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A Reorganization in the Continuity of Subject Matter in Mathematics

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ABSTRACT

A REORGANIZATION IN THE CONTINUITY OF THE SUBJECT MATTER IN MATHEMATICS

This thesis considers a reorganization in the order of arrangement of certain topics in elementary and undergraduate mathematics, i.e., arithmetic, algebra, plane geometry, solid geometry, trigonometry, analytic geometry, and calculus. Two terms important in the discussion are reorganization, the process of changing the relative position of topics or proofs in mathematics to an earlier or later place in the development of subject matter, and continuity, the logical order of topics arranged according to the need of one to explain the other.

The purpose of the thesis is two-fold: First, to show what arrangement of topics may be desirable, and, Second, to justify the proposed changes by showing that such a reorganization will make it possible to give a simpler and more complete presentation of mathematics without affecting the logical sequence of topics.

The discussion reviews the recent changes in elementary mathematics during the past forty years. These changes, in

general, may be thought of as either of a general character indicating a trend or of a special character indicating a re-arrangement in the order of particular topics.

The general arrangement of the thesis is somewhat as follows. It is observed that propositions in elementary mathematics have been proved by methods of analytic geometry and calculus. Proofs of certain propositions in plane geometry are possible by coordinate methods. When they are presented in algebra, these proofs are not only simple but provide further understanding of topics in algebra, such as graphs, ratio and proportion, and the operations of algebra. Proofs of certain propositions, or formulas, from elementary mathematics are possible by means of integration. Such proofs by calculus are too difficult to be presented in algebra. These proofs should be postponed to calculus where the simple method of integration justifies the omission of any earlier type of proof of these propositions in elementary mathematics.

In the conclusion of this discussion a rearrangement of topics in elementary mathematics (seventh year mathematics, eighth year mathematics, first year algebra, second course in algebra, and plane geometry) with special attention to the continuity of subject matter is given. Such a rearrangement, of necessity, implies changes in the order of some of the topics in later mathematics.

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CHAPTER I

Although in recent years there has been a tendency to reorganize the subject matter in mathematics according to its value to the pupil, this study will be a discussion of a reorganization in the order of arrangement of topics and not a résumé of the teaching practices involved in adapting the subject matter of mathematics to the individual.

Since the study makes use of the ideas of continuity and reorganization, it is desirable to give some explanation of the meaning of these terms as they are used in this thesis. Continuity in subject matter in mathematics is the logical order of topics arranged according to the need of one to explain the other. For example, in geometry the proposition[1] that the areas of two similar polygons have the same ratios as the squares of their radii depends upon the proofs of preceding propositions, definitions, and assumptions. Such preceding propositions in logical order are:

a. Regular polygons of the same number of sides are similar.

b. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii.

c. The sum of the interior angles of a polygon having n sides is $(n - 2)$ straight angles.

d. Each angle of a regular polygon having n sides is $(n - 2)/n$ straight angles.

It is seen that the foregoing propositions include such topics

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[1] W. Wells and W.W. Hart, Modern Plane Geometry, p. 244.

as perimeters, radii of circles, areas of polygons, ratio and proportion, and the measurement of angles.

The term reorganization as used in this discussion means the process of changing the relative positions of topics or proofs in mathematics in order of arrangement either to an earlier or to a later place in the development of the subject matter. Some changes have already been made.[2] For example, in geometry the proposition that the ratio of the circumference of a circle to its diameter is a constant, π, formerly depended upon a more difficult proof, a proof which is now postponed to a later time. A statement of the dependent proposition and a brief explanation of the proof follow.[3]

> Given p, the perimeter of a regular inscribed polygon of a certain number of sides, and P, the perimeter of a regular circumscribed polygon of the same number of sides, to find p' and P', the perimeters of the regular inscribed and circumscribed polygons having double the number of sides.

Then $p' = \sqrt{p \times P'}$ and $P' = \frac{2P \times p}{P + p}$

In the proof of this proposition it is found that the limit approached through increasing the number of sides of both inscribed and circumscribed polygons is a circle. If the diameter of the circle is 1, both p' and P' would be equal approximately to the circumference of the circle. Since the proof involves the idea of limits, it is probably too difficult for beginners. This proof is now presented in geometry

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[2] There will be further discussion of recent changes in Chapter II.
[3] W. Wells, Plane Geometry, pp. 208-209.

as an optional topic[4] or is omitted entirely.[5] In a geometry omitting the proof, the only required explanation is an illustion to show that the relation between the perimeter and the diameter of an inscribed polygon changes as the number of sides in the polygon is increased. The perimeter approaches the circumference of the circle, and in the limit the circumference is equal to a certain number times the diameter. This number is called pi, π.

The purpose of this thesis is two-fold: First, to show what arrangement of topics may be desirable; and, Second, to justify the proposed changes by showing that such a reorganization will make it possible to give a simpler and more complete presentation of mathematics without affecting the logical sequence of topics.

In Chapter II we shall point out some of the outstanding changes that have been made in subject matter in recent years. Chapter III consists of proofs by coordinate geometry of certain propositions in plane geometry which may be introduced into the algebra. Chapter IV is in three parts which show the applications of calculus to elementary mathematics: first, the value of the theory of limits in algebra and geometry; second, proofs from the calculus which are more simple than proofs of corresponding topics in Euclidean geometry; and third,

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4 Wells and Hart, op. cit., pp. 262-264.
5 A.M. Welchons and W.R. Krickenberger, Plane Geometry, p. 315; D. Reichgott and L.R. Spiller, Today's Geometry, p. 235.

an illustration of the value of curve tracing in algebra. Chapter V, as a summary, sets forth places in the study of mathematics where the suggested reorganization is possible and shows that in such a reorganization, making a simpler and more complete presentation of mathematics, a logical order of topics is still maintained.

CHAPTER II

Mathematics has two general divisions: elementary mathematics and higher mathematics. Elementary mathematics includes subjects in the following order: arithmetic, algebra, plane geometry, solid geometry, and trigonometry. Higher mathematics includes analytic geometry, calculus, and more advanced theories.

In the past forty years certain changes have taken place in the order and content of some of these subjects. Although many books and articles have been written about these changes, it is not our purpose to present an exhaustive study. An effort has been made to select comments of writers and illustrations in textbooks to indicate outstanding changes.

In algebra it was considered important to include topics from the arithmetic. Early writers of algebras, more familiar with textbooks in arithmetic, followed the order of topics in the arithmetic. Thus the simple processes in arithmetic, addition, subtraction, multiplication, division, and the like, were discussed in the first few chapters of algebra before the simple equation was given. In 1908 W.J. Milne pointed out in the introduction to his algebra that the "basis of algebra is found in arithmetic".[1] In the list of topics on page 23, it is readily seen that he more or less followed the order of arithmetic in arranging his material. The topics which precede the

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[1] W.J. Milne, Standard Algebra, p. iii.

the simple equation are addition, subtraction, multiplication, division, factoring, highest common factor, lowest common multiple, and fractions. Milne's purpose was to show the relation of algebra to arithmetic rather than the dependence of algebra upon arithmetic. For example, after explaining the procedure in finding the square root of a polynomial, he reviews the similar process of finding the square root of a number.[2]

On the other hand today several other topics of algebra precede the four fundamental operations in algebra. An algebra written in 1936 by U.G. Mitchell and H.M. Walker shows this change.[3] A list of topics as given in this book shows the equation at the beginning of the book.

1. The Shorthand of Algebra
2. Use of Equations
3. Equations in Two Unknowns
4. Graphs
5. Directed Numbers
6. The Formula
7. Linear Equations Involving Negative Numbers
8. Multiplication of Binomials; Quadratic Equations
9. Fundamental Operations with Fractions
10. Exponents and Radicals
11. Fractional Equations[4]

Since square root and logarithms are first explained by numbers, another suggestion for elementary mathematics was an advanced course in special numerical computations. In 1916 David Eugene Smith discussed the desirability of a separate

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[2] W.J. Milne, Ibid., pp. 218-233.
[3] U.G. Mitchell and H.M. Walker, Algebra: A Way of Thinking.
[4] Fundamental operations discussed with various topics.

course in arithmetic for the high school.[5] His comment was that topics from the arithmetic should be studied simultaneously with the related topics of algebra and geometry. Today the algebra includes not only the four fundamental operations but also approximate computation in measurement, numerical operations such as square root, fractions, etc., and the numerical computations of trigonometry.[6]

An explanation of approximation in computation is necessary. Since many computations involve large decimals, it is desirable to determine the number of places beyond the decimal point to be retained in a result. This requires an understanding of significant figures. Upon agreeing that all the digits in a number including 0, except in special instances, are significant, we can show by simple rules the number of figures to retain in an approximate quotient, approximate sum, or approximate product.

The importance of approximate measurement in elementary mathematics has already been justified. The above ideas are also used in elementary statistics, another topic recently introduced into elementary mathematics. A further discussion of statistics will be found on page 26. In the algebra, topics in statistics include the possible error of a measurement and the

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5 D.E. Smith, "Arithmetic in the High School", _The Mathematics Teacher_, vol. viii, (1916), p. 155.
6 W. Betz, _Algebra for Today_, pp. 402-404; Mitchell and Walker, _op. cit._, pp. 190-209; R. Schorling, J.R. Clark, and R.R. Smith, "Measurement and Approximate Numbers", _Modern-School Mathematics, Book III_, pp. 129-138.

percentage of error in a measurement.[7] We shall now discuss other fundamental ideas of statistics which might very well be given in elementary mathematics.

Although the possible error of a measurement is important, the computations of statistics are more concerned with relative errors.[8] The <u>relative error</u> in any measurement is the ratio of the possible error to the measured value. Three simple theorems about possible and relative errors follow.

> <u>Theorem I</u>. The possible error in the sum or difference of two measurements is equal to the sum of the possible errors in the individual measurements.
>
> <u>Theorem II</u>. The relative error in the product of two measurements is equal approximately to the sum of the relative errors of the measurements.
>
> <u>Theorem III</u>. The relative error in the quotient of two measurements is equal approximately to the sum of the relative errors of the measurements.

The proofs of the above theorems depend upon certain elementary operations in algebra and require a knowledge of signed numbers, the laws of equations, fractional equations, multiplication of binomials, and the division of a binomial by a binomial. We assume that the above topics have been presented. Then in each of the three theorems the letters a and b will be taken as two measurements, and the letters e_1 and e_2 as their respective errors. The three theorems may be expressed as follows:

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7 Schorling, Clark, and Smith, *ibid*.
8 C.H. Richardson, *An Introduction to Statistical Analysis*, pp. 11-13.

<u>Theorem I</u>. $(a+e_1)+(b+e_2) = (a+b)+(e_1+e_2)$

$(a-e_1)-(b+e_2) = (a-b)-(e_1+e_2)$

where (e_1+e_2) is the sum of the possible errors.

<u>Theorem II</u>. $(a+e_1)(b+e_2) = ab+ae_2+be_1+e_1e_2$

$(a-e_1)(b-e_2) = ab-ae_2-be_1+e_1e_2$

where e_1e_2 is a very small decimal. Then the possible error is $(ae_2 \quad be_1)$ and the relative error is $(ae_2+be_1)/ab = e_1/a+e_2/b$ approximately.

<u>Theorem III</u>. $(a+e_1)/(b-e_2) = a/b+(ae_2+be_1)/[(b)(b-e_2)]$

in which $(b-e_2)$ is approximately **b**, and the absolute error is $(ae_2+be_1)/b^2$, and the relative error $\frac{ae_2+be_1}{b^2} / \frac{a}{b}$ $e_1/a+e_2/b$ approximately.

These theorems and the simple algebraic proofs could very well be presented in elementary mathematics in a second course in algebra.

Another change in elementary algebra has been to include topics from higher mathematics. The theory of determinants was one of these. In "The Teaching and History of Mathematics in the United States" Florian Cajori makes the following suggestion.[9] "It seems quite plain that the elements of determinants should form a part of algebra and should be taught early in

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[9] F. Cajori, "The History and Teaching of Mathematics in the United States", <u>Circular No. 3</u>, (1890), p. 293.

the course in order that they may be used in the study of coordinate geometry." Today determinants are a part of advanced algebra, being used to solve equations with two, three, four, and five unknowns.[10] Other attempts have been made to place this topic in elementary mathematics. A second year algebra written in 1929 by F. Engelhardt and L.D. Haertter included determinants as a supplementary topic as an additional method of solving simultaneous equations in two unknowns.[11] A recent thesis demonstrated the simplicity of the use of determinants from the eighth year throughout elementary mathematics.[12] It was shown that determinants readily followed operations with signed numbers and proved to be a simple method in solving problems. The use of determinants still remains a supplementary topic in elementary mathematics and may easily be postponed to advanced algebra. In the advanced algebra it has a definite use as an introduction to the solution of complicated systems of equations in advanced analytic geometry.

The presentation and the inclusion of geometry with algebra have also been considered; both Euclidean geometry and analytic geometry have been attempted in elementary algebra. The comparative value of the two geometries (Euclidean and analytic) will now be discussed in relation to many of the

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[10] H.R. Willard and N.R. Bryan, College Algebra, pp. 184-202.
[11] F. Engelhardt and L.D. Haertter, Second Course in Algebra, pp. 101-104.
[12] M.B. Milburn, Determinants, History, and Developments in Transitional Mathematics.

changes that have been suggested. As early as 1890 Florian Cajori recommended that the new idea of the study of geometry be extended throughout all the courses in mathematics.[13] This seemed to be desirable because the Euclidean geometry contained too large a number of new concepts, definitions, and propositions to be covered in a single course. If more time could be spent on them, they would be better understood. An attempt was then made to present geometry with the course in algebra through quadratics. There still remained, however, a wide difference between the algebra and plane geometry. Propositions by Euclid's method were proved by spatial magnitudes without the use of numbers. Algebra consisted of operations with numbers and letters. The early correlation of algebra with such a geometry was not evident. The purpose of presenting Euclidean geometry with algebra had no other reason or value, therefore, than that of extending the length of time in which to study geometry.

The appearance of René Descartes' coordinate, or analytic geometry in 1637 led to a great change in mathematics.[14] It was Descartes' idea to explain algebra by means of concepts and intuition in geometry. In other words he conceived an explanation of the equation by means of a graph. Showing first that any point in a plane is determined by two coordinates, x and y, he designated the equation $F(x,y) = 0$ as an expression true of every point on a curve which the equation represented. Se-

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13 F. Cajori, <u>op. cit.</u>, p. 293.

14 D.E. Smith, <u>History of Mathematics</u>, vol. i, p. 375.
B. Russell, <u>Principles of Mathematics</u>, p. 158.

lecting values for x and y, he could by the principle of one-to-one correspondence transfer numerical values to the curve.

Analytic geometry was not immediately accepted in Euclidean geometry. The method of simplifying the ancient geometry by numbers and algebra only began to appear in textbooks forty years ago. In the preface of Plane and Solid Geometry, published in 1899, by W.W. Beman and D.E. Smith, there was this sentence:[15]

> There is a growing belief among many teachers that such of the notions of modern geometry as naturally simplify the ancient should find a place in elementary textbooks. Accordingly they (the authors) have not hesitated to introduce the ideas of one-to-one correspondence, of negative magnitudes ...

The authors explained these two "notions" within the text.

The principle of one-to-one correspondence showed that one symbol, one operation, one result., of algebra was related to one symbol, one operation, one result, etc., of geometry. For example, a number in algebra corresponds to a line-segment in geometry. A more complete discussion of one-to-one correpondence shows that integers, fractions, and irrational numbers of the number system may be transferred to points on a line. The second "notion", negative magnitudes, shows that the authors recognized negative direction of a line-segment. Thus negative values of integers, fractions, and irrational numbers may also be transferred to a line.

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15 W.W. Beman and D.E. Smith, Plane and Solid Geometry, p. iii.

The ideas of Descartes indicate a true relation between algebra and geometry through transferring the equation to points on a plane. Today by means of analytic geometry the illustrations of equations whose graphs are straight lines, circles, parabolas, hyperbolas, and ellipses are possible. Sometimes the work in algebra also includes the graphic representation of the cubic equation. The methods of analytic geometry, therefore, have a logical place in the algebra.

Other changes have taken place in the geometry. It was considered desirable to study plane geometry and solid geometry simultaneously. By the traditional arrangement, geometry was studied in nine books: five books of plane geometry and four of solid geometry. Since plane geometry was concerned with lines on a plane and solid geometry was concerned with planes in space, the relation of the subjects was obvious. Today there are two ways of correlating these two geometries: first, by dividing the whole geometry into informal intuitive geometry and formal demonstrative geometry; and second, by alternating propositions of plane geometry with the related propositions of solid geometry.

Intuitive geometry includes concepts and definitions of geometry that can be presented informally. These concepts and definitions include angles, planes, solids, measurement of plane figures, and simple constructions of plane figures. This type of geometry has been placed in seventh and eighth year mathematics. Topics of intuitive geometry include the following:

the circle,
the right triangle, drawing perpendiculars,
constructing rectangles, squares, and parallelograms,
bisecting a line and dividing a line into equal parts,
bisecting an angle,
constructing regular octagons, equilateral triangles, and regular hexagons,
definitions of similar and congruent triangles, symmetry,
formulas for the circumference of a circle, area of a circle; areas of a square, rectangle, and trapezoid; volumes of a cube, prism, cone, cylinder, and sphere; surfaces of a cylinder, sphere, and prism.

Demonstrative geometry includes formal proofs of propositions of geometry. The content of demonstrative geometry has had several changes. By actual count, the number of propositions requiring actual formal proof has decreased over a period of years. The Wells' _Plane and Solid Geometry_ of 1894 included about 275 propositions. In 1927 the combined number of propositions in Wells and Hart's _Modern Plane Geometry_ and _Modern Solid Geometry_ was 207. One of the recent geometry textbooks, _Today's_ Geometry by Reichgott and Spiller, published in 1938, contains 32 theorems requiring proof. The first reduction of the number occurred with the omission of the ninth book of Euclid on conic sections. Euclid found the conic sections by passing a plane through a cone of two nappes. The cutting plane in various positions made with the surface of the cone certain curves of intersection. Euclid then proved that the curves were a parabola, an ellipse, a circle, or a hyperbola according to the position of the cutting plane. After it was discovered that conics could be represented on a simple graph by means of coordinates, equations of conics were discussed in the algebra. The geometric proof of the properties of conics was postponed to higher mathematics.

Other proofs have also been eliminated from the geometry. Proofs from Euclidean geometry of the mensuration of figures have been replaced by explanations depending upon the theory of limits. Such proofs include the areas, volumes, and surfaces which contain the incommensurable value pi. It has already been seen that the circumference of a circle is the limit of the perimeter of an inscribed polygon of an increasing number of sides. Similarly it has been demonstrated that the area of a sphere is a limit. If a semi-polygon is inscribed in a semicircle, the figure generated by the semi-polygon about its diameter would be inscribed in a sphere formed by revolving the semicircle about the diameter of a circle. As the number of sides of the polygon is increased, the limit of the area of the resulting solid would be the area of a sphere.

As a result of the decrease in the number of propositions to be proved in geometry, there is a corresponding increase in the number of postulates. A postulate is a statement that is accepted as true without any form of proof. (In advanced mathematics, a postulate may be a statement accepted as a basis for discussion and proof. The statement need not be accepted as true.) In many textbooks propositions that formerly were given formal proof ~~were~~ are now treated as postulates. In a comparison of propositions in a textbook published in 1899, _New Plane and Solid Geometry_ by W.W. Beman and D.E. Smith, with the propositions in several more recent textbooks shows that the following propositions are considered as postulates:

Postulates	Propositions
Wells and Hart, Modern Plane Geometry, (1926)	Corresponding reference in Beman and Smith (1899)
Post. 4. A straight line-segment has one and only one midpoint.	Prop. VII, p. 17.
Post. 6. An angle has one and only one bisector.	Prop. VIII, p. 17.
Post. 7. All right angles are equal.	Prop. I, p. 13.
Welchons and Krickenberger, Plane Geometry, (1933)	
Post. 10. At any point in a straight line one perpendicular, and only one, can be drawn to the line.	Prop. II, p. 14.
Post. 16. A straight line cannot intersect a circle in more than two points.	Prop. VI, Corollary, p. 122
Post. 24. The locus of points at a given distance from a given point is a circle with the given point as the center and the given distance as radius.	Prop. XL, p. 81.
Clark, Smith, and Schorling, Modern-School Geometry, (1938)	Prop. I, p. 25.
Post. 1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.	

Postulates (continued)	Propositions (continued)
Clark, Smith, Schorling, Modern-School Geometry, (1938)	Corresponding reference in Beman and Smith (1899)
Post. 2. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, they are congruent.	Prop. II, p. 26.
Post. 3. If two triangles have three sides of one equal respectively to three sides of the other, they are congruent.	Prop. XII, p. 40.
Reichgott and Spiller, Today's Geometry, (1938)	
Postulates 1-5. The areas of a rectangle, parallelogram, triangle, rhombus, and trapezoid.	Prop. II, Corollaries, pp. 201 and 202.
Postulates 1-13. The measurement of solids: the lateral areas of a right prism, regular pyramid, frustum of a regular pyramid;	Prop. VII, Cor., p. 298. Prop. XIV, Cor., p. 309. Prop. XIV, p. 309.
the lateral area of a cylinder of revolution;	omitted
the lateral areas of a cone of revolution, frustum of a cone of revolution;	Prop. III, Cor., p. 324. Prop. III, p. 324.
the area of a sphere;	Prop. XXV, p. 355.
the volumes of a prism and pyramid;	Prop. XIII, Cor., p. 307. Prop. XVI, Cor., p. 313.
the volume of the frustum of a pyramid;	omitted
the volumes of a cylinder of revolution, cone of revolution, frustum of a cone of revolution.	Prop. IV, Cor., p. 325. Prop. IV, p. 325.

There are three justifications for considering some propositions as postulates. First, proofs of some propositions formerly depended upon a _reductio_ _ad_ _absurdum_ method. For example, the proof of the proposition that "a straight line-segment has one and only one midpoint" leads only to an absurd proof. By definition, the midpoint of a line could not be any other point on a line. Second, proofs of other propositions are evident by definition. The proof of the proposition that "if two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent" depends upon the definition of congruence. Third, proofs of propositions may depend upon an interpretation of the word _postulate_. In _Today's_ _Geometry_ by Reichgott and Spiller a _postulate_ is explained as a statement which _must_ be accepted as true. "These statements may be compared to the rules of a game" which one _must_ follow. The first justification of accepting a proposition as a postulate is adequate. Obvious truths in geometry are needlessly proved. In the second reason, the proofs of theorems concerning congruent triangles cannot satisfactorily be considered as postulates. Many other propositions and original exercises in geometry depend upon these theorems. Proving other propositions requires a thorough understanding of the method of proof of the fundamental theorems. The third justification or reason concerns the measurement of plane figures and solids. The formulas of measurement become familiar enough by intuitive geometry in the seventh and eighth year mathematics and first year algebra to be easily accepted as postulates in plane geometry or solid geometry. Moreover, proofs of some of these formulas by Euclidean geometry are difficult.[16] Proofs of some of these propositions by other methods have been discovered; for example, the method of integration in the calculus for finding the area of a circle. The value of proofs by other methods will be further discussed in Chapters III and IV.

In elementary mathematics the place of trigonometry has also been changed. Like the geometry, the concepts and definitions of trigonometry are now spread over a longer period of time. The informal ideas of the right triangle, the tangent, and similar triangles have been included in seventh and eighth year mathematics. As a topic in algebra, trigonometry serves a purpose by showing the use of formulas. In plane geometry the proof of the Pythagorean theorem and further discussions of

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[16] In the discussion of changes in geometry the following articles and textbooks have been used. The articles are in indicated volumes of <u>The Mathematics Teacher</u>.

J.B. Reynolds, "Finding Plane Areas by Algebra", vol. xxi, (1928), pp. 197-203.
A.S. Wannemacher, "Geometry Aids for Elementary Algebra", vol. xxii, (1929), pp. 49-57.
D.E. Ziegler, "Concerning Orientation and Application in Geometry", vol. xxii, (1929),
K. Blank, "Functions of Intuitive Geometry and Demonstrative Geometry", vol. xxii, (1929), pp. 31-37.
W.P. Good and H.H. Chapman, "The Teaching of Proportion in Plane Geometry", vol. xxi, (1928), pp. 462 and 463.
J.C. Stone, "One Year Course in Plane and Solid Geometry", vol. xxiii, (1930), pp. 236-242.
W.D. Reeves and His Students, "Tenth Year Mathematics Outline", vol. xxiii, (1930), pp. 343-357.

H.E. Slaught and N.J. Lennes, <u>Solid Geometry</u>, (1911).
Other textbooks are mentioned within the study.

similar triangles lay the foundations for the formulas of trigonometry.[17]

The most outstanding change was the introduction of the function concept in all the subjects of elementary mathematics. In 1920 E.R. Hedrick prepared a discussion of the function concept in secondary school mathematics for the National Committee on Mathematical Requirements. He pointed out that the function concept is not to be considered as a theory or new idea of mathematics. It emphasizes the idea that many topics in mathematics are concerned with the relation and dependence of variables and quantities.[18] Topics already in elementary mathematics in which the function concept is implicit were the familiar terms ratio, proportion, variation, formula, graph, and congruence. For example, in a ratio the relation between two quantities is considered. The importance of the function concept is that it shows that many topics of mathematics are unified by this idea of dependence. The function idea came from calculus and led to the suggestions that more calculus could be included in elementary mathematics.[19]

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17 In the discussion of changes in trigonometry the following articles from The Mathematics Teacher, vol. xxiii, (1930), were used: A. Hans, "Geometric Proofs for Trigonometric Formulas", pp. 321-326; J.B. Orleans, "Experiment with Mathematics of the Eleventh Year", pp. 447-488. Also R. Schorling, J.R. Clark, and R.R. Smith, Modern-School Mathematics, Books I and II.

18 "The Reorganization of Mathematics in Secondary Education", Bulletin, 1921, no. 32, (Bureau of Education, Washington).

19 In The Mathematics Teacher, vol. xxii, (1929): J.O. Hassler, "Teaching Geometry into Its Rightful Place", p. 155; J.A. Swenson, "Newer Type of Mathematics", pp. 337-338.

A further discussion of the value of calculus in elementary mathematics will be found in Chapter IV.

The function concept and other changes already mentioned in this thesis were part of a general reorganization of mathematics between the years 1912 and 1920 as the following statements and comments show:

> The traditional round of mathematics in high school, to wit: elementary algebra, plane and solid geometry, trigonometry, and advanced algebra, must be revised both as to organization and content.[20]

> Suggestions for changes:[21]
> a. Omission of some of the more difficult techniques from algebra.
> b. Informal, intuitive approach to plane geometry with perhaps some abbreviation of the subject omitting unnecessary propositions and difficult originals.
> c. Introduction of elementary trigonometry into required courses.
> d. Introduction of the function concept near the beginning of a required course and its use in the sequel both for its own sake and as a basis of unification.
> e. Unification of a required course around the function concept and a notion of the coordinate system including elements of analytic geometry and calculus.

In all of these changes it should be noticed that the traditional order of subjects is maintained. The study of topics from arithmetic, solid geometry, trigonometry, algebra, and calculus, for example, in the plane geometry, merely contributes to the understanding of that one subject and does not minimize or eliminate later value, or earlier value, in the study

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20 H.C. Morrison, "Reconstructed Mathematics in the High School", The Mathematics Teacher, vol. vii, (1914-1915), pp. 141-152.
21 A.D. Pitcher, "The Reorganization of the Mathematics Curriculum in Secondary Schools", The Mathematics Teacher, vol. viii, (1916), p. 15.

of these topics.

That the order of topics in high school textbooks has changed in the past forty years will be indicated by comparison of two textbooks in algebra. The first of these books is Standard Algebra published by W.J. Milne in 1908. The other is Algebra for Today written in 1929 by W. Betz. Although Milne prepared his textbook for the study of elementary and intermediate algebra, i.e., a more advanced algebra, we have selected the two books because each represents an elementary approach to the subject. In examining the topics of the two textbooks the procedure will be as follows: first, to list the topics in both textbooks in the order that they are presented; second, to compare the relative order of topics in the places where there is definite change; and third, to point out the noticeable omissions in the discussions of some topics and important additions to other topics. The list of topics will be found on page 23 and the remainder of the discussion on the following pages.

1. List of topics in two textbooks in algebra.

Standard Algebra, Milne (1908)

1. Definitions and notations
2. Positive and negative numbers
3. Addition
4. Subtraction
5. Multiplication
6. Division
7. Factoring
8. Highest common factor
9. Lowest common multiple
10. Fractions
11. Simple equation
12. Simultaneous simple equations
13. Graphic solutions
14. Involution
15. Evolution
16. Theory of exponents
17. Radicals
18. Imaginary numbers
19. Quadratic equations
20. Graphic solution of quadratic equations in x.
21. Properties of quadratic equations
22. Inequalities
23. Ratio and proportion
24. Variation
25. Progressions
26. Interpretations of results: 0/0, a·0, a/0
27. Binomial theorem
28. Logarithms
29. Permutations and combinations
30. Complex numbers

Algebra for Today, Betz (1929)

1. How letters are used in algebra
2. Use of formulas
3. Making of formulas
4. Equation
5. Problems solved by equations
6. Graphs
7. Signed numbers
8. Fundamental operations
9. Equations of the first degree in one unknown
10. Equations of the first degree in two unknowns
11. Special products and fractions
12. Fractions
13. Fractional equations
14. How quantities change together
15. Numerical trigonometry
16. Square root and radicals
17. Equation of the second degree in one unknown

2.a. <u>Relative position of topics in two textbooks in algebra</u>

Name of Topic	Order of Topic in Betz, *Algebra for Today* (1929)	Order of Topic in Milne, *Standard Algebra* (1908)
Simple equation	Precedes discussion of positive and negative numbers and the four fundamental operations, Topic 4.	Follows the four fundamental operations, factoring, highest common factor, lowest common multiple, and fractions, Topic 11.
Equations of the first degree in one unknown	Precedes factoring and fractions, Topic 9	Follows factoring and fractions, Topic 11.
Graphic solution of equations	Topic 6. Also under solutions of equations in two unknowns, Topic 10.	Topic 13.

2.b. <u>Uniting topics in a recent algebra</u>

Name of topic	Treatment of Topic in Betz, *Algebra for Today*, (1929)	Treatment of Topic in Milne, *Standard Algebra*, (1908)
Four fundamental operations	Topic 8.	Addition, Subtraction, Multiplication, and Division, Topics 3-6.
Special products and factoring	A special product alternated with a case in factoring, Topic 11.	Multiplication of binomials, Topic5; Factoring, Topic 7.
Fractions	Topic 12.	Highest common factor, Lowest common multiple, Fractions, Topics 8-10.
How quantities change together	Ratio, proportion, and variation, Topic 14.	Ratio and proportion, Topic 23; Variation, Topic 24.

2.b. Uniting topics in a recent algebra (continued)

Name of Topic	Treatment of Topic in Betz, Algebra for Today (1929)	Treatment of Topic in Milne, Standard Algebra (1908)
Square root and radicals	Topic 16.	Involution, Evolution, Theory of Exponents, Radicals, Topics 14-17.

3.a. Topics defined, briefly discussed, or omitted in a recent algebra

Name of Topic	Treatment of Topic in Betz, Algebra for Today (1929)	Treatment of Topic in Milne, Standard Algebra (1908)
Cube root	Definition	Method of extracting the cube root of polynomials and arithmetic numbers
Factor theorem	Statement of the theorem	Proof of the theorem
Theory of exponents	Simple laws of exponents	Advanced theory
Finding roots higher than the third root	Omitted	Several methods
Graphic solutions of conics and general equations of the conics	Omitted	General equations of the parabola, ellipse, and hyperbola

3.b. Topics added in a recent algebra

Both Milne and Betz point out the relation of algebra to arithmetic particularly in the square root, the four fundamental operations, and signed numers. Betz adds trigonometry to his algebra. Trigonometry combines topics from arithmetic and algebra in numerical solutions of the triangle.

Statistics

The final change to be considered is the introduction of theories from statistics into elementary mathematics. Statistics is the science of the collection and classification of facts on the basis of relative number or occurrence as a ground for induction. At least two kinds of facts are related in such a collection or classification. The facts are usually assembled on a graph by a one-to-one correspondence of values. Induction is then the process of reasoning from this graph about the relation of one set of facts to the other. Statistical graphs are seen to be another example of dependence.

The place of statistics in mathematics has been gradually accepted. Some of the first statistical graphs were published by William Playfair in 1801. By simple line graphs he illustrated exports, imports, and the national debt of England. The methods of representing facts by graphs were new at that time, but William Playfair's graphs are easily read in the first year algebra today. The place of statistical graphs in elementary mathematics today depends upon the earlier use of graphs for illustrating formulas. Later numerous forms of representing facts by picture diagrams, circle graphs, and bar graphs were added. These had no particular mathematical significance. In recent textbooks some of the definitions and concepts of advanced theory of statistics have been added; for example, the mean, median, mode, frequency, the symbol Σ, random sampling, normal curve, and the coefficient of correlation by the rank order method. Each of these terms is explained in the algebra in a very simple way. The mean, median, and mode are called averages, a term from the arithmetic. The mean is sometimes explained by the formula, $M = \Sigma X / N$ where M is the mean; ΣX, the sum of all the values; and N, the number of values. The median is defined as the middle score of a group of scores. The mode is the most frequent score of a group of scores. Either the derivations or the definitions of the most difficult formulas for the mean, median, and mode are not possible or necessary in elementary mathematics. Frequency, random sampling, and the normal curve are easily illustrated. The coefficient of correlation by the rank order method is presented as a formula in which values are substituted. It is readily seen that statistics has a place in elementary mathematics. Following graphic representation and the idea of dependence, it contributes to the fundamental concepts of mathematics.[22]

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22 ~~The~~ In the discussion of statistics in elementary mathematics the following sources were used:

Webster's New International Dictionary, second edition
Engelhardt and Haertter, *op. cit.*, "statistical graph", pp. 9-19;
Mitchell and Walker, *op. cit.*, "Statistics", pp. 211-282.
Schorling, Clark, and Smith, *op. cit.*,"Statistics and Their Use", pp. 145-172.

Summary

There have been many changes in elementary mathematics during the past forty years. General changes in the order of subjects include:

1. the subordination of arithmetic to algebra in seventh and eighth year mathematics and in algebra;

2. emphasis of the importance of arithmetic in approximate measurement in algebra;

3. introduction of the theory of determinants and calculus in elementary mathematics;

4. correlation of algebra and geometry by graphic methods;

5. extending the concepts of geometry throughout the course in elementary mathematics by means of intuitive and demonstrative geometry;

6. correlation of topics in plane and solid geometry;

7. extending trigonometry throughout the courses in elementary mathematics and eliminating the separate course.

Special changes in the order of topics include:

1. in algebra, the combining of some topics, the elimination of others, and the addition of still others;

2. in geometry, the elimination of the section on conics, the reduction of the number of propositions proved by formal demonstration;

3. the centering of topics about the idea of dependence;

4. the addition of elementary statistics to algebra.

CHAPTER III

From the study of analytic geometry we have noticed that certain simple proofs of propositions from plane geometry are possible by coordinate methods. It will be the purpose of this section of the thesis to point out their relation and aid to elementary algebra and their value to plane geometry.

From any standard algebra the following concepts, definitions, and procedures may be studied:[1]

definition of horizontal and vertical axes,
the zero point,
location of points on the graph of a formula,
dependence (relative) and the change of quantities particularly in ratio and proportion,
use of graphs in the solution of problems,
signed numbers and the direction of line-segments,
x-axis, y-axis, and the origin,
location of points as abstract numbers,
graph of a straight line,
parallel lines,
perpendicular lines,
similar triangles.

From a preceding discussion on changes in elementary mathematics it is seen that these topics are taken from elementary analytic geometry and plane geometry.

Another fundamental topic from analytic geometry is the division of a line-segment into any ratio. A discussion of this topic follows.

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[1] Among the textbooks used in this discussion are
W. Betz, Algebra for Today, (1929);
F. Engelhardt and L.D. Haertter, Second Course in Algebra, (1929);
U. G. Mitchell and H.M. Walker, Algebra: A Way of Thinking, (1936).

In the graphic representation of a line-segment, the scale of signed numbers and the corresponding directions are combined.

In Fig. 1 a line is divided by means of a point called a zero point. The points at distances of equal units to the right are the positive values, +1, +2, +3, etc. The points at distances of equal units to the left are the negative values, -1, -2, -3, etc.

In Fig. 2 the direction of the line-segment AB is taken as positive; the direction of the line-segment BA, as negative.

Combining Fig. 1 and Fig. 2 in Fig. 3, we can show that the following line-segments have these directions: DB positive, DA negative, AD positive, BD negative. Assigning numerical values to DB and DA, DB = 2 and DA = -1, we show that

$$BA = BD + DA = -2-1; \qquad AB = AD + DB = 1 + 2.$$

Fig. 1

Fig. 3

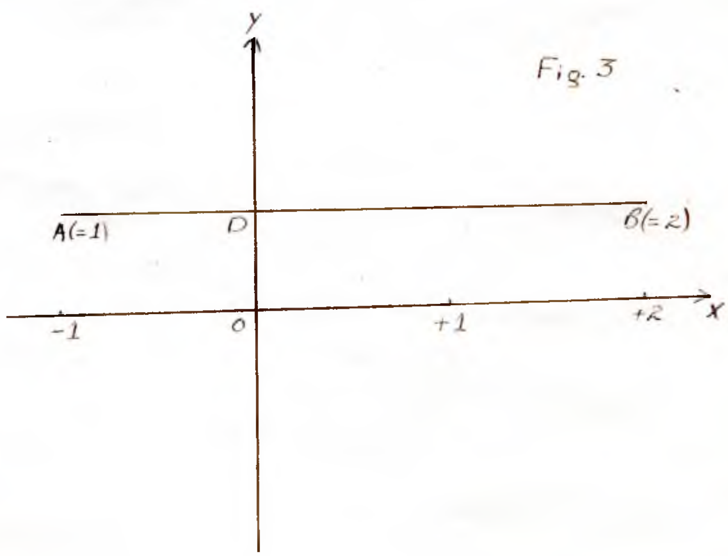


Fig. 2

By locating the points A and B as (r,s) and (m,n) respectively, we can show that in Fig. 4 $BA = r + m$ where r is negative. In Fig. 5 $BA = s + n$ where s is negative.

Fig. 4

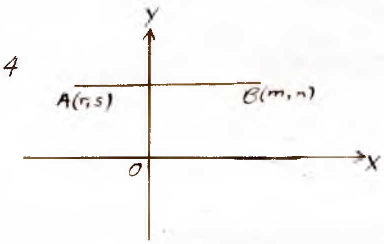
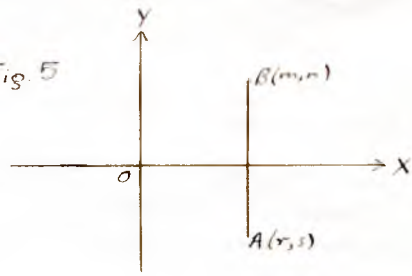


Fig. 5

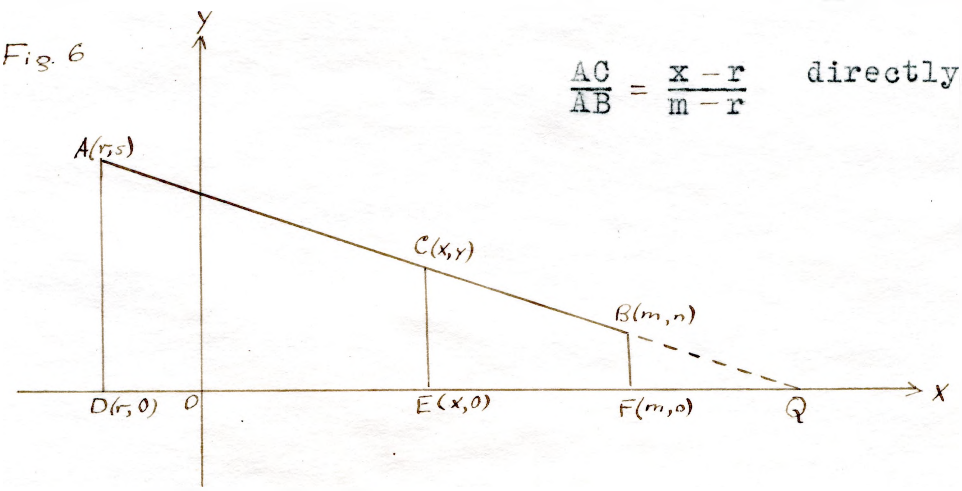


In Fig. 4 AB is parallel to the x-axis. In Fig. 5 AB is parallel to the y-axis. Suppose we take line-segments AB that are not parallel to either axis. In Fig. 6 C is any point (x,y) on any line AB not parallel to, but above, the x-axis. Points $D(r,0)$, $E(x,0)$, and $F(m,0)$ correspond on the x-axis to $A(r,s)$, $C(x,y)$, and $B(m,n)$ respectively. $DE = x - r$ and $DF = m - r$. If AB is extended to meet the x-axis, the quadrilateral ABFD is seen to be part of a right triangle AQD. Then by ratio, proportion, and similar triangles

$$\frac{AC}{AB} = \frac{DE}{DF} = \frac{x - r}{m - r}, \quad \text{or}$$

$$\frac{AC}{AB} = \frac{x - r}{m - r} \quad \text{directly.}$$

Fig. 6



In Fig. 7, points D(O,s), E(O,y), and F(O,n) correspond on the y-axis to A(r,s), C(x,y), amd B(m,n) respectively.

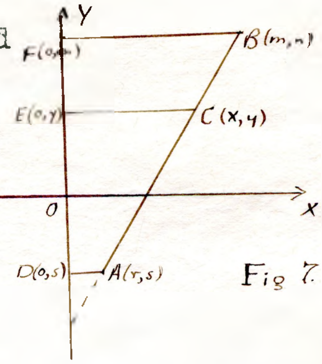


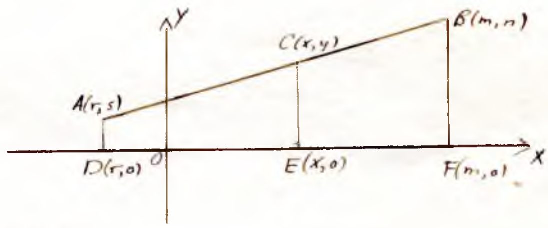
Fig 7.

Therefore $\frac{AC}{AB} = \frac{DE}{DF} = \frac{y-s}{n-s}$, or $\frac{AC}{AB} = \frac{y-s}{n-s}$ directly.

The midpoint formulas from analytic geometry may be explained in algebra. In this case we wish to find the coordinates of the midpoint of a line AB. Let C(x,y) be the midpoint of the line. The ratio of AC:AB is 1:2. By proportion, as before, we have in Fig. 8

$$\frac{AC}{AB} = \frac{DE}{DF} = \frac{x-r}{m-r} = \frac{1}{2}.$$

Fig 8.



Then by simple algebra

$$x-r = \frac{1}{2}(m-r) \quad \text{or} \quad x = \frac{m+r}{2}.$$

Similarly in Fig. 9 we can show

$$\frac{AC}{AB} = \frac{DE}{DF} = \frac{y-s}{n-s} = \frac{1}{2}$$

Hence $y-s = \frac{1}{2}(n-s)$ or $y = \frac{n+s}{2}.$

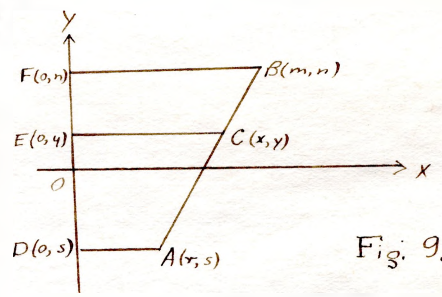


Fig. 9.

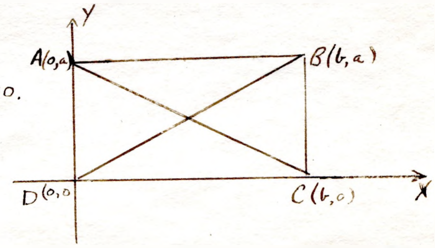
From the above illustration it may be stated that the x-coordinate of the midpoint is equal to one half the sum of the x-coordinates of the end points of the line. The y-coordinate of the midpoint is equal to one half the sum of the y-coordinates of the end points of the line.

The formula for the distance between two points may also be presented in algebra. It closely follows the formula for the Pythagorean theorem and, like it, is valuable in showing the dependence of one value (distance) upon others (the location of points). The distance formula briefly stated is $d = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$, where (x_2,y_2) and (x_1,y_1) are the ends of the line segment.

In harmony with the above ideas we shall now indicate by methods of analytic geometry the proofs of several propositions from plane geometry that may be introduced into the algebra.

A. Prove that the diagonals of a rectangle are equal.[2]

In rectangle ABCD with points (0,a),(b,a), (b,0), and (0,0) respectively as vertices, we may prove that the diagonals AC and BD are equal.



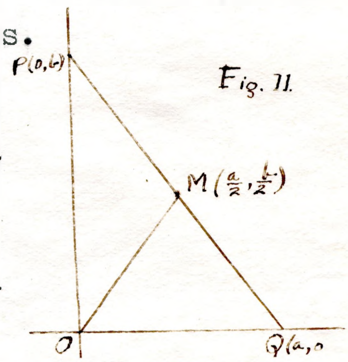
Then by the distance formula

$AC = \sqrt{b^2+a^2}$ and $BD = \sqrt{b^2+a^2}$.

Hence, AC = BD.

Proposition A. shows the relation of geometry to algebra by representation of a rectangle by letters and points in a plane. It uses the simple laws of equations in arriving at the proof.

B. Prove that the midpoint of the hypotenuse of a right triangle is equidistant from all the vertices.



In the right triangle located by the points P(o,b), Q(a,0), and O(0,0), the midpoint, M, of the hypotenuse by the midpoint formulas is seen to be $(\frac{a}{2}, \frac{b}{2})$. It is necessary to prove that OM = PM = MQ.

---

[2] Statements of the propositions are from original exercises in (note continued on following page)

Then by the distance formula

$$OM = \sqrt{(a/2)^2 + (b/2)^2}\ ; \quad PM = \sqrt{(a/2)^2 + (b/2 - b)^2}\ ;$$

and $QM = \sqrt{(a/2 - a)^2 \quad (b/2)^2}$. Therefore OM = PM = MQ.

A special value of Proposition B. is that it furnishes practice ~~that it gives~~ in the multiplication of fractions and fractional equations.

C. Prove that the line joining the midpoints of any two sides of a triangle whose vertices are (0,8), (-2,-1), and (3,7) is equal in length to half the third side.[5]

In the given triangle ABC, the midpoint, M, of the side AB is $(-1, \frac{7}{2})$, since

$$x = (0-2)/2 = -1 \quad \text{and} \quad y = (8-1)/2 = 7/2.$$

The midpoint, N, is $(\frac{1}{2}, 3)$, since

$$x = (3-2)/2 = 1/2 \quad \text{and} \quad y = (7-1)/2 = 3.$$

It is necessary to prove that $MN = \frac{1}{2}AC$.

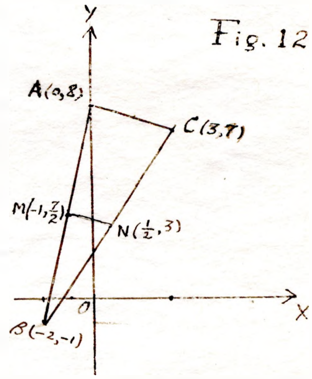


Fig. 12

Proposition C combines arithmetic and geometry. Although arithmetic numbers are used to locate the five points of the triangle, the generality of a proposition concerning midpoints may also be proved.

Then $MN = \sqrt{(1/2 - (-1))^2 + (3 - 7/2)^2} = \sqrt{9/4 + 1/4}$,

and $AC = \sqrt{(3)^2 + (7-8)^2} = \sqrt{9+1}$.

Hence, $MN = \frac{1}{2}\ AC$.

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[2](continued) in I.A. Barnett, Analytic Geometry. Corresponding statements may be found in any plane geometry.

3 Ibid., Ex. 12, p. 28.

4 Ibid., Ex. 14, p. 28.

5 Ibid., Ex. 9, p. 14.

D. Prove that the distance between the middle points of the non-parallel sides of a trapezoid is equal to one half the sum of the parallel sides.[6]

In the trapezoid ABCD with points (0,0), (a,0), (d,c), and (b,c) respectively, the midpoint, M, is easily seen to be $(\frac{b}{2}, \frac{c}{2})$, and the midpoint, N, $(\frac{a+d}{2}, \frac{c}{2})$. Then, since DC and MN are parallel to the x-axis, it is necessary to prove that $MN = \frac{1}{2}(DC + AB)$. By the distance formula $MN = \frac{a+d}{2} - \frac{b}{2}$; $DC = d-b$; and $AB = a$. Hence, $MN = \frac{1}{2}(\frac{a+d-b}{1}) = \frac{1}{2}(AB + DC)$.

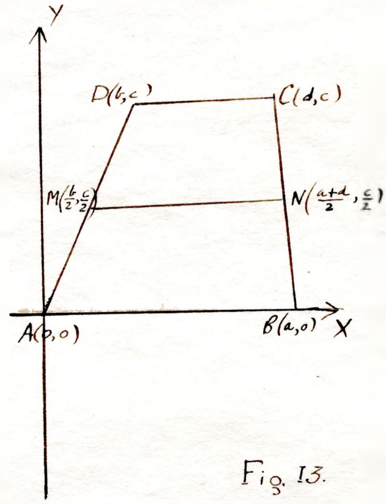

Fig. 13.

E. In a parallelogram $P_1(2,1)$, $P_2(6,5)$, $P_3(0,0)$, and $P_4(4,4)$ prove that the line which joins the middle point, M, of the side P_1P_2 to the vertex P_3, and the diagonal P_1P_4 trisect each other.[7] In the given parallelogram the midpoint and point of trisection are seen to be as follows:

midpoint M (4,3) where $x = \frac{2+6}{2} = 4$

and $y = \frac{1+5}{2} = 3$;

point N $(\frac{8}{3}, 2)$ where by the point of division formula $x = \frac{0-4}{3} + 4$ and $y = \frac{0-3}{3} + 3$. It is seen that

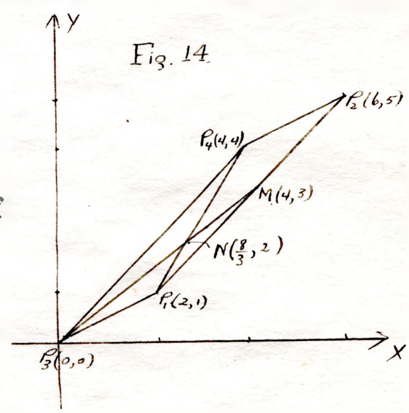

Fig. 14

6 Ibid., Ex. 20, p. 28.
7 Ibid., Ex. 12, p. 14.

$$MN = \sqrt{(4-\frac{8}{3})^2 + (3-2)^2} = \sqrt{\frac{16}{9} + 1} = \frac{5}{3}$$

and $MP_3 = \sqrt{16 + 9} = 5.$ Therefore $MN = \frac{1}{3}MP_3$

Similarly $P_1N = \sqrt{(2-\frac{8}{3})^2 + (1-2)^2} = \sqrt{\frac{4}{9} + 1} = \frac{\sqrt{13}}{3}$ and

$$P_1P_4 = \sqrt{(2-4)^2 + (1-4)^2} = \sqrt{4 + 9} = \sqrt{13}.$$

Therefore $P_1N = \frac{1}{3}P_1P_4.$

Although arithmetic numbers are used here, as in Proposition D., the generality of the proposition may also be proved. Propositions D. and E. provide further exercises in literal and fractional equations.

In some present day algebras[8] the equation of a line and an explanation of the slope of a line are presented. Further geometric propositions could be added by means of the slope formulas: $m = \frac{y_1 - y_2}{x_1 - x_2}$, $m_1 = m_2$ (test for parallel lines),

$m_1 m_2 = -1$ (test for perpendicular lines).

Some of these propositions are:

> The diagonals of a square are perpendicular to each other.[9]
>
> The diagonals of a rhombus are perpendicular to each other.[10]

Both of these propositions can be proved by showing that the product of the slopes of the two diagonals equals -1.

> The lines joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.[11]

In the proof of this proposition use is made of the principle ~~is used~~ that the lines having equal slopes are parallel.

In a list of propositions that may be proved in algebra as an introduction to geometry, it will be necessary to exclude those propositions in geometry which are needed to establish the proof of the forty-seventh problem of Euclid; i.e., "In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides". Proving these propositions by the distance formula which in turn depends upon the above forty-seventh problem would result in reasoning in a circle. The propositions which must be excluded from the proofs by coordinate methods are as follows:[12]

1. The area of a rectangle is equal to the product of its base and altitude.
2. The area of a parallelogram is equal to the product of its base and altitude.
3. The area of a triangle is equal to one half the product of its base and altitude.
4. A triangle is half a parallelogram having the same base and latitude.
5. Two triangles are congruent if three sides of one are equal respectively to three sides of the other.
6. Two triangles are congruent if two sides and an included angle of one are equal to two sides and an included an gle of the other.
7. Construct a perpendicular to a line from a point not in the line.
8. At a point in a line, construct a perpendicular to the line.

Notes pp. 36 and 37.
8 U.G. Mitchell and H.M. Walker, Algebra: A Way of Thinking, pp. 294-297.
A.F. Leary, "Analytic Geometry in the High School", The Mathematics Teacher, vol. xxxiii, (1940), p. 60.
H. Gottschalk, "Analytic Geometry and Trigonometry in Second Year Algebra", The Mathematics Teacher, vol. xxxiii, (1940), pp. 82-83.
9 I.A. Barnett, Analytic Geometry, Ex. 1, p. 25.
10 Ibid., Ex. 24, p. 29.
11 Ibid., Ex. 15, p. 28.
12 W. Wells and W.W. Hart, Modern Plane Geometry, p. 216.

A summary of this discussion brings out several points. Some proofs from analytic geometry are not too difficult to present in algebra. They rpovide further understanding of graphs, ratio and proportion, and the operations of algebra. They present a simple introduction to propositions in geometry. Such an introduction to geometry is valuable because by making use of algebraic methods and by involving geometric ideas , we bring into closer relation these two subjects in mathematics.

CHAPTER IV

As we have already mentioned, the emphasis of the function concept in explaining the topics of algebra showed that calculus might have a place in elementary mathematics. In this section we shall point out three examples of the relation of calculus to algebra; namely, theory of limits, formulas by integration, and curve tracing.

Theory of Limits

By actual definition, differential calculus is a method of "analysis dealing with the rate of change of a variable function".[1] One of the first ideas introduced into the differential calculus is the theory of limits. "A limit is a fixed value or form which a varying value or form may approach indefinitely but cannot reach."[1] Or, in certain cases the variable may reach its limit. In algebra the theory of limits appears in a definition of infinite series. An infinite series is contrasted with a finite series, where a finite series is an enumeration of figures that has a definite end.

In geometry the theory of limits may be brought out in several propositions. H.E. Slaught and N.J. Lennes in Solid Geometry, published in 1911, followed the method of approach, or limit, carefully in proving the formulas for the areas and volumes of the cylinder, cone, and sphere. Certain geometric axioms were used as bases of proofs. Some of these axioms are:

1 Webster's New International Dictionary

Axiom VIII. The lateral surface of a convex cone has a definite area, and the cone incloses a definite volume, which are less respectively than those of any circumscribed pyramid and greater than those of any inscribed pyramid.[2]

In the proof of the proposition that the area of the lateral surface of a right circular cone is equal to one half the product of its slant height and the circumference of the base, it was shown that

L (lateral area) can be neither less nor greater than 1/2 sc, where s is the slant height and c is the circumference of the base. Therefore the lateral area must be equal to 1/2 sc.

Axiom IX. A sphere has a definite area and incloses a definite volume which are less respectively than the surface and volume of any circumscribed figure and greater than those of any inscribed convex figure.[3]

One of the theorems proved by the latter axiom was the formula for the area of a sphere, $A = 4\pi r^2$. In a procedure similar to that in the preceding proposition, the authors proved the formula by showing that the area of a sphere is neither less nor greater than $4\pi r^2$.

The above explanations of limits appeared to be both awkward in expression and in the method of proof. In recent textbooks theorems on limits are also used, but the manner of expressing them is more simple. Two of the theorems are[4]

1. If a variable x approaches a limit k and if c is a constant, then cx approaches ck as a limit, and x/c approahes k/c as a limit.
2. If two variables are always equal while approaching their respective limits, their limits are equal.

These theorems on limits are accepted as true without proof.

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[2] H.E. Slaught and N.J. Lennes, op. cit., pp. 84 and 85.
[3] Ibid. pp. 148-149.
[4] A.M. Welchons and W.R. Krickenberger, Plane Geometry, p. 310.

Propositions depending upon these theorems involve equalities; for example, "the circumference of two circles have the same ratio as their radii",[4a] and "two reiangular pyramids having equal altitudes and equivalent bases are equivalent".[5]

Elementary Formulas by Integration

A second relation of calculus to elemantary mathematics is through integral calculus. Integral calculus is a method of analysis concerned with "the theory and application of integrals, their evaluation, etc.[6] The proofs of several simple formulas of algebra or intuitive geometry are found in integral calculus. Proofs of these formulas follow.

A. Show that the circumference of a circle is $2\pi r$.[7a] By integration and the formula for the arc-length along a curve $y = f(x)$, the proof may be developed as follows. From the equation of a circle, $x^2 + y^2 = r^2$, the curve is $y = \sqrt{r^2-x^2}$. Then $s = \int_a^b \sqrt{1+(dy/dx)^2}\, dx = \int_{-r}^{r} \sqrt{1+x^2/(r^2-x^2)}\, dx = r\int_{-r}^{r} dx/\sqrt{r^2-x^2}$;

and $s = r \text{ arc sin } x/r\Big]_{-r}^{r} = r \text{ arc sin } (1) - r \text{ arc sin } (-1)$.

Hence $s = r\,(\pi/2) - r\,(-\pi/2) = \pi r$, or half the circumference.

B. Show that the area of a circle of radius r is πr^2.[7b] Similarly by integration and the formula for the area of a curve $y = \sqrt{r^2-x^2}$, the proof may be developed as follows.

$$A = \int_{-r}^{r} \sqrt{r^2-x^2}\, dx = \Big[(1/2)x\sqrt{r^2-x^2}\, dx + 1/2 \text{ arc sin } (x/r)\Big]_{-r}^{r}$$

[7a] W. B. Ford, A First Course in Differential and Integral Calculuc, Ex. 5, p. 248; [7b] Ex. 7, p. 250.

Hence $A=(r^2/2)(\pi/2) - (r^2/2)(-\pi/2) = \pi r^2$.

<u>C</u>. Show that the volume of a sphere generated by revolving the circle $x^2+y^2=r^2$ about its diameter is $(4/3)\pi r^3$.[8a]
From the use of the formula for the volume generated by revolving a circle about its diameter, it is seen that

$$V=\pi\int_a^b y^2\,dx = \pi\int_{-r}^{r}(r^2-x^2)\,dx = \pi\left[r^2x-x^3/3\right]_{-r}^{r}$$

Then $V=\pi\{[r^3-(r^3/3)] - [-r^3+(r^3/3)]\} = (4/3)\pi r^3$.

<u>D</u>. Find by integration the area of the surface of the sphere generated by revolving the circle $x^2+y^2=r^2$ about its diameter.[8b]
Similarly by integration and the use of the formula for the surface generated, we may show that

$$S=2\pi\int_a^b y\sqrt{1+(dy/dx)^2}\,dx = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+x^2/(r^2-x^2)}\,dx.$$

Hence $S=2\pi rx\Big]_{-r}^{r} = 2\pi r^2 - (-2\pi r^2) = 4\pi r^2$.

The proofs of the preceding propositions, or formulas, are seen to be clear and simple by means of integration. As proofs to be presented in algebra they would be difficult without a thorough course in elementary calculus. Such a reversal of subjects is not feasible. A possible suggestion that the formulas of integration used in these problems might

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[8a] Ford, <u>op</u>. <u>cit</u>., Ex. 2, p. 270; [8b] Ex. 1, p. 275.

be introduced in algebra also fails. The best conclusions to the discussion, therefore, are: *first*, that *the proofs* of these propositions *should be postponed to the calculus*, and *second*, that the *simplicity of these proofs in calculus justifies the omission* of any other type of proof of these propositions in elementary mathematics. The formulas for the circumference, area, and volume of plane figures and solids involving the incommensurable value pi, , should be presented as definitions or rules in algebra and as postulates in geometry.

Curve Tracing

"A curve is said to be traced when the general form of its several parts or branches is determined and the position of those (parts or branches) which are unlimited in extent is indicated."[9] Elementary analytic geometry and calculus have simple ~~considerations~~ methods which combined enable one to trace a curve quickly. ~~Steps which might be used in tracing a curve quickly.~~ Steps which might be used in tracing a curve of the third, or higher, degree in first or second year algebra are these:[10]

1. Determine if possible the quadrants, within which the curve must lie.
2. Determine if possible the points, if any, at which the curve intersects the x-axis and the y-axis.
3. Determine whether the curve is symmetrical with respect to the x-axis, to the y-axis, or to the origin.
4. Determine the maximum and minimum points, if any.
5. Determine the points of inflection, if any.

Solve the following equation graphically:[11] $x^3-x^2-6x=0$. In algebra the process of solving such an equation would be the two steps of forming a table and plotting all points in the table on a graph, as follows:

Step 1. Table.

x	-3	-2	-1	0	1	2	3
y	-21	0	4	0	-6	-8	0

Fig. 1

In finding the values for the table, we set the equation equal to y and substitute values for x.

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9 W.W. Johnson, Curve Tracing in Cartesian Coordinates, p. 1.
10 W.B. Ford, A First Course in Differential and Integral Calculus, p. 146-147.
11 W. Betz, Algebra for Today, First Course, Ex. 22, p. 459.

Step 2. Graph.

In plotting the values from the table on a graph, we can show that the curve looks something like this (Fig. 2) :

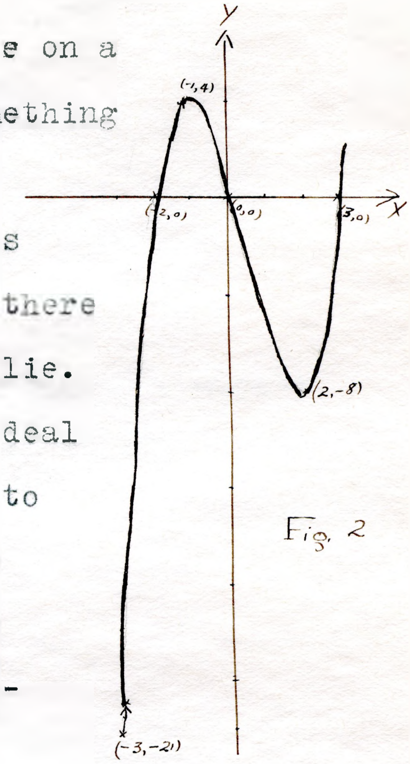
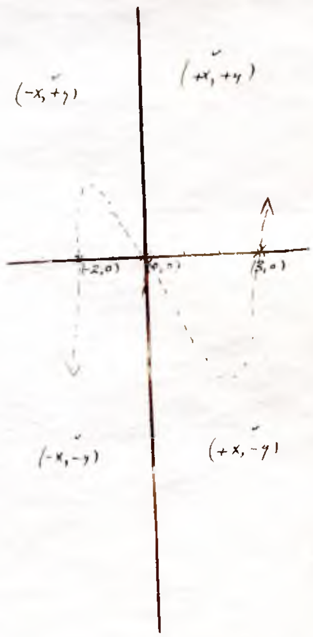


Fig. 2

There are several weak points in this solution. Before the table is constructed there is no way of guessing where the curve will lie. Substituting values for x consumes a great deal of time. The resulting curve is not known to be accurate. It is merely assumed that the curve takes the indicated directions. Some method should be used that would be more accurate.

By the method of curve tracing in calculus, the determining of the curve would be as follows:

1. <u>Determine the quadrants within which the curve must lie.</u> $y = x^3 - x^2 - 6x$. By observation it is seen that substituting any real values whatsoever for x, the value of y is always a real number. Therefore parts of the curve will lie in each of the four quadrants. A further statement about the quadrants may be made in the next step.



2. <u>Determine the points at which the the curve intersects the x-axis; the y-axis.</u> At the point where the curve intersects the

x-axis, the value of y must be 0. Then by simple algebra $x^3-x^2-6x=0$; $x(x^2-x-6)=0$; $x=0$, $x=3$, and $x=-2$. The curve crosses the x-axis at 0,3,and -2. Values of $x=0$ are used to determine where the curve crosses the x-axis. In this case, substituting 0 for x, y has one value, 0. Taking a value between $x=0$ and $x=-2$, we may show that a definite segment of the curve lies in the second quadrant. Similarly a value between $x=0$ and $x=3$ shows us that there is another definite segment in the fourth quadrant. Beyond $x=-2$ the curve recedes in the third quadrant towards $-\infty$. To the right of $x=3$ the curve proceeds in the first quadrant toward $+\infty$. (See Fig. 3)

3. Determine whether the curve is symmetrical to the x-axis or the y-axis. The symmetry of a curve is determined by substituing -x for x and -y for y to determine whether the curve is unchanged. In this case if -x is substituted for x, the form of the equation is changed:

$$y=(-x)^3-(-x)^2-6(-x)=-x^3-x^2+6x .$$

If -y is substituted for y, the form of the equation is changed: $-y=x^3-x^2-6x$.

Therefore the curve is not symmetrical to either axis.

4. Determine the maximum and minimum points, if any. Steps 4. and 5. involve the use of the first and second derivative which are easily explained by rule.

Let $f(x) = x^3-x^2-6x$. Then the first derivative of the equation will be $f'(x) = 3x^2-2x-6$. For the value of x that will make $f'(x) = 0$, there may be a maximum value, a minimum value, both, or neither, where the maximum value give the ordinate at ~~is~~ the high point of the curve and the minimum value give the ordinate of ~~is~~ the low point of the curve. By simple algebra

$3x^2-2x-6 = 0$. Then $x = 1.8$ and -1.2.

Substituting in the original equational equation, the corresponding values of y are -8.8 and 6.8. Hence the maximum point is (-1.2, 6.8) and the minimum point is (1.8,-8.8).

5. <u>Determine the points of inflection, if any.</u>

We find the second derivative of the equation to be $f''(x) = 6x-2$. The value of x that will make $f''(x) = 0$ will give the point of inflection of the curve. Then $x = 1/3$. Substituting this value in the original equation, the corresponding value of y is -2.1. Therefore there is a point of inflection at the point (.3,-2.1). The resulting curve is shown in Fig. 4.

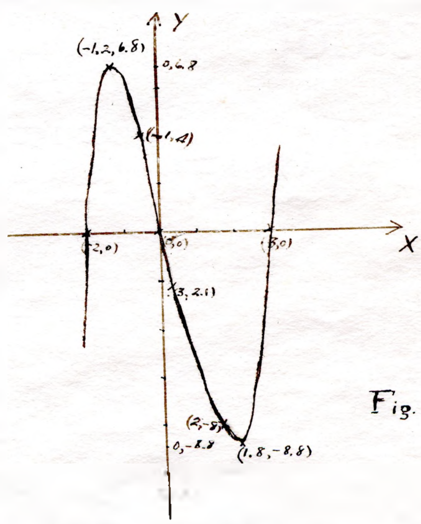


Fig. 4

The above exercise may easily be presented in elementary mathematics. It involves a knowledge of algebra including the solution of the quadratic equation. There is exercise in computation. The resulting curve is sufficiently different from the first solution to illustrate the accuracy of the last method. In elementary mathematics maximum point, minimum point, and point of inflection, may be explained as high point, low point, and point ~~of change in concave~~ where the curve changes direction, respectively.

CHAPTER V

Certain changes in the content of elementary mathematics have been taking place over a period of forty years. These changes are bringing about a reorganization of the continuity of subject matter.

In this chapter we shall summarize the changes mentioned in the preceding pages. In addition we shall point out places in the subject matter where the reorganization should be made. The procedure will be as follows:

1. to list the topics in elementary mathematics;
2. to indicate in some cases the special content of these subjects;
3. to show reorganization in the continuity of subject matter;
4. to show that such a reorganization makes the presentation of the subject more simple and complete without affecting the logical order of the topics.

Topics with Specific Content	Reasons for Placement in Order
Seventh Year Mathematics	
1. Fundamentals of Arithmetic a. Addition b. Subtraction c. Multiplication d. Division e. Table and line graph	Thorough treatment of the fundamental operations in arithmetic is necessary. Algebra, geometry, and trigonometry contain topics involving these operations. Simple relation of a table to a graph gives a foundation for a point system.
2. Fractions and Decimals a. Fractions b. Ratio c. Decimals d. Percent e. Average	Knowledge of the simple laws of fractions, multiplying with fractions, and dividing by fractions is also essential to algebra. A ratio is recognized as a fraction. Decimals, percent, and average are a basis for numerical computations in statistics.
3. Measurement a. Distances Perimeter and line-segment b. Angles Acute, obtuse, right and straight angle, Sum of the angles of a triangle $x^\circ + y^\circ + z^\circ = 180^\circ$	Introduction to both plane and analytic geometry through the measurement of lines and angles. The sum of the angles of a triangle is expressed and designated as an equation.
4. Constructions in geometry a. The circle Center, diameter, and radius b. Arc, isosceles triangle, and equilateral triangle c. Perpendicular lines Perpendicular bisector, right triangle, rectangle, and square	Since constructions are based on the circle and the arc these are presented first.

Topics with Specific Content	Reasons for Placement in Order
Seventh Year Mathematics (continued)	
5. Measurement with simple formulas a. Areas of circle, triangle, rectangle, and square b. Volume of sphere, pyramid, rectangular solid, and cube	This topic not only is an introduction to equations but also shows the measurement of figures constructed in Topic 4. The volumes selected correspond to the plane figures.

6.-7. Since the study of elementary arithmetic is continued in the seventh year, it is expected that other topics of arithmetic , such as, interest, banking, taxes, insurance, investments, and other social applications will be included. The social applications, except for some computation and simple substitution in formulas, are more or less outside the field of mathematics. Since this study deals with the mathematical content of these courses, no further comment on social applications will be made except in relation to statistics. The discussion of statistics, however, will also remain within the limits of mathematics.

Topics with Specific Content	Reasons for Placement in Order
Eighth Year Mathematics	
8. Fundamentals of arithmetic a. Fundamental operations of arithmetic b. Fractions and approximate numbers	Although this is a review of arithmetic, there should be special relation to algebra. There should be treatment of each of the fundamental opertions as a numerical equation. Operations have heretofore been carried on in vertical columns of numbers. Positive and negative numbers should be mentioned without a number scale. Study of fractions should include percentage of increase and decrease. Approximate numbers should indicate the inaccuracy of measurement.
9. Simple dependence a. Use of table and formula b. Graph Point system, direction of a line, locating points in a plane c. Transferring a formula to a graph	Fundamentals of analytic for the relation of algebra to geometry. Interdependence of table, formula, and graph.
10. Simple equation a. Four principles of equations b. Algebraic addition and subtraction c. Fractional equations	Point out the removal of the ÷ sign in division, use of ( ) parentheses, both small points fundamental in operations of algebra.
11. Measurement a. Angles, triangles, lines b. Additional formulas: volumes of pyramid, cylinder, and cone c. Construction of plane figures and location of parts of section a. above and section c. on a graph	Topics of intuitive geometry are interpreted on a graph. Positions of acute, obtuse, and right angles with vertex at (0,0) compared. Comparison of parallel and perpendicular lines, bisector of a line on a graph. Discover midpoint formula for lines parallel to axes.

Topics with Specific Content	Reasons for Placement in Order
Eighth Year Mathematics (continued)	
12. Indirect measurement a. Similar triangles (proposition stated) b. Congruent triangles (proposition stated) c. Symmetry Axis of symmetry	In similar and congruent triangles several propositions are stated: similarity is observed in relation to angles, in relation to sides, and in relation to ratios of two corresponding sides and the included angle. Congruency is observed both by two sides and the included angle and by two angles and the included side. Representation of figures on a graph with reference particularly the axis of symmetry.
13. Introduction to trigonometry a. Right triangle Tangent ratio, table of elevation and depression b. Indirect measurement Angles and triangles, Pythagoras' rule, square root	Early introduction to trigonometry follows knowledge of angles and triangles.

In the mathematics of the seventh and eighth year four purposes are evident:

1. A foundation of arithmetic is maintained.

2. The simplest form of the equation is introduced.

3. The fundamentals of geometry and trigonometry are presented.

4. There is special attention to elementary analytic geometry in which a connection is made in every instance between plane geometry and algebra by means of the graph.

Topics with Specific Content	Reasons for Placement in Order
Algebra, First Course	
14. The Equation a. Numerical equation b. Simple equations of letters c. Review of four principles of equations d. Fractional and decimal equations e. Review of algebraic addition and subtraction	Simple forms of equations continue facility in numerical and algebraic operations. Algebraic addition and subtraction leads both to more complicated equations and to directed numbers.
15. Directed numbers a. Review of graph including graphs of four quadrants b. Four fundamental operations and directed numbers including signed polynomials, brackets, braces c. Equations	Early understanding of directed numbers in not difficult with numerical and graphical foundation. This is the second simple step with the equation.
16. Equations with two unknowns a. Single equation with two unknowns plotted on a graph Variables, constant, intersection of x- and y-axes b. Algebraic solution of simultaneous equations by several methods c. Graphic solutions of pairs of equations Simultaneous equations, parallel lines, equivalent equations	There is emphasis both on dependence and on graphic representation. The equivalent equation compares the process of algebraic solution with graphic solution.
17. Multiplication of binomials and factoring a. Multiplication of binomials b. Shorter method by rule c. Special products (four)	The simpler processes of algebra continue with four special products, namely, product of a sum, product of a difference, product of sum and difference, product of the type $(a+b)(a+c)$

Topics with Specific Content	Reasons for Placement in Order
Algebra, First Course (continued)	
18. Quadratic Equation a. Graphic solution Symmetry, intersection with axes b. Equation of condition c. Solution by factoring d. Solution by completing the square (formula)	Observation of algebraic processes is continued by use of a graph.
e. Simple treatment of exponents and radicals Laws of exponents including fractional exponents Reading squares from a graph Square root of a product Square root of a fraction	More difficult problems in the second course will show further use of exponents and radicals.
19. Fundamental operations with fractions a. Laws concerning fractions b. Review of ratio and proportion 20. Fractional equations a. Least common denominator b. Equations in one unknown c. Equations in two unknowns, simultaneous equations	Possibility of placing fractions near the beginning of the course defeated in two ways: necessity of a foundation in algebraic operations; connection with the following topic involving fractional equations entirely.
21. Introduction to plane geometry by graphic methods a. Division of a line into any ratio b. Midpoint formulas c. Propositions by coordinate methods i. The diagonals of a rectangle are equal.	This topic makes for a better correlation of algebra and geometry.

Topics with Specific Content	Reasons for Placement in Order

Algebra, First Course (continued)

21. Introduction to plane geometry by graphic methods (continued)
    - ii. The line joining the midpoints of any two sides of a triangle is equal in length to half the third side.
    - iii. The distance between the middle points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.
    - d. Slope of a line
      Slope formula, parallel lines, perpendicular lines, equation of a line
    - e. Propositions by coordinate methods
      - iv. Diagonals of a square are perpendicular.
      - v. Diagonals of a rhombus are perpendicular.
      - vi. The line joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.
    - f. Finding simple loci by graph — *These loci are merely observation of parallel lines and perpendicular lines in another way.*
      - vii. The locus of points at a gicen distance from a given line is a pair of lines that are parallel to the given distance from the given line.

Topics with Specific Content	Reasons for Placement in Order

Algebra, First Course, (continued)

21. Introduction to plane geometry by graphic methods (continued)
    - viii. The locus of points equidistant from two insecting lines is the bisector of the angles formed by the lines.
    - ix. The locus of points equidistant from two given points is the perpendicular bisector of the line joining the points.

The purpose of the order of topics in the first course in algebra is to present thoroughly the fundamentals of algebra. Constant reference to graphic solutions of equations leads finally to an introduction to plane geometry by simple graphic, or coordinate, methods.

Topics with Specific Content	Reasons for Placement in Order
Plane Geometry	
1. Constructions and review of intuitive geometry a. Circle, bisecting a line b. Perpendiculars c. Parallels d. Angles e. Triangles f. Quadrilaterals, polygons g. Related geometry of three dimensions Perpendicular to a plane, parallel lines in a plane h. Coordinate system related to plane geometry Suggestion of coordinate system in solid geometry i. Axioms and general postulates Axioms expressed algebraically	Simple constructions and observation of lines and figures precedes formal proof. Plane and solid geometry compared. Plane and solid geometry interpreted by graph. Since many proofs will contain algebra, algebraic expression of axioms is used.
2. Triangles a. Propositions: angles of a triangle b. Propositions: congruence of triangles by informal and formal proof c. Pythagorean theorem d. Distance formula (graph) e. 30°-60° triangle, 45° triangle f. Proposition: right triangle	The propositions related to congruent triangles are not considered as postulates in order that the method of Euclidean proof may be illustrated. Graphic representation is continued briefly to show the relation of plane and analytic geometry. Introduction to trigonometry is continued early in the course.
3. Parallel lines and quadrilaterals a. Propositions: parallel lines b. Sum of the angles of a polygon c. Propositions: parallelograms	

Topics with Specific Content	Reasons for Placement in Order
Plane Geometry (continued)	
3. Parallel lines and quadrilaterals (continued) d. Parallel planes	Correlation of plane and solid geometry continued.
4. Areas a.-e. Rectangle, parallelogram, triangle, rhombus, trapezoid. Unit proof: rectangle Propositions: other areas Recall that"the diagonals of a rhombus are perpendicular to each other" from the proof by coordinate geometry f. Surface of a rectangular solid, prism, and pyramid by analogy g. Area of any triangle, s-formula through the Pythagorean theorem h. Area of equilateral triangle by Pythagorean theorem and the relations of the 30^o-60^o triangle	Areas of quadrilaterals and triangles follow propositions concerning these figures. Relation of analytic geometry to plane geometry pointed out. Informal proof of measurement of surfaces of related solids continues introduction to solid geometry.
5. Circles a. Postulates of the circle b. Propositions of the circle: arcs, chords, tangents, secants	
c. Measurement of the circle Circumference, area (without proof) d. Measurement of solids related to the circle Area of sphere, lateral areas of cone and cylinder	Measurement of the circle and related solids continued by definitions and illustrations.
6. Propositions: regular polygons and the circle	Propositions of inscribed and circumscribed polygons readily follow study of the circle.

Topic with Specific Content	Reasons for Placement in Order
Plane Geometry (continued)	
7. Loci by informal proof a. Plane figures Circle, parallel lines, perpendicular bisector, intersection of lines, angle bisector b. Solids Sphere, parallel planes, perpendicular planes, intersection of planes, cylinder	It is important that loci by graphic representation be reviewed and related to loci by plane geometry.
8. Relationships a. Ratio and proportion b. Propositions: similar polygons c. Propositions: inequalities	Relationships are analyzed through the use of similar figures.
9. Numerical trigonometry a. Ratios of the general right triangle b. Ratios of 30°-60° and 45° triangles c. Interpolation and use of tables of trigonometric functions	Except for the study of solutions of the right triangle by logarithms in the second course in algebra, the study of trigonometry is concluded here.

In the course in plane geometry there were several purposes in reorganizing the continuity of topics:

1. presenting the fundamental theorems both informally and formally for proof;

2. introducing solid geometry by informal proofs at the places where it corresponds to plane geometry;

3. Continuing the study of algebra through axioms, proofs, and numerical trigonometry;

4. representing the application of the Pythagorean theorem to analytic geometry with plane geometry;

5. omitting proofs of formulas involving pi;

6. enumerating formulas in solid geometry related to the circle;

7. introducing elements of trigonometry wherever it is possible in the study of triangles before the short course in trigonometry is given.

The courses in trigonometry and solid geometry are not included in this thesis except as these courses have been incorporated into other courses in elementary mathematics.

Topics with Specific Content	Reasons for Placement in Order
Algebra, Second Course	
1. Measurement and computation a. Measurement Accuracy of measurement, significant figures, rounding off a number	This topic include a review of numerical computation and introductory ideas for the following topic, statistics.
b. Errors Possible error, Theorem I: Possible error of a sum Percent of error Relative error, Theorems II and III: Relative errors of product and quotient	This includes a brief review of algebraic operations.
2. Statistics (mathematical significance only in this thesis) a. Statistical graphs Line graph, ratio chart, percentage graphs	Fundamentals of statistical representation and computation are introduced.
b. Frequency distribution c. Averages Mean, median, mode d. Simple formulas	Treatment of the topic is by simple definitions and illustrations, no formal proofs.
3. Solution of equations by graphs	Facility with the graph leads to equations of the conic sections.
4. Special products and factoring a. All products including the sum and difference of cubes	This is a basis for the topic directly following.
5. Solutions of equations by methods other than the graph, quadratic equations a. Linear equations Simultaneous b. Quadratic equations Simultaneous	

Topics with Specific Content	Reasons for Placement in Order
Algebra, Second Course (continued)	
6. Theory and graph of quadratic equations, cubic equation a. Circle at the origin and otherwise b. Ellipse c. Parabola d. Hyperbola e. General conic f. Cubic equation Solution by tracing the curve	Representation of conics on a graph emphasized.
7. Exponents and radicals a. Laws of exponents Zero and negative exponent b. Fundamental operations with radicals c. Radical equation d. Introduction to imaginary numbers Graphic representation	Imaginary numbers included in elementary mathematics to complete the number system.
8. Logarithms and numerical trigonometry a. Fundamentals of logarithms b. Logarithmic solution of the right triangle	Study of trigonometry continued from plane geometry. Facility of computations by logarithms emphasized.

The purpose of this course is to continue the study of algebra to more advanced topics. The introductory work in statistics serves as a review of the fundamentals of arithmetic and algebra at the same time that elementary statistics is presented. Special correlation of arithmetic, algebra, plane geometry, and trigonometry brought in the last topic serves to unite the courses in elementary mathematics.

Conclusion

In this section we have listed the topics of the following subjects of elementary mathematics: seventh grade mathematics, eighth grade mathematics, first course in algebra, second course in algebra, and plane geometry. We have ~~mad~~ made this list for two purposes: first, to give a summary of recent changes in mathematics; second, to present a suggested reorganization of topics in mathematics. Recent changes include such ideas as:

1. an interrelation of the topics of arithmetic with algebra;
2. elementary statistics in algebra;
3. the use of the function concept and relationships to unite the ideas of elementary mathematics;
4. geometry in the seventh and eighth grades as a result of the division of the course into intuitive and demonstrative geometry;
5. informal proofs from solid geometry in plane geometry;
6. graphic representation in elementary mathematics;
7. applications of calculus in elementary mathematics.

Suggestions for reorganization include:

1. proofs of propositions from plane geometry by methods of analytic geometry to correlate algebra and plane geometry;
2. postponing to the calculus proofs of propositions concerning areas of plane and solid figures involving pi;

3. including a simpler way of plotting the curve of a third degree equation by a method of calculus.

It has been pointed out that the proposed changes in the reorganization of the subject matter of mathematics will furnish both a more satisfactory continuity of topics and a simpler and more complete presentation of subject matter. Moreover, the logical order of the branches of mathematics indicated at the beginning of the study is still maintained.

APPENDIX

Further Propositions from Plane and Solid Geometry Proved by Methods of Analytic Geometry and Calculus

In addition to the proofs of propositions in plane geometry suggested in Chapter III by methods of analytic geometry and in Chapter IV by methods of calculus, there are many others. Without considering the exact place that they may be presented in the subjects of elementary mathematics, we shall list these proofs here. Since most of the statements are taken from textbooks in analytic geometry and calculus, it is assumed that the proofs of the propositions are very easily shown. Because of the limitations of this thesis they will not be presented here.

It has already been stated that there are certain prerequisites in analytic geometry necessary for proofs by coordinate methods. The following proofs, then, will be listed under the headings of the prerequisite concepts, definitions, and formulas of analytic geometry.

A. Prerequisite: explanation of the point system, projections of a directed segment on the coordinate axes, point of division formulas, distance formula.

1. The opposite sides of a parallelogram are equal.
2. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.
3. The diagonals of a parallelogram bisect each other.
4. The diagonals of a rhombus bisect each other.

B. Prerequisite: the midpoint formula.

5. The medians of a triangle intersect at a point two thirds of the distance from the vertex to the midpoint of the opposite side.

C. Prerequisite: slope of a line, tests for parallelism and perpendicularity, $m = \frac{y_2 - y_1}{x_2 - x_1}$, $m_1 = m_2$, $m_1 m_2 = -1$, respectively.

6. If a line divides two sides of a triangle proportionally, it is paralle to the third side.
7. Two straight lines perpendicular to the same straight line are parallel.
8. The lines joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.
9. If in a triangle the line joining any vertex to the middle point of the opposite side is equal in length to half that side, the figure is a right triangle with the right angle at the vertex.
10. A line perpendicular to one of two parallel lines is perpendicular to the other also.
11. The diagonals of a square are equal.
12. The lines joining the midpoints of the adjacent sides of a rectangel form a rhombus.

D. Prerequisite: the point slope formula; $y - y_1 \quad m(x - x_1)$.

13. The bisectors of the interior angles of a triangle meet in a point.
14. The bisectors of two exterior angles of any triangle and of the third interior angle meet in a point.
15. The bisectors of the angles formed by any two lines are perpendicular.
16. The perpendicular bisectors of the sides of a triangle meet in a point.
17. The medians of a triangle meet in a point.
18. The altitudes of a triangle meet in a point.
19. The line joining the points of intersection of the pe perpendicular bisectors of the sides of a triangle, the altitudes, of a triangle, and the medians of a triangle is a straight line.

E. Proofs from solid geometry using the above fundamentals in analytic geometry.

20. A line parallelto one of two parallel planes is parallel to the other.
21. A line perpendicular to one of two parallel planes is perpendicular to the other.
22. Bisector planes are perpendicular.
23. Three planes intersect in a straight line.

E. Proofs from solid geometry using the above fundamentals in analytic geometry (continued).

24. The lines joining the midpoints of a tetrahedron bisect each other and hence meet in a common point.

25. If a straight line is perpendicular to one plane, any plane which contains the line is perpendicular to the plane. [1]

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[1] Sources of statements of the propositions that may be proved by methods of analytic geometry are as follows:

I.A. Barnett, Analytic Geometry,
Propositions 16, 17, 18, Ex. 11, p. 54.
Proposition 2, Ex. 28, p. 29.

W.F. Osgood and W.C. Grausten, Plane and Solid Analytic Geometry,
Proposition 24, Ex. 20, p. 419.

P.F. Smith and A.S. Gale, Analytic Geometry,
Proposition 3, Ex. 11, p. 34,
Proposition 11, Ex. 10, p. 90.
Propositions 13 and 14, Ex. 15, p. 114.

W.A. Wilson and J.E. Tracey, Analytic Geometry,
Proposition 8, Ex. 16c, p. 21.
Proposition 12, Ex. 16d, p. 21.
Proposition 19, Ex. 14, p. 63.

__________ "The Reorganization of Mathematics in Secondary Education", Bulletin, 1921, No. 32,
Propositions 1, 3-8, 10-13, 21-23, 16-18, 25.
(list of theorems in plane and solid geometry pp.48-52).

Several interesting proofs may be found in J.S. Stewart, Elementary Geometry Propositions Proved by Coordinate Methods.

In Chapter IV proofs of formulas from elementary mathematics by integration in calculus were seen to be possible. Several other proofs may be added. We shall list statements of these formulas and indicate the prerequisite formulas of integration in each case.[2]

A. Prerequisite: volume of solids of revolution, where $V=\pi\int_a^b y^2\,dx$.

1. The volume of a cylinder generated by revolving about the x-axis the line-segment joining a point $(0,h)$ to the point (r,h) is $\pi r^2 h$.
2. The volume of a right cone generated by revolving about the x-axis the line-segment joining the origin to the point (p,q) is $(1/3)\pi pq^2$.
3. The volume of the frustum of a cone generated by revolving about the x-axis the line-segment joining a point $(0,b)$ to the point (h,r), where $b<r$ is $(1/3)\pi h(r^2+b^2+rb)$.

B. Prerequisite: area of surfaces of revolution, where $S=2\pi\int_a^b y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx$

4. The lateral area of a right circular cylinder of radius r and altitude h is $2\ rh$.
5. The lateral area of a right cone generated by revolving about the x-axis a line joining the origin to the point (a,b) is $\pi b(a^2+b^2)^{\frac{1}{2}}$.
6. The lateral area of the frustum of a cone generated by revolving the line from the point $(0,b)$ to the point (h,r) about the x-axis is $\pi(r+b)(h^2+(r-b)^2)^{\frac{1}{2}}$.

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[2] Sources of statements of the propositions that may be proved by methods of integration in calculus are as follows:

W.B. Ford, _A First Course in Differential and Integral Calculus_,
Formula A., p. 268. Formula B., p. 273.
Proposition 2, Ex. 1, p. 270.
Proposition 4, Ex. 5, p. 275.
Proposition 5, Ex. 3, p. 275.

BIBLIOGRAPHY

History and Theory of Mathematics

Cajori, Florian, "The Teaching and History of Mathematics in the United States", Circular of Information No. 3, 1890, Bureau of Education, Washington.

Hamley, H.R., Relational and Functional Thinking in Mathematics, Bureau of Publications, Teachers College, Columbia University, New York, 1934.

Milburn, M., Determinants, History, and Developments in Transitional Mathematics, Master's Thesis, University of Maine, Orono, 1939.

Russell, Bertrand, Principles of Mathematics, W.W. Norton and Company, Incorporated, New York, 1938.

Smith, David Eugene, History of Mathematics, vol. i, Ginn and Company, New York, 1923.

Stewart, John E., Elementary Geometry Propositions Proved by Coordinate Methods, Master's Thesis, University of Maine, Orono, 1927.

__________ "The Reorganization of Mathematics in Secondary Education", Bulletin, 1921, No. 32, Bureau of Education, Washington.

Textbooks

Barnett, I.A., Analytic Geometry, John Wiley and Sons, Incorporated, New York, 1928.

Bemans, W.W., and Smith, D.E., Plane and Solid Geometry, Ginn and Company, Boston, 1899.

Betz, W., Algebra for Today, First Course, Ginn and Company, New York, 1929.

Brink, R.W., Plane Trigonometry, The Century Company, New York, 1928.

Clark, J.R., and Otis, A.S., Modern Solid Geometry, World Book Company, New York, 1928.

Engelhardt, F., and Haertter, L.D., Second Course in Algebra, John C. Winston Company, Chicago, 1929.

Ford, W.B., A First Course in Differential and Integral Calculus, Henry Holt and Company, New York, 1928.

Milne, W.J., Standard Algebra, American Book Company, New York, 1906.

Mitchell, U.G., and Walker, H.M., Algebra: A Way of Thinking, Harcourt, Brace, and Company, New York, 1936.

Reichgott, D., and Spiller, L.R., Today's Geometry, Prentice Hall, Incorporated, New York, 1938.

Schorling, R., Clark, J.R., Smith, R.R., Modern School Mathematics, Book One, Book Two, Book Three, World Book Company, New York, 1935.

Slaught, H.E., and Lennes, N.J., Solid Geometry, Allyn and Bacon, Boston, 1911.

Smith, P.F., and Gale, A.S., Analytic Geometry, Ginn and Company, 1905.

Smith, P.F., and Granville, Elementary Analysis, Ginn and Company, New York, 1910.

Welchons, A.M., and Krickenberger, W.R., Plane Geometry, Ginn and Company, New York, 1935.

Wells, W., Plane Geometry, Leach, Shewell, and Sanborn, Boston, 1894.

Wells, W., and Hart, W.W., Modern Plane Geometry, D.C. Heath and Company, New York, 1926.

Wells, W., and Hart, W.W., Modern Solid Geometry, D.C. Heath and Company, New York, 1926.

Willard, H.R., and Bryan, N.R., College Algebra, planograph reprint, Spaulding-Moss Company, Boston, 1935.

Wilson, W.A., and Tracey, J.L., Analytic Geometry, D.C. Heath and Company, New York, 1925.

Articles

The following articles are from indicated volumes of *The Mathematics Teacher*.

Blank, K., "Functions of Intuitive and Demonstrative Geometry, vol. xxii, (1929), pp. 31-37.

Good, W.P., and Chapman, H.H., "The Teaching of Proportion in Plane Geometry", vol. xxi, (1928), pp. 462-463.

Gootschalk, W.M., "Analytic Geometry and Trigonometry in Second Year Algebra", vol. xxxiii, (1940), pp. 82-83.

Hans, A., "Geometric Proofs for Trigonometric Formulas", vol. xxxiii, (1930), pp. 321-326.

Hassler, J.O., "Teaching Geometry into Its Rightful Place", vol. xxii, (1929), pp. 337-338.

Leary, A.F., "Analytic Geometry in the High School", vol. xxxiii, (1940), pp. 70-72.

Orleans, J.B., "Experiment with Mathematics of the Eleventh Year", vol. xxiii, (1930), pp. 447-488.

Reeves, W.D., and His Students, "Tenth Year Mathematics Outline", vol. xxiii, (1930), pp. 343-357.

Reynolds, J.B., "Finding Plane Areas by Algebra", vol. xxi, (1928)

Stone, J.C., "One Year Course in Plane and Solid Geometry", vol. xxiii, (1930), pp. 236-242.

Smith, D.E., "Arithmetic in the High School", vol. viii, (1916), p. 115.

Swenson, J.A., "Newer Type of Mathematics", vol. xxii, (1929), p. 255.

Wannemacher, A.S., "Geometry Aids for Elementary Algebra", vol. xxii, (1929), pp. 49-57.

Zeigler, D.G., "Concerning Orientation and Application in Geometry", vol. xxii, (1929), pp. 109-116.

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