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The Category of Modules Over a Leavitt Path Algebra

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THE CATEGORY OF MODULES OVER A LEAVITT PATH ALGEBRA

By

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An Abstract of the Thesis Presented
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In this thesis we establish the technique used to construct universal algebras, and apply this technique to construct Leavitt path algebras. We then establish the basic language of category theory. From there we look at the category of modules over Leavitt Path Algebras over a finite graph, and establish a functorial classification of this category.

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CHAPTER 1

INTRODUCTION

Leavitt path algebras have been the object of much attention recently. To find some of this attention, one can look at [1] and look at the references within. A Leavitt path algebra is a type of universal algebra with relations and generators associated with a directed graph and they generalize a family of algebras first studied by W. G. Leavitt in his study of the invariant basis number property . However, the meaning of the relations is not immediately transparent upon first examination. In this thesis however, we will not be concerned so much with the algebras themselves but with the category of modules over them.

We say two algebras are *Morita equivalent* whenever their module categories are the equivalent. For certain graphs it remains an open question whether the Leavitt path algebras given by those graphs are Morita equivalent. For example, one may consider the Cuntz splice as in [1, 7.3]. This motivates a study of the category of modules over Leavitt path algebras.

A functorial picture of these module categories was previously established in [3, Proposition 3.2]. In particular, we will show that the category of modules over a Leavitt path algebra can be considered as a collection of vector spaces associated with the vertices of the graph, and a collection of linear maps associated with the edges, satisfying a coproduct condition that will be expanded on later. A straightforward extension of this is to consider instead associations of the graphs of other categories, satisfying the same coproduct condition. This is hoped to be useful in regarding the question of the Morita equivalence of the Cuntz splice and other open questions of Morita equivalence.

CHAPTER 2

CONSTRUCTION OF UNIVERSAL ALGEBRAS

2.1 Defining Modules and Algebras

Definition 2.1.1 (Module)**.** Let R be a ring (not necessarily commutative nor with 1). A *left R-module* or a *left module over R* is a set M together with

1. a binary operation $+$ on M under which M is an abelian group, and

2. an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

   (a) $(r+s)m = rm + sm$, for all $r, s \in R, m \in M$,

   (b) $(rs)m = r(sm)$, for all $r, s \in R, m \in M$, and

   (c) $r(m+n) = rm + rn$, for all $r \in R, m, n \in M$.

   If the ring R has 1, then we call M a *unital* R-module, and if it satisfies the additional axiom:

   (d) $1m = m$, for all $m \in M$.

A *right R-module* is similarly defined, except with the action of R being a right action. In this text we will take unital left modules to be the default, and will simply call them modules unless context requires otherwise.

Examples 2.1.2.

1. Let $\mathbb{F}$ be any field. Then any module over $\mathbb{F}$ is in fact an $\mathbb{F}$-vector space. In fact, the vector space axioms are almost identical to the module axioms - the only difference is that a vector space is over a field.

2. Let M be any abelian group, and for $n \in \mathbb{Z}$ and $m \in M$, define $nm = m + \cdots + m$ (with n copies of m). It is straightforward to check that M satisfies the module axioms under this action. Hence every abelian group is a $\mathbb{Z}$-module. Since modules are required to be abelian groups, $\mathbb{Z}$-modules and abelian groups are in fact the same thing.

3. Let R be any ring. Then we can take R to be a (left) R-module by defining the action of R on itself to be the usual multiplication in R. In this case, the module axioms become identical to the ring axioms. If S is any subring of R, then R is an S-module under multiplication by elements of S.

4. Let R be any ring, M any R-module, and S any set. Then we define $\mathrm{Hom}(S, M)$ to be the set of functions from S to M. For $\phi, \varphi \in \mathrm{Hom}(S, M)$, $r \in R$, and $s \in S$, we define

   (a) $(\phi + \varphi)(s) = \phi(s) + \varphi(s)$,

   (b) $(r\phi)(s) = r(\phi(s))$.

   Under these operations, $\mathrm{Hom}(S, M)$ is an R-module. We can also define $\mathrm{Hom}_{group}(G, M)$ in case G is some group. In this case we require the functions to be homomorphisms of groups. Similary, if N is an R-module, we can take $Hom_R(N, M)$ to the set of all R-module homomorphisms from $N \to M$ (where an R-module homomorphism takes on the natural definition).

Given the similarity of modules and vector spaces, we will often refer to elements of the ring R as scalars. We now introduce a type of module that is of particular importance to this text.

Definition 2.1.3 (Algebra)**.** Let R be a commutative ring with identity. An R-*algebra* is a ring A with identity that is also a unital R-module such that

$$r(ab) = (ra)b = a(rb)$$

for all $r \in R$ and $a, b \in A$.

Examples 2.1.4.

1. Let R be a commutative ring with identity, and define $R[x]$ to be the set of R-polynomials in a single indeterminate. Then $R[x]$ is an R-algebra under the standard polynomial operations.

2. Let R be a commutative ring with identity. Then R is an R-module, and since R is commutative, an R-algebra.

3. Let R be a commutative ring with identity, and for some $k \in \mathbb{N}$, define $R[x_1, \dots, x_k]$ to be the ring of commutative R-polynomials in k indeterminates. By commutative polynomials, we mean that $x_m x_n = x_n x_m$ for all $m, n \in 1, \dots, k$. Then $R[x_2, \dots, x_k]$ is an algebra over R under the standard multivariable polynomial operations.

4. Let R be a commutative ring with identity, and let $M_n(R)$ be the set of $n \times n$ matrices over R. Then $M_n(R)$ is an R-algebra under the usual matrix multiplication, addition, and scalar multiplication. Note that in this example, matrix multiplication is not commutative, and so $M_n(R)$ is (very creatively) called a non-commutative algebra.

5. Let V be any $\mathbb{F}$-vector space. We will write $\mathrm{End}_{\mathbb{F}}(V)$ to denote the set of linear endomorphisms (linear maps from V to V) of V. If we take addition to be given by $(f+g)(v) = f(v) + g(v)$, and multiplication to be composition of maps, then $Hom_{\mathbb{F}}(V, V)$ is an $\mathbb{F}$-algebra.

6. Let R be a commutative ring with identity, and M an R-module. Then we can generalize by the previous example by defining $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$. As in previous examples, we know that $\mathrm{End}_R(M)$ is a R-module. We can turn $\mathrm{End}_R(M)$ into a ring by defining multiplication as function composition. Since the identity endomorphism will clearly work as a multiplicative identity, and function composition is associative, we only need to verify the distributive law. For $\phi, \varphi, \psi \in \mathrm{End}_R$, we have

$$\phi(\varphi + \psi)(m) = \phi(\varphi(m) + \psi(m)) = (\phi\varphi)(m) + (\phi\psi)(m).$$

Under this multiplication, we can see that $\mathrm{End}_R(M)$ is a R-algebra.

7. Let R be any commutative ring with identity, and define $R[x^{-1}, x]$ to be the set of *Laurent polynomials* over R. These are the polynomials with coefficients in R in the two indeterminates x^{-1}, x with the relation $xx^{-1} = x^{-1}x = 1$. Under normal polynomial operations, this gives an algebra over R.

2.2 Basic Theory of Algebras

Definition 2.2.1 (Algebra homomorphism and isomorphism)**.** Let A, B be R-algebras. Then an *algebra homomorphism* $A \to B$ is a ring homomorphism $\varphi : A \to B$ such that for any $r \in R$ and $a \in A$, we have $\varphi(ra) = r\varphi(a)$, and $\varphi(1_A) = 1_B$. An *algebra isomorphism* is a ring isomorphism satisfying the same conditions. If there exists an algebra isomorphism $A \to B$, then we say A and B are *isomorphic* and denote this with $A \cong B$.

Examples 2.2.2.

1. Consider the $\mathbb{R}$-algebras $\mathbb{R}$ and $\mathbb{C}$. The injection $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = x$ is trivially an algebra homomorphism.

2. Consider $\mathbb{C}$ as a $\mathbb{R}$-algebra. Then the complex conjugation function is an algebra isomorphism. When an algebra isomorphism is a function from the algebra to itself, we call it an automorphism.

3. Let $M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over some field $\mathbb{F}$. Then $M_n(\mathbb{F}) \cong End_{\mathbb{F}}(\mathbb{F}^n)$, where $\mathbb{F}^n$ is the usual n-dimensional vector space over $\mathbb{F}$.

Definition 2.2.3 (Ideal)**.** Let A be an algebra over some ring R. Then an *ideal* of A is an additive subgroup I of A such that for all $a \in A$ we have $aI \subseteq I$ and $Ia \subseteq I$. If S is some subset of A, then the set $(S) := \{a_1 s a_2 \mid a_1, a_2 \in A, s \in S\}$ is an ideal of A, and we call it the ideal generated by S.

Remark: We note that ideals are closed under scalar multiplication. This is because for any $\alpha \in R$, and any $x \in I$, we have $\alpha x = (\alpha 1_A)x \in I$.

Definition 2.2.4 (Quotient Algebra)**.** Let A be an R-algebra, and I some ideal of A. The *quotient algebra* A/I is the set of cosets given by the equivalence relation $a \sim b$ whenever $a - b \in I$. The operations in this algebra are the natural extensions of the operations of A.

There is a similar concept of a quotient for both rings and modules. Since an algebra is both, it is possible to construct quotient rings and quotient modules of an algebra. The fact that these quotients are compatible follows from the theorem below.

Theorem 2.2.5 (First Isomorphism Thoerem)**.** *[2] Let A and B both be R-algebras, and $\varphi : A \to B$ an algebra homomorphism. If we denote the kernel of φ with $\ker(\varphi)$, then $\ker(\varphi)$ is an ideal of A and $A/\ker(\varphi) \cong \operatorname{im}(\varphi)$.*

We note that the proof of this theorem is essentially the same as any proof of the first isomorphism theorem a reader will encounter in any text on groups or rings, and so we omit it. The following theorem establishes a useful fact about algebras.

Theorem 2.2.6 (Homomorphism definition of algebras)**.** *Let R be a commutative ring with identity, and A a ring with identity. Then A is an R-algebra if and only if there exists a (ring) homomorphism $\varphi : R \to A$ such that $Im(\varphi)$ is contained within the center of A, and $\varphi(1) = 1$.*

Proof. We begin by showing that if such a homomorphism exists, we can use it to obtain an action that turns A into an R-algebra. For $r \in R$ and $a \in A$, we define $ra := \varphi(r)a$. Since $\varphi(r)$ is an element of the ring A, the action satisfies the module axioms as an immediate consequence of the ring axioms. Furthermore, $\varphi(R)$ lies in the center of A, telling us that $r(ab) = (ra)b = a(rb)$ for $r \in R$ and $a, b \in A$. This establishes that A is an R-algebra.

Now we prove the other direction of implication. Assume that A is an R-algebra. Then we define $\varphi : R \to A$ to be the function $\varphi(r) = r \cdot 1$ (where $\cdot$ denotes the action of R on A).

Now we verify that φ is a ring homomorphism. Let $r, s \in R$. Then we see

$$\varphi(r+s) = (r+s)\cdot 1 = r\cdot 1 + s\cdot 1 = \varphi(r) + \varphi(s),$$

and

$$\varphi(rs) = (rs)\cdot 1 = r\cdot(s\cdot 1) = (r(1\cdot 1))(s\cdot 1) = 1(r\cdot 1)(s\cdot 1) = \varphi(r)\varphi(s).$$

Next we show that the image of φ lies in the center of A. Let $r \in R$ and $a \in A$. Then

$$\varphi(r)a - a\varphi(r) = (r\cdot 1)a - a(r\cdot 1) = (r\cdot a) - (r\cdot a) = 0,$$

as desired. This proves the equivalence. □

In this text we are primarily concerned with algebras over fields (i.e. the ring R is a field). We now use the following proposition to allow for a convenience of notation:

Proposition 2.2.7 (Fields are embedded in algebras)**.** *Let $\mathbb{F}$ be a field, and A an $\mathbb{F}$-algebra. Then A contains within its center a field that is canonically isomorphic to $\mathbb{F}$.*

Proof. Let φ be the homomorphism from $\mathbb{F} \to A$ provided by Theorem 2.2.6. Since $\mathbb{F}$ is a field, it has no non-trivial proper ideals and hence the kernel of φ is trivial. This immediately gives the desired result. □

The above proposition allows us to speak of elements of $\mathbb{F}$ as belonging to A, and we will do so in this paper. Note that if we consider algebras over rings instead, we are unable to do so. For example, the canonical homomorphism from $\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})$ is clearly not injective.

2.3 Free and Universal Algebras

Definition 2.3.1 (Free algebra)**.** We say that an $\mathbb{F}$-algebra A is *free on the subset S of A* (and called a *free $\mathbb{F}$-algebra*) if F satisfies the following universal property: For any $\mathbb{F}$-algebra B and any function $f : S \to B$, there exists a unique homomorphism $\varphi_f : F \to A$ such that the restriction of φ_f to S is equal to f. The existence and uniqueness of these homomorphisms is called the *universal property of free algebras*. The next proposition tells us that free algebras are determined uniquely up to a canonical isomorphism by the field $\mathbb{F}$ and the set S.

Proposition 2.3.2 (Free algebra uniqueness)**.** *Let $\mathbb{F}$ be a field and S some set. If $\mathbb{F}$-algebras A and B are both free over S, then there exists a unqiue isomorphism between A and B whose restriction to S is the identity.*

Proof. Let A and B both be free $\mathbb{F}$-algebras over S. Then there exist unique homomorphisms $\psi : A \to B$ and $\varphi : B \to A$ that extend the identity function $s \mapsto s$ for $s \in S$. Then the composition $\varphi \circ \psi : A \to A$ is a homomorphism from A to A that extends the identity function on S. The universal property of free algebras tells us that only one such homomorphism exists. However, the identity function on A is such a homomorphism, and hence $\varphi \circ \psi = Id_A$. An analogous argument tells us that $\psi \circ \varphi = Id_B$, and this tells us that ψ and φ are isomorphisms. $\square$

Remark: Since this result tells us that the information of the field $\mathbb{F}$ and the set S is enough to denote a free algebra, we will denote a free algebra from now on by $\mathbb{F}\langle S \rangle$.

An observant reader will notice that we have not yet shown that free algebras exist. The following proposition takes care of that.

Proposition 2.3.3 (Existence of free algebras)**.** *Let $\mathbb{F}$ be a field and S any set. Then there exists a free $\mathbb{F}$-algebra over S.*

Proof. The basic motivation for constructing a free algebra is to try to define the most "general" algebra containing the elements of S. In particular, we need multiplication, addition, and scalar multiplication. To obtain multiplication, we introduce the idea of a "word". This can be considered as a finite string of elements of S, or an element of the set

$$W(S) = \emptyset \cup \left(\bigcup_{k \in \mathbb{N}} S^k \right),$$

where S^k is the cartesian product of k copies of S. Notice that $W(S)$ contains S as a subset, and hence we can talk about elements of S as elements of $W(S)$. For

$(s_1, \ldots, s_j), (t_1, \ldots, t_k) \in W(S)$, we define their formal product to be $(s_1, \ldots, s_j, t_1, \ldots t_k)$, i.e. the concatenation of the two tuples. If we have the empty word $()$, then we define $(s_1, \ldots, s_k)() = ()(s_1, \ldots, s_k) = (s_1, \ldots, s_k)$.

We now define the free algebra over S to be the set

$$\mathbb{F}\langle S\rangle = \{f : W(S) \to \mathbb{F} \mid f(w) = 0 \text{ for all but finitely many } w \in W(S)\}.$$

For $w \in W(S)$, we can associate it with the function that sends w to 1 and everything else to 0. This gives us an $\mathbb{F}$-vector space under pointwise addition and scalar multiplication of functions, and $W(S)$ forms a basis of $\mathbb{F}\langle S\rangle$. Hence we introduce the notation

$$f := \sum_{w \in W(S)} f(w)w,$$

whenever $f \in \mathbb{F}\langle S\rangle$, and point out that this is just writing out elements of $\mathbb{F}\langle S\rangle$ explicitly as $\mathbb{F}$-linear combinations of elements of $W(S)$. We now define the product of elements of $\mathbb{F}\langle S\rangle$ to be the convolution of the sum in $\mathbb{F}\langle S\rangle$ and the formal product in $W(S)$:

$$fs = \left(\sum_{w \in W(S)} f(w)w\right)\left(\sum_{v \in W(S)} g(v)v\right) = \sum_{u \in W(S)}\left(\sum_{wv=u} f(w)g(v)\right)u.$$

It is both simple and tedious to verify that these definitions of multiplication (taking the empty word to be the multiplicative identity), scalar multiplication, and addition satisfy the axioms for an algebra, and so we leave it as an exercise for the reader.

Having constructed this algebra, we now demonstrate that it is free. Let A be any $\mathbb{F}$-algebra and $f : S \to A$ be any function. Since $\mathbb{F}\langle S\rangle$ and A are both $\mathbb{F}$-vector spaces, we may define φ_f to be the unique linear map defined on basis elements by $(s_1, \ldots, s_k) \mapsto f(s_1) \cdot \cdots \cdot f(s_k)$ and $() \mapsto 1_A$ whenever $(s_1, \ldots, s_k) \in W(S)$. As this mapping is linear, it respects addition and scalar multiplication. The defininition of φ_f makes it obvious that it respects multiplication on basis elements. Since the multiplication in $\mathbb{F}\langle S\rangle$ is bilinear, φ_f respects multiplication on all elements. $\square$

Readers may notice that multiplication in $\mathbb{F}\langle S\rangle$ is similar to multiplication of polynomials. This is not coincidental: elements of $\mathbb{F}\langle S\rangle$ may be considered as non-commutative polynomials with coefficients in $\mathbb{F}$ and unknowns in S. Monomials then, are elements of $W(S)$. To reconcile the two seemingly disparate notions, we point out to the reader that we may rewrite, for example, the tuple (s,t,t,v,t) as st^2vt. This now visually resembles a monomial, and allows us to see that the multiplication given for $W(S)$ coincides with that of (non-commutative) monomials. Under this interpretation, the homomorphism given by the free algebra's universal property may be thought of as polynomial evaluation. The reader will also note that if S has one element, then $\mathbb{F}\langle S\rangle$ is just the algebra $\mathbb{F}[x]$ (up to isomorphism).

Definition 2.3.4 (Relation)**.** When context requires it, we will refer to elements of a free algebra $\mathbb{F}\langle S\rangle$ as *relations*. Let A be a $\mathbb{F}$-algebra and f a function from S to A. Then we say that f *satisfies the relation* r if $r \in \ker(\varphi_f)$, where $\varphi_f : \mathbb{F}\langle S\rangle \to A$ is the homomorphic extension of f given by the universal property of $\mathbb{F}\langle S\rangle$.

Examples 2.3.5.

1. In the free algebra $\mathbb{R}[x]$, the only mappings from $\{x\}$ to $\mathbb{R}$ satisfying the relation x^2-2 are $x \mapsto \pm\sqrt{2}$. The reader will note that this coincides with saying that $\pm\sqrt{2}$ are the roots of x^2-2 in $\mathbb{R}$.

2. There are no maps from $\{x\}$ to $\mathbb{R}$ that satisfy the relation x^2+1. However, the homomorphisms extending $x \mapsto \pm i$ from $\mathbb{R}[x] \to \mathbb{C}$ both satisfy this relation. In fact, it is the fundamental theorem of algebra that for any relation in $\mathbb{R}[x]$ (except for scalar multiples of the identity), there exists some evaluation homomorphism $\varphi : \mathbb{R}[x] \to \mathbb{C}$ satisfying that relation.

3. It is also possible to describe properties of an algebra through a relation. Let A be some $\mathbb{F}$-algebra containing elements a, b that commute with each other, i.e. $ab = ba$. Taking $\mathbb{F}\langle x, y\rangle$ to be the free algebra over two elements, we observe that the mapping $x \mapsto a$, $y \mapsto b$ satisfies the relation $xy - yx$. In fact, A is commutative if and only if every mapping from $\{x, y\}$ to A satisfies this relation.

Definition 2.3.6 (Universal Algebra)**.** Let $\mathbb{F}$ be some field, S some set. Then for $R \subset \mathbb{F}\langle S\rangle$, we call the quotient $\mathbb{F}\langle S\rangle/(R)$ the *universal* $\mathbb{F}$*-algebra* with *generators* S and *relations* R.

Theorem 2.3.7 (Universal Property of Universal Algebras)**.** *Let A be some $\mathbb{F}$-algebra, S some set, and let $f : S \to A$ satisfy each relation $r \in R$ for some $R \subset \mathbb{F}\langle S\rangle$. Then there exists a unique homomorphism $\phi : \mathbb{F}\langle S\rangle/(R) \to A$ whose restriction to $S + (R)$ is f. Since this will sometimes lead to elements of S being associated with one another, this sometimes leads to a slight abuse of notation when considering S as a subset of $\mathbb{F}\langle S\rangle/(R)$. However, this will never lead to issues.*

Proof. This is an immediate consequence of the first isomorphism theorem. □

Remark. For any $s \in S$, we will denote the element $s + (R)$ of $\mathbb{F}\langle S\rangle/(R)$ just by s. Observe that these elements generate $\mathbb{F}\langle S\rangle/(R)$ as an algebra - every element of $\mathbb{F}\langle S\rangle/(R)$ can be written as a linear combinations of products of these elements.

Examples 2.3.8 (Universal Algebras)**.**

1. Let A be some $\mathbb{F}$-algebra, and let $\mathbb{F}\langle A\rangle$ be the free $\mathbb{F}$-algebra over A. Then the universal property of free aalgebras proves a homomorphism $\varphi : \mathbb{F}\langle A\rangle \to A$ extending the identity map on A. Clearly $\mathbb{F}\langle A\rangle/\ker\varphi \cong A$, meaning A is a universal algebra. Thus we have shown that every $\mathbb{F}$-algebra can be considered a universal algebra.

2. Let $\mathbb{F}$ be any field, and $S = \{x_1, \ldots, x_k\}$ be a finite set. The reader will recall that $\mathbb{F}\langle S\rangle$ is the set of non-commutative polynomials in the indeterminates in S and and coefficients in $\mathbb{F}$. In order to obtain the commutative polynomials, we can take our set of relations to be

$$R = \{x_m x_n - x_n x_m \mid m, n \in 1, \ldots, k\}.$$

   We then obtain $\mathbb{F}\langle S\rangle/(R) \cong \mathbb{F}[x_1, \ldots, x_k]$.

3. Now consider the free algebra $\mathbb{F}[x]$, the set of single variable polynomials over some field $\mathbb{F}$. Then we may take $f(x)$ to be any irreducible polynomial in $\mathbb{F}[x]$. If we do so, the quotient $\mathbb{F}[x]/(f(x))$ is isomorphic to the extension field $\mathbb{F}[r]$, where r is any root of f. For example, if we take our field to be $\mathbb{R}$, and our polynomial to be x^2+1, we obtain

$$\mathbb{R}[x]/(x^2+1) \cong \mathbb{R}[i] = \mathbb{C}.$$

This shows that every field extension of $\mathbb{R}$ is a universal algebra.

4. Consider $M_n(\mathbb{F})$, the algebra of $n \times n$ matrices. As a vector space, it has basis $\beta = \{\epsilon_{i,j} \mid i,j \in 1,\ldots,n\}$, where $\epsilon_{i,j}$ is the matrix with a 1 in position i,j and zeroes elsewhere. We can also represent $M_n(\mathbb{F})$ as a universal algebra with generators β, and relations $R := \{\epsilon_{i,j}\epsilon_{k,l} - \delta_{j,k}\epsilon_{i,l} \mid i,j,k,l \in 1,\ldots,n\} \cup \{\sum_{k=1}^{n} \epsilon_{k,k} = 1\}$ (where $\delta_{j,k}$ is the function that is 1 when $j=k$, and 0 otherwise - this function is known as the *Kronecker delta function*). To see why, first we observe that standard matrix multiplication tells us $\epsilon_{i,j}, \epsilon_{k,l} = \delta_{j,k}\epsilon_{i,l}$ in $M_n(\mathbb{F})$. Hence we see that $\epsilon_{i,j}\epsilon_{k,l} - \delta_{j,k}\epsilon_{i,l} = 0$ is in the kernel of the homomorphism from $\mathbb{F}\langle\beta\rangle \to M_n(\mathbb{F})$ that extends the identity. Furthermore, the sum of the diagonal matrices $\epsilon_{i,j}$ is the identity matrix in $M_n(\mathbb{F})$. This tells us that every relation in R is satisfied by this homomorphism, and hence there exists a unique homomorphism $\varphi : \mathbb{F}\langle\beta\rangle/(R) \to M_n(\mathbb{F})$ such that $\varphi(\epsilon_{i,j}) = \epsilon_{i,j}$ for each $\epsilon_{i,j} \in \beta$. Notice that we have established a surjective mapping onto a basis of $M_n(\mathbb{F})$ that respects both addition and scalar multiplication. This implies that φ is surjective on $M_n(\mathbb{F})$. Now we note that since $\epsilon_{i,j}\epsilon_{k,l} - \delta_{j,k}\epsilon_{i,l} = 0$ for any $\epsilon_{i,j}, \epsilon_{k,l} \in \beta$, the product of any two of these elements is an element of β. Since theses elements generate $\mathbb{F}\langle S\rangle/(R)$ as an algebra, this tells us that they span $\mathbb{F}\langle\beta\rangle/(R)$ as a $\mathbb{F}$ -vector space. Hence we observe that if $\sum_{k=1}^{n}\sum_{j=1}^{n}\varphi(\alpha_{j,k}\epsilon_{j,k}) = 0$, then linear independence in $M_n(\mathbb{F})$ indicates that each $\alpha_{j,k}$ must be zero. This tells us that φ has a trivial kernel and hence is injective. So φ is an isomorphism, as is desired.

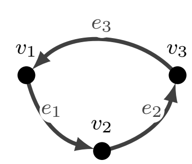
CHAPTER 3

LEAVITT PATH ALGEBRAS

We now introduce the central topic of this paper, Leavitt path algebras. A Leavitt path algebra is a type of universal algebra whose relations are defined based on a directed graph. We begin by defining a directed graph, following the terminology used in [1, Definition 1.2.2].

Definition 3.0.1 (Directed Graph)**.** Let E^0 and E^1 be a disjoint pair of finite[1] sets, equipped with functions $r : E^1 \to E^0$ and $s : E^1 \to E^0$. Then we call the collection $E = (E^0, E^1, r, s)$ a *directed graph.* We call the elements of E^0 the *vertices* of E, and the elements of E^1 *edges.* For a given $e \in E^1$, we call $s(e)$ the *source* of e, and $r(e)$ the *range.* Note that it is possible for multiple edges to have both the same soure and range.

Example 3.0.2. Although directed graphs are defined abstractly, they are usually represented by diagrams. In the diagram below, the vertices are represented by the dots,



and the edges by the arrows. Thus we have $E^0 = \{v_1, v_2, v_3\}$ and $E^1 = \{e_1, e_2, e_3\}$. The source of an edge is the vertex that the edge is pointing away from, and its range is the vertex it is pointing to. As such, we have $s(e_1) = v_1$ and $r(e_1) = v_2$.

The following is a minor adaptation of the definition of a Leavitt path algebra from the one found in [1, Definition 1.2.3]. This modification is explained in the remark following the definition.

[1]It is possible to define directed graphs without requiring these sets to be finite. However, allowing for these sets to be infinite allows the algebras we will be defining with these graphs to become more complicated (in particular, it allows for algebras without a multiplicative identity). It should be possible to extend the results of this paper to infinite graphs, but for the sake of simplicity we will restrict to the finite case.

Definition 3.0.3 (Leavitt path algebra)**.** [1] Let $E = (E^0, E^1, r, s)$ be some directed graph, and $\mathbb{F}$ some field.

We will be using this graph to generate an algebra, but we wish to have two distinct generators associated with each edge of the graph. To this end, we define $(E^1)^* := \{e^* \mid e \in E^1\}$ to be the set of 'ghost edges' of E and note that it is simply a set with the same cardinality as E^1, and whose elements are indexed by those of E^1. Then the *Leavitt path algebra of* E *over* $\mathbb{F}$ is the quotient of the free algebra $\mathbb{F}\langle E^0 \cup E^1 \cup (E^1)^* \rangle$ by the ideal generated by the relations

1. $vv' = \delta_{v,v'}v$, for all $v, v \in E^0$,
2. $s(e)e = er(e) = e$, for all $e \in E^1$,
3. $e^*s(e) = r(e)e^* = e^*$, for all $e \in E^1$,
4. $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$,
5. $v = \sum_{s(e)=v} ee^*$, for each $v \in E^0$ that is the source of at least one edge, and
6. $\sum_{v \in E^0} v = 1$

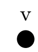
For a given graph E, we denote the Leavitt path algebra of E over $\mathbb{F}$ by $L_{\mathbb{F}}(E)$.

Remark. The sixth relation is only required since we have required universal algebras to have identity. In the more usual definition, where universal algebras do not have identities, it ends up being a consequence relations 1, 2, and 3.

Remark. Recall that we initially defined relations to be elements of a universal algebra, which seems distinct from our usage of them here. However, when constructing the universal algebra, we end up setting the relations equal to zero (they are used to generate the kernel of the canonical homomorphism), and hence saying $r_1 = r_2$ is the same as saying $r_1 - r_2 = 0$.

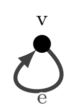
Examples 3.0.4.

1. Let E be the graph with a single vertex and no edges, i.e. the graph below:



Then we claim that $L_{\mathbb{F}}(E) \cong \mathbb{F}$ as algebras. To see why, notice that our free algebra is just $\mathbb{F}[v]$, and the Leavitt path algebra relations end up collapsing down to $v^2 = v$ and $v = 1$ in this case. In particular, the ideal (R) of $\mathbb{F}[v]$ that we are dealing with is just $(v^2 - v, v - 1)$. Since $v - 1$ divides $v^2 - v$, this ideal is just $(v - 1)$ and we have $L_{\mathbb{F}}(E) \cong \mathbb{F}[x]/(v - 1) \cong \mathbb{F}$.

2. Now let E be the graph below:



We note that $E^0 \cup E^1 \cup (E^1)^* = \{v, e, e^*\}$, and we claim $L_{\mathbb{F}}(E) \cong \mathbb{F}[x^{-1}, x]$ (the algebra of Laurent polynomials over $\mathbb{F}$).The reader will observe that the function sending v to 1, e to x, and e^* to x^{-1} satisfies the Leavitt path algebra relations, and hence gives us a unique homomorphism $\varphi : L_{\mathbb{F}}(E) \to \mathbb{F}[x^{-1}, x]$.

We now show that this mapping is an isomorphism of algebras. Note that $\mathbb{F}[x^{-1}, x]$ is generated by 1, x, and x^{-1} meaning that the homomorphism φ is surjective. Now the relations 4 and 5 ensure that every element $a \in L_{\mathbb{F}}(E)$ is of the form

$$a = \alpha v + \sum_{j=1}^{\infty} \beta_j e^j + \sum_{k=1}^{\infty} \gamma_k (e^*)^k,$$

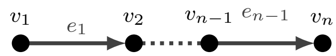
with only finitely many of the β_j's and γ_k's being non-zero.

This tells us that

$$\varphi(a) = \alpha + \sum_{j=1}^{\infty} \beta_j x^j + \sum_{k=1}^{\infty} \gamma_k x^{-k}.$$

Since the powers of x form a basis for the Laurent polynomials, linear independence tells us that $\varphi(a) = 0$ if and only if $a = 0$, giving the desired result.

3. Let $\mathbb{F}$ be some field, and E the graph given below:



We claim that $L_{\mathbb{F}}(E)$ is isomorphic to $M_n(\mathbb{F})$. The proof of this fact allows us to demonstrate an important usage of the universal property of universal algebras.

We start by ensuring that we may use this property. The reader will recall from Example 2.3.8 number 2 the equivalent definition of $M_n(\mathbb{F})$ as $\mathbb{F}\langle\beta\rangle/(R)$ where $\beta = \{\epsilon_{i,j} \mid 1 \leq i, j \leq n\}$ and $R = \{\epsilon_{i,j}\epsilon_{k,l} - \delta_{j,k}\epsilon_{i,l} \mid 1 \leq i, j, k, l \leq n\} \cup \{\sum_{k=1}^{n} \epsilon_{k,k} = 1\}$. Let $\psi : \mathbb{F}\langle\beta\rangle \to L_{\mathbb{F}}(E)$ be the homomorphism extending the mapping

$$\epsilon_{i,j} \mapsto \begin{cases} e_i e_{i+1} \cdots e_{j-1} & \text{if } i < j, \\ e_{i-1}^* e_{i-2}^* \cdots e_j^* & \text{if } i > j, \\ v_i & \text{if } i = j. \end{cases}$$

Then we wish to show that this homomorphism satisfies each relation in R. First we note that relations 2 and 3 from Definition 3.0.3 ensure that

$$\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}) = \psi(\epsilon_{i,j})v_j v_k \psi(\epsilon_{k,l}) = \delta_{j,k}\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}).$$

Since this trivially gives us the desired result $\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}) = \delta_{j,k}\psi(\epsilon_{i,l})$ whenever $j \neq k$, let us now assume $j = k$. We make note of the fact that each vertex in E is the source of at most one edge, and hence relation 5 simplifies down to

$$e' e^* = \delta_{e',e} s(e).$$

Now we claim that

$$\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}) = \begin{cases} e_i e_{i+1} \cdots e_{l-1}, \text{ if } i < l, \\ e_{i-1}^* e_{i-2}^* \cdots e_l^*, \text{ if } i > l, \\ v_l, \text{ if } i = l. \end{cases}$$

To verify this claim, there are several case to check. For example, when $i < j = k < l$, we have

$$\psi(\epsilon_{i,j})\psi_{\epsilon_{k,l}} = e_i e_{i+1} \dots e_{j-1} e_k e_{k+1} \dots e_{l-1} = \psi(\epsilon_{i,l}).$$

On the other hand, if $i < l < k$, then we have

$$\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}) = e_i e_{i+1} \dots e_{j-1} e_{k-1}^* e_{k-2}^* \dots e_l^* = \psi(\epsilon_{i,l-1}).$$

Verification of all the other cases use similar arguments, and which we will omit.

This tells us that we have $\psi(\epsilon_{i,j})\psi(\epsilon_{k,l}) = \delta_{j,k}\psi(\epsilon_{i,l})$ as is desired.

Now we construct an algebra homomorphism in the other direction. Let $\phi : \mathbb{F}\langle E^0 \cup E^1 \cup (E^1)^* \rangle \to M_n(\mathbb{F})$ be the homomorphism extending the mappings $v_j \mapsto \epsilon_{j,j}$, $e_j \mapsto \epsilon_{j,j+1}$ and $e_j^* \mapsto \epsilon_{j+1,j}$. We show that ϕ satisfies relations 1-6 in Definition 3.0.3. For relation 1, we see

$$\phi(v_k v_j) = \phi(v_k)\phi(v_j) = \epsilon_{k,k}\epsilon_{j,j} = \delta_{j,k}\epsilon_{k,j} = \delta_{j,k}\epsilon_{k,k} = \phi(\delta_{j,k}v_k).$$

For relation 2, we have

$$\phi(e_j) = \epsilon_{j,j+1} = \epsilon_{j,j}\epsilon_{j,j+1} = \phi(v_j e_j) = \phi(s(e_j))\phi(e_j).$$

Similar arguments work for $\phi(e_j) = \phi(e_j r(e))$ and the entirety of relation 3. For relation 4, we have

$$\phi(e_j^* e_k) = \epsilon_{j+1,j}\epsilon_{k,k+1} = \delta_{j,k}\epsilon_{j+1,k+1} = \delta_{j,k}\epsilon_{k+1,k+1} = \phi(\delta_{j,k}r(e_k)).$$

Since each vertex is the source of at most one edge, the sum in relation 5 has at most one term. Hence an identical argument shows that relation 5 is satisfied. Hence ϕ satisfies the Leavitt path algebra relations.

By the universal properties of $L_{\mathbb{F}}(E)$ and $M_n(\mathbb{F})$, the maps defined above induce homomorphisms $\varphi : L_{\mathbb{F}}(E) \to M_n(\mathbb{F})$ and $\psi : M_n(\mathbb{F}) \to L_{\mathbb{F}}(E)$. Since the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are both homomorphisms that are the identity on the generators of each algebra, the universal properties of these algebras ensure that they are the identity homomorphism. In particular, both φ and ψ are isomorphisms, as is desired.

CHAPTER 4

CATEGORY THEORY

4.1 The Basic Language of Category Theory

We now introduce the basic language of category theory, mostly following [4].

Definition 4.1.1 (Category)**.** A *category* C consists of a collection $Ob(C)$ of *objects*, and for each pair X, Y of objects a set $\mathrm{Hom}(X, Y)$ of *morphisms* satisfying the following:

1. If X, Y, Z are objects, then for each $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$, there is given a unique morphism $g \circ f \in \mathrm{Hom}(X, Z)$ called the *composition* of g with f such that or any triplet f, g, h we always have $(f \circ g) \circ h = f \circ (g \circ h)$ whenever both $f \circ g$ and $g \circ h$ exist.

2. For each object X, there is a morpism $Id_X \in \mathrm{Hom}(X, X)$ with the property that for all objects Y, and all morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ we have $f \circ Id_X f$ and $Id_X \circ g = g$.

Examples 4.1.2.

1. There is a category called SET. In this category we take the objects to be sets, and the morphisms to be functions between sets. Composition of morphisms in this category is defined to be composition of functions.

2. Many of the structures encountered in algebra classes also form categories. In particular, there are categories of monoids, groups, rings, fields, modules, vector spaces, and algebras. In each of these categories, the morphisms are the various types of homomorphism (and linear maps for vector spaces).

3. Let C be a category with one object X such that for any $f \in Hom(X, X)$, there exists some $f^{-1} \in Hom(X, X)$ such that $f \circ f^{-1} = f^{-1} \circ f = Id_X$. Since we have

associativity of composition, inverses, and an identity element, the morphisms of this category form a group. Given an arbitrary group G, we may also form a category by defining the object to be a single point, and the morphisms to be elements of the group. The reader may verify that the definition of a group easily ensures that the category axioms are satisfied.

4. Let X be some partially ordered set. Then we may define a category C by taking $Ob(C) = X$. In this category, we say there exists a morphism from $x \to y$ if and only if $x \leq y$. Since $x \leq x$ for each $x \in X$, we see that each object has the required identity morphism. Since there is at most one morphism between any two objects, and a partial ordering is transitive, we have only one choice for composition of morphisms, and this composition satisfies the axioms for a category.

5. There is a category of categories. In this category, we take the objects to be categories themselves. We will define the morphisms of this category in Definition 4.1.7

6. Let E be some finite directed graph. Then we can define the category $Path(E)$. In this category, we take the objects to be the vertices of E. A morphism of this category is a *path*, which one may consider as an operdered tuple $(e_1e_2 \ldots e_n)$ of edges such that $r(e_k) = s(e_{k+1})$ for all $1 \leq k < n$. We also take a single vertex to be a path. Then we define the identity morphism for a vertex v to be the path (v), and we define composition of morphisms to be the concatenation of paths (i.e. $e_1 \cdot e_2 = e_1e_2$).

Definition 4.1.3 (Isomorphism)**.** Let X and Y be objects of some category C. We say X and Y are *isomorphic* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$. Whenever f and g fulfill this conditions, we call them *isomorphisms*, and say that f and g are *inverses*.

Examples 4.1.4.

1. Let C be the category SET. Then objects $X, Y \in C$ are isomorphic if and only if there exists a bijection between X and Y.

2. Readers will also have encountered examples of isomorphisms in their studies of other mathemtacial structures, such as groups, rings, and vector spaces.

As mentioned before, there is a category of categories. The morphisms in this category are called functors. They are defined as follows:

Definition 4.1.5 (Functor)**.** Let C and D be categories. Then a *functor* from C to D, denoted by $F : C \to D$, is a mapping that sends each object X of C to some object $F(X)$ of D and each morphism f of C to some morphism $F(f)$ of D, satisfying the following:

1. For $X, Y \in Ob(C)$, and $f \in Hom(X, Y)$, we have $F(f) \in Hom(F(X), F(Y))$,

2. For any $X \in Ob(C)$, we have $F(Id_X) = Id_{F(X)}$, and

3. For any morphisms f, g of C that compose, we have $F(f \circ g) = F(f) \circ F(g)$.

Examples 4.1.6.

1. Let C and D be two partially ordered sets, treated as categories as in Example 4.1.2. we may take F to be a functor between these two categories. Then we observe that if $X \leq Y$, we must have $F(X) \leq F(Y)$. This tells us that a functor between these categories is the same thing as a function that preserves the partial ordering of their elements.

2. Let G and H be groups. As in Example 4.1.2 number 3, when we treat these groups as categories, the morphisms are just the elements of the group and composition is multiplication. If we have a functor $F : G \to H$, then we see that for any $a, b \in G$ we have $F(ab) = F(a \circ b) = F(a) \circ F(b) = F(a)F(b)$. This tells us that a functor between groups is just a group homomorphism.

3. In the initial examples of categories, we mentioned many categories based on sets (such as groups, rings, vector spaces, and the like). In each of these examples, we

may define *forgetful functors* from these categories to the category of sets. These functors map each object of the category to the set containing its elements, and each morphism to the function that gives the same action. For example, if G is the group $\mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{Z}/4\mathbb{Z}$, and $\varphi : H \to G$ is the homomorphism such that $\varphi(1) = 1$, and F is the forgetful functor from the category of groups to the category of sets, then $F(G)$ is the set containing the elements of $\mathbb{Z}/2\mathbb{Z}$, $F(H)$ containing the elements of $\mathbb{Z}/4\mathbb{Z}$, and $F(\varphi)$ is still just φ, only now it is considered a map of sets.

4. Now let us define C to be the category whose objects are vector spaces over some fixed field $\mathbb{F}$, and whose morphisms are linear maps between these vector spaces. For a given vector space V, we define its dual space to be
$V^* := \{\text{linear maps from } V \to \mathbb{F}\}$. We may also define its double dual,
$V^{**} := \{\text{linear maps from } V^* \to \mathbb{F}\}$. If V and W are $\mathbb{F}$-vector spaces, and f a linear map between them, then we define

$$f^{**} : V^{**} \to W^{**} :$$

$$f^{**}(v^{**})(w^*) = v^{**}(w^* \circ f), \text{ for all } v^{**} \in V^{**} \text{ and } w^* \in W^*.$$

Then the mapping

$$F : C \to C$$

$$F(V) = V^{**}$$

$$F(f) = f^{**}$$

is a functor from $C \to C$, called the *double dual functor.*

Definition 4.1.7 (Natural Transformation)**.** Let C and D be categories, and let $F, G : C \to D$ be functors. A *natural transformation* $\tau : F \to G$ is a function giving each object $X \in C$ a morphism $\tau_X : F(X) \to G(X)$ of D such that for each morphism $f : X \to Y$ in C, the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[d, "f"] & F(X) \arrow[r, "\tau_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
Y & F(Y) \arrow[r, "\tau_Y"] & G(Y)
\end{tikzcd}
$$

If τ is such a function, then we say that the morphisms τ_X are *natural* in X. If each τ_X is also an isomorphism, we say τ is a *natural isomorphism*. If F is naturally isomorphic to G, we denote the situation with $F \cong G$.

Since we have defined a category of categories, it seems natural to think in terms of isomorphisms of categories. Following the general notion of an isomorphism, this would require us to have functors $F : C \to D$ and $G : D \to C$ such that $F \circ G = Id_D$ and $G \circ F = Id_C$. However, it turns out that in many cases where we want to talk about categories being the same, this idea of isomorphism is too restrictive. To resolve this problem, we introduce the idea of equivalence of categories.

Definition 4.1.8. [Categorical Equivalence] Let C and D be categories. Then we say C and D are *equivalent* if there exist functors $F : C \to D$ and $G : D \to C$ such that $F \circ G \cong Id_D$ and $G \circ F \cong Id_C$.

Examples 4.1.9.

1. Let C be any category, and let $Id_C : C \to C$ be the identity functor. Then clearly for any $X, Y \in C$ and morphism $f : X \to Y$ the following diagram commutes:

$$
\begin{tikzcd}
c \arrow[d, "f"] & c \arrow[r, "Id_c"] \arrow[d, "f"'] & c \arrow[d, "f"] \\
c' & c' \arrow[r, "Id_{c'}"] & c'
\end{tikzcd}
$$

Since the identity morphism is always an isomorphism, this tells us that each category is equivalent to itself.

2. Let C be the category of finite dimensional vector spaces over some field $\mathbb{F}$, let $F : C \to C$ be the double dual functor defined in Example 4.1.6 part 4, and let $Id_C : C \to C$ be the identity functor. Since $Id_C \circ F = F \circ Id_C = F$, it just remains to show that F is naturally isomorphic to Id_C. So for a given vector space V, let $\tau_V : V \to F(V)$ be the map such that $\tau_V(v)(w^*) := w^*(v)$. As in [2, Section 11.3 Theorem 19], we see that this map is an isomorphism. It is also naturally isomorphic to the identity, since for each $v \in V$ and each $w^* \in W^*$ we have

$$\tau_W(f(v))(w^*) = w^*(f(v)) = \tau_V(v)(w^* \circ f) = f^{**}(\tau_V(v))(w^*).$$

4.2 The Category of Modules Over an Algebra

Before we approach the main theorem of the text, we will establish a useful result about the category of modules over a general algebra.

Definition 4.2.1. Let A be some $\mathbb{F}$-algebra with identity. Then we define $\mathrm{Mod}(()A)$ to be the category whose objects are unital A-modules, and whose morphisms are homomorphisms of A-modules.

Definition 4.2.2. Let A be some $\mathbb{F}$-algebra with identity. Then we take $M(A)$ to be the category whose objects and morphisms are defined as follows.

Objects: An object of $M(A)$ consists of a pair (V, φ) of an $\mathbb{F}$-vector space V and an identity preserving algebra homomorphism $\varphi : A \to End_{\mathbb{F}}(V)$, where $End_{\mathbb{F}}(V)$ is the algebra of linear maps from V to V, as in Example 2.1.4 part 6.

Morphisms: For objects (V, φ) and (W, ψ), a morphism $T : (V, \varphi) \to (W, \psi)$ consists of a linear map $t : V \to W$ such that for each $a \in A$, we have $t \circ \varphi(a) = \psi(a) \circ t$.

Theorem 4.2.3. *Let A be any $\mathbb{F}$-algebra with identity. Then the categories $M(A)$ and $Mod(A)$ are equivalent.*

Proof. Let A be an algebra over some field $\mathbb{F}$ and let $M \in \mathrm{Mod}(A)$. For $\alpha \in \mathbb{F}$ and $m \in M$, we define $\alpha m := (\alpha 1_A)m$. The module and algebra axioms ensure that M is a $\mathbb{F}$-vector space under this definition of scalar multiplication. Now we define the map

$$\varphi_M : A \to \mathrm{End}_{\mathbb{F}}(M)$$

(where $\mathrm{End}_{\mathbb{F}}(M)$ is the set of linear endomorphisms from M to M when considered as a vector space) to be given by

$$\varphi_M(a)(m) = am,$$

for $a \in A$ and $m \in M$.

Now let $M, N \in \mathrm{Mod}(A)$ and let $f : M \to N$ be a module homomorphism. For $a \in A$ and $m \in M$, we clearly have $f(am) = af(m)$. If we now consider M as a vector space, this tells us $f(\varphi_M(a)m) = \varphi_N(a)f(m)$. If we define

$$F : \mathrm{Mod}(A) \to M(A)$$

to be the map sending

$$M \mapsto (M, \varphi_M), \text{ and}$$

$$f \mapsto f,$$

then F preserves the composition and identities of morphisms (as it is the identity on them), and hence is a functor.

Now let (V, φ) be some object of $M(A)$. Then for $v \in V$ and $a \in A$, we can define the action $av := \varphi(a)(v)$. As $\varphi(a)$ is linear, we observe that this action makes V a module over A. Note that for any morphism

$$T : (V, \varphi) \to (W, \psi)$$

we have

$$t(av) = t(\varphi(a)v) = \psi(a)t(v) = at(v).$$

Along with the fact that t is linear, this ensures that t is a morphism from $V \to W$ in $\mathrm{Mod}(A)$. If we define

$$G : M(A) \to \mathrm{Mod}(A)$$

to be the map sending

$$(V, \varphi) \mapsto V, \text{ and}$$

$f \mapsto f$, then we note that G preserves the composition and identities of morphisms and is therefore a functor.

Now we note that for $M \in \mathrm{Mod}(A)$, we clearly have $G \circ F(M) = M$, for $f \in \mathrm{Mod}(A)$, we have $G \circ F(f) = f$. Hence $G \circ F = Id_{\mathrm{Mod}(A)}$. Similarly $F \circ G = Id_{M(A)}$. Since the identity functor is definitely naturally isomorphic to itself, this establishes the desired equivalence of categories. $\square$

CHAPTER 5

THE CATEGORY OF MODULES OVER LEAVITT PATH ALGEBRAS

Definition 5.0.1 (Cuntz-Krieger-Leavitt Category)**.** Let E be some finite directed graph and let Paths(E) be the path category of E, as in Example 4.1.2 . Then we define the category $CKL_{\mathbb{F}}(E)$ to be as follows:

The objects of the category are functors $M : \text{Paths}(E) \to Mod(\mathbb{F})$ such that if $v \in E^0$ is the range of at least on edge, then the map

$$\sum_{r(e)=v} M(e) : \bigoplus_{r(e)=v} M(s(e)) \to M(v)$$

given by

$$(x_{s(e)}|r(e) = v) \mapsto \sum_{r(e)=v} M(e)(x_{s(e)})$$

is an isomorphism of vector spaces.

The morphisms of $CKL_{\mathbb{F}}(E)$ are natural transformations of functors, and the composition of morphisms is composition of functors.

Note that although we are defining the objects of this category as functors, it is not something categorical in nature. Objects of this category end up looking like a collection of vector spaces and linear maps between them, up to the above direct sum condition. The purpose of defining this as a functor is that it allows for this idea to be expressed in a compact manner. Under this consideration, a morphism of this category is a collection of linear maps from the vector spaces of one object to another, such that the structure of the linear maps of each object is in some sense the same.

Examples 5.0.2.

1. Let E be the graph with a single vertex v, and let $\mathbb{F}$ be some field. Then an object of $\text{CKL}_{\mathbb{F}}(E)$ consists of a functor that sends v to some vector space V, and Id_v to Id_V. For objects M and N of $\text{CKL}_{\mathbb{F}}(E)$, a morphism $T : M \to N$ consists of a linear map

$T_v : M(v) \to N(v)$ such that $T_v \circ Id_{M(v)} = Id_{N(v)} \circ T_v$. Clearly this applies for every linear map T_v, and so a morphism is just any linear map from $M(v) \to N(v)$.These considerations show that the functor $CKL_F(E) \to \text{Mod}(()F)$ defined on objects by $M \mapsto M(v)$, and on morphisms by $T \mapsto T_v$, is an equivalence of categories.(We have not quite demonstrated this, but finding the inverse functor here consists of essentially the same argument.)

2. Let E be the graph with two vertices, v and w, and no edges between them, and let $\mathbb{F}$ be some field. Then an object of $\text{CKL}_{\mathbb{F}}(E)$ consists of a functor sending v and w each to some vector spaces, and the identity morphisms to the identites of their respective vector spaces. For objects M and N of $\text{CKL}_{\mathbb{F}}(E)$, a morphism $T : M \to N$ is a collection $T = \{T_v, T_w\}$ of linear maps such that $T_v \circ Id_{M(v)} = Id_{N(v)} \circ T_v$ and $T_w \circ Id_{M(w)} = Id_{N(w)} \circ T_w$. Just as in the last example, this will hold for any pair of linear maps.

   This allows us to take a functor from $\text{CKL}_{\mathbb{F}}(E) \to \text{Mod}(\mathbb{F} \times \mathbb{F})$ sending $M \mapsto (M(v), M(w))$, $T : M \to N$ to (T_v, T_w). As before, this gives us an equivalence of categories.

Definition 5.0.3. Let $E = (E^0, E^1, r, s)$ be some graph. Then we define it's opposite graph, to be the graph $E^{\text{opp}} := (E^0, E^1, s, r)$. Put into the common interpretation of directed graphs, this is the graph with the same vertices and edges as E, but with the arrows pointing in the opposite direction.

The following theorem is essentially [3, Proposition 3.2]. We shall give a proof of this result that fills in some of the omitted details.

Theorem 5.0.4. *Let E be some finite directed graph and E^{opp} its opposite. Then $CKL_{\mathbb{F}}(E) \cong Mod(\mathrm{L}_{\mathbb{F}}(E^{opp}))$.*

Proof. Let E be some finite directed graph, and $\mathbb{F}$ some field. For a given $M \in \text{CKL}_{\mathbb{F}}(E)$, we define

$$V_M := \bigoplus_{v \in E^0} M(v).$$

If we denote elements of V_M by $(x_v | v \in E^0)$, then we define π_{M,v_0} to be the map sending $(x_v | v \in E^0) \mapsto x_{v_0}$, and $\chi_{M,v} : M(v) \to V_M$ to be the map sending $M(v)$ to its naturally isomorphic copy within the direct suma V_M.

Now let $v \in E^0$ be a vertex that is the range of at least one edge, i.e. $r^{-1}(v) \neq \emptyset$. Recall from Definition 5.0.1 that the map

$$\sum_{r(e)=v} M(e) : \bigoplus_{r(e)=v} M(s(e)) \to M(v)$$

given by

$$(x_{s(e)} | r(e) = v) \mapsto \sum_{r(e)=v} M(e)(x_(s(e)))$$

is an isomorphism. For each edge $e_0 \in r^{-1}(v)$ we define

$$\pi'_{M,e_0} : M(r(e_0)) \to M(s(e_0))$$

to be the map

$$\pi'_{M,e_0} = \pi_{M,s(e_0)} \circ \left(\sum_{r(e)=v} M(e) \right)^{-1}.$$

Then we take F to be the functor from $\mathrm{CKL}_{\mathbb{F}}(E) \to M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$ defined as follows: For a given object M of $\mathrm{CKL}_{\mathbb{F}}(E)$, we take $F(M) = (V_M, \varphi_M)$, where φ_M is the algebra homomorphism from $\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}) \to \mathrm{End}_{\mathbb{F}}(V)_M$ such that for each $x \in V_M$, we have

$$\varphi_M(v)(x) = (\chi_{M,v} \circ \pi_{M,v})(x) \text{ whenever } v \in E^0, \tag{5.0.5}$$

$$\varphi_M(e)(x) = (\chi_{M,r(e)} \circ M(e) \circ \pi_{M,s(e)})(x) \text{ whenever } e \in E^1, \text{ and} \tag{5.0.6}$$

$$\varphi_M(e^*)(x) = (\chi_{M,s(e)} \circ \pi'_{M,e} \circ \pi_{M,r(e)})(x) \text{ whenever } e^* \in (E^1)^*. \tag{5.0.7}$$

The fact that this is enough to define a homomorphism follows from the universal property of $\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})$, so long as we are able to verify that φ_M satisfies the necessary relations. To do so, we make note of the following identities:

$$\pi_{M,v} \circ \chi_{M,w} = \delta_{v,w} Id_w, \text{ and} \tag{5.0.8}$$

$$\pi'_{M,e} \circ M(e) = Id_{s(e)}. \tag{5.0.9}$$

An observant reader may notice that strictly speaking, the first identity isn't always well defined - if $v \neq w$, the codomains are different. However, in this case the delta function sends everything to zero anyways, and so we decide to abuse notation and have the zero live wherever we need it to. It will not end up making a difference.

The first identity holds as a simple observation that $\pi_{M,v}$ sends elements of $M(v)$ to themselves, and everything else to zero. The second identity is similar. For a given $x \in M(s(e))$, we are sending it to $\pi_{M,s(e)}(e_0) \circ \left(\sum_{r(e')=v} M(e')\right)^{-1} \circ M(e)$. Hence we see that an element of $M(s(e))$ will first get sent to $M(r(e))$. Then it will get sent to itself, but now living in the direct sum given in Definition 5.0.1, and then will get sent to itself living in $M(s(e))$, as is desired.

Now we verify the relations in defintion 3.0.3 (recall that we considering the Leavitt path algebra over the opposite graph of E, and so the various s's and r's are reversed). The reader will note that we will always send elements of $E^0 \cup E^1 \cup (E^1)^*$ to compositions whose leftmost element is one of our χ functions, and whose rightmost element is one of our π functions. This allows for immediate vertification of the first four relations. For example, we see in the first relation, we have

$$\varphi_M(v)\varphi_M(v') = (\chi_{M,v} \circ \pi_{M,v})(\chi_{M,v'} \circ \pi_{M,v'}) = \delta_{v,v'}\chi_{M,v}\pi_{M,v'} = \delta_{v,v'}\varphi_M(v).$$

Dealing with relations $2-4$ follow arguments that are essentially the same. The fifth relation is a bit trickier, and so we handle it in more detail. Recalling the definition of $\pi'_{M,e}$

we observe

$$\sum_{r(e)=v} \varphi_M(e) \circ \varphi_M(e^*) = \sum_{r(e)=v} \chi_{M,r(e)} \circ M(e) \circ \pi_{M,s(e)} \circ \chi_{M,s(e)} \circ \pi'_{M,e} \circ \pi_{M,r(e)} \tag{5.0.10}$$

$$= \sum_{r(e)=v} \chi_{M,r(e)} \circ M(e) \circ \pi'_{M,e} \circ \pi_{M,r(e)} \tag{5.0.11}$$

$$= \chi_{M,v} \circ \left(\sum_{r(e)=v} M(e)\pi'_{M,e} \right) \circ \pi_{M,v} \tag{5.0.12}$$

$$= \chi_{M,v} \circ \pi_{M,v} \tag{5.0.13}$$

$$= \varphi_M(v). \tag{5.0.14}$$

For relation 6, we see that for any $(x_v|v \in E^1) \in V_M$, we have

$$\sum_{v \in E^0} M(v)(x_v|v \in E^1) = \sum_{v \in E^0} (\chi_{M,v} \circ \pi_{M,v})(x_v|v \in E^1) = \sum_{v \in E^1} \chi_{M,v}(x_v) = (x_v|v \in E^1).$$

This shows that we have satisfied the universal property of $\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})$. This verifies φ_M is an algebra homomorphism, and so we have confirmed that F sends each object of $\mathrm{CKL}_{\mathbb{F}}(E)$ to an object of $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$.

Now we wish to define the action of F on a morphism of $\mathrm{CKL}_{\mathbb{F}}(E)$. For a pair M, N in $\mathrm{CKL}_{\mathbb{F}}(E)$, a morphism from M to N is a collecion $\tau = \{\tau_v \mid v \in E^0\}$ of linear maps $\tau_v : M(v) \to N(v)$ such that for any $e \in E^1$, we have $N(e) \circ \tau_{s(e)} = \tau_{r(e)} \circ M(e)$. Then we define $F(\tau) : F(M) \to F(N)$ to be the linear map $T : V_M \to V_N$ given by $T = \bigoplus_{v \in E^0} \tau_v$. This gives us the following identities:

$$\pi'_{N,e} \circ \pi_{N,r(e)} \circ T = \tau_{s(e)} \circ \pi'_{M,e} \circ \pi_{M,r(e)}, \tag{5.0.15}$$

$$T \circ \chi_{M,v} = \chi_{N,v} \circ \tau_v, \text{ and} \tag{5.0.16}$$

$$\pi_{N,v} \circ T = \tau_v \circ \pi_{M,v}. \tag{5.0.17}$$

If $v \in E^0$, then we have

$$T \circ \varphi_M(v) = T \circ \chi_{M,v} \circ \pi_{M,v} \tag{5.0.18}$$

$$= \chi_{N,v} \circ \tau_v \circ \pi_{M,v} \tag{5.0.19}$$

$$= \chi_{N,v} \circ \pi_{N,v} \circ T \tag{5.0.20}$$

$$= \varphi_N(v) \circ T. \tag{5.0.21}$$

If $e \in E^1$, then we have

$$T \circ \varphi_M(e) = T \circ \chi_{M,r(e)} \circ M(e) \circ \pi_{M,s(e)} \tag{5.0.22}$$

$$= \chi_{N,r(e)} \circ \tau_{r(e)} \circ M(e) \circ \pi_{M,s(e)} \tag{5.0.23}$$

$$= \chi_{N,r(e)} \circ N(e) \circ \tau_{s(e)} \circ \pi_{M,s(e)} \tag{5.0.24}$$

$$= \chi_{N,r(e)} \circ N(e) \circ \pi_{N,s(e)} \circ T \tag{5.0.25}$$

$$= \varphi_M(e) \circ T \tag{5.0.26}$$

Finally, if $e^* \in (E^1)^*$, then

$$T \circ \varphi_M(e^*) = T \circ \chi_{M,s(e)} \circ \pi'_{M,e} \circ \pi_{M,r(e)} \tag{5.0.27}$$

$$= \chi_{N,s(e)} \circ \tau_v \circ \pi'_{M,e} \circ \pi_{M,r(e)} \tag{5.0.28}$$

$$= \chi_{N,s(e)} \circ \pi'_{N,e} \circ \pi_{N,r(e)} \circ T \tag{5.0.29}$$

$$= \varphi_N(e^*) \circ T. \tag{5.0.30}$$

We have verified that $T \circ \varphi_M(s) = \varphi_N(s) \circ T$ for each $s \in E^0 \cup E^1 \cup (E^1)^*$. Since these elements generate $L_{\mathbb{F}}(E^{opp})$, this tells us that $T \circ \phi_M(a) = \phi_N(a) \circ T$ for every $a \in L_{\mathbb{F}}(E^{opp})$, and so $T = F(\tau)$ is a morphism in the category $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$. We also note that since each τ_v is the identity map whenever T is the identity morphism, clearly $F(Id_M) = Id_{F(M)}$ for any $M \in \mathrm{CKL}_{\mathbb{F}}(E)$.

Finally, if $S : M \to N$ given by $S = \{\sigma_v \mid v \in E^0\}$ and $T : N \to O$ given by $T = \{\tau_v \mid v \in E^0\}$ are two morphisms in $\mathrm{CKL}_{\mathbb{F}}(E)$, then their composition is

$T \circ S = \{\tau_v \circ \sigma_v \mid v \in E^0\}$. Since $F(T) \circ F(S) = \bigoplus_{v\in E^0} \tau_v \circ \bigoplus_{v\in E^0} \sigma_v = \bigoplus_{v\in E^0} \tau_v \circ \sigma_v$, we see that F respects the composition of morphisms and is therefore a functor.

Now we wish to define a second functor, $H : M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})) \to \mathrm{CKL}_{\mathbb{F}}(E)$. Recall that an object of $\mathrm{CKL}_{\mathbb{F}}(E)$ is a functor from $\mathrm{Paths}(E) \to \mathrm{Mod}(\mathbb{F})$. Let (V, φ) be an object of the category $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$. Then for a given $v \in E^0$, we take $H(V, \varphi)(v)$ to be $Im(\varphi(v))$. (We will consider this image as its own vector space, instead of as a subspace living in V.) For an edge $e \in E^1$, we define $H(V, \varphi)(e) : Im(\varphi(s(e))) \to Im(\varphi(r(e)))$ to be the restriction of the map $\varphi(e) : V \to V$ to the subspace $Im(\varphi(s(e)))$. Note that the relation $e = r(e)e$ in $\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})$ ensures that the image of $\varphi(e)$ is contained in the image of $\varphi(r(e))$. For a path $e_1e_2 \ldots e_k$, we define

$$H(V, \varphi)(e_1e_2 \ldots e_k) := H(V, \varphi)(e_k) \circ \cdots \circ H(V, \varphi)(e_2) \circ H(V, \varphi)(e_1).$$

Now let $v \in E^0$ be a vertex such that $r^{-1}(v) \neq \emptyset$. Then we define the map

$$\psi_v : \bigoplus_{r(e)=v} Im(\varphi(s(e))) \to Im(\varphi(v))$$

to be given by

$$\psi_v(x_e|r(e) = v) = \sum_{r(e)=v} \varphi(e)(x_e).$$

We claim that ψ_v is an isomorphism of vector spaces. For a given $x \in Im(\varphi(v))$, we have via relation 3 in Definition 3.0.3 $\varphi(e^*)(x) = \varphi(s(e)) \circ \varphi(e^*)(x)$ (recall that we are in the opposite graph of E). This ensures that $\psi_v(e^*)(x)|r(e) = v)$ a well-defined element of $\bigoplus_{r(e)=v} Im(\varphi(s(e)))$, and relation 5 tells us that ψ_v sends this element to x.

To see that ψ_v is injective, we begin by assuming $\psi_v(x_e|r(e) = v) = 0$ for some $(x_e|r(e) = v) \in \bigoplus_{r(e)=v} Im(\varphi(s(e)))$. Relation 4 of Definition 3.0.3 tells us $0 = \varphi(f^*) \circ \psi_v(x_e \mid r(e) = v) = \sum_{r(e)=v} \varphi(f^*e)x_e = \phi(s(f))x_f$, for each edge $f \in E^1$ with $r(f) = v$. Since $\varphi(s(f))$ is idempotent (via relation 1), we know that it must act as the identity on $Im(\varphi(s(f))$ and hence we have

$$x_f = \varphi(s(f))(x_f) = 0.$$

Since this holds for each f with $r(f) = v$, this tells us the kernel of ψ_v is trivial and hence ψ_v is injective. Therefore $H(V, \varphi)$ is an object of the category $\mathrm{CKL}_{\mathbb{F}}(E)$.

Now we define the action of H on morphisms. Let $T : (V, \varphi) \to (W, \psi)$ be some morphism in $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$. Recall that this morphism T consists of a linear map $t : V \to W$ with the property that $t \circ \phi(a) = \psi(a) \circ t$ for every $a \in \mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})$. Then for each vertex $v \in E^0$, we have $t \circ \varphi(v) = \psi(v) \circ t$, and so we can restrict t to the map $\tau_v : Im(\phi(v)) \to Im(\psi(v))$. Since this map satisfies $\tau_v \circ \phi(e) = \psi(e) \circ \tau_v$ for each edge $e \in E^1$, the collection $(\tau_v | v \in E^0)$ constitutes a natural transformation $H(V, \phi) \to H(W, \psi)$.

Now we wish to show that H respects the identity. If T is the identity morphism, then t is the identity linear map, and hence each restriction of t_v of t is the identity on the subspace it is identified with. We also see that H respects the composition of functions. This is because the action of H on morphisms is restricting functions, which respects the composition of functoins as well. This proves that H is a functor from $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}})) \to \mathrm{CKL}_{\mathbb{F}}(E)$.

Now we wish to show that the functors F and H establish an equivalence of categories. To do this, we must show that both $F \circ H$ and $H \circ F$ are naturally isomorphic to the identity. We first show this for $H \circ F$.

Let M be an object of $\mathrm{CKL}_{\mathbb{F}}(E)$. Then $H \circ F(M) = H(V_M, \varphi_M)$, where V_M and φ_M are defined as earlier in this proof. For a vertex $v \in E^0$, we have $H(V_M, \varphi_M)(v) = Im(\varphi_M(v)) = Im(\chi_{M,v} \circ \pi_{M,v})$, where the image is treated as its own vector space. Recall that we defined $\chi_{M,v}$ to be the natural injection $M(v) \to V_M$. Since $Im(\pi_{M,v})$ is clearly just $M(v)$, this tells us $H(V_M, \varphi_M)(v) = M(v)$.

Now let $e_1 \in E^1$ be an edge of E. Then

$$H(V_M, \varphi_M)(e) : Im(\varphi_M(s(e))) \to Im(\varphi_M(r(e)))$$

is the restriction of $\varphi_M(e)$ to the subspace $Im(\varphi_M(s(e)))$ (again with both treated as their own vector spaces). We previously proved that $Im(\varphi_M(s(e))) = M(s(e))$ and $Im(\varphi_M(r(e))) = M(r(e))$. Furthermore, for a given $x \in M(s(e))$, we have

$\varphi_M(e)(x) = \chi_{M,r(e)} \circ M(e) \circ \pi_{M,s(e)}(x)$. Since we are dealing with the restriction to $M(s(e))$, this tells us $\varphi_M(e)(x) = M(e)(x)$, and hence $H(V_M, \varphi_M)(e) = M(e)$. Since both functors have been shown to respect multiplication of edges, we may extend this to any path of E. This tells us that for any object $M \in \mathrm{CKL}_{\mathbb{F}}(E)$, we have $M = H \circ F(M)$. In particular, we have shown that $H \circ F = Id_{\mathrm{CKL}_{\mathbb{F}}(E)}$.

Now we consider $F \circ H$. First we note that for a given object (V, φ) of $M(\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}))$, $F \circ H(V, \varphi)$ will give us a vector space and an identity respecting algebra homomorphism from $\mathrm{L}_{\mathbb{F}}(E^{\mathrm{opp}}) \to End_{\mathbb{F}}(V)$. First we note that, by direclty applying the definitions of F and H, this vector space will be

$$\bigoplus_{v \in E^0} H(V, \varphi)(v) = \bigoplus_{v \in E^0} Im(\varphi(v)).$$

Relation 6 in definition 3.0.3 tells us that this sum is a natural isomorphism.

Considering the image of φ under $F \circ H$, we see

$$(\chi_{H(V,\varphi),v} \circ \pi_{H(V,\varphi),v})(x_v | v \in E^0) = \varphi(v)(x_v) \text{ whenever } v \in E^0, \quad (5.0.31)$$

$$(\chi_{H(V,\varphi),r(e)} \circ H(V,\varphi)(e) \circ \pi_{H(V,\varphi),s(e)})(x_v | v \in E^0) = \varphi(e)(x_v) \text{ whenever } e \in E^1, \text{ and} \quad (5.0.32)$$

$$(\chi_{H(V,\varphi),s(e)} \circ \pi'_{H(V,\varphi),e} \circ \pi_{H(V,\varphi),r(e)})(x_v | v \in E^0) = \varphi(e^*)(x_v) \text{ whenever } e^* \in (E^1)^*. \quad (5.0.33)$$

This tells us that $F \circ H$ is the identity. Thus we have demonstrated an equivalence of categories, as is desired. $\square$

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