E_4|\psi\rangle|^2$). These can generally be refuted by additionally looking at the expressions chosen for the *probability* concept, as shown in Figure 5.7. The general trends observed within Figure 5.6 still hold here, despite this prompt asking for probability. This suggests that students may not see a very clear conceptual distinction between *probability* and *probability amplitude*. This difficulty distinguishing between these related concepts was noted during our interviews (see Chapter 3), and has been observed before by other studies of students' conceptual understanding in quantum mechanics (Marshman & Singh, 2017). This suggests that even if they were misinterpreting *probability amplitude* as *probability*, the observed trends still hold. It can also be observed from Figure 5.7 that the wave functions-first students' larger preference for expressions without complex squares holds, even for expressions that do not have matching symbols within the inner product. This taken in conjunction with the previous point about the seeming ambiguity between *probability* and *probability amplitude* suggests that this preference is real, and thus lends credence to the potential for more-strongly-developed symbolic forms containing complex squares for these concepts for spins-first students. The strength of these symbol templates could potentially be explained by these students starting with Dirac notation, which has high levels of compactness and symbolic support for computation when it comes to inner products, according to Gire and Price's *structural features* framework (Gire & Price, 2015). Potentially, these features of the notation they began the course using allowed for a symbol template such as $|\langle \quad | \quad \rangle|^2$ (see Chapter 3) to form, aspects of which (e.g., the $| \quad |^2$) they may then carry forward as an expectation for symbolic forms within other notations but with similar conceptual schemata.

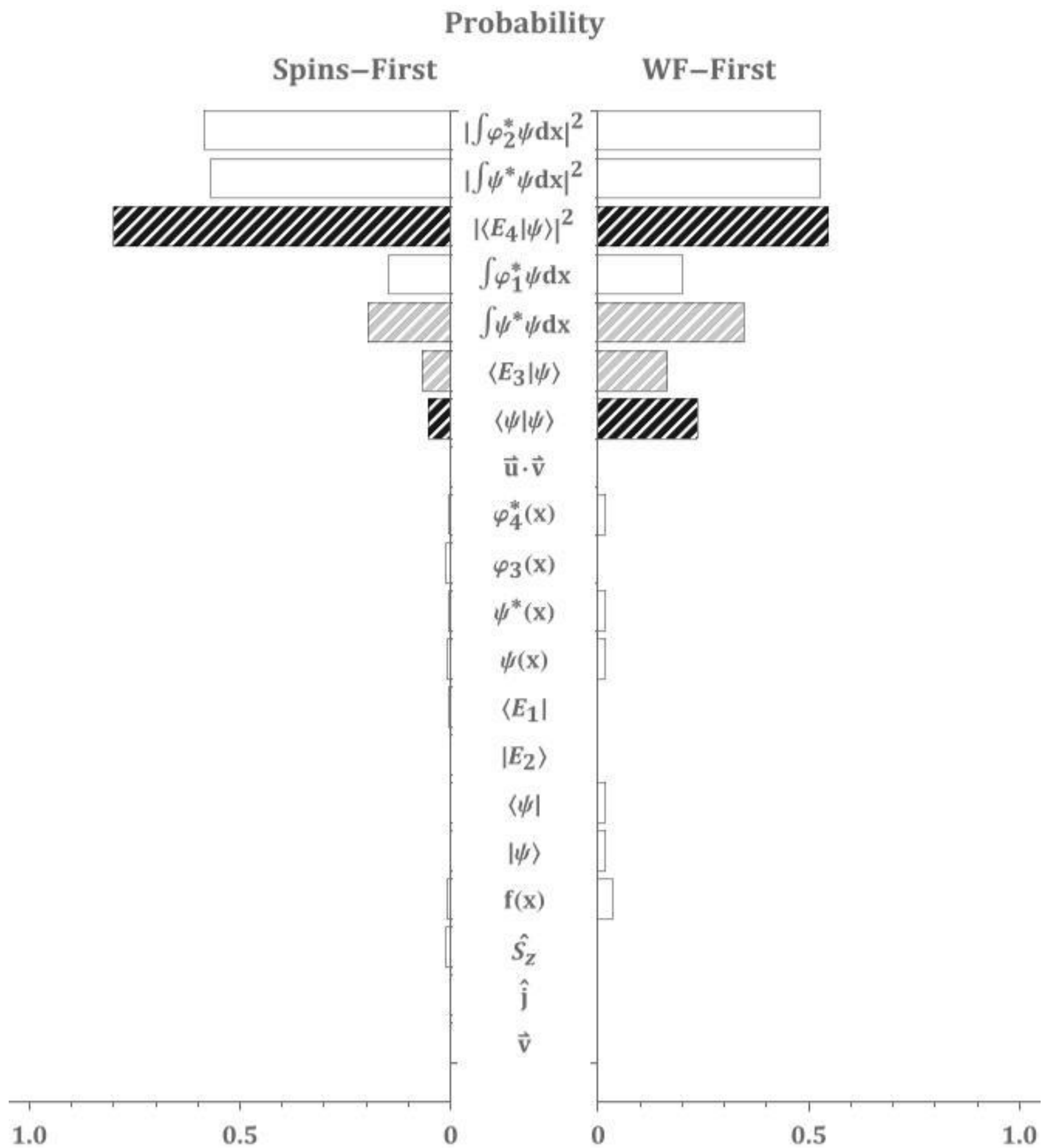


Figure 5.7: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

5.4.3 Principal Component Analysis

A principal component analysis was performed to determine the primary differentiators between students' survey responses. To this end, DeltaCon distances were calculated for every pair of students' individual survey response networks and tabulated into an $N \times N$ matrix. A singular value decomposition was then applied to this matrix to change bases and allow for an examination of principal components. The student data projected onto the first two principal components (i.e., the two that explained the most variation between the students' individual survey response networks) is shown in Figure 5.8(a), and the cumulative percentage of variance between the individual student networks explained by the first several principal components is shown in Figure 5.8b. As can be seen in Figure 5.8(b), PC1 explains the vast majority (~97%) of the variance between the student networks, and the first two together encompass approximately 99% of the variance. Because the subsequent principal components add incrementally to the variance, we will focus on the first two. It is then our task to find *a posteriori* what traits correspond to these primary principal components, and thus which aspects of each student's network vary the most across all students.

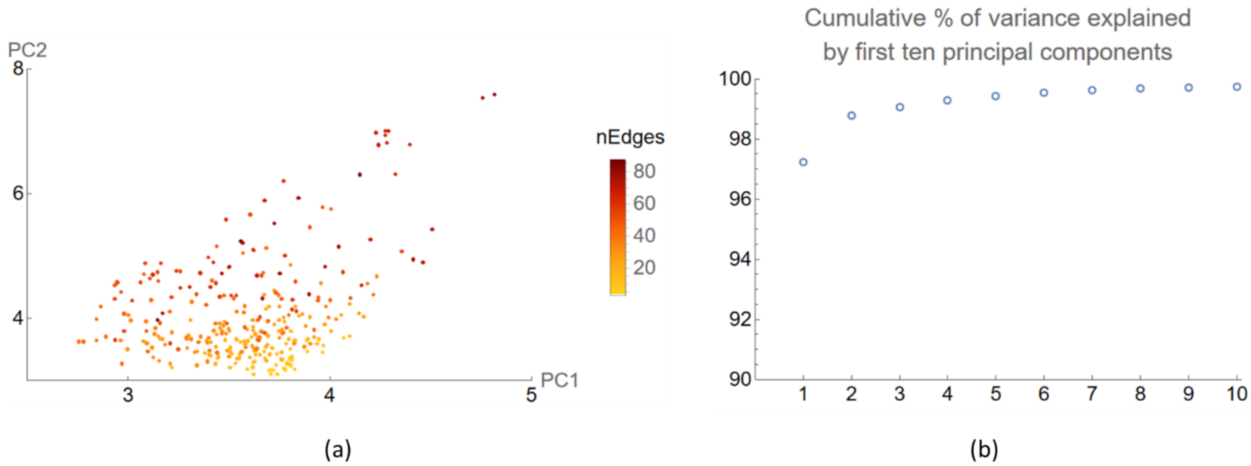


Figure 5.8: (a) Principal component analysis plot of students' survey response networks, projected to show the first two principal components along the two axes. Each point is shaded based on the total number of edges within each students' network. The values along either axis come from a transformation of the DeltaCon distances used to create the plot, and thus have no inherent meaning themselves. (b) A plot of the cumulative percentage of the variance between student networks that can be explained by each principal component for the first ten principal components.

5.4.3.1 Characterizing the Principal Components

Each point on the plot in Figure 5.8(a) represents a student's individual network and is colored by the total number of edges that lie within each network. It may seem obvious from inspection that PC2 is likely a measure of the number of edges within each network (the data points seem to become more red the higher up on the graph they appear). This can be determined more quantitatively by plotting the number of edges within a given student's network vs. that student network's value for PC2, as seen in Figure 5.9(a). These variables were found to correlate strongly (Pearson's correlation coefficient $r = 0.75$). The least-squares fit line is overlayed on the plot in Figure 5.9(a), and the standardized residuals are shown in Figure 5.9(b). The second principal component, then, is effectively separating students by the number of expressions they selected as representative of different concepts on the survey. While this is potentially a measure of students' levels of interconnectedness between various expressions and concepts, it is also likely representative of different students' online survey-taking preferences.

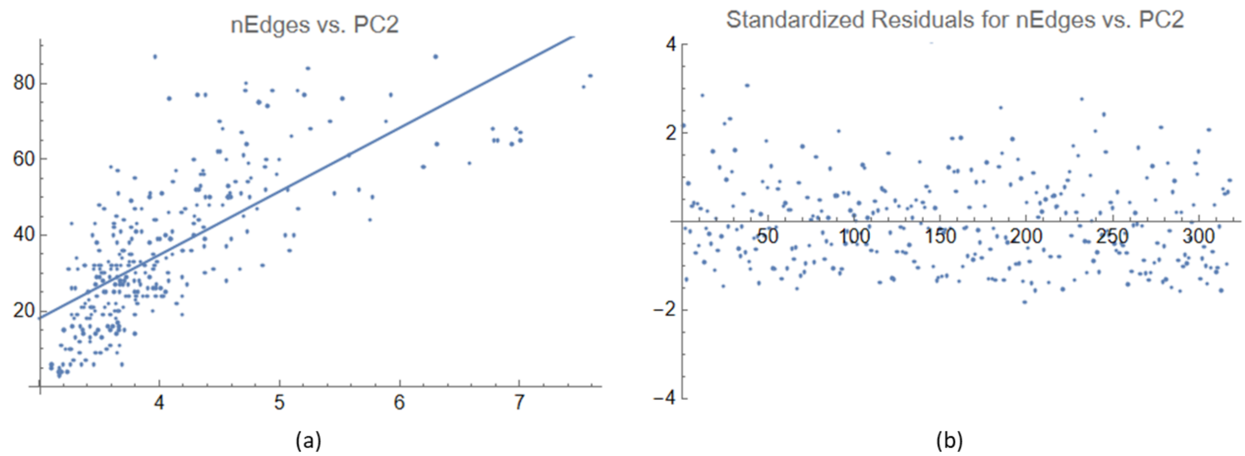


Figure 5.9: (a) Plot of student networks' number of edges vs. their calculated values of PC2. Overlaid is the least-squares line of best fit, showing the linear correlation between the two variables (r=0.75). (b) Standardized residuals for the plot, showing the variance away from the line of best fit for each data point. This shows that a linear fit is appropriate.

Determining what trait PC1 corresponds to (and thus what trait explains the vast majority of the variance in student response networks) is somewhat less straightforward. To get an idea of what this could be measuring, we treat the student networks with the highest and lowest PC1 scores—more specifically, the networks among the first and fourth quartiles of PC1 scores—as separate populations. Superposing the individual networks within these populations will emphasize any consistent structural features, which can then be compared between the two populations. This is similar to how the initial cumulative network of all student survey results was constructed, except only the students' networks within the upper 25% and lower 25% of PC1 scores are superposed to form a 1st quartile network and a 4th quartile network, respectively. These networks are shown in Figure 5.10, with exemplary individual student networks from the upper and lower quartile of PC1 (with very similar PC2 scores) shown in Figure 5.11.

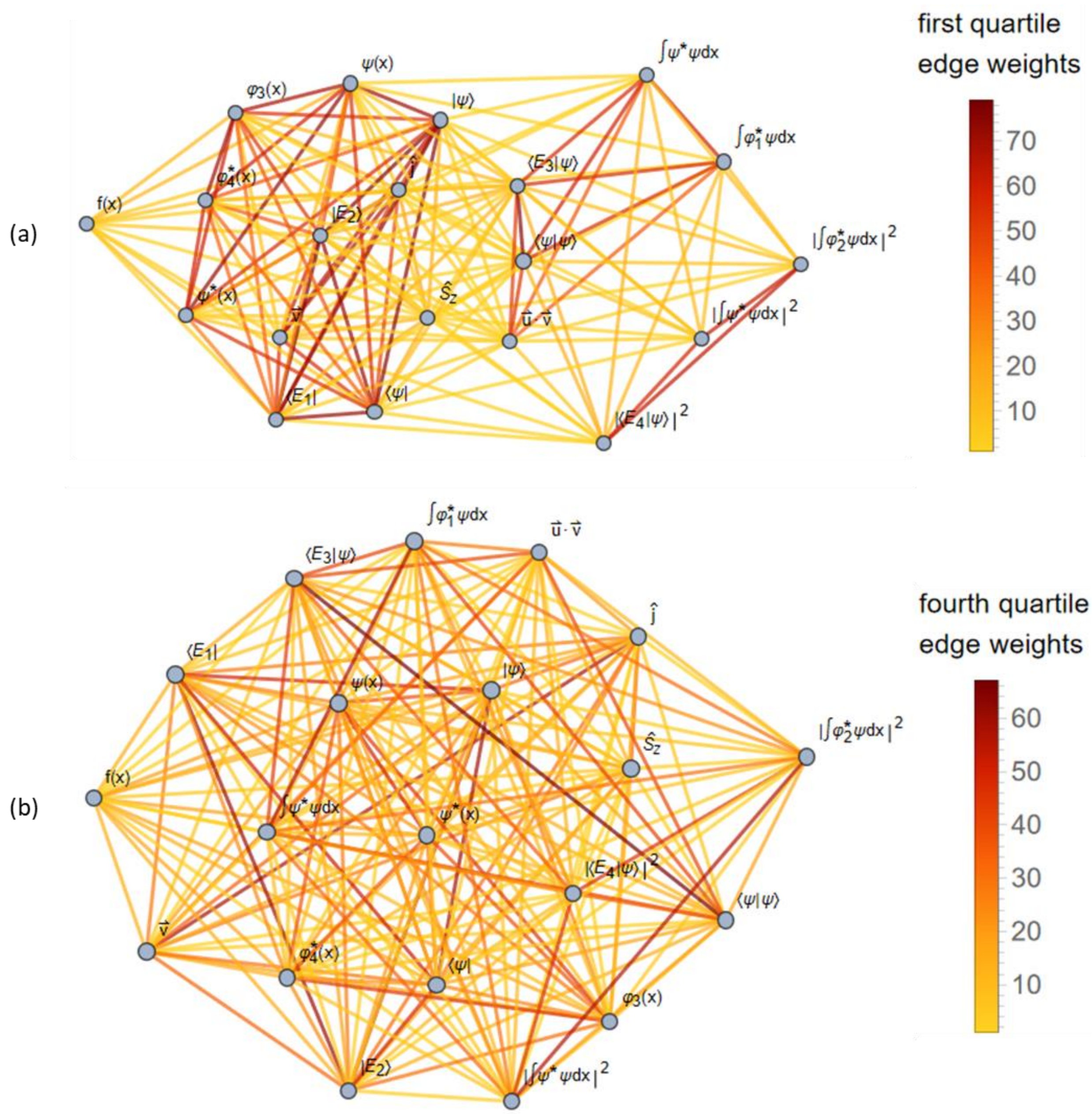
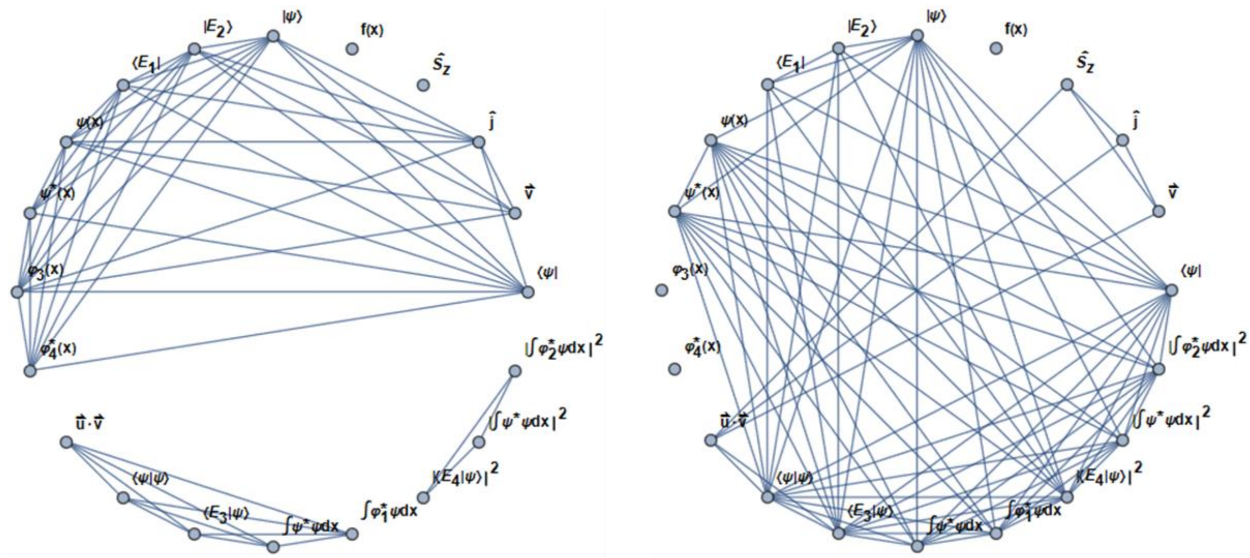


Figure 5.10: The networks formed by the superposition of the individual student networks with (a) the lowest quarter of PC1 values and (b) the highest quarter of PC1 values. The edge weights represent the number of students that used the two expressions simultaneously, and is shown by the shading applied to the edges.



(a) (b)

Figure 5.11: (a) Example student individual network from (a) the lowest (first) quartile of scores for PC1 and (b) the highest (fourth) quartile of scores for PC1. Both networks had similar values for PC2. There is a much clearer community structure present within (a), whereas (b) has nodes connected much more randomly.

There appears to be a clear structural difference between student networks with high and low values of PC1. In particular, students' networks with low values of PC1 (Figure 5.10(a)) appear to consistently have three fairly distinct communities: a community of Dirac bras and kets, wave functions, and generic vector expressions; a community of inner products; and a community of the squares of inner products. These all showed up as clear communities within the community detection dendrograms in Section 5.4.1 (Figure 5.2). The network formed from students with high values for PC1 (Figure 5.10(b)) shows much less of that community structure. This also manifests in the distribution of edge weights for the two networks, as seen in Figure 5.10. While many edges within both networks have relatively small edge weights—meaning not many students within that population made the same connections—the distribution is somewhat bimodal for those within the first quartile of PC1, with a large number of edges with much higher weights. This can be seen on the networks in Figure 5.10; while both networks have many low-weight edges, the stronger communities present in the first quartile network manifests as a selection of edges with abnormally high edge weights respective to the rest of the network. The lower quartile network lacks a group of edges with abnormally-large edge weights. This is representative of students that do not have as clear a shared predilection for connecting certain expressions together. PC1 therefore would seem to be an inverse measure of a student's likelihood to connect the expressions seen in the communities in Figure 5.11(a) together. Given that the expressions within these communities share quite a bit of conceptual meaning, and the survey prompts effectively ask participants to sort expressions by their conceptual meaning, lower values of PC1 may correlate with more normative conceptions of the expressions present on the survey.

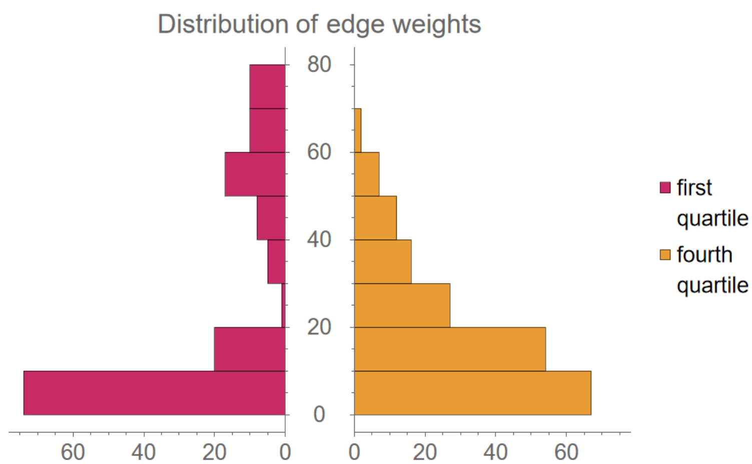


Figure 5.12: Histograms showing the distribution of edge weights among the edges present in the networks for the lower (left) and upper (right) quartiles of PC1.

5.4.3.2 Comparing Principal Components Across Curricula

Now that the two primary principal components have been characterized, it is worthwhile to see whether students in spins-first and wave functions-first curricula are separable along either of these principal components. Normalized distributions of PC1 and PC2 values for both curricula are shown in Figure 5.13. Neither of the curricula appear to be significantly higher or lower in either of these metrics. Based on our findings for PC2, this suggests that students in both curricula have similar survey-taking behavior, in that the numbers of students that selected many expressions simultaneously and that selected very few expressions simultaneously were in roughly equal proportions, irrespective of curriculum. This is to be expected, as survey-taking behavior is not likely to depend on the type of quantum mechanics course one is enrolled in, and in fact should be independent of both content and conceptual understanding. While the distributions for PC1 may suggest that wave functions-first students may have a slightly higher average PC1 score (and thus slightly less expert-like connections between conceptually similar expressions), this is not a statistically significant finding. While there is potential that collecting more data from wave functions-first courses may make this trend statistically significant, currently there does not appear to be a difference between the two curricula when it comes to this general community-like behavior. This inability to differentiate the two populations by this metric that correlates with rough community structure does align with the dendrograms and community structures discussed in Section 5.4.1 if viewed with a larger grain size, as the three larger communities observable higher up on the dendrograms do appear in the data from both curricula.

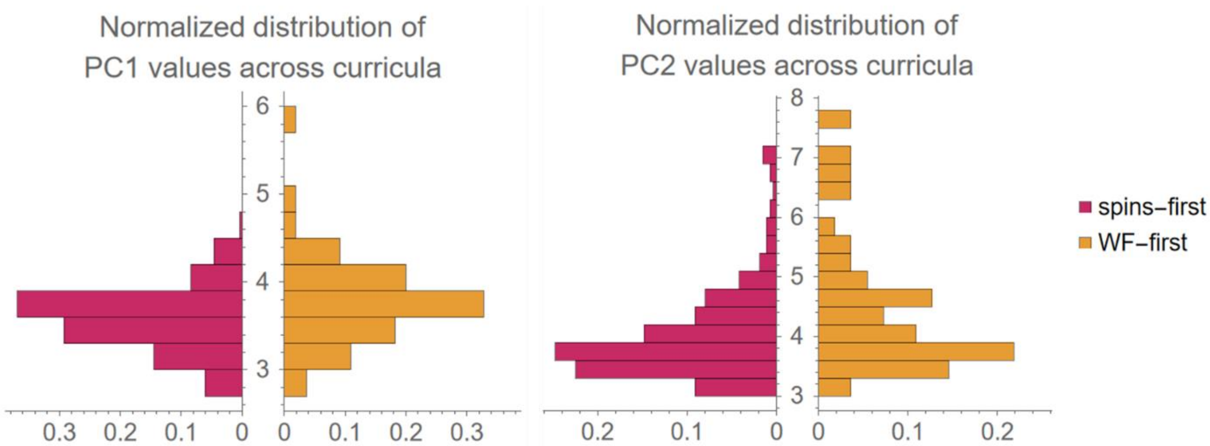


Figure 5.13: Normalized distributions of students' scores for PC1 and PC2, separated by curriculum.

5.5 Conclusions

In summary, while students in spins-first and wave functions-first courses share many similarities in the ways they conceptualize common expressions in quantum mechanics, there are some interesting differences that point to potential curricular effects.

Community detection of expression networks suggests that the initial focus of spins-first courses on linear algebra and geometric interpretations of Dirac expressions causes students to conceptualize Dirac expressions more strongly in vector-like terms than students enrolled in wave functions-first courses.

Examining the conceptual interpretations of the various expressions on the survey also highlighted some interesting differences between the curricula, with potential pedagogical explanations. One of the biggest distinguishing factors between the two populations was their conceptualization of Dirac bras and kets. While Sections 5.4.2.1 and 5.4.2.3 showed that students in spins-first courses tended to view these expressions as more representative of vectors and quantum states than the wave functions-first students did, Section 5.4.2.2 showed that wave functions-first students thought of bras and kets far more as representing wave functions than the spins-first students did. These interpretations all have merit, of course, as Dirac bras and kets *are* vectors, *do* represent quantum states, and can contain the same information as (and more than) wave functions. That students in these different curricula each view them in all of these terms is encouraging, though the degree to which these interpretations are salient to the students is seemingly affected by their curriculum. The prevalence of wave functions-first students treating Dirac bras and kets as representing wave functions is reasonable, given their text's preference for discussing kets as representative of functions as seen in Section 2.1.4.

It was also seen in Section 5.4.2.4 that students in both curricula seemed to view "probability" and "probability amplitude" as somewhat synonymous—or at the very least conceptually ambiguous—as evidenced by their responses for the expressions representative of both concepts being largely similar. It was also noteworthy that spins-first students selected inner product expressions with complex squares as representative of these concepts more often than wave functions-first students, while the opposite was true for inner products without complex squares. This is potentially evidence of different symbolic forms having been developed in the different curricula, as Chapter 3 showed that spins-first students develop strong symbol templates containing complex squares as representing probability concepts.

Our principal component analysis was a means to determine the primary causes of variability among students' survey responses. While student networks generally varied significantly, the primary causes of their variability appear to be due to general survey response behavior differences (which transcend curriculum), and general levels of normative conceptual understanding (which also transcends curriculum). While on an individual level, the principal component analysis of individual student networks caught onto larger variables, once aggregated as in Sections 5.4.1 and 5.4.2 the summative differences hidden within the “noise” of survey response behavior and level of general conceptual understanding could emerge as general trends throughout the larger student populations. Overall, it can be seen that curricular focuses do in fact influence students’ conceptualizations of different symbolic representations. It appears that each curriculum strengthens students’ conceptual interpretations in different ways, and so a choice on a curriculum should come down to instructor preference as to which conceptions of the various expressions they wish their student to best understand and internalize. Spins-first courses appear to make students think of Dirac bras and kets as representing vectors and quantum states more than wave functions-first courses do, while wave functions-first courses (perhaps unsurprisingly) appear to make them more strongly connect bras and kets to wave functions, and not as strongly to vector-like concepts.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

Quantum mechanics is a notoriously challenging topic for students, and the abstract mathematics required is no small contributor to that challenge (Gire & Price, 2015; B.P. Schermerhorn et al., 2019; Singh, 2001, 2008; Singh & Marshman, 2015; Wawro et al., 2020). The goal of the work presented in this dissertation was thus to better understand the ways that students understand the two primary mathematical notations used in upper-division quantum mechanics: Dirac notation and wave function notation. For the purposes of this dissertation, we limited our focus to expressions (and pieces of expressions) that are used to calculate probabilities; in a quantum mechanical context, this means we studied inner product expressions (and their components) in both Dirac and wave function notation. While these two notations share analogous or equivalent expressions and can be used interchangeably for many tasks, each has its strengths and weaknesses (Gire & Price, 2015) and thus each is more efficient for some tasks than the other (Schermerhorn et al., 2019). Unsuccessful or incomplete translation between notations has been shown to cause students to struggle when generating expressions for quantities such as probabilities (Wan et al., 2019).

To better understand the ways that students interpret and use expressions in these two notations, we conducted think-aloud interviews and administered surveys to hundreds of students at several institutions around the US. The breadth of the coverage afforded by our surveys also allowed for a comparative study of the two predominant curricular styles of upper-division quantum mechanics courses. This is of particular interest for this project because the primary difference between these two curricular styles is the order in which the two notations in question are introduced. To examine the student interviews (all of which were conducted at an institution that taught Dirac notation first and wave function notation later), we found Sherin's symbolic forms framework (Sherin, 2001) to be a good fit for this analysis, as the different notations each share analogous expressions, and so it may be expected for conceptual schemata to be applied to different symbol templates, depending on the notation used. We also deemed this a useful framework for the quantum mechanical context in particular, due to the multifaceted possible conceptualizations of many expressions used within this topic. For example, Dirac bras and kets are treated mathematically as vectors—with all of the rules and features belonging to that type of mathematical object—while representing quantum mechanical systems. The symbolic forms framework thus proved helpful in distinguishing the different concepts that students could view quantum mechanical expressions as representing. Within this chapter, we recapitulate the primary conclusions of this work and discuss the broader ideas these conclusions inform as well as how they relate to prior research in the field. We will conclude by discussing potential avenues for continuing this work in the future.

6.1 Summary of Conclusions

Symbolic forms analysis of students' think-aloud interviews while constructing expressions for probability revealed numerous normative symbolic forms, within both Dirac and wave function notations. This included interpretations of bras, kets, and functions as representations of vectors and quantum states, and both Dirac brackets and inner product-style integrals as vector dot products and probability-related concepts. One interesting result of this analysis is the strong connection students made between a complex square and a probability. This requirement for a complex square in symbolic forms for probability did on occasion lead students astray when a complex square is implicit, such as with $\int \psi^*(x)\psi(x)dx$. While there was generally agreement that a complex square should be present, there was some confusion when determining what quantity was squared. For example, when calculating the probability for measuring a particle within an energy state, students expressed confusion as to whether the wave function expression should be $|\int \varphi_n^*(x)\psi(x)dx|^2$ or $\int|\varphi_n^*(x)\psi(x)|^2dx$. One interesting point to note is that the two complex square-based instances of confusion generally manifested more often for expressions written in wave function notation. While this could be due to a lack of expertise using this notation in a spins-first curriculum, the students had been using this notation extensively for several weeks before the interviews were conducted. We believe it is more likely that this is an example of Dirac notation's higher degrees of *individuation* and *symbolic support for computation* (Gire & Price, 2015). In essence, the notation lends itself toward viewing inner products (and thus the squares of inner products) as an elemental, easy-to-recognize form. Within the symbolic forms framework, this could be interpreted as students being more readily able to recall and recognize the symbol templates (and thus the symbolic forms) for the Dirac versions of these expressions.

The nature of these identified symbolic forms also helps to explain aspects of our results of network analysis of the conceptual connections survey administered in both spins-first and wave-functions-first courses. We found that students with a spins-first background conceptually connected Dirac bras and kets, generic vector expressions, and wave function expressions. This is explainable due to the shared conceptual schemata associated with the symbolic forms identified for Dirac bras and kets and functions, in particular the *ket/bra as quantum state*, *ket/bra as vector*, *function as state*, and *function as vector* symbolic forms. Interestingly, upon applying community detection methods, the Dirac expressions were found to connect more strongly with the generic vector expressions than with the wave function expressions. This suggests a number of possibilities regarding the prevalence of these different symbolic forms. For example, it is possible that the *bra/ket as vector* symbolic form is more common and/or more strongly instantiated than *bra/ket as quantum state*, and/or that *wave function as vector* is less common and/or less strongly instantiated than *wave function as state*. It is likely that several of these (or other) potential explanations combined to cause the Dirac bras and kets to be more strongly conceptually connected to the generic vector expressions than to their physically analogous wave function expressions for spins-first students.

This strong association between Dirac bras and kets and vector ideas is likely due in part to the spins-first background of these students, as the community structure observed in the data from wave functions-first courses was different. For these students, the Dirac bras and kets were more closely connected to their wave function analogs, with the generic vector expressions being more weakly related to both. Looking at the conceptual breakdown of the different expressions provides a potential explanation for these different community structures as well, as spins-first students selected bras and kets as representative of vector concepts more often than wave functions-first students; while both populations of students chose wave function expressions as representative of wave function concepts, the wave functions-first students were more likely to select the bras and kets as representative of wave function concepts as well. One potential explanation for this prevalence of wave function terminology being used to describe Dirac bras and kets is that due to their courses beginning with the Schrödinger equation and using wave functions to describe quantum states, students in these courses came to closely tie the concepts of “quantum state” and “wave function,” and thus every different representation of a quantum state that was introduced later in the semester was then also connected to their conception of wave functions. This extra “connective tissue” between bras, kets, and wave functions for wave functions-first students and between bras, kets, and vector expressions for spins-first students likely helps to explain this difference in their community structure. These conceptual structures are not absolute, of course—recall Bilbo (a spins-first student) referring to wave functions as representing vectors—but they are nonetheless generally the case.

In general, one common distinguishing factor between the two populations was a seeming preference for selecting expressions in the notation that was used first in their curriculum for many concepts present on the survey. In other words, spins-first students tended to select Dirac notation expressions more frequently than wave function expressions for concepts related to vectors and quantum states, and vice versa. This suggests a level of comfort or perhaps simply better conceptual understanding of the notation with which the students have worked the longest. Some concepts did buck this trend, however, such as the “wave function” concept, where very few spins-first students selected Dirac expressions, but many more wave functions-first students did. Marshman and Singh (2017) had previously found some evidence of students struggling to distinguish the different “flavors” of probability concepts (i.e., probability vs. probability amplitude vs. probability density), and our survey responses seem to corroborate and support this finding. Many students appeared to treat “probability” and “probability amplitude” similarly on the survey, suggesting that this difficulty in parsing these admittedly nuanced distinctions between these concepts is a general challenge that affects students regardless of curriculum.

Another conclusion from these results is that while the survey was successful in teasing out differences between students in spins-first and wave functions-first curricula through network analysis techniques conducted on their respective cumulative networks, other factors that were independent of curriculum (such as variance in student' survey-taking behaviors and levels of expert-like thinking) dominated the comparative analysis of individual students' networks conducted via principal component analysis (PCA). In other words, while the populations differed when taken as a whole, the resolution of a PCA was not sufficient to detect the signals distinguishing the two populations through the noise of domain-general properties.

While our symbolic forms analysis relied on interview data that was only collected within a spins-first context, it is possible to speculate as to the symbolic forms that may have developed by wave functions-first students based on the curriculum, our findings from spins-first students, and our survey results. It is reasonable to assume that, due to the difference in curricular focus and notational usage, some of the differences observed between curricula in our survey data are attributable to differences in symbolic forms developed within a wave functions-first curriculum (via either different symbolic forms entirely or differences in the relative prevalence of the forms discussed in Chapter 3). One example of a symbolic form that may arise from an analysis of interview data with wave functions-first students would be potential *ket/bra as (wave) function* symbolic forms, wherein the symbol templates $| \quad \rangle$ and $\langle \quad |$ are paired with conceptual schemata that capture some specific property of wave functions that may be specific to students in wave functions-first courses. Whether this wave function concept is a distinguishable concept from that of a quantum state for this population is an interesting question in its own right, as was discussed above. Also, it is possible that the *(conjugate) function as vector* symbolic forms would be observed more often or with more coherence with wave functions-first students, as our survey data showed a statistically significantly larger portion of wave functions-first students selecting wave function expressions when prompted for vector concepts. It would be interesting to see if these speculations based on our analyses would bear out in a symbolic forms-based analysis on students within wave functions-first curricula.

In general, this research suggests that students successfully developed many normative symbolic forms throughout a spins-first quantum mechanics course, and thus learned how to generate and interpret expressions representing quantum states and probabilities written in both Dirac and wave function notation. There does appear to be some difficulty when it comes to generating and interpreting expressions for probability in wave function notation, often relating to the presence or location of a complex square. It would be interesting to see whether this difficulty persists in a wave functions-first context, in which the students would potentially be more comfortable and skilled with that notation, and whether they would instead find working with expressions in Dirac notation more challenging.

Network analysis techniques were shown to be useful in determining the relative strengths of conceptual connections between expressions for students in different curricula, and for distinguishing those curricula by the differences in the strength of these conceptual connections. As may be expected, students in these curricula with very different notational focuses do appear to learn to interpret expressions in these notations differently as well. As these differences in conceptual interpretations of these notations are quite prevalent, this difference in understanding should be a factor in determining which approach an instructor chooses to teach.

6.2 Thoughts on the Use of Network Analysis Techniques

To our knowledge, we adapted and made use of standard network analysis techniques in a unique way in this study. While network analysis has seen extensive use in the study of social communities in discipline-based education research (among other uses, as discussed in Section 2.5), this study represents the first attempt to use these techniques to probe students' representational understanding. As such, much of our work was in determining which techniques were appropriate for a study of this kind—this section will hopefully serve as a helpful record of this process, such that future work can build on this work more expediently. Due both to the number of analogous expressions across wave function and Dirac notation as well as the prevalence of expressions with very subtle distinctions within each notation, we determined that community detection techniques would prove useful in analyzing how expressions in these two notations relate. Our desire both to measure these connections along conceptual lines and to do so for a large number of students at multiple institutions with different curricular focuses gave cause for the creation of the card-sorting survey described in Section 4.4.1.

A multitude of options for community detection methods have been developed within network analysis research (Fortunato, 2010), and in fact we originally made use of a different method than the betweenness-based method used in our final work. We initially made use of a modularity maximization-based method (Newman & Girvan, 2004), which was helpful in that it determines the number of communities that are “best” as well as which nodes should belong within each community. This avoids the need for a separate procedure for doing so yourself—in our case, our modified statistical bootstrapping procedure discussed in Section 4.3.2. Unfortunately, this method proved troublesome for networks of our size, as modularity maximization has an inherent limitations on its resolution for communities of a small number of nodes (Fortunato & Barthélemy, 2007b). Other techniques such as those based on hierarchical clustering were passed over as well due to the reasons discussed in Section 4.3.1, and we eventually settled on our combined betweenness-bootstrapping method for determining the best communities for our networks. Unless the networks in question are much larger than ours, we would suggest others studying similar types of networks adapt the methods we used to avoid spending too much time testing out several different methods. If the networks and the communities of interest are much larger than the networks studied here, we would suggest the use of modularity maximization techniques instead. This is primarily due to the computation time for betweenness-based methods scaling aggressively with the number of nodes and edges within a network and become unfeasible for very large networks, as was discussed in Section 4.3.1.

While the PCA based on network distance metrics adapted from Wolf et al. (2012) and discussed in Section 5.4.1 did allow for discussion of the primary causes for variability among all survey participants’ networks, the high levels of variability in student survey-taking behavior dominated much of the analysis. The second principal component did allow for a potential categorization of students based on expert-like thinking, though after PC1 it accounted for much less of the variability among participants’ survey responses. As such, applying a PCA to survey-based network data in this way is perhaps most advisable if the high level of variability in the number of answer selections seen in our data can be mitigated by survey design. If this can be done, then this approach would be very illuminating as to the primary distinctions among survey participants. This would be especially useful if used for a similar purpose to that in Section 5.4.3, when comparing two different student populations that could be expected to respond differently for a given data collection apparatus.

Overall, a significant portion of the work conducted throughout this project concerned determining appropriate methods for analyzing the unique type of networks our survey results generated. Not every method used yielded definitive results—modularity maximization and our PCA being two examples—but it proved instructive. We hope that our discussion of the various techniques available will prove useful for future research, as every method has a dataset for which it will be the best option. We determined that the methods described in Chapters 4 and 5 were the optimal approaches for our data, given the number of students, the size of our networks, and the specific questions we were asking.

6.3 Future Work and Implications for Instruction

While this work answered many questions concerning students' representational understanding in quantum mechanics, it raised many more in the process, all of which suggest fruitful ground for future studies. First, while we limited our investigation to Dirac and wave function notation, Gire and Price (2015) and Schermerhorn et al. (2019) included vector-matrix notation expressions in their work. We are not familiar with any prior symbolic forms analysis of matrix notation expressions, but as it is another way to symbolically express physical relationships, students likely develop symbolic forms for these expressions as well in a linear algebra context, and perhaps different symbolic forms within a quantum mechanical context.

Additionally, while our investigation was wholly concerned with expressions representing probabilities and their constituent expressions, there are many other expressions that students must grapple with in this context. In particular, expressions for outer products, expectation values, and eigenvalue equations in quantum mechanics all are worthy of study. Students likely need to develop entirely new symbolic forms to interpret all of these expressions within this context, as several of them are most likely entirely novel to the students (e.g., outer products and expectation values expressed as $\langle\psi|\hat{A}|\psi\rangle$ or $\int \psi^*(x)\hat{A}\psi(x)dx$) or require an entirely novel interpretation (e.g., eigenvalue equations, as discussed by Dreyfus et al. (2017) and Pina et al. (2023)). We also believe that a comparative analysis between spins-first and wave functions-first students would likely highlight substantive differences in their interpretations and comfort levels with these expressions as well.

While collecting and analyzing more survey data would also potentially assist in gaining further clarity in our comparative analysis between students in different curricula, another fruitful avenue would be to extend this survey to instructors of these courses. This would allow for an analysis of experts' conceptual interpretations of these expressions, and even for a potential "expert" network that could be compared to individual students. There are several ways analysis of these results could be illuminating, the first of which would be determining the variance between experts' responses, and whether any variance could be explained by instructional approach and/or the multiple interpretations of quantum mechanics concepts. Another interesting potential outcome of this extension of our work would be as a potential diagnostic tool. For example, students' or whole classes' collective response networks could be compared to the "expert" network as a means of determining how successful a course has been in developing expert-like conceptions of these expressions.

The network analysis techniques as adapted for this project could also be used to investigate conceptual connections between mathematical entities well beyond this context. For example, these techniques could be used to study the ways students interpret mathematical relationships both in symbolic and graphical forms, or to study interpretations of physical laws and whether they vary depending on the way they are written. For example, a similar survey could be designed with each question proposing a graph with several different symbolic algebraic expressions or physical scenarios that they would select as potentially being represented by the graph. Relatedly, recent work has been done by chemistry education researchers to attempt to adapt the symbolic forms framework into explaining students' developing a vocabulary of graphical curves, fittingly named *graphical forms* (Rodriguez, Bain, & Towns, 2020). Within PER, some preliminary work has found that while expert physicists display a command of such a vocabulary of graphical functional forms—often stating that a physical behavior "goes like" some of these common functions (e.g., linear, quadratic, sinusoidal relationships)—students have not been shown to develop such a collection of familiar graphical relationships (Zimmerman et al., 2020). A potential survey and resultant network analysis could thus study the shared conceptual nature of various "graphical forms." The community detection methods described in Chapter 4 would be useful anytime one is curious about whether two things (be it expressions, graphs, etc.) are viewed as potentially connected by a shared concept or context. As such, we believe that there is much room for this or similar methods to be used to study such conceptual or contextual connections. Also, these methods represent a small foray into the broad and growing field of network analysis, and thus it is likely that future studies could greatly improve and iterate on this design and implementation.

We would like to conclude with some implications our results suggest regarding quantum mechanics instruction. In general, we found that spins-first and wave functions-first curricula both appear to have merit insofar as each appears to promote students' representational understanding of expressions for probability concepts. The primary differentiator between the two curricula appears to be the relative level of familiarity with the notation in their respective courses, as one could expect. Wave functions-first students appear to treat wave functions as the fundamental concept and physical/mathematical construct, while spins-first students appear to treat state vectors (especially when represented as kets) in this way. An instructor should thus take this difference into account when choosing a curricular focus for their course. If the goal for the course is to prepare students for graduate study in physics, we might suggest using a spins-first curriculum, as much of graduate-level quantum mechanics makes extensive use of Dirac notation, and thus increased familiarity and comfort with this notation may prove helpful for students that continue on to graduate study. One more concrete suggestion for instructors of spins-first courses, based on our interview findings, is to emphasize the appropriate use and placement of complex squares when calculating probabilities, particularly when using wave function notation. This was a difficult task for students in our interviews; our interview data is likely representative of students with above-average understanding of the coursework due to the inherent selection bias of asking for volunteers. Thus, if the proper placement of the complex square for these expressions proved challenging for our interviewees, it is likely that this difficulty is even more prevalent for the average student. The fact that Dirac expressions are so individuated and supportive of computation due to the symmetry of the notation itself (Gire & Price, 2015), we would also suggest that leveraging these benefits and emphasizing translating expressions for probability from Dirac into their equivalent wave function notations may be helpful. Overall, we wish to emphasize that the notations and expressions used in upper-division quantum mechanics courses are rife with nuanced distinctions and subtle variations, and thus great care should be taken to assist students in developing the conceptual schemata that they will learn to apply to the various expressions in these courses.

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APPENDICES

APPENDIX A: COMPARISON OF NOTATION INTRODUCTIONS BETWEEN SPINS-FIRST AND WAVE FUNCTIONS-FIRST TEXTBOOKS

This appendix will discuss the ways in which Dirac and wave function notation are introduced in spins-first and wave functions-first courses.

A.1 Spins-first Introduction of Dirac Notation

There were two spins-first texts used by the students in this study: *Quantum Mechanics* by McIntyre (2012) and *A Modern Approach to Quantum Mechanics* by Townsend (2000). The majority of the spins-first students studied used McIntyre's text (all interviewees used McIntyre, and 182 McIntyre students participated in the online survey vs. 81 Townsend students), so we will focus on the language and notation used by McIntyre in this section, though Townsend's notational choices are broadly similar. These courses begin by using the results of the Stern-Gerlach experiment to motivate the fundamental postulates of quantum mechanics while simultaneously introducing students to Dirac notation. After first explaining the Stern-Gerlach experiment and the requisite classical physics, the text introduces Dirac kets in the following way:

> In [**Error! Reference source not found.**], the input and output beams are labeled w ith a new symbol called a **ket**. We use the ket $|+\rangle$ as a mathematical representation of the quantum state of the atoms that exit the upper port corresponding to $S_z = +\hbar/2$. The lower output beam is labeled with the ket $|-\rangle$, which corresponds to $S_z = -\hbar/2$, and the input beam is labeled with the more generic ket $|\psi\rangle$. The kets are representations of the quantum states (McIntyre, 2012, p. 4).

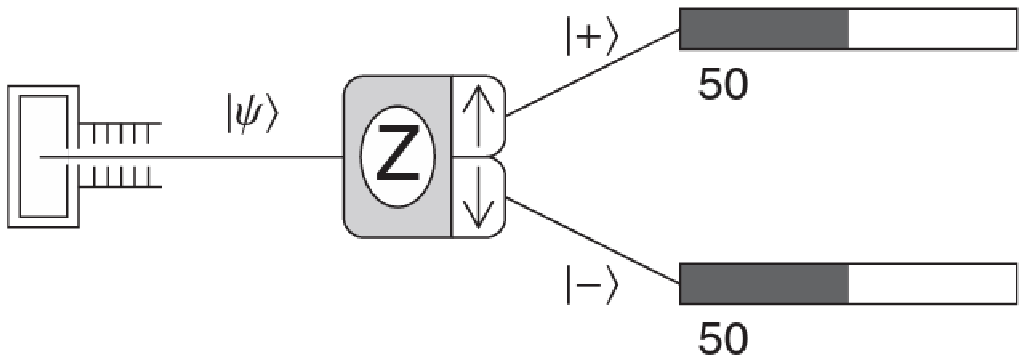

Figure A.1: Schematic from Figure 1.2 in McIntyre (2012), showing results from a Stern-Gerlach experiment. On the left is the oven releasing silver atoms described by the ket $|\psi\rangle$, in the middle is the Stern-Gerlach apparatus oriented in the z-direction, and on the right are the counts of silver atoms detected to have $S_z = +\hbar/2$ and $S_z = -\hbar/2$, described by the kets $|+\rangle$ and $|-\rangle$, respectively.

Once the author has discussed all of the various configurations of the Stern-Gerlach experiments—and thus showcasing the incompatibility of S_x, S_y, and S_z states—he describes the mathematics governing these kets. He does this by drawing analogies to spatial vectors: "These kets are abstract entities that obey many of the rules you know about ordinary spatial vectors. Hence they are called **quantum state vectors**" (McIntyre, 2012, p. 10). He then draws direct analogies to Cartesian unit vectors, introducing ideas such as normalization, orthogonality, and completeness, summarizing them as

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1 \qquad \textit{normalization}$$

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = 0 \qquad \textit{orthogonality}$$

$$\boldsymbol{A} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}} \qquad \textit{completeness}$$

and stating that

> we require that these same properties (at least conceptually) apply to quantum mechanical basis vectors. For the S_z measurement, there are only two possible results, corresponding to the states $|+\rangle$ and $|-\rangle$, so these two states comprise a

complete set of basis vectors. [...] The completeness of the basis kets $|\pm\rangle$ implies that a general quantum state vector $|\psi\rangle$ is a linear combination of the two basis kets: $|\psi\rangle = a|+\rangle + b|-\rangle$ (McIntyre, 2012, p. 11).

Once he has defined kets in analogy to Cartesian spatial vectors and discussed how completeness appears in Dirac notation, the author moves on to explain the other two features, combined together into orthogonality, which inherently required use of dot products for the Cartesian unit vectors discussed prior. Before discussing orthonormality, however, he first discusses complex numbers—as the a and b in the quote above are described as complex scalar numbers—and draws an analogy to vectors that show up in electromagnetism:

> When using complex numbers to describe classical vectors like electric and magnetic fields, the definition of the dot product is generalized slightly, such that one of the vectors is complex conjugated. A similar approach is taken in quantum mechanics. The analog to the complex conjugated vector of classical physics is called a **bra** in the Dirac notation of quantum mechanics. Thus corresponding to a general ket $|\psi\rangle$, there is a bra, or bra vector, which is written as $\langle\psi|$. If a general ket $|\psi\rangle$ is specified as $|\psi\rangle = a|+\rangle + b|-\rangle$, then the corresponding bra $\langle\psi|$ is defined as $\langle\psi| = a^*\langle+| + b^*\langle-|$, where the bras $\langle+|$ and $\langle-|$ correspond to the basis kets $|+\rangle$ and $|-\rangle$, respectively, and the coefficients a and b have been complex conjugated (McIntyre, 2012, pp. 11-2).

Here McIntyre defines the Dirac bra in direct analogy to a simple complex conjugate of a vector in classical physics. He then immediately moves on to discuss the scalar product of a bra and ket and discusses normality in analogy to Cartesian unit vectors once more:

> The scalar product in quantum mechanics is defined as the product of a bra and a ket taken in the proper order—bra first, then ket second: $(\langle bra|)(|ket\rangle)$. [...] [this] is written in shorthand as $\langle bra|ket\rangle$. [...] So how do we calculate the inner

> product $\langle + | + \rangle$? We do it the same way we calculated the dot product $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}$. […] So the normalization of the spin-1/2 basis vectors is expressed in this new notation as $\langle + | + \rangle = 1$ and $\langle - | - \rangle = 1$ (McIntyre, 2012, p. 12).

With normality covered, he then immediately moves on to discuss orthogonality, again in analogy to Cartesian spatial vectors:

> The spatial unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ used for spatial vectors are orthogonal to each other because they are at 90° with respect to each other. That orthogonality is expressed mathematically in the dot products $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = 0$. For the spin basis kets $| + \rangle$ and $| - \rangle$, there is no spatial geometry involved. Rather, the spin basis kets are orthogonal in the mathematical sense, which we express with the inner product as $\langle + | - \rangle = 0$. […] Though there is no geometry in this property for quantum mechanical basis vectors, the fundamental idea of orthogonality is the same, so we use the same language—if a general vector "points" in the direction of a basis vector, then there is no component in the "direction" of the other unit vectors (McIntyre, 2012, p. 12).

The final step to laying out the relevant formalism for our purposes is the discussion of the probabilistic interpretation of Dirac expressions. He first lays out the postulate that the complex square of the inner products $\langle + | \psi \rangle$ and $\langle - | \psi \rangle$ are the "[probabilities] that the state $| \psi \rangle$ is found to be in the [states] $| + \rangle$ [and $| - \rangle$] when a measurement of S_z is made, [respectively]" (McIntyre, 2012, p. 14). We will end this discussion of a spins-first approach at introducing Dirac notation with McIntyre's discussion of Dirac brackets being probability amplitudes:

> Because the quantum mechanical probability is found by squaring an inner product, we refer to an inner product, $\langle + | \psi \rangle$ for example, as a **probability amplitude** or sometimes just an **amplitude**; much like a classical wave intensity is

> found by squaring the wave amplitude. Note that the convention is to put the input or initial state on the right and the output or final state on the left: $\langle out|in\rangle$, so one would read from right to left in describing a problem (McIntyre, 2012, p. 15).

We would like to draw attention to the way that the author discusses these expressions by providing a sort of literal translation or way to read inner product expressions aloud. We now will discuss the way that these spins-first texts (using McIntyre as an example) introduces wave function notation and its analogous expressions.

A.2 Spins-first Introduction of Wave Function Notation

When first introducing expressions in wave function notation, McIntyre directly connects them to their Dirac notation analogs:

> The spatial functions we use to represent quantum states are called **wave functions** and are generally written using the Greek letter ψ as $\psi(x)$. The wave function is a representation of the abstract quantum state, so we can use our representation notation to write $|\psi\rangle \doteq \psi(x)$. We call this representation the **position representation**, which means that we are using the position eigenstates as the preferred basis […] For clarity, we will use the Greek letter ψ when referring to generic quantum states and other Greek letters to denote specific eigenstates. For example, in the case of the energy eigenstates, we write the wave functions representing them as $|E_i\rangle \doteq \varphi_{E_i}(x)$ to distinguish them as specific eigenstates (McIntyre, 2012, p. 111).

The ”representation notation” he describes is in reference to using the $\doteq$ symbol as meaning "is represented by," rather than "is equivalent to." This convention was introduced earlier in the text, and is used here because while $\psi(x)$ is the *analog* to $|\psi\rangle$ in wave function notation, they are not equivalent. When attempting to more rigorously define the position wave function, he draws parallels to representations of the ket $|\psi\rangle$ as a column vector in various bases, including a discrete (but infinite-dimensional) energy basis:

$$|\psi\rangle \doteq \begin{pmatrix} \langle E_1|\psi\rangle \\ \langle E_2|\psi\rangle \\ \langle E_3|\psi\rangle \\ \vdots \end{pmatrix}.$$

In analogy to this, he represents the ket $|\psi\rangle$ as a column vector in a discretized position basis:

$$|\psi\rangle \doteq \begin{pmatrix} \langle x_1|\psi\rangle \\ \langle x_2|\psi\rangle \\ \langle x_3|\psi\rangle \\ \vdots \end{pmatrix},$$

where $\langle x_i|\psi\rangle$ is "the probability amplitude for the state $|\psi\rangle$ to be measured in the position eigenstate $|x_i\rangle$" (McIntyre, 2012, p. 113). He then goes on to describe the necessary leap from discrete bases to continuous bases:

> Experiment tells us that the physical observable x is not quantized. Rather, *all values of position x are allowed*. […] For a continuous variable like position, the column vector representation is not convenient because we cannot write down the infinite number of components. Even if the number were infinite but large, say 100, then we would find a column vector cumbersome. Instead, we might choose to represent the 100 discrete numbers $\langle x_i|\psi\rangle$ as points in a graph, such as shown in [Figure A.2(a)]. However, because the position spectrum is continuous, there is an infinite continuum of the probability amplitudes $\langle x|\psi\rangle$, and the natural way of to represent such a continuous set of numbers is as a continuous function, as shown in [Figure A.2(b)]. This function is what we call the quantum mechanical wave function $\psi(x)$. The wave function is the collection of numbers that represents the quantum state vector in terms of the position eigenstates, in the same way that the column vector used to represent a general spin state is a collection of numbers that represents the quantum state vector in terms of the spin eigenstates (McIntyre, 2012, p. 113).

Here McIntyre uses column vector representations of the $|\psi\rangle$ ket to lead to an understanding of the wave function $\psi(x)$ being a collection of inner products $\langle x|\psi\rangle$ for every value of position x.

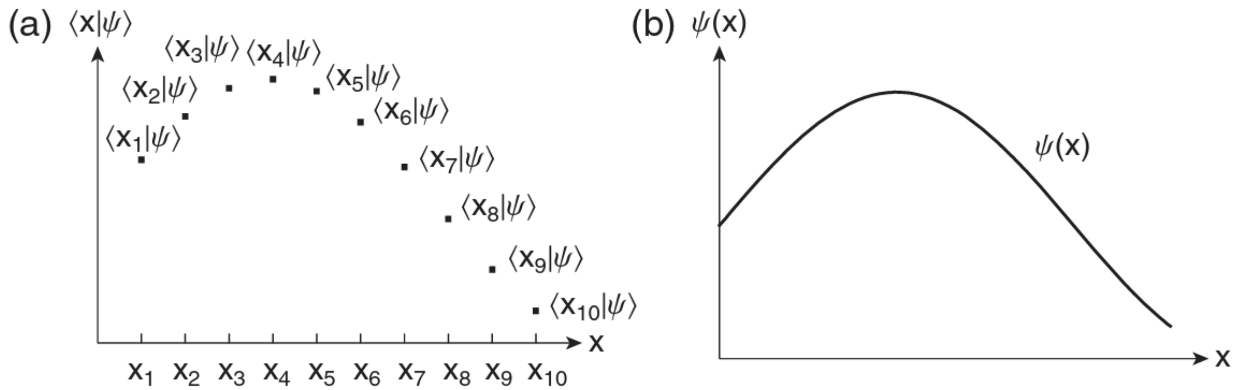

Figure A.2: Plots used by McIntyre (Fig. 5.4) to show the leap from (a) a representation for probability amplitudes for a discrete basis, and (b) a plot of probability density amplitude for a continuous basis.

In determining the appropriate expression for probabilities in wave function notation, McIntyre again returns to analogies in Dirac:

> Continuing with the analogy to the [discrete examples used prior], we expect that the probability of measuring a particular value of position is obtained by taking the absolute square of the projection $\langle x|\psi\rangle$, as was done […] for spin and energy representations. However, because the projection $\langle x|\psi\rangle$ is the continuous wave function $\psi(x)$, the absolute square yields a continuous probability function (actually a probability density, as we'll find in a moment), which we write as $\mathcal{P}(x)$ […] In wave function notation, this new probability function is $\mathcal{P}(x) = |\psi(x)|^2$ (McIntyre, 2012, p. 114).

He eventually uses the normalization condition for a continuous basis to show that $\mathcal{P}(x)$ is necessarily a probability density, and thus introduces the probability for a particle within a (one-dimensional) region from $a \to b$ as $\mathcal{P}_{a<x<b} = \int_a^b |\psi(x)|^2 dx$. He then explicitly connects the Dirac and wave function expressions for the normalization conditions by rewriting the wave

function version to "look more like the bra-ket form" (McIntyre, 2012, p. 115): $\langle\psi|\psi\rangle = 1$ and $\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = 1$, emphasizing the visual parallels between the two notations. Perhaps controversially, he then introduces four "rules" for translating Dirac expressions into wave function notation, as seen in Figure A.3. He uses these rules to generate the wave function equivalent of the probability of a particle to be measured with a single value of an observable:

> Using the rules for translating bra-ket notation to wave function notation, a general state vector projection or probability amplitude expressed in wave function language is $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}\phi^*(x)\psi(x)dx$. The square of this probability amplitude is the probability that the state $\psi(x)$ is measured to be in the state $\phi(x)$,
>
> $$\mathcal{P}_{\psi\to\phi} = |\langle\phi|\psi\rangle|^2 = \left|\int_{-\infty}^{\infty}\phi^*(x)\psi(x)dx\right|^2.$$
>
> Technically, we should say that this is the probability that the system prepared in state $\psi(x)$ is measured to have the physical observable for which $\phi(x)$ is the eigenstate, because we measure observables, not states. But the looser language is common and does not create any ambiguity in the calculation (McIntyre, 2012, pp. 116-7).

Here McIntyre is attempting to substantiate a visual logic for translating familiar (Dirac) expressions in from earlier in the course into their wave function notation equivalents.

1) Replace ket with wave function $|\psi\rangle \rightarrow \psi(x)$

2) Replace bra with wave function conjugate $\langle\psi| \rightarrow \psi^*(x)$

3) Replace bracket with integral over all space $\langle|\rangle \rightarrow \int_{-\infty}^{\infty} dx$

4) Replace operator with position representation $\hat{A} \rightarrow A(x)$

Figure A.3: The four rules laid out by McIntyre for translating a Dirac expression into its wave function equivalent.

McIntyre does notably also caution about the similarities between the expression for the probability of measuring a single discrete eigenvalue ($\mathcal{P}_{\psi\rightarrow\phi} = \left|\int_{-\infty}^{\infty} \phi^*(x)\psi(x)dx\right|^2$) and the probability of measuring a particle to be within a range of positions ($\mathcal{P}_{a<x<b} = \int_a^b |\psi(x)|^2 dx$):

> Note that [$\mathcal{P}_{a<x<b} = \int_a^b |\psi(x)|^2 dx$] and [$\mathcal{P}_{\psi\rightarrow\phi} = \left|\int_{-\infty}^{\infty} \phi^*(x)\psi(x)dx\right|^2$] look similar but have important differences. In [$\mathcal{P}_{a<x<b} = \int_a^b |\psi(x)|^2 dx$] we integrate the probability density (wave function complex squared) over a finite range of position in order to sum the probabilities of measuring many different positions. In [$\mathcal{P}_{\psi\rightarrow\phi} = \left|\int_{-\infty}^{\infty} \phi^*(x)\psi(x)dx\right|^2$] we integrate the product of two wave functions over all space to determine their mutual overlap, *and then we complex square* that result to get the probability of measuring a single result (McIntyre, 2012, p. 117).

This confusion about the similarities between these two different types of probability should be noted here, as it will appear within student data in the following chapters.

As can be seen by reading the excerpts above, McIntyre very much chooses to leverage the vector-like identity used for Dirac expressions in his introduction to wave function notation, including treating the wave function as a column vector containing elements and relating the inner product integral to ideas of projection. We will now look at the ways that the common text used in wave functions-first courses introduces these two notations, starting with its introduction to wave function notation.

A.3 Wave Functions-first Introduction of Wave Function Notation

The most ubiquitous text for wave functions-first courses is undoubtedly Griffiths' *Introduction to Quantum Mechanics* (Griffiths, 1995). This text begins by first discussing the way that classical mechanics would determine the position of a particle constrained to move along the x-axis while subject to some force, $x(t)$, by drawing upon Newton's second law (for conservative systems), $m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}$. He contrasts this with quantum mechanics:

> Quantum mechanics approaches this same problem very differently. In this case what we're looking for is the particle's **wave function**, $\Psi(x,t)$, and we get it by solving the **Schrödinger equation**: $i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial t^2} + V\Psi$. […] The Schrödinger equation plays a role logically analogous to Newton's second law: Given suitable initial conditions (typically $\Psi(x,0)$), the Schrödinger equation determines $\Psi(x,t)$ for all future time, just as, in classical mechanics, Newton's law determines $x(t)$ for all future time (Griffiths, 1995, pp. 1-2).

As can be seen with this excerpt, Griffiths immediately begins his book by analogizing the Schrödinger equation to Newton's second law, and the wave function as the solution to this foundational equation. He then goes on to describe the wave function, at least insofar as its role for determining probabilities for position ranges:

> But what exactly *is* this "wave function," and what does it do for you once you've *got* it? After all, a particle, by its nature, is localized at a point, whereas the wave function (as its name suggests) is spread out in space (it's a function of x, for any given time t). How can such an object represent the state of a *particle*? The

answer is provided by Born's **statistical interpretation** of the wave function, which says that $|\Psi(x,t)|^2$ gives the *probability* of finding the particle at point x, at time t—or, more precisely,

$\int_a^b |\psi(x)|^2 dx =$ {probability of finding the particle between a and b, at time t.}

Probability is the *area* under the graph of $|\Psi|^2$ (Griffiths, 1995, p. 2).

This introduction to the wave function and probabilities for position ranges is markedly different from the way spins-first courses introduce wave function notation—it is based entirely on positing rules and equations and observing the mathematical effects of those rules and equations. Of course, this mirrors much closer the way that spins-first courses introduce Dirac notation—whichever notation is introduced first is in the unfortunate position of needing to be explained without having another (quantum mechanical) description to compare it to, after all. While spins-first courses were able to use the nature of vectors and geometry to analogize the Dirac formalism, wave function notation does not have as obvious a supportive analogy, aside from an understanding of solutions to differential equations.

A.4 Wave Functions-first Introduction of Dirac Notation

After spending a chapter discussing wave function solutions to various potential wells and their allowed energy levels, Griffiths begins to draw parallels between these solutions and describe general mathematical properties of quantum physics:

Quantum theory is based on two constructs: *wave functions* and *operators*. The state of a system is represented by its wave function, observables are represented by operators. Mathematically, wave functions satisfy the defining conditions for abstract **vectors**, and operators act on them as **linear transformations**. So the natural language of quantum mechanics is **linear**

algebra. But it is not, I suspect, a form of linear algebra with which you are immediately familiar. In an N-dimensional space it is simplest to represent a vector, $|\alpha\rangle$, by the N-tuple of its components, $\{a_n\}$, with respect to a specified orthonormal basis:

$$|\alpha\rangle \rightarrow \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

The **inner product**, $\langle\alpha|\beta\rangle$, of two vectors (generalizing the dot product in three dimensions) is the complex number, $\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_N^* b_N$. […] But the "vectors" we encounter in quantum mechanics are (for the most part) *functions*, and they live in *infinite*-dimensional spaces. […] We define the **inner product of two functions**, $f(x)$ and $g(x)$, as follows: $\langle f|g\rangle \equiv \int_{-\infty}^{\infty} f(x)^* g(x) dx$ (Griffiths, 1995, pp. 93-5).

Griffiths initially defines what look to be expressions in Dirac notation in the context of discretized vector spaces, but immediately moves to describe these expressions (not yet referred to as kets or brackets) as relating to functions.

He also very briefly touches on the coefficients of expansion of the wave function into the eigenfunctions for a given potential well being related to probability when first discussing the wave function solutions for the infinite square well potential. He returns to fully discuss probabilities of measuring values of observables other than position here, now that he has introduced inner products.

If you measure an observable $Q(x,p)$ on a particle in the state $\Psi(x,t)$, you are certain to get *one of the eigenvalues* of the Hermitian operator $\hat{Q}(x, -i\hbar d/dx\,)$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting a particular eigenvalue q_n associated with the orthonormalized eigenfunction $f_n(x)$ is

$$|c_n|^2, \quad \text{where} \quad c_n = \langle f_n|\Psi\rangle.$$

If the spectrum is continuous, with real eigenvalues $q(z)$ and associated Dirac-orthonormalized eigenfunctions $f_z(x)$, the probability of getting a result in the range dz is

$$|c(z)|^2 dz \quad \text{where} \quad c(z) = \langle f_z|\Psi\rangle$$ (Griffiths, 1995, p. 106).

After some discussion of the completeness and orthonormality of these eigenfunctions, Griffiths then discusses the coefficients c_n, writing the wave function as a (discrete) weighted sum of the eigenfunctions as $\Psi(x,t) = \sum_n c_n f_n(x)$. He then recapitulates the expression for the coefficient and discusses its probabilistic interpretation:

$$c_n = \langle f_n|\Psi\rangle = \int f_n(x)^* \Psi(x,t)dx$$

Qualitatively, c_n tells you "how much f_n is contained in Ψ," and given that a measurement has to return one of the eigenvalues of $\hat{Q}$, it seems reasonable that the probability of getting the particular eigenvalue q_n would be determined by the "amount of f_n" in Ψ. But because probabilities are determined by the absolute *square* of the wave function, the precise measure is actually $|c_n|^2$ (Griffiths, 1995, p. 107).

It isn't until later in this chapter that Griffiths directly discusses Dirac notation, and later still that he introduces the terms "bra" and "ket." He first introduces the state vector:

[The state of a system in quantum mechanics] is represented by a *vector*, $|\mathcal{S}(t)\rangle$, that lives "out there in Hilbert space," but we can *express* it with respect to any number of different *bases*. The wave function $\Psi(x,t)$ is actually the coefficient in the expansion of $|\mathcal{S}\rangle$ in the basis of position eigenfunctions:

$$\Psi(x,t) = \langle x|\mathcal{S}(t)\rangle,$$

(with $|x\rangle$ standing for the eigenfunction of $\hat{x}$ with eigenvalue x) […] Or we could expand $|\mathcal{S}\rangle$ in the basis of energy eigenfunctions (supposing for simplicity that the spectrum is discrete):

$$c_n(t) = \langle n|\mathcal{S}(t)\rangle$$

(with $|n\rangle$ standing for the nth energy eigenfunction of $\hat{H}$) […] but it's all the same state; the function Ψ […] and the collection of coefficients $\{c_n\}$ contain exactly the same information—they are simply three different ways of describing the same vector (Griffiths, 1995, p. 119).

Interestingly, in a footnote, Griffiths discusses his earlier description of Hilbert spaces as the set of square-integrable functions as being "too restrictive, committing us to a specific representation (the position basis). I want now to think of it as an abstract vector space, whose members can be expressed with respect to any basis you like" (Griffiths, 1995, p. 119).

It is later that he introduces the terms "bra" and "ket":

Dirac proposed to chop the bracket notation for the inner product, $\langle\alpha|\beta\rangle$, into two pieces, which he called **bra**, $\langle\alpha|$, and **ket**, $|\beta\rangle$ […] The latter is a vector, but what exactly is the former? […] In a function space, the bra can be thought of as an instruction to integrate:

$$\langle f| = \int f^*[\cdots]\,dx,$$

with the ellipsis $[\cdots]$ waiting to be filled by whatever function the bra encounters in the ket to its right. In a finite-dimensional vector space, with the vectors expressed as columns,

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

the corresponding bra is a row vector:

$$\langle\alpha| = (a_1^* \quad a_2^* \quad \cdots \quad a_n^*).$$

The collection of all bras constitutes another vector space—the so-called **dual space**. The license to treat bras as separate entities in their own right allows for some powerful and pretty notation (though I shall not exploit it in this book) (Griffiths, 1995, p. 122).

As the author highlights in that closing parenthetical, this is essentially the extent to which Dirac notation is used in his text (and thus presumably the extent to which Dirac notation is used in wave functions-first courses). It is not used much for the rest of the text.

It is notable that, much like spins-first courses use Dirac notation and its vector-like interpretation to help their students transition to using wave functions, Griffiths first describes inner products (with the appearance of Dirac brackets but without the name at that point) in terms of wave functions and overlap integrals. Understandably, both texts seek to leverage the knowledge and notational understanding developed over the first several chapters to interpret and make sense of the newly introduced notation. Aside from a very quick dalliance in treating kets (again, not named as such initially) as vectors, they are largely taught as representing functions. It is not until the very end of the chapter aimed at discussing and introducing Dirac formalism that bras and kets are named—and in the case of bras, that they are discussed at all.

APPENDIX B: QUANTUM MECHANICS TEXTBOOK CONTENTS

Below are pasted the tables of contents for the three texts used by students surveyed. First is the table of contents for the text by McIntyre, then Townsend, then Griffiths.

Figure B.1: Table of contents for the textbook by McIntyre (McIntyre, 2012).

Figure B.2: Contents of chapters 1-3 of Townsend's text (Townsend, 2000).

Figure B.3: Contents of chapters 4-6 of Townsend's text (Townsend, 2000).

Figure B.4: Contents of chapters 7-9 of Townsend's text (Townsend, 2000).

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Figure B.5: Contents of chapters 10-13 of Townsend's text (Townsend, 2000).

Figure B.6: Contents of chapter 14 and the appendices of Townsend's text (Townsend, 2000).

PART I THEORY

Figure B.7: Contents of chapters 1-5 of Griffiths' text (Griffiths, 1995).

PART II APPLICATIONS

Figure B.8: Contents of chapters 6-10 of Griffiths' text (Griffiths, 1995).

Figure B.9: Contents of chapters 11, 12, and the appendices of Griffiths' text (Griffiths, 1995).

APPENDIX C: CONCEPT-EXPRESSION CHARTS ACROSS CURRICULA

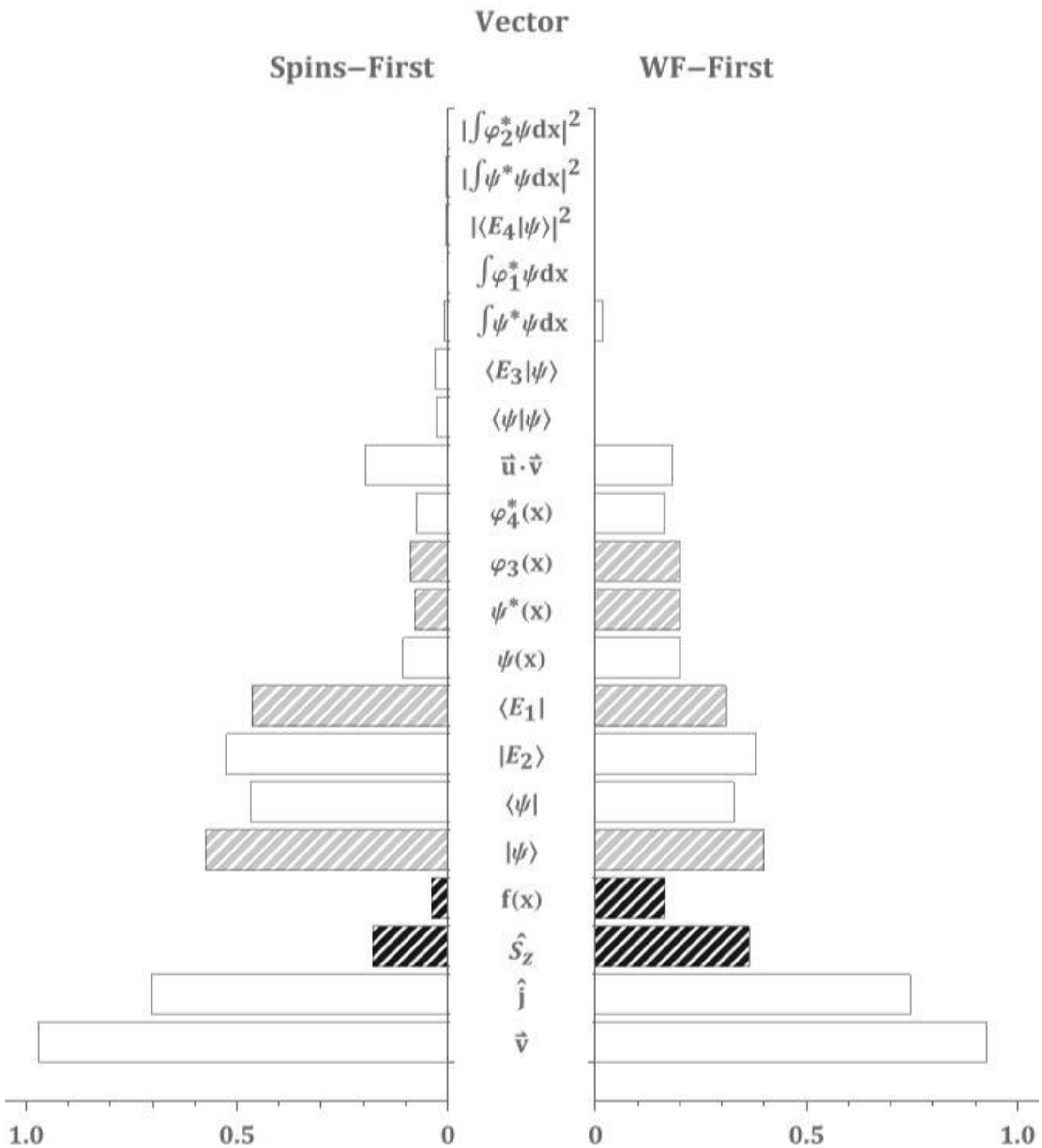

Figure C.1: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "vector" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

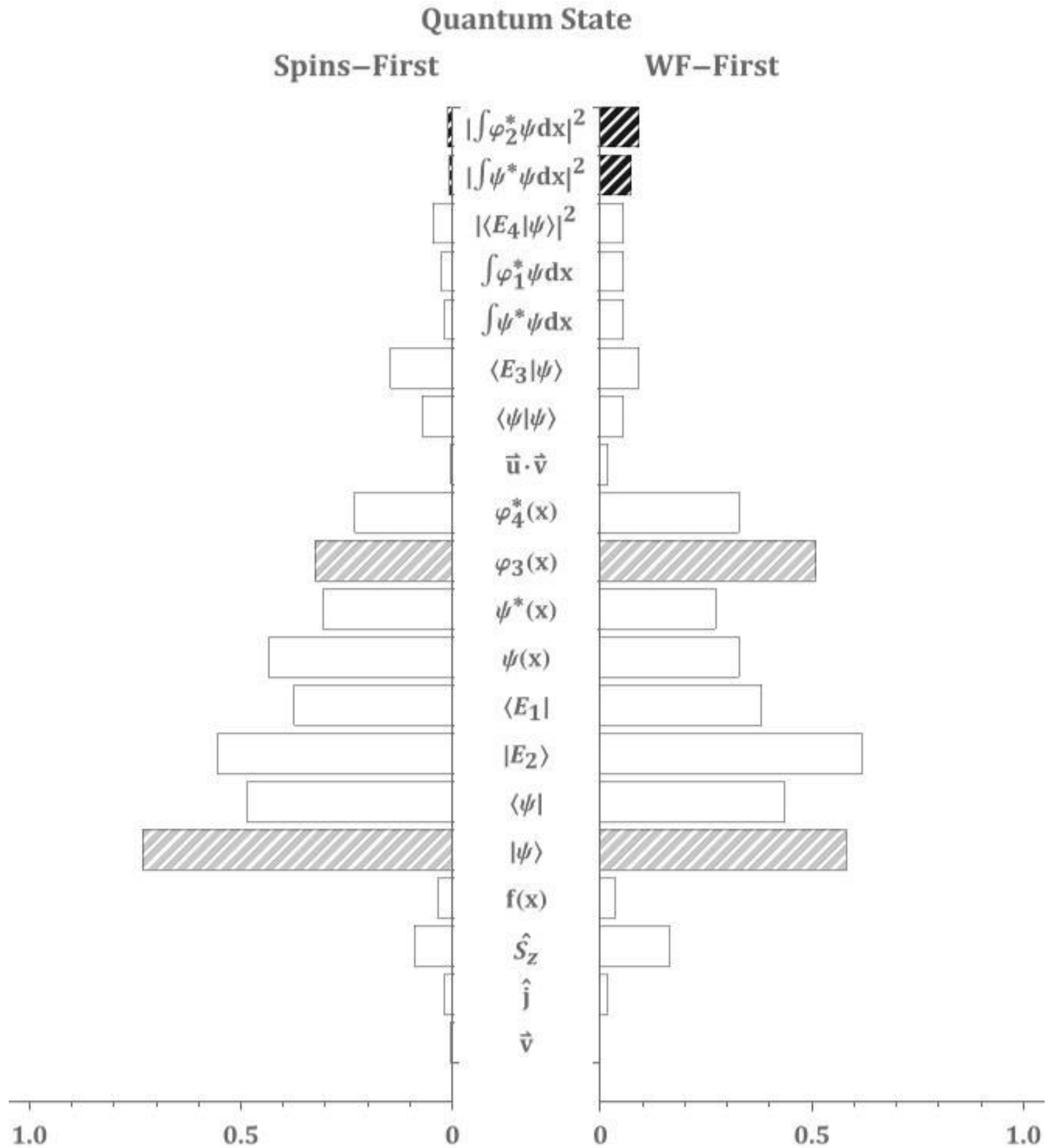

Figure C.2: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "quantum state" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

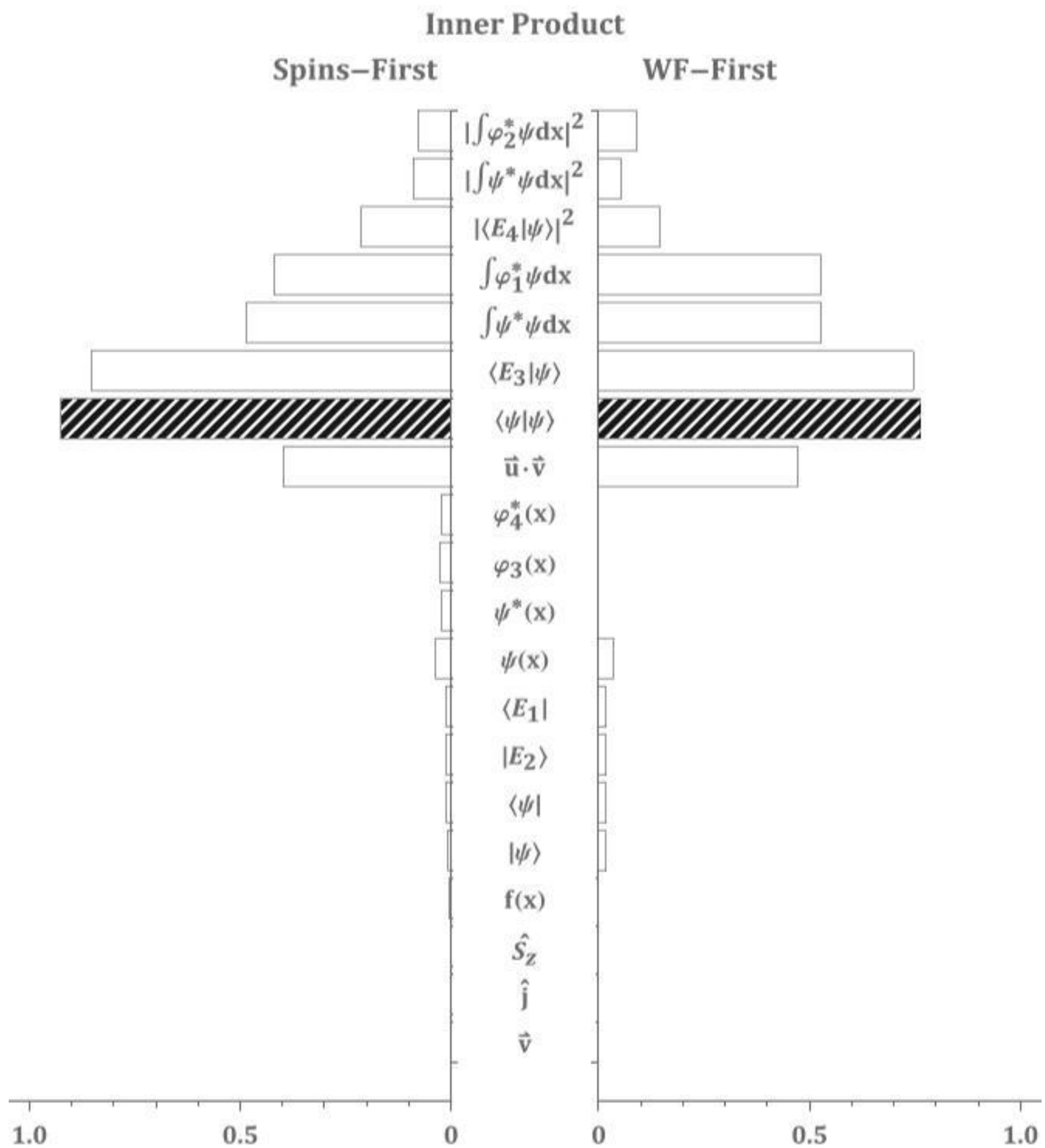

Figure C.3: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "inner product" concept. The black-shaded bar represents a statistically significant difference between the two populations' responses to a p-value of <0.01. The statistically significant difference here has 0.1<ϕ<0.3 (small effect size).

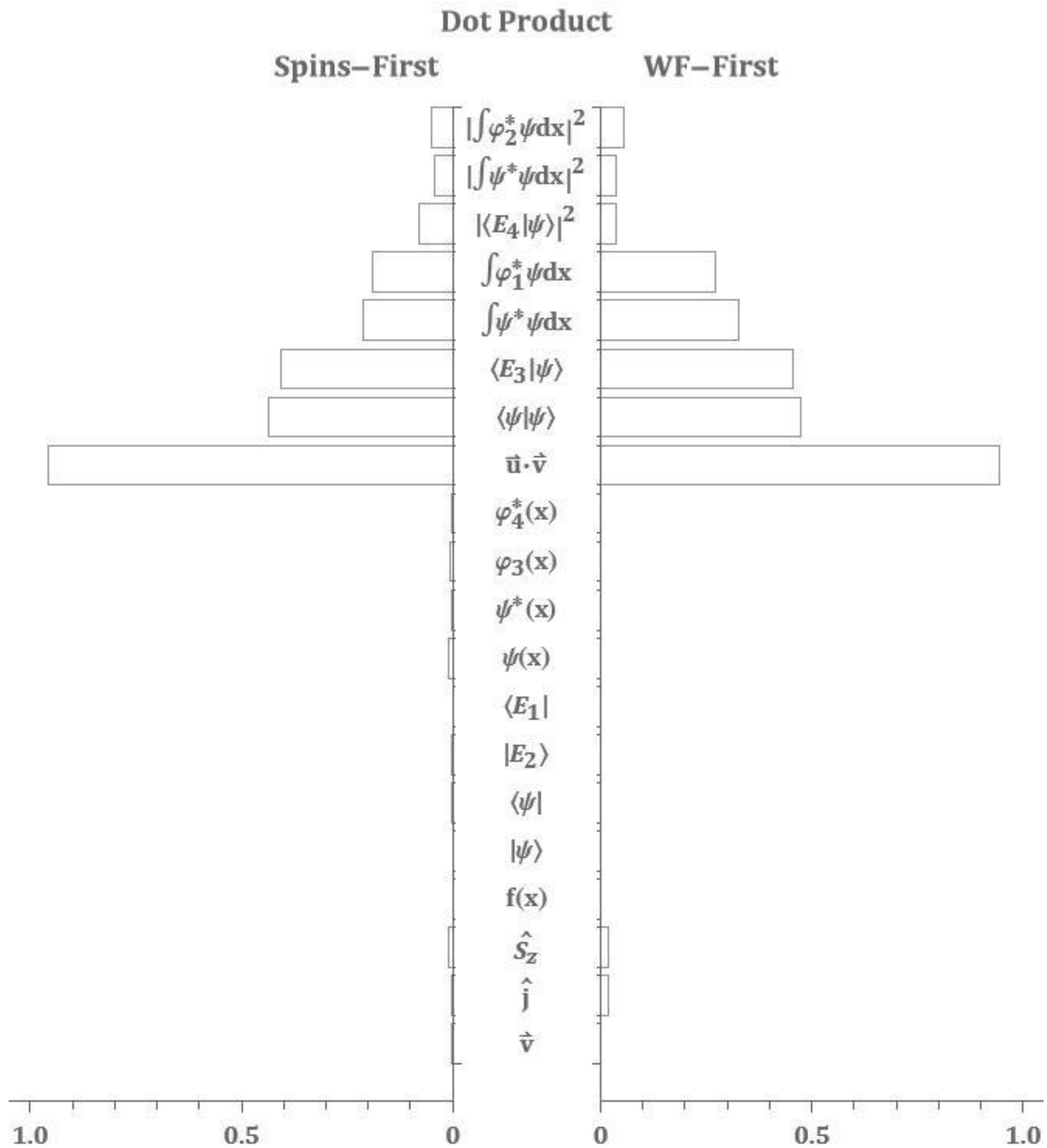

Figure C.4: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "dot product" concept.

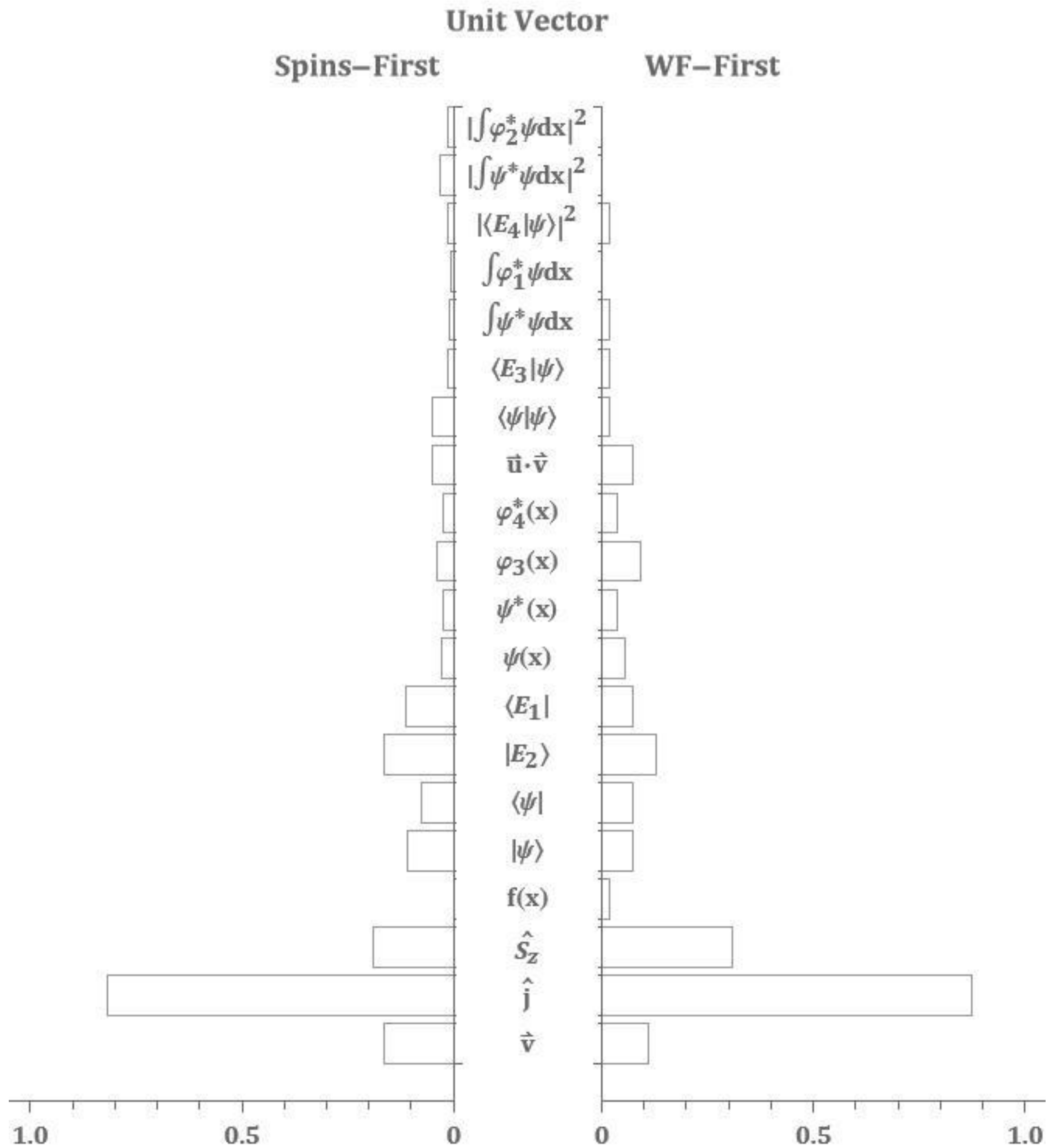

Figure C.5: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "unit vector" concept.

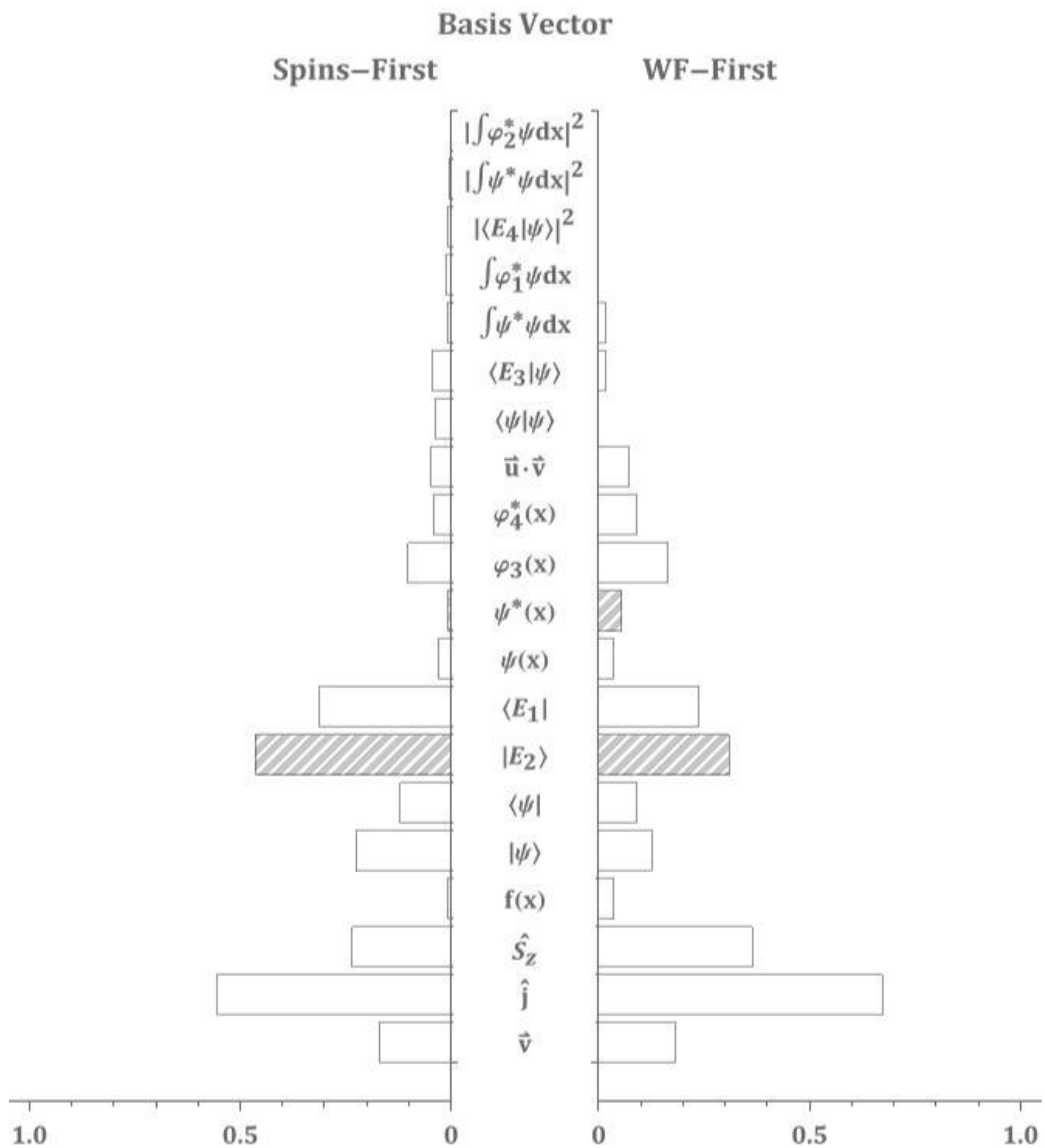

Figure C.6: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "basis vector" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

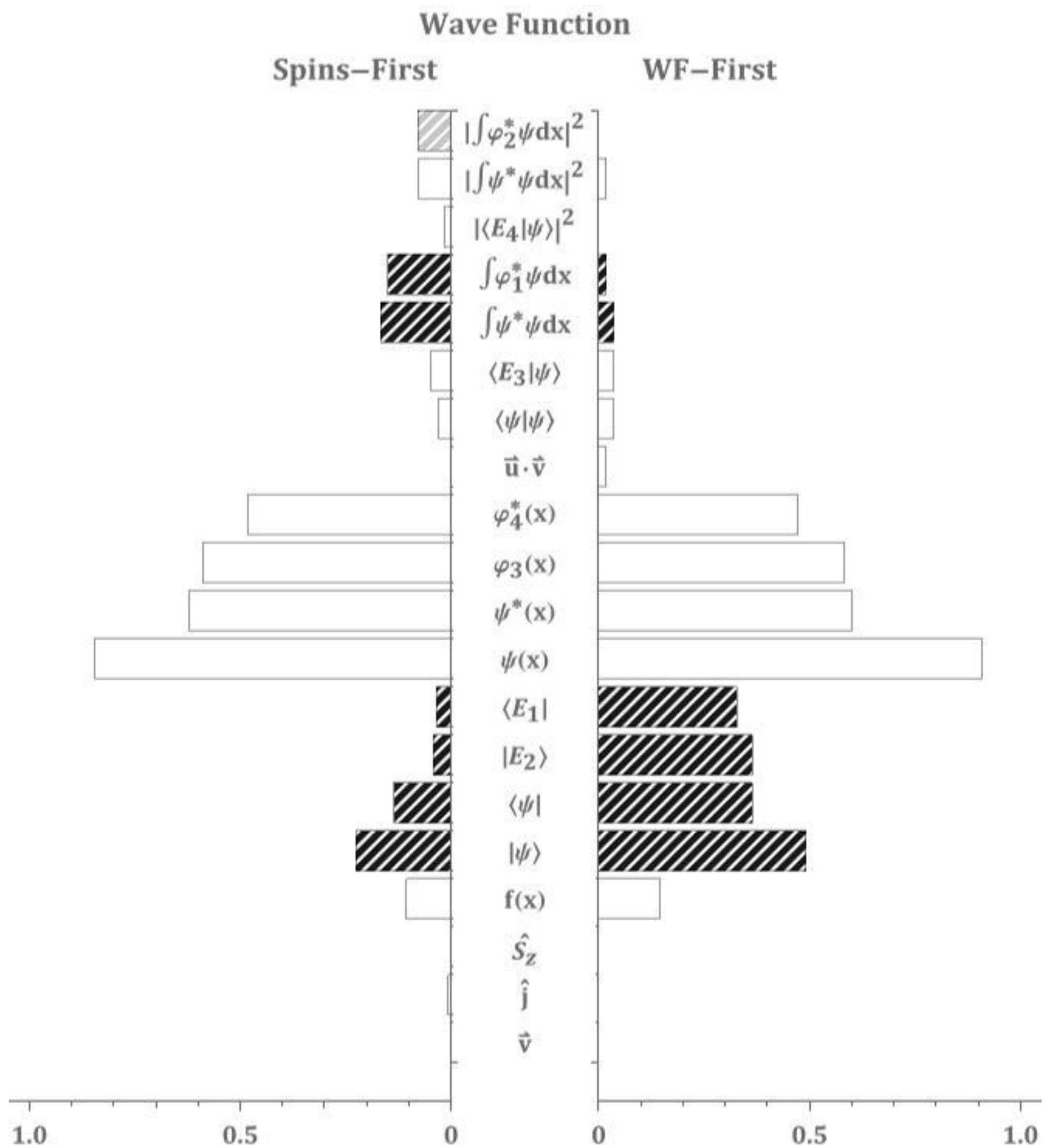

Figure C.7: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "wave function" concept. The gray-shaded bar represents a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. For $\langle E_1|$ and $|E_2\rangle$, 0.3<ϕ<0.5 (medium effect size), and the other statistically significant differences have 0.1<ϕ<0.3 (small effect size).

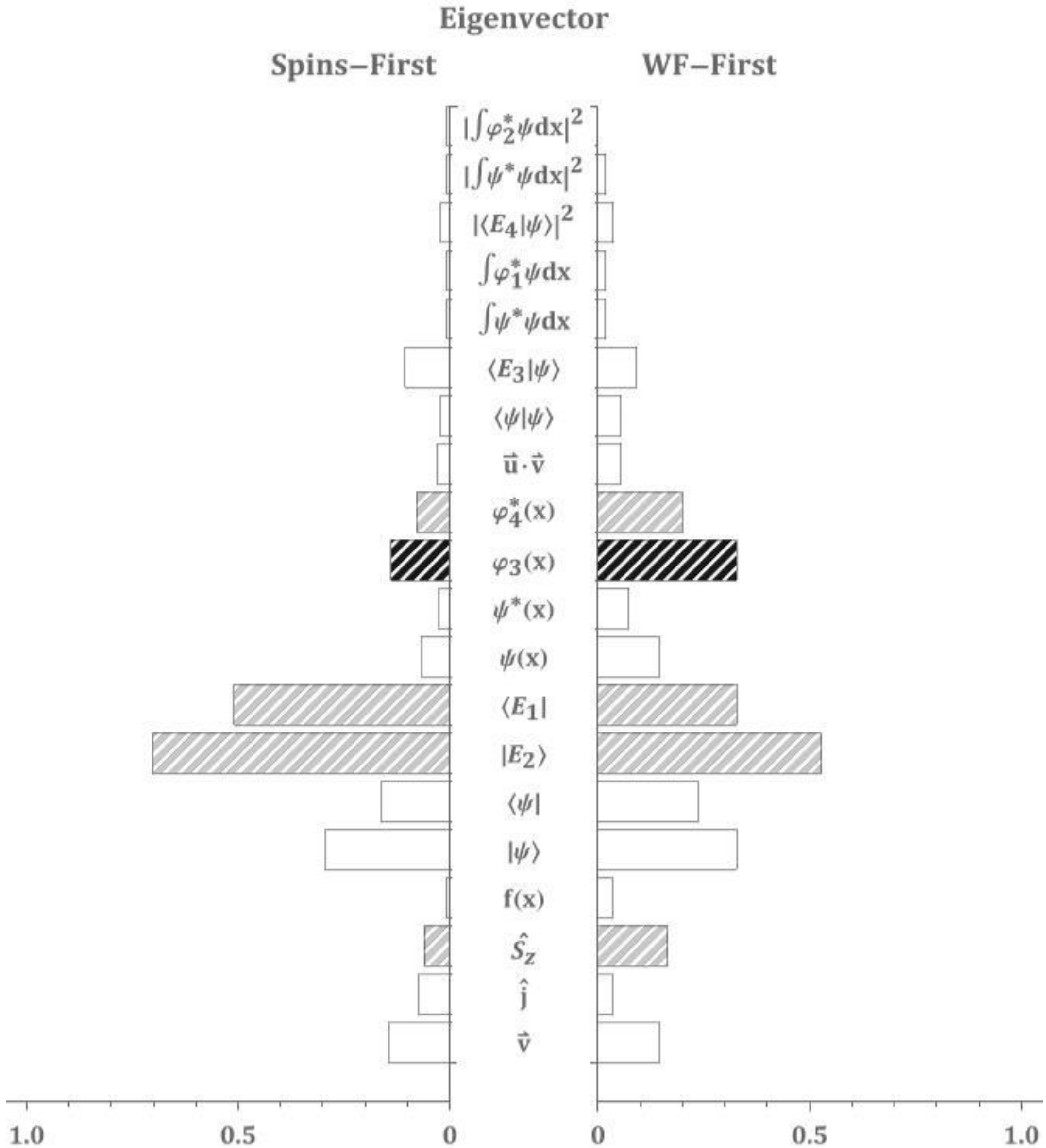

Figure C.8: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "eigenvector" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bar represents a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

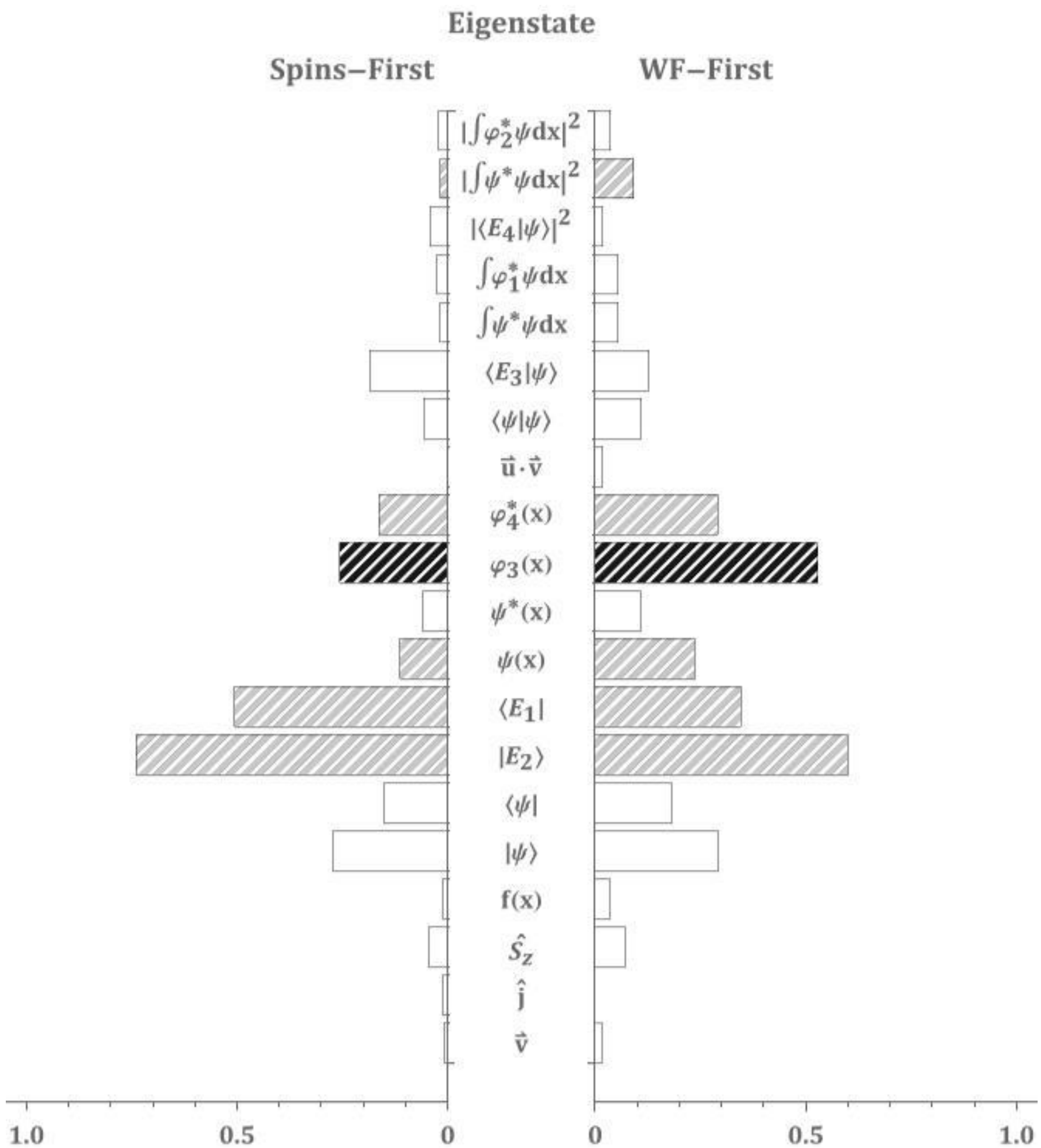

Figure C.9: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "eigenstate" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bar represents a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

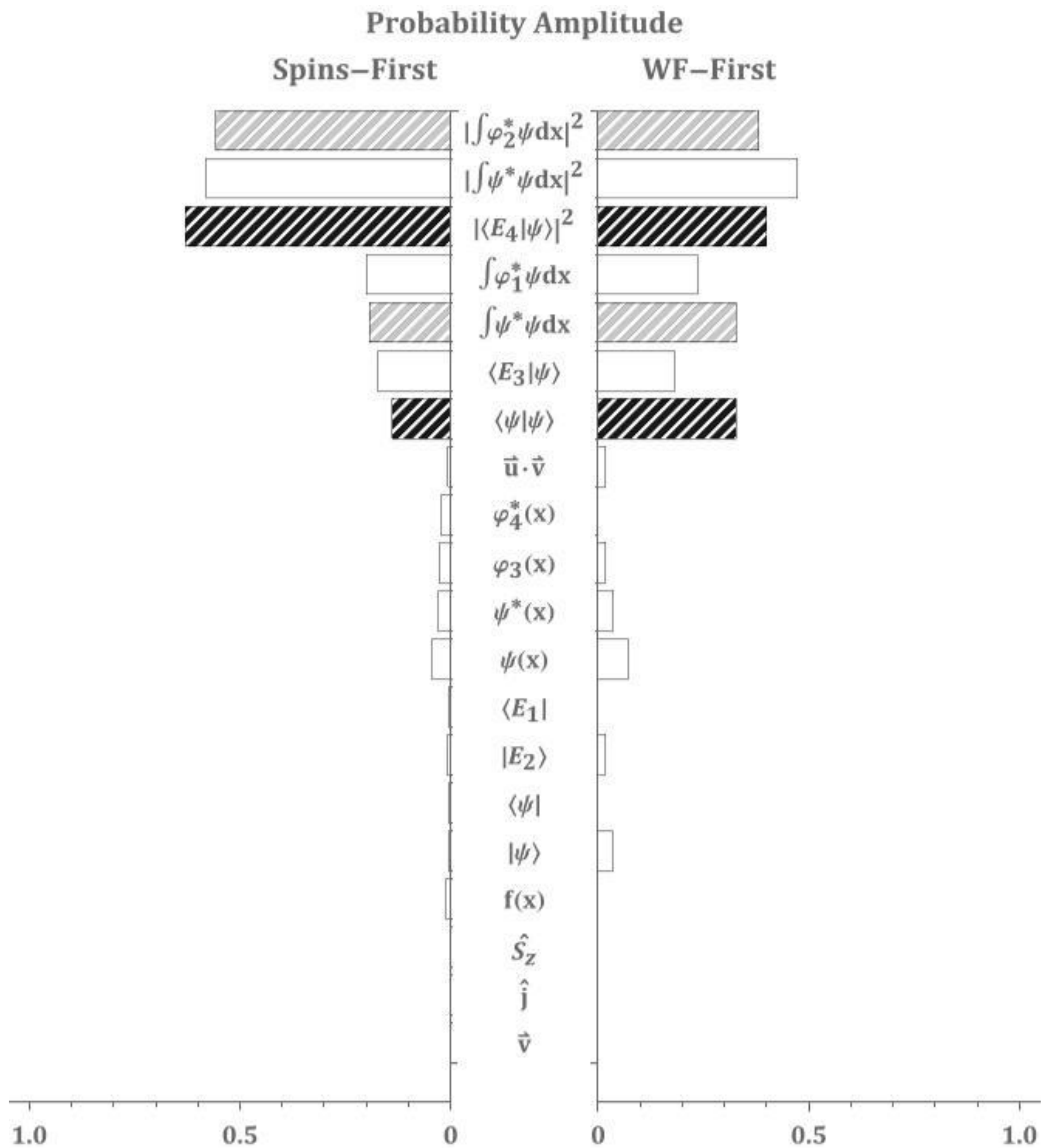

Figure C.10: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability amplitude" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

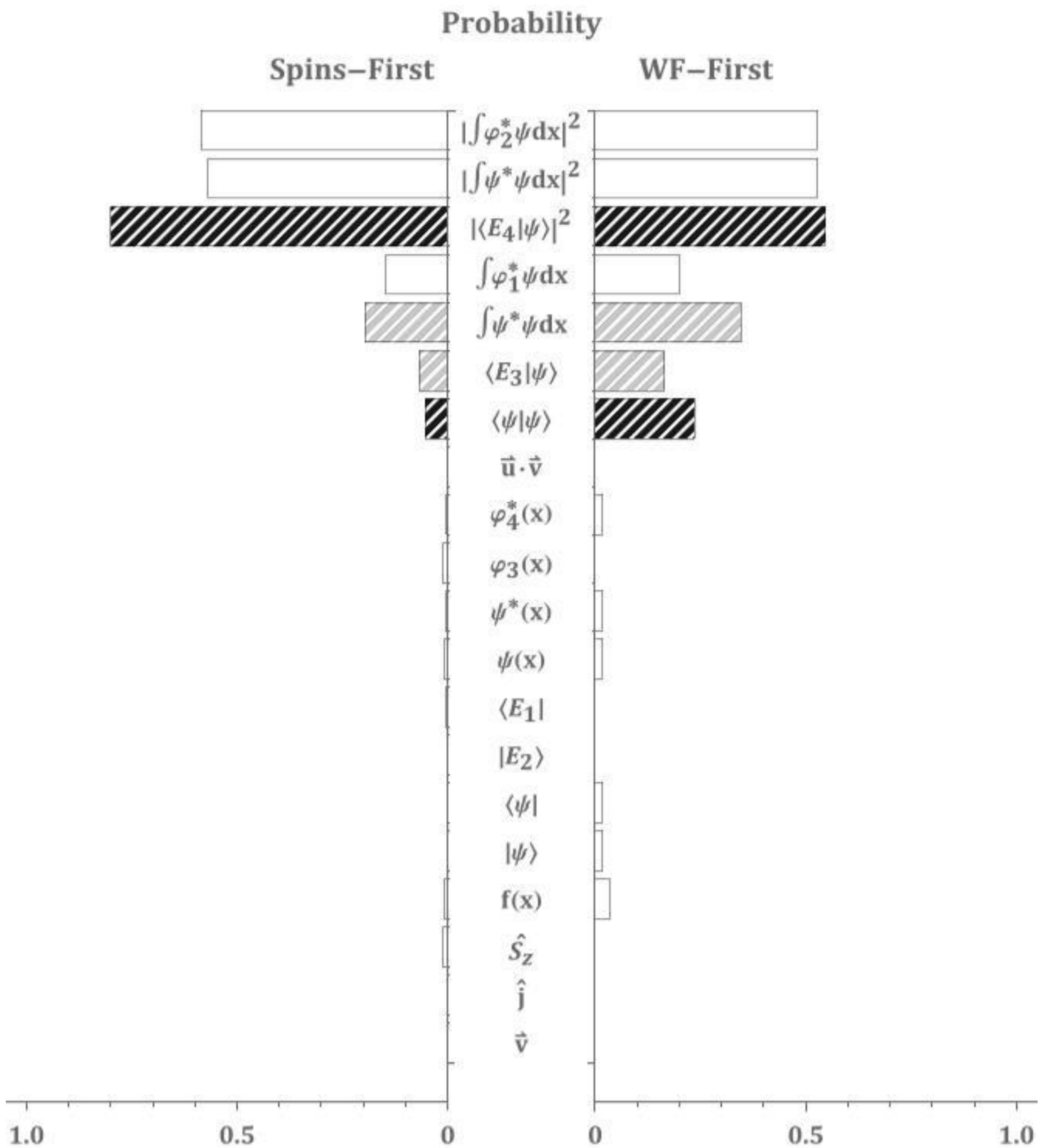

Figure C.11: Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability" concept. The gray-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.05, while the black-shaded bars represent a statistically significant difference between the two populations' responses to a p-value of <0.01. All statistically significant differences here have 0.1<ϕ<0.3 (small effect size).

APPENDIX D: FISHER'S EXACT SCORES AND EFFECT SIZES COMPARING EXPRESSIONS CHOSEN FOR CONCEPTS ACROSS CURRICULA

Table D.1: Data table containing the p-values calculated from the Fisher's Exact test comparing the two curricula's expression selection for each concept. Cells containing p-values less than 0.05 are shaded, and include the ϕ-values (effect sizes) in parentheses beneath the p-values for those cells. $\phi > 0.5$ is representative of a large effect size, $0.3 < \phi < 0.5$ is a medium effect size, and $0.1 < \phi < 0.3$ is a small effect size.

	Vector	Quantum State	Inner Product	Dot Product	Unit Vector	Basis Vector	Wave Function	Eigenvector	Eigenstate	Probability Amplitude	Probability
$\vec{v}$	0.133	1	N/A	1	0.412	0.845	N/A	1	0.433	N/A	N/A
$\hat{j}$	0.625	1	N/A	0.315	0.433	0.134	1	0.391	1	N/A	N/A
$\hat{S}_z$	0.003 (0.17)	0.14	N/A	0.532	0.067	0.061	N/A	0.022 (0.15)	0.492	N/A	1
$f(x)$	0.002 (0.2)	1	1	N/A	0.172	0.138	0.48	0.138	0.205	1	0.138
$\|\psi\rangle$	0.025 (0.13)	0.034 (0.12)	0.433	N/A	0.625	0.144	<0.001 (0.22)	0.63	0.743	0.078	1
$\langle\psi\|$	0.073	0.554	0.532	1	1	0.647	<0.001 (0.22)	0.24	0.545	1	1
$\|E_2\rangle$	0.075	0.455	0.532	1	0.683	0.037 (0.12)	<0.001 (0.41)	0.017 (0.14)	0.048 (0.12)	0.433	N/A
$\langle E\|$	0.037 (0.12)	1	0.532	N/A	0.476	0.332	<0.001 (0.4)	0.017 (0.14)	0.038 (0.12)	1	1
$\psi(x)$	0.068	0.176	1	1	0.385	0.684	0.292	0.098	0.027 (0.14)	0.492	0.433
$\psi^*(x)$	0.012 (0.15)	0.747	0.595	1	0.63	0.037 (0.14)	0.762	0.101	0.236	0.684	0.315
$\varphi_3(x)$	0.03 (0.13)	0.013 (0.15)	0.608	1	0.15	0.238	1	0.002 (0.22)	<0.001 (0.22)	1	1
$\varphi_4^*(x)$	0.066	0.168	0.595	1	0.63	0.165	1	0.012 (0.13)	0.035 (0.13)	0.595	0.315

Table D.1 Continued.

$\vec{u}\cdot\vec{v}$	1	0.315	0.296	0.715	0.507	0.507	0.172	0.41	0.172	0.433	N/A
$\langle\psi\|\psi\rangle$	0.608	1	0.001 (0.2)	0.655	0.478	0.221	0.684	0.188	0.225	0.002 (0.19)	<0.001 (0.25)
$\langle E_3\|\psi\rangle$	0.359	0.389	0.071	0.55	1	0.705	1	1	0.435	0.847	0.031 (0.13)
$\int \psi^*\psi dx$	0.433	0.142	0.657	0.078	0.532	0.433	0.01 (0.14)	0.433	0.142	0.032 (0.12)	0.02 (0.14)
$\int \varphi_1^*\psi dx$	N/A	0.385	0.179	0.196	1	1	0.004 (0.15)	0.433	0.385	0.583	0.314
$\|\langle E_4\|\psi\rangle\|^2$	1	0.729	0.275	0.381	1	1	1	0.63	0.699	0.002 (0.18)	<0.001 (0.22)
$\left\|\int \psi^*\psi dx\right\|^2$	1	0.009 (0.18)	0.593	1	0.359	1	0.143	0.433	0.016 (0.16)	0.179	0.654
$\left\|\int \varphi_2^*\psi dx\right\|^2$	N/A	0.005 (0.19)	0.787	0.744	1	N/A	0.032 (0.12)	1	0.63	0.018 (0.13)	0.456

APPENDIX E: INTERVIEW PROTOCOLS

Below is an example of the interview protocols used for the in-person interviews. The tasks used for the virtual interviews are shown in full in Section 3.3.1.

Prompt: "How would you express the probability for an electron within a potential well to be measured as having the ground state energy of that well?"

A1: $|\langle E_1|\psi\rangle|^2$

Q1: And why that? What do the E_1 and ψ represent?

A1: The E_1 is the state with the ground state energy, and the psi is the initial state.

Q1: Why did you write them like that? What are the angles on the sides, or the lines in the middle/on the sides? Why is it squared?

A1: The angles show that it is an inner product between the two states, E_1 and psi, and the square is because we want a probability.

A2: It's an inner product between the two states, E_1 and psi.

Q1: What do you mean by "states"?

A1: E_1 is the state of a particle in the ground state, and psi is the actual state of the electron.

Q2: What do you mean by "inner product"?

A1: Like a dot product, where you do the projection of one along another.

Q1: I see. Dot products usually are done with vectors. Is that the case here?

A1: Yeah, in QM we treat different quantum states as vectors, and the inner products between them are probabilities.

Q3: What do you mean by the "states" E_1 and psi?

A1: Like, if the particle were measured with the ground state energy, it would be in the E_1 state, and before we know what state it is in, we call it psi.

Q2: What are the angles on the sides, or the lines in the middle/on the sides? Why is it squared?

A1: The angles show that it is an inner product between the two states, E_1 and psi, and the square is because we want a probability.

A2: $\langle E_1|\psi\rangle$

Q1: And why that? What do the E_1 and ψ represent?

A1: The E_1 is the state with the ground state energy, and the psi is the initial state.

Q1: Why did you write them like that? What are the angles on the sides, or the lines in the middle/on the sides? Why is it squared?

A1: The angles show that it is an inner product between the two states, E_1 and psi, and the square is because we want a probability.

A2: It's an inner product between the two states, E_1 and psi.

Q1: What do you mean by "states"?

A1: E_1 is the state of a particle in the ground state, and psi is the actual state of the electron.

Q2: What do you mean by "inner product"?

A1: Like a dot product, where you do the projection of one along another.

Q1: I see. Dot products usually are done with vectors. Is that the case here?

A1: Yeah, in QM we treat different quantum states as vectors, and the inner products between them are probabilities.

Q3: What do you mean by the "states" E_1 and psi?

A1: Like, if the particle were measured with the ground state energy, it would be in the E_1 state, and before we know what state it is in, we call it psi.

A3: $|\int \varphi_1^*(x)\psi(x)dx|^2$

Q1: And why that? What do the $\varphi_1^*(x)$ and $\psi(x)$ represent?

A1: $\varphi_1^*(x)$ is the state with the ground state energy, and $\psi(x)$ is the initial state.

Q1: Why are they in an integral?

A1: Because that's just how it works?

A2: Because that's what inner products look like in position.

A4: $\int \varphi_1^*(x)\psi(x)dx$

A5: $\hat{H}|\psi\rangle$

A6: $\hat{H}|E_1\rangle$

A7: $\langle E_1|\hat{H}|\psi\rangle$

Prompt: "Let's say we have an electron in a potential well—perhaps an infinite square well. If we know that it has an even 33% chance of having any of the three lowest possible measurable energy values for that well, how could you express its current quantum state mathematically?"

A1: $|\psi\rangle = \frac{1}{\sqrt{3}}|E_1\rangle + \frac{1}{\sqrt{3}}|E_2\rangle + \frac{1}{\sqrt{3}}|E_3\rangle$

Q1: Why are you adding these together?

A2: $\psi(x) = \sqrt{\frac{2}{3L}} sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{2}{3L}} sin\left(\frac{2\pi x}{L}\right) + \sqrt{\frac{2}{3L}} sin\left(\frac{3\pi x}{L}\right)$

Prompt: Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. It is currently in the superposition state described by (for McIntyre/Townsend students):

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$$

for Griffiths students:

$$|\Psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|\psi_1\rangle + |\psi_2\rangle + 2|\psi_3\rangle\right)$$

PROMPT 1: How would you go about finding the probability of measuring that particle to be in the left half of the square well?

A1: $\psi(x) = \frac{1}{2\sqrt{\mathbf{2}}}\left(\sqrt{3}\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L}\right) \;\rightarrow\; \int_0^{L/2} |\psi(x)|^2 dx$

A2: $\psi(x) = \frac{1}{2\sqrt{\boldsymbol{L}}}\left(\sqrt{3}\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L}\right) \;\rightarrow\; \int_0^{L/2} |\psi(x)|^2 dx$

A3: $\psi(x) = \frac{1}{2\sqrt{\mathbf{2}}}\left(\sqrt{3}\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L}\right) \;\rightarrow\; \int_0^{L/2} \psi^*(x)\psi(x)dx$

A3: $\psi(x) = \frac{1}{2\sqrt{\boldsymbol{L}}}\left(\sqrt{3}\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L}\right) \;\rightarrow\; \int_0^{L/2} \psi^*(x)\psi(x)dx$

A4: $\psi(x) = \frac{1}{2\sqrt{\mathbf{2}}}\left(\sqrt{3}\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L} + 2\sin\frac{3\pi x}{L}\right) \;\rightarrow\; \left|\int_0^{L/2} \psi^*(x)\psi(x)dx\right|^2$

A5: 50%, because half of the well

PROMPT 2: How would you go about finding the probability of measuring that particle to be in the lowest energy state?

A1: $|\langle E_1|\psi\rangle|^2 = {}^3/_8 = 37.5\%$

A2: $\langle E_1|\psi\rangle = \frac{\sqrt{3}}{2\sqrt{2}}$

A3: $|\int \varphi_1^*(x)\psi(x)dx|^2$

A4: $\int \varphi_1^*(x)\psi(x)dx$

Prompt: Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. The particle is described by the following wave function (for McIntyre/Townsend students):

$$\psi(x) = \frac{4}{\sqrt{5L}} \sin^3\left(\frac{4\pi x}{L}\right)$$

For Griffiths students:

$$\Psi(x) = \frac{4}{\sqrt{5L}} \sin^3\left(\frac{4\pi x}{L}\right)$$

PROMPT 1: How would you go about finding the probability of measuring that particle to be in the lowest energy state?

A1: $|\int \varphi_1^*(x)\psi(x)dx|^2$

A2: $\int \varphi_1^*(x)\psi(x)dx$

A3: $|\langle E_1|\psi\rangle|^2 \;\rightarrow\; |\int \varphi_1^*(x)\psi(x)dx|^2$

A4: $\langle E_1|\psi\rangle \;\rightarrow\; \int \varphi_1^*(x)\psi(x)dx$

PROMPT 2: How would you go about finding the probability of measuring that particle to be on the left half of the square well?

A1: $\int_0^{L/2} |\psi(x)|^2 dx$

A2: $\int_0^{L/2} \psi^*(x)\psi(x)dx$

A3: $\left|\int_0^{L/2} \psi^*(x)\psi(x)dx\right|^2$

Bonus Prompt: How would you write the wave function $\psi(x)$ in Dirac Notation?

Bonus Prompt: How would you write the state vector $|\psi\rangle$ in the position representation?

APPENDIX F: SURVEY TASK

Below is a screenshot of a single question asked on the online surveys. All other questions were identical to this, with the only differences being that the order of the expressions were randomized and the concepts would differ, also being asked in random order throughout the survey. The first two images are screenshots of the first two blocks on the survey, which consisted of an informed consent slide and a slide to explain the structure of the questions used on the survey. After that, we will list example questions from the three slightly different versions of the survey. The first example is from the survey used for students enrolled in a course using the McIntyre text, the second with the Townsend text, and the third with the Griffiths text. The only difference between these three surveys were the exact formulation of the expressions—we attempted to most closely match the texts' choices for representations of the expressions chosen.

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You are invited to participate in a research project being led by John Thompson, a faculty member in the Department of Physics and Astronomy at the University of Maine. The purpose of this research is to better understand how students think about and reason with certain mathematical concepts in physics in order to improve student learning in physics courses, including the development of research-based instructional resources that support the development of both conceptual understanding and mathematical reasoning skills in physics contexts. You must be at least 18 years of age to participate.

What Will You Be Asked to Do?
If you decide to participate, you will be asked to answer questions involving physics and/or related topics. It will take approximately ten minutes of your time to participate.

Risks

- Except for your time and inconvenience, there are no risks to you from participating in this study.

Benefits

- There is no direct benefit to you for participating.
- Researchers and physics instructors will benefit by learning how to improve student learning by supporting student mathematical reasoning in physics and by using the research-based instructional resources developed through this project.

Confidentiality
This study is anonymous. Please do not write your name on the survey. There will be no records linking you to the data. Data will be kept on a password-protected computer indefinitely.

Voluntary
Participation is voluntary. If you choose to take part in this study, you may stop at any time. You may skip any questions you do not wish to answer.

Contact Information
If you have any questions about this study, please contact us at (207) 581-1015 or at thompsonj@maine.edu. If you have any questions about your rights as a research participant, please contact the Office of Research Compliance, University of Maine, (207) 581-2657 (or e-mail umric@maine.edu).

Figure F.1: Informed consent slide used for all online surveys.

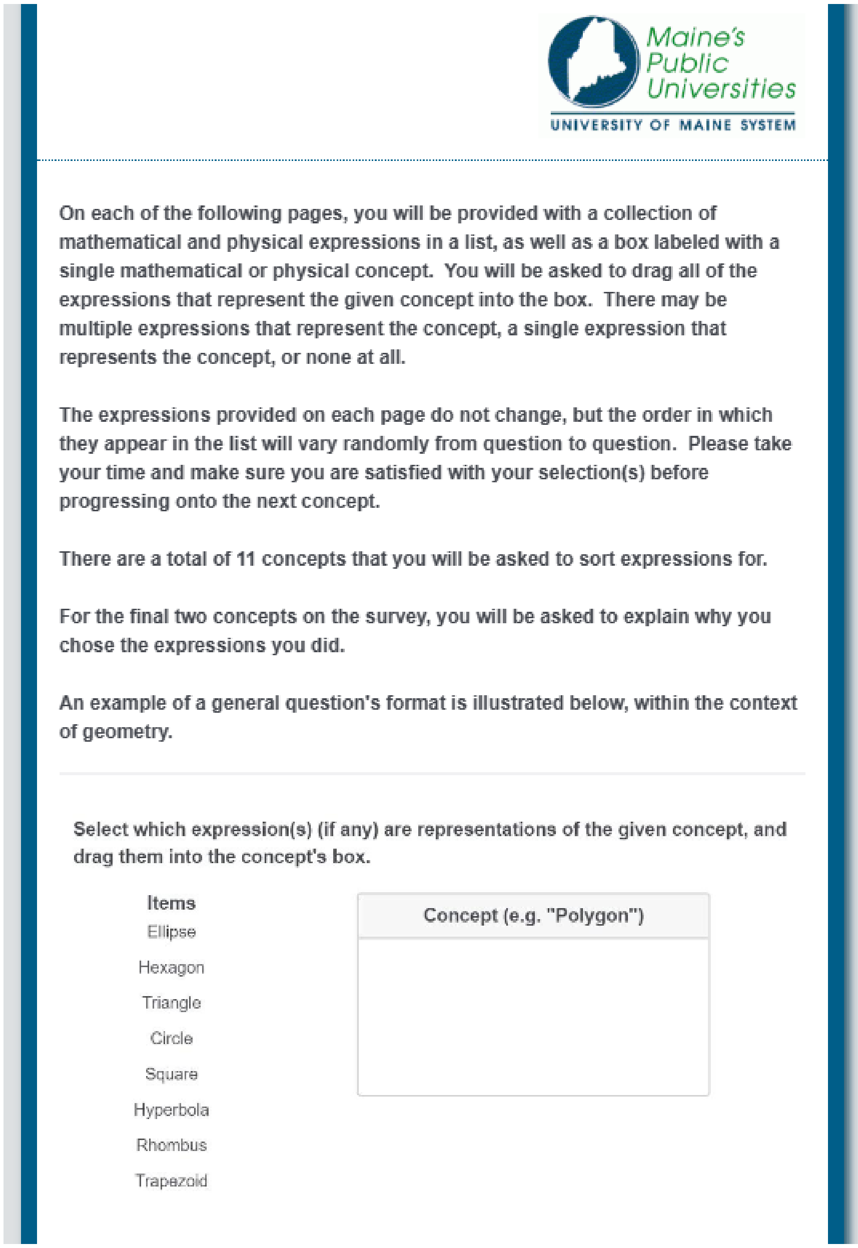


Figure F.2: Second slide on the online survey, informing participants about the tasks they will be responding to throughout the survey

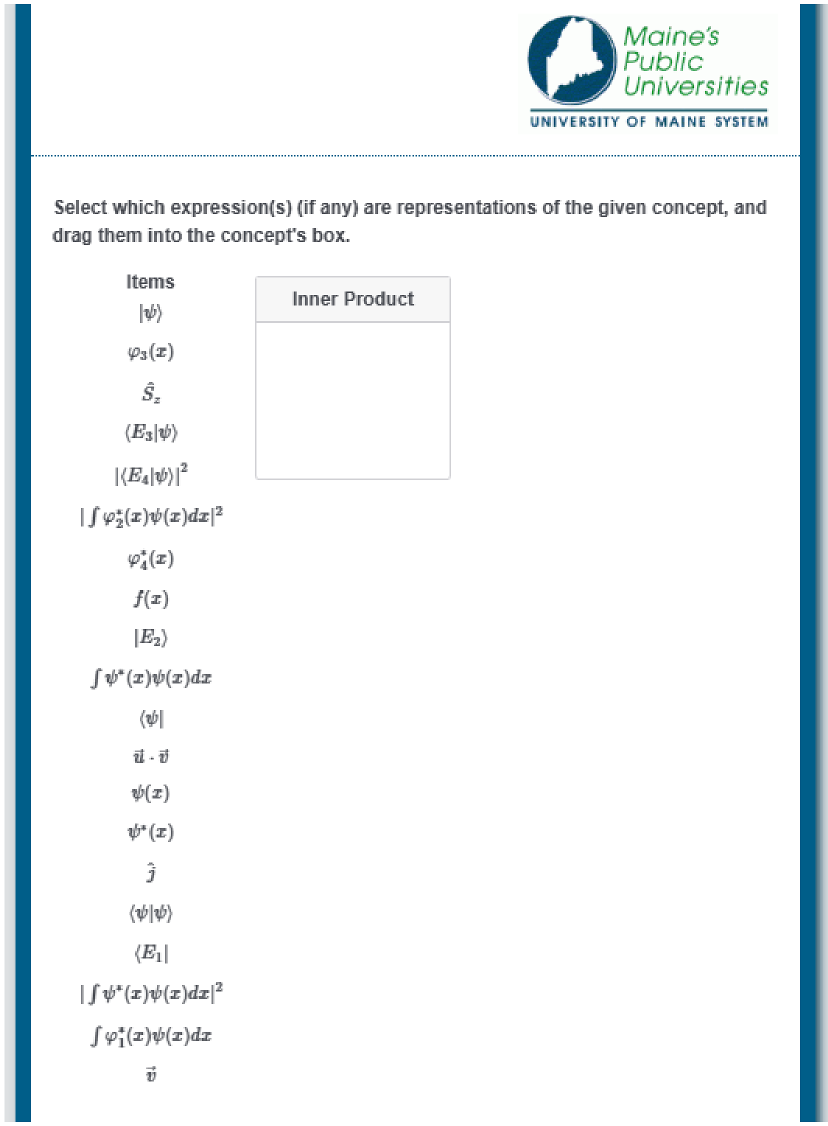


Figure F.3: Example of a survey question given to participants enrolled in courses using McIntyre's textbook (McIntyre, 2012).

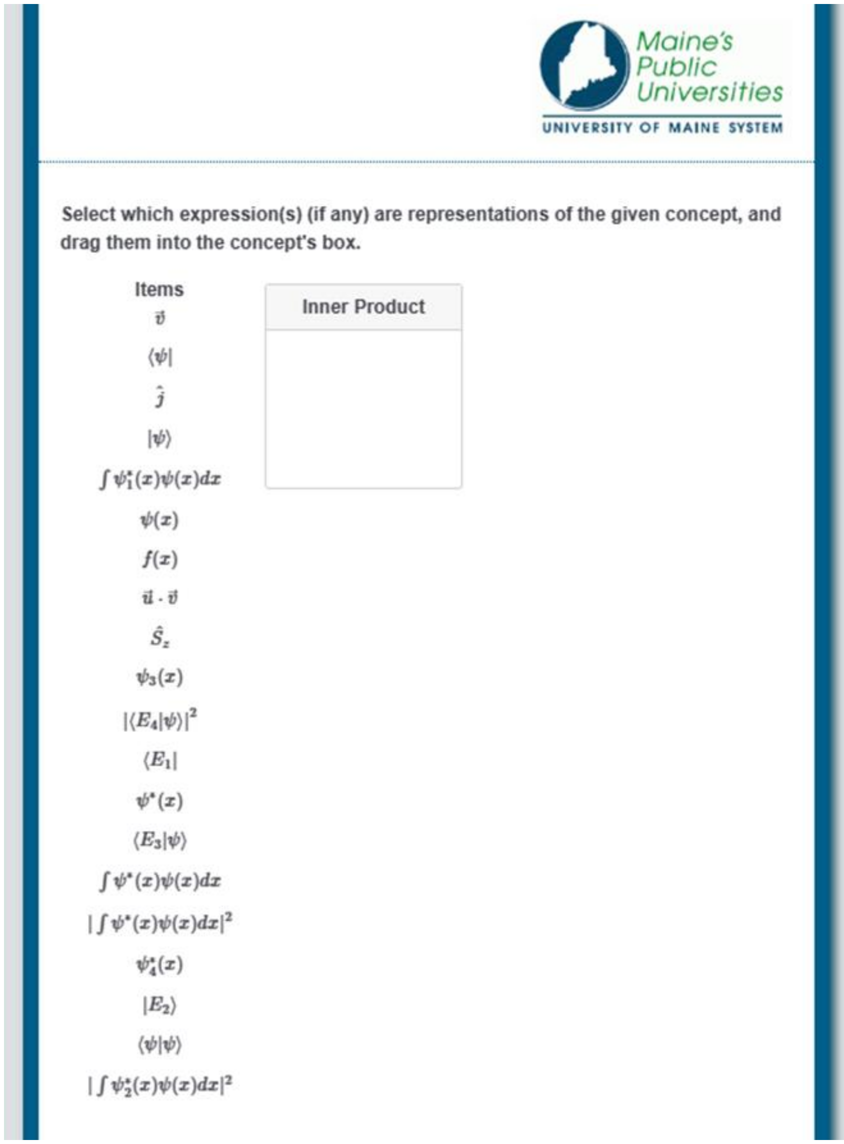
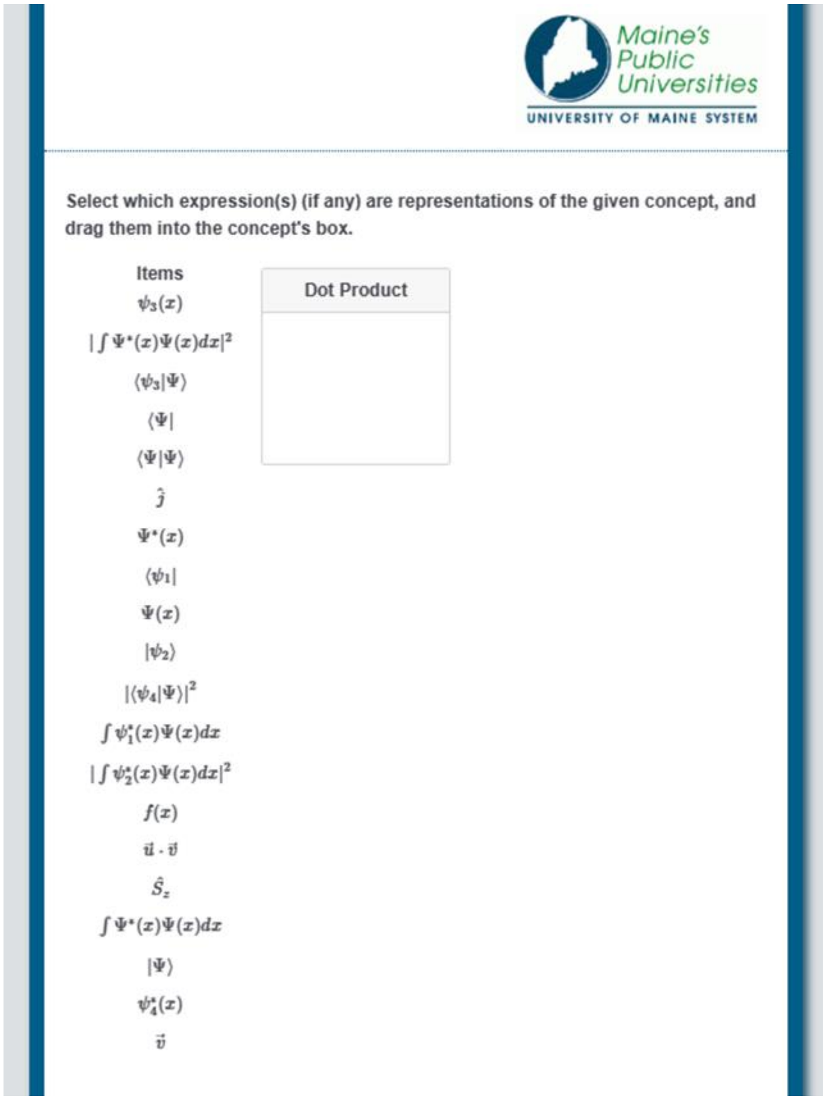


Figure F.4: Example of a survey question given to participants enrolled in courses using Townsend's textbook (Townsend, 2000).



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Select which expression(s) (if any) are representations of the given concept, and drag them into the concept's box.

Items	Dot Product
$\psi_3(x)$	
$\lvert \int \Psi^*(x)\Psi(x)dx \rvert^2$	
$\langle \psi_3 \vert \Psi \rangle$	
$\langle \Psi \vert$	
$\langle \Psi \vert \Psi \rangle$	
$\hat{j}$	
$\Psi^*(x)$	
$\langle \psi_1 \vert$	
$\Psi(x)$	
$\vert \psi_2 \rangle$	
$\lvert \langle \psi_4 \vert \Psi \rangle \rvert^2$	
$\int \psi_1^*(x)\Psi(x)dx$	
$\lvert \int \psi_2^*(x)\Psi(x)dx \rvert^2$	
$f(x)$	
$\vec{u} \cdot \vec{v}$	
$\hat{S}_z$	
$\int \Psi^*(x)\Psi(x)dx$	
$\vert \Psi \rangle$	
$\psi_4^*(x)$	
$\vec{v}$	

Figure F.5: Example of a survey question given to participants enrolled in courses using Griffiths' textbook (Griffiths, 1995)*.*

APPENDIX G: COMPUTER CODE FOR NETWORK ANALYSIS

Pasted below is a link to a GitHub repository where examples of the code used are accessible for the curious reader. Due to the number of similar files used to analyze data from multiple institutions or curricula, there is only one file from each institution/curriculum, to avoid overwhelming the reader with almost identical files. The goal with sharing this code is to assist anyone who wishes to replicate or utilize our methods in their own work. The code is all in Mathematica, and is fairly well-commented. If the reader has questions about the code used in this analysis beyond what can be gleaned from the comments in the code, the author is more than willing to assist if reached out to.

https://github.com/WillRiihiluoma/Riihiluoma_Dissertation_Code

BIOGRAPHY OF THE AUTHOR

William Riihiluoma was born in Cloquet, Minnesota on January 26, 1995. He was raised in that town and enjoyed his time in school immensely due to an amazing collection of friends, teachers, coaches, and mentors. In particular, he sang in choir and participated in knowledge bowl with his friends from middle school through graduation. A particular joy of his was theatre, and he enjoyed performing in many a play and musical. He graduated from Cloquet High School in 2013, and from there attended Gustavus Adolphus College, where he was lucky enough to continue to sing (and even do a little theatre!) throughout college. It is at Gustavus where his joy for physics and Latin were nurtured and truly ignited into passions. He was lucky enough to conduct research within several different physics subfields while a student. This included a summer doing acoustics research with his good friend Ian under Dr. Tom Huber and working with a visiting professor, Dr. Dan Young, on his physics education research (PER) project during his senior year. He had never heard of PER before, but found the field fascinating. He eventually graduated in 2017 with a B.A. in Physics and a minor in Classical Studies and planned to attend the University of Maine to get his PhD in Physics, with the goal of becoming a professor at a liberal arts college like his alma mater. Before arriving at graduate school, however, he spent a summer working with his good friend Ian once again. This time, they swapped out the optical breadboards and ultrasonic transducers for ziplines, high-rope courses, and hiking tours at Big Sky Resort in Montana.

Once his summer fun was had (and his physics brain fully recharged), he arrived at the University of Maine eager to learn more physics and develop his skills as a researcher (and continue to sing as a member of the Black Bear Men's Chorus!). Will was welcomed with open arms into the Physics Education Research Laboratory (PERL), and made many lasting friendships throughout Bennett Hall. At least a portion of his job each year (until receiving a full research assistantship award for his final year) was spent as a teaching assistant in the introductory physics courses, where he had far too much fun getting to know his students and on occasion teaching them some physics. While working with Dr. John Thompson, he was lucky enough to attend multiple conferences to present their findings and meet amazing people (the PER community is a menagerie of wonderful weirdos), including at the final Foundations and Frontiers of Physics Education Research conference held in Bar Harbor, Maine, several meetings of the American Association of Physics Teachers, Physics Education Research Conferences, Conferences for Research in Undergraduate Mathematics Education, and an April Meeting of the American Physical Society. Of course, the COVID-19 pandemic hit during the spring of Will's third year at UMaine, but despite losing a year's worth of data, he considered himself lucky: he had a stable job while the outside world collapsed around him, and had wonderful roommates to allay the isolation and fear of living in a world diseased, particularly during the early months where it wasn't well understood and nothing felt certain. After a few more years of research, conferences, and living in the beautiful summers and warm(er than in Minnesota) winters of Maine, he finished up his work and prepared to defend his dissertation (this very document!). After receiving his degree, Will looks forward to spending a year at Dickinson College as a visiting assistant professor, where he is excited to teach introductory mechanics for life science majors and thermodynamics and statistical mechanics for upper-division physics majors in a setting similar to his alma mater.

Will is a candidate for the Doctor of Philosophy degree in Physics from the University of Maine in August 2023.