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Hopf Algebras in the Representation Theory of Combinatorial//Families of Groups

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HOPF ALGEBRAS IN THE REPRESENTATION THEORY OF COMBINATORIAL // FAMILIES OF GROUPS

By

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Thesis Advisor: Dr. Tyrone Crisp

An Abstract of the Thesis Presented
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The main result of this thesis is that there exists a positive, self-adjoint Hopf (PSH) algebra structure in the representation theory of a certain family of groups. This new construction is inspired directly by Andrey Zelevinsky's discovery of such a structure in the representation theory of the symmetric groups. Zelevinsky's work *Representations of finite classical groups: a Hopf algebra approach* gives an account of this. We will walk through Zelevinsky's work in this field in detail, and then follow up with the construction on the groups in question. We will develop the necessary theory along the way, with the reader assumed to be familiar with the basic properties of groups and rings. The notion of categories, functors, and Grothendieck groups will be useful, but knowledge of these concepts is not necessary for the reading of this thesis.

DEDICATION

To my mother, who has always led by example.

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The vast majority of the progress I've made while attending the University of Maine was made possible only with the assistance of my professors, advisors, and friends. Chief among these is Dr. Tyrone Crisp, whose self control in the face of repeated questions, misunderstandings, and lapses in discipline is beyond admirable.

My friends Jeremy Juybari and Kristin Tenney were a support system I never deserved.

Finally, my Jiu Jitsu family kept me honest and always reminded me that quitting is never an option. Get you some.

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CHAPTER 1

INTRODUCTION

The representation theory of finite groups allows us to study the ways finite groups act as linear symmetries. The representation theory of the symmetric groups S_n is well understood, with the construction of the irreducibles coming from considering the entire family S_n simultaneously (for an account of this, see [4, section 5.12]). This method of analyzing the representations of symmetric groups suggests that insight about the representation theory of other well-behaved combinatorial families of groups might be gained by considering the whole family at once.

Andrey Zelevinsky introduced the structure of a positive, self-adjoint Hopf algebra (PSH algebra) on the representation theory of the family of symmetric groups in [11]. In that same work he also shows that the construction could be done with other groups; namely the wreath products $S_n \ltimes G^{\{1,\dots,n\}}$ and the general linear groups $\mathrm{GL}(n, \mathbf{F}_q)$ over the finite fields. In what follows we will recreate the construction of a PSH algebra from the representation theory of the family S_n, followed by proving the construction is possible for a new family of groups G_n which contain the symmetric groups.

Along the way, we draw parallels between the behavior of our groups G_n and that of the symmetric groups S_n. These parallels come in the form of Lemmas 4.2.5 and 4.2.6, which give tools for calculating products and intersections of different groups G_n.

This begs the question of which characteristics of a family of groups are necessary in order to exhibit a PSH algebra structure in their representation theory. This question is answered for groups expressed as wreath products in [3], which provides sufficient conditions for the construction of a PSH algebra in the representation theory of a family of wreath products. The family of groups we study is expressed as a semidirect product $S_n \ltimes M_n(\mathbb{Z}/p\mathbb{Z})$, and the fact that $M_n(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\{1,\dots,n\}^2}$, the abelian group of $\mathbb{Z}/p\mathbb{Z}$-valued functions on $\{1, \ldots, n\}^2$, allows us to think of this family of groups as a generalization of Zelevinsky's family $S_n \ltimes G^{\{1,\ldots,n\}}$ with $G = \mathbb{Z}/p\mathbb{Z}$.

Although not purely combinatorial in origin, Hopf algebras appear in many combinatorial fields of study. The feature distinguishing Hopf algebras from algebras is the unary comultiplication operation. Multiplication gives us methods of understanding how objects may be combined, whereas comultiplication allows us to study the various ways of decomposing objects. Hopf algebras arise when the ways of "sticking objects together" and "pulling objects apart" are compatible with one another. This will be made rigorous in chapter 3.

CHAPTER 2

BACKGROUND

2.1 Representation Theory

We will be studying the representation theory of a family of combinatorial semidirect products. This section gives basic background, and provides results of representation theory that will be used later. Later, we will also discuss algebras, coalgebras, and Hopf algebras.

We'll begin with representations of groups. All groups considered will be finite.

Definition 2.1.1. Given a group G, a *representation* of G on a finite-dimensional, complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ denotes the group of invertible linear maps $V \to V$. In the future we will refer to either ρ, V, or (ρ, V) as the representation. For $g \in G$ and $v \in V$, we will denote $gv := \rho(g)v$. All representations we'll consider will be finite-dimensional.

An equivalent definition of a representation of G which we will make frequent use of is considering the space V as a left, unital $\mathbb{C}G$-module, where

$$\mathbb{C}G = \left\{ \sum_{g \in G} a_g g \,\middle|\, a_g \in \mathbb{C} \right\}.$$

This is known as the *group ring* or *group algebra* of G. The action of this algebra (i.e., ring) on the vector space (i.e., additive abelian group) V is given by $(\sum_g \alpha_g g)v := \sum_g \alpha_g (gv)$.

Definition 2.1.2. Given two representations ρ and φ of G on V and W, respectively, a *morphism* of representations from V to W is a linear map $T : V \to W$ satisfying $Tgv = gTv$ for all $g \in G, v \in V$. Two representations are said to be *isomorphic* if there exists a morphism between them which is an isomorphism of vector spaces. Given two representations V, W of G, we denote by $\mathrm{Hom}_G(V, W)$ the set ($\mathbb{C}$-vector space) of all morphisms $V \to W$.

This notion of morphism allows us to endow the collection of all representations of some fixed group G with the structure of a category. Indeed, for every representation V, the identity map 1_V is a morphism $V \to V$; furthermore, composition of linear maps is associative. We will denote the category of finite dimensional complex representations of a group G by $\mathrm{Rep}(G)$. For a more complete exposition of the theory of categories and functors which we will be conforming to, see [9]

Definition 2.1.3. Let $f : H \to K$ be a homomorphism of groups. This induces a functor $f^* : \mathrm{Rep}(K) \to \mathrm{Rep}(H)$ given by $\rho \mapsto \rho \circ f$ called the *pullback* of f. This functor sends a morphism $T : V \to W$ of K representations to the same morphism T of H representations.

Example 2.1.4. Let H be a subgroup of a finite group G. Then for each fixed $x \in G$ the map $\mathrm{Ad}_x : H \to xHx^{-1}$ defined by $h \mapsto xhx^{-1}$ is an isomorphism. This induces a functor $\mathrm{Ad}_x^* : \mathrm{Rep}(xHx^{-1}) \to \mathrm{Rep}(H)$ given by sending the representation $\rho : xHx^{-1} \to \mathrm{GL}(V)$ to the representation $\rho \circ \mathrm{Ad}_x : H \to xHx^{-1} \to \mathrm{GL}(V)$. That is, the element $h \in H$ is defined to act on V via the element xhx^{-1}.

As is often the case, given some object, we'd like to find subobjects to make our study more digestible. We do this with representations via the following.

Definition 2.1.5. Given a representation $\rho : G \to \mathrm{GL}(V)$, we say a subspace $W \subseteq V$ is *G-invariant* if given any $g \in G, w \in W$, we have $gw \in W$. It follows that $\rho|_W : g \mapsto \rho(g)|_W \in \mathrm{GL}(W)$ is a representation of G. We call W a *subrepresentation* of V.

Remarks 2.1.6. Given representations V and W of G, the direct sum of vector spaces $V \oplus W$ is given the structure of a representation of G by the formula $g(v,w) := (gv, gw)$. Additionally, if $V \in \mathrm{Rep}(G)$ and $U \in \mathrm{Rep}(H)$, then $V \otimes U \in \mathrm{Rep}(G \times H)$, with the action on elementary tensors given by $(g,h)(v \otimes u) := gv \otimes hu$.

Definition 2.1.7. A representation V of G is said to be *irreducible* if $V \neq 0$, and the only G-invariant subspaces of V are 0 and V. Denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible members of $\mathrm{Rep}(G)$.

Lemma 2.1.8 (Schur's Lemma). *If V is an irreducible representation of G, then* $\dim \mathrm{Hom}_G(V,V) = 1$.

Proof. Suppose $T \in \mathrm{Hom}_G(V,V)$. Then $\ker(T)$ and image(T) are G-invariant subspaces of V. Hence T is either an isomorphism of representations, or $T = 0$. If T is an isomprohism, let $\lambda \in \mathbb{C}$ be an eigenvalue of T (i.e., a root of the characteristic polynomial of T which exists since we're working over $\mathbb{C}$). Then $T - \lambda 1_V \in \mathrm{Hom}_G(V,V)$. But this operator has a nonzero kernel U which is G-invariant. By irreducibility, $U = V$; therefore $T - \lambda 1_V = 0$, and so $\mathrm{Hom}_G(V,V) = \mathbb{C}1_V$. □

Theorem 2.1.9 (Maschke). *Every finite dimensional representation of a finite group G is a direct sum of irreducible representations of G.*

This is Theorem 4.1.1 of [4], which is accompanied by a proof.

Proposition 2.1.10. *[4, Corollary 4.2.2] The number of irreducible representations of a finite group G is equal to the number of conjugacy classes in G.*

Corollary 2.1.11. *Let $V_1, \ldots, V_m$ be a complete set of irreducible representations of a group G. Let W be a representation of G, and suppose W decomposes as*

$$W \cong \sum_{i=1}^{m} n_i V_i.$$

Then for each $j = 1, \ldots, m$, $\dim \mathrm{Hom}_G(V_j, W) = n_i$.

Proof. First, rewrite

$$\mathrm{Hom}_G(V_j, W) \cong \mathrm{Hom}_G(V_j, \sum_{i=1}^{m} n_i V_i) \cong \sum_{i=1}^{m} n_i \,\mathrm{Hom}_G(V_j, V_i).$$

Taking dimensions and applying Schur's Lemma (2.1.8), the corollary is proved. □

Proposition 2.1.12. *[4, Theorem 5.6.1] Let G, H be two finite groups. Then there is a natural bijection* $\mathrm{Irr}(G) \times \mathrm{Irr}(H) \xrightarrow{(V,W) \mapsto V \otimes_{\mathbb{C}} W} \mathrm{Irr}(G \times H)$.

2.1.1 Examples of Representations

We now give some examples of representations.

Example 2.1.13. Given any finite group G, the group algebra is a representation of G with $h \in G$ acting by $h \cdot \sum_{g\in G} a_g g := \sum_{g\in G} a_g hg$. Another way to view this is that $\mathbb{C}G$ is itself a $\mathbb{C}G$-module. This is called the regular representation and is actually isomorphic to

$$\bigoplus_{V\in\mathrm{Irr}(G)} \dim(V)V$$

as representations of G (this is [4, Theorem 4.1.1]).

Example 2.1.14. Let $\mathbb{C}^\times$ denote the group of nonzero complex numbers under multiplication. Any group homomorphism $G \to \mathbb{C}^\times$ is a representation via the identification $\mathbb{C}^\times \cong \mathrm{GL}(\mathbb{C})$. By Cayley's Theorem (see [7, Theorem II.4.5]), G embeds as a subgroup of S_G, the permutation group of G. The sign homomorphism $\mathrm{sgn} : S_G \to \mathbb{C}^\times$ which takes the value 1 on even permutations and -1 on odd permutations may then be restricted to G to give a one-dimensional representation of G. If G happens to be contained in the alternating group A_G consisting of even permutations of G, then the sign representation is the trivial representation.

Example 2.1.15. Suppose G acts on a finite set X. The *permutation representation* of this action is

$$\mathbb{C}X = \left\{ \sum_{x\in X} a_x x \,\middle|\, a_x \in \mathbb{C} \right\}$$

where $g \in G$ acts by $g \cdot \sum_{x\in X} a_x x := \sum_{x\in X} a_x (g \cdot x)$. When $X = G$ this gives the regular representation of G.

Example 2.1.16. The map $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^2)$ given by

$$(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$(123) \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}$$

is an irreducible representation. It's also true (though we won't have need to prove it here) that if $V_1, \ldots, V_m$ are the irreducible representations of a group G with dimensions $d_1, \ldots d_m$, then $\sum_i d_i^2 = |G|$. Hence this representation, along with the trivial representation and the sign representation are an exhaustive list of the irreducible representations of S_3.

2.1.2 Characters, class functions, and their connection with representations

Representations have equivalent notions which will be useful in computations. These are called characters and class functions: complex-valued functions on the group in question which encode all the pertinent information from the representation.

Definition 2.1.17. Given a representation $\rho : G \to \mathrm{GL}(V)$ of a group, we define the *character* of V to be the function $\chi_V : G \to \mathbb{C}$ given by $g \mapsto \mathrm{Tr}_V(g)$ (when V is clear from the context, we will often write just χ).

A basic property of Tr is that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ (see [6], § 56). From this we calculate that for invertible a, that $\mathrm{Tr}(aba^{-1}) = \mathrm{Tr}(a^{-1}ab) = \mathrm{Tr}(b)$. This shows that a character χ of G is a *class function*. That is, χ is constant on conjugacy classes of G. We denote the $\mathbb{C}$-vector space of all such class functions by $Cl(G)$.

It follows immediately from $\mathrm{Tr}(aba^{-1}) = \mathrm{Tr}(b)$ that isomorphic representations have equal characters. This allows us to define a function $\mathrm{Rep}(G) \to Cl(G)$ by $V \mapsto \chi_V$ which is constant on isomorphism classes. Define $\mathrm{R}(G)$ to be the free abelian group with a basis consisting of the isomorphism classes of irreducible representations of G.

Theorem 2.1.18. *The characters of the irreducible representations of G form an orthonormal basis for $Cl(G)$, with the inner product on $Cl(G)$ defined by*

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

For a proof of 2.1.18, see [10, Theorem 2.5.6]. We will utilize this fact as follows.

Corollary 2.1.19. *1. The map $\psi : \mathrm{Irr}(G) \to Cl(G)$ defined by sending $V \mapsto \chi_V$ induces an injective $\mathbb{Z}$-linear map $\tilde{\psi} : \mathrm{R}(G) \to Cl(G)$.*

2. *For any functor $F : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ which commutes with direct sums, there is a unique linear map $\tilde{F} : Cl(G) \to Cl(H)$ making the diagram*

$$\begin{array}{ccc} \mathrm{R}(G) & \xrightarrow{V \mapsto \chi_V} & Cl(G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\ \mathrm{R}(H) & \xrightarrow[W \mapsto \chi_W]{} & Cl(H) \end{array}$$

commute.

Proof. 1. The equivalence classes of irreducible representations of G form a basis for $\mathrm{R}(G)$ by definition. By 2.1.18, the characters of these representations are linearly independent. For injectivity, suppose

$$\sum_{V \in \mathrm{Irr}(G)} n_V V \in \ker(\tilde{\psi}),$$

so that

$$\sum_{V \in \mathrm{Irr}(G)} n_V \chi_V = 0.$$

By the linear independence guaranteed by Theorem 2.1.18, it must be that $n_V = 0$ for all V. Hence $\tilde{\psi}$ is injective.

2. Define $\tilde{F}$ on the image of $\mathrm{Irr}(G)$ by $\chi_V \mapsto \chi_{F(V)}$, and extend linearly to $Cl(G)$ using Theorem 2.1.18.

□

Corollary 2.1.20. *Given two functors F_1, F_2 as in 2.1.19, we may consider $F_1, F_2 : \mathrm{R}(G) \to \mathrm{R}(H)$, and calculate $\tilde{F_1}$ and $\tilde{F_2}$. If $\tilde{F_1} = \tilde{F_2}$, then $F_1 = F_2$ as maps $\mathrm{R}(G) \to \mathrm{R}(H)$.*

Proof. Indeed, $[F_1 - F_2](V) = 0$ for all V if and only if $\chi_{[F_1 - F_2](V)} = 0$ for all V. By the commutativity of the diagram in 2.1.19, this is equivalent to saying $[\tilde{F_1} - \tilde{F_2}](\chi_V) = 0$ for all V. □

2.2 First Tensor Product Formulas and Consequences

In this section we will develop a viewpoint which will be useful in much of the remainder of this paper. This relies on the definition of representations of G as $\mathbb{C}G$-modules. Recall that a representation $\rho: G \to \mathrm{GL}(V)$ gives a $\mathbb{C}G$-module structure on V given by $gv := \rho(g)v$, and extended to all of $\mathbb{C}G$ by linearity. We will exhibit formulas for functors between categories of representations which are given in terms of tensor products of modules. This allows us to make the following mental identifications: functors correspond to taking tensor products with bimodules, composition of functors corresponds to a tensor product of bimodules, a direct sum of functors corresponds to a direct sum of bimodules, and natural transformations of functors correspond to morphisms of bimodules. These correspondences can be made rigorous by expressing them as an equivalence between categories of functors and bimodules, but such things are outside the scope of this thesis.

For a definition and properties of tensor products, see [7, IV.5]. For A and B right and left R-modules, respectively, we will write $A \otimes_R B$ for the tensor product of A with B over R, writing $A \otimes B$ if R is understood. Recall some of the elementary properties of tensor products:

- $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$
- $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$
- If A is a left S-module, then so is $A \otimes B$ via $s(a \otimes b) := sa \otimes b$

Theorem 2.2.1. *Let R, S, and T be rings with unit. Suppose M is a unital left (R, T)-bimodule, N is a unital (S, R)-bimodule, and Q is a unital left S-module. Then*

$$\Psi : \mathrm{Hom}_S(N \underset{R}{\otimes} M, Q) \to \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, Q))$$

given by $\Psi(\varphi)(m)(n) := \varphi(n \otimes m)$ is an isomorphism of T-modules.

Remarks 2.2.2. With the assumptions of Theorem 2.2.1, the T-module structure on $\mathrm{Hom}_S(N \underset{R}{\otimes} M, Q)$ is given by $(tf)(x) := f(xt)$; the T-module structure on $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, Q))$ is defined similarly.

For a proof of Theorem 2.2.1, see [8, Theorem X.10.23]. We will refer to this result as the tensor-hom adjunction.

Given a subgroup H of G, we'd like to study how representations of H relate to representations of G. We accomplish this by defining functors for inducing and restricting representations between subgroups and supergroups. With the above viewpoint in mind, we exhibit the bimodules and tensor products which correspond to these functors.

Until the next section, we will define an algebra over a commutative ring R to be an R-module A with a linear map

$$A \otimes_R A \xrightarrow{(a\otimes b)\mapsto ab} A$$

satisfying $(ab)c = a(bc)$, and containing an element 1 such that $1a = a1 = a$.

Definition 2.2.3. Let $H \subseteq G$ be a subgroup, and let $\rho : G \to V$ be a representation of G. We define the functor $\mathrm{Res}_H^G\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ by sending ρ to the representation $\rho|_H$ of H.

It follows, then, that a representation of a group is thus a representation of any of its subgroups. This definition is somehow the most natural, and should therefore serve as the basis for defining relations between $\mathrm{Rep}(G)$ and $\mathrm{Rep}(H)$ where $H \subseteq G$. Now we define a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$ and prove an adjunction known as Frobenius reciprocity.

Definition 2.2.4. Let $H \subseteq G$ be a subgroup as above. Define the functor $\mathrm{Ind}_H^G\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ by $V \mapsto \{f : G \to V | f(hx) = hf(x) \text{ for all } h \in H, x \in G\}$, with the action of G given by right translation; i.e. for $g \in G$ and $f \in \mathrm{Ind}_H^G V$, set $(g \cdot f)(x) := f(xg)$ for each $x \in G$.

Remarks 2.2.5. The set of functions $\mathrm{Ind}_H^G V$ is easily seen to be a representation of G. Indeed, if $f_1, f_2 \in \mathrm{Ind}_H^G V$, then $(f_1 + f_2)(hx) = f_1(hx) + f_2(hx) = hf_1(x) + hf_2(x) = h(f_1 + f_2)(x)$; so $\mathrm{Ind}_H^G V$ is a vector space. To see that the given action of G satisfies the necessary associativity property, let $g, k \in G$, and $f \in \mathrm{Ind}_H^G V$. We have $[gk \cdot f](x) = f(xgk) = [k \cdot f](xg) = [g \cdot (k \cdot f)](x)$, as needed.

Theorem 2.2.6 (Frobenius reciprocity). *For $V \in \operatorname{Rep}(G)$ and $W \in \operatorname{Rep}(H)$, we have*

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W).$$

This theorem states that the functor Ind_H^G is right-adjoint to Res_H^G. In order to prove it, we will need some lemmas utilizing the more concrete view of representations as modules. In what follows we will view $\mathbb{C}G$ as a $(\mathbb{C}G, \mathbb{C}H)$-bimodule, and a $(\mathbb{C}H, \mathbb{C}G)$-bimodule. These structures are given by multiplication in $\mathbb{C}G$.

Lemma 2.2.7. *Let V be a representation of G, and $H \subseteq G$ be a subgroup. Then, as $\mathbb{C}H$-modules, $\operatorname{Res}_H^G V \cong \mathbb{C}G \otimes_{\mathbb{C}G} V$.*

Proof. Define $\varphi : \operatorname{Res}_H^G V \to \mathbb{C}G \otimes_{\mathbb{C}G} V$ by $v \mapsto 1 \otimes v$. To see that this map is H-equivariant, let $h \in H$ and $v \in V$. Then $\varphi(hv) = 1 \otimes hv = h \otimes v = h(1 \otimes v) = h\varphi(v)$. Next define $\psi : \mathbb{C}G \otimes_{\mathbb{C}G} V \to \operatorname{Res}_H^G V$ by $g \otimes v \mapsto gv$ for the elementary tensor $g \otimes v$. This map is well-defined since $\psi(g \otimes g'v) = gg'v = \psi(gg' \otimes v)$ by associativity of multiplication in $\mathbb{C}G$. It's also the case that $\psi = \varphi^{-1}$. To prove this, observe first that $\varphi(\psi(g \otimes v)) = \varphi(gv) = 1 \otimes gv = g \otimes v$; additionally, $\psi(\varphi(v)) = \psi(1 \otimes v) = 1v = v$, as desired. □

Lemma 2.2.8. *Let $W \in \operatorname{Rep}(H)$. Then $\operatorname{Ind}_H^G W \cong \operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, W)$ as left $\mathbb{C}G$-modules.*

Proof. The function $\varphi\colon \operatorname{Ind}_H^G W \to \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G, W)$ given by extending a function f from its domain G to the domain $\mathbb{C}G$ is an isomorphism. □

The two preceding lemmas, when considered alongisde the tensor-Hom adjunction, prove Theorem 2.2.6. That is, the functors Res and Ind are adjoints. Now one last formula for Ind which will be useful in what comes.

Lemma 2.2.9. *Let G and H be as above, and $W \in \operatorname{Rep}(H)$. Then $\operatorname{Ind}_H^G W \cong \mathbb{C}G \otimes_{\mathbb{C}H} W$.*

Proof. Define $\varphi : \operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, W) \to \mathbb{C}G \otimes_{\mathbb{C}H} W$ by

$$f \mapsto \sum_{Hg \in H\backslash G} g^{-1} \otimes f(g).$$

This map doesn't depend on our choice of coset representatives. Indeed,

$$(hg)^{-1} \otimes f(hg) = g^{-1}h^{-1} \otimes hf(g) = g^{-1} \otimes f(g),$$

as desired.

To see that this map is G-equivariant, suppose $g \in G$ and $f \in \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, W)$. Then $\varphi(k \cdot f) = \sum_{Hg} g^{-1} \otimes [k \cdot f](g) = \sum_{Hg} g^{-1} \otimes f(gk)$. Make the change of variables $x = gk$, so that $g^{-1} = kx^{-1}$ and $g = xk^{-1}$. Then we have
$\varphi(k \cdot f) = \sum_{Hxk^{-1}} kx^{-1} \otimes f(x) = k \sum_{Hxk^{-1}} x^{-1} \otimes f(x)$, which sums over the same set. Hence $\varphi(k \cdot f) = k\varphi(f)$.

To see that φ is injective, suppose now that $\varphi(f) = \varphi(f')$, so that $\sum_{Hg \in H\backslash G} g^{-1} \otimes f(g) = \sum_{Hg \in H\backslash G} g^{-1} \otimes f'(g)$. If we consider one fixed g, we arrive at $f(g) = f'(g)$. Since f and f' are determined by their values on representatives of the cosets Hg, we know $f = f'$ on Hg. Since g was chosen arbitrarily, $f = f'$ everywhere.

Let $\delta_{x=y} : \mathbb{C}G \to \mathbb{C}$ be the indicator function with $\delta_{x=y}(x) = 1$ when $x = y$ and 0 otherwise. For an elementary tensor $k \otimes w \in \mathbb{C}G \otimes_{\mathbb{C}H} W$, define
$\psi : \mathbb{C}G \otimes_{\mathbb{C}H} W \to \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, W)$ by

$$\psi(k \otimes w)(g) := \sum_{h \in H} \delta_{kg=h} hv.$$

Now we compute the composition $\varphi \circ \psi$ to show that φ is surjective:

$$\varphi(\psi(k \otimes w)) = \sum_{Hx \in H\backslash G} x^{-1} \otimes \psi(k \otimes w)(x) = \sum_{\substack{Hx \in H\backslash G, \\ h \in H}} x^{-1} \otimes \delta_{xk=h} hw$$

$$= x^{-1} \otimes hw = x^{-1} \otimes xkw = x^{-1}xk \otimes w = k \otimes w.$$

This shows that the map φ is an isomorphism.

□

Next we'll prove two technical lemmas which will be used later to help with an associativity check: the transitive and multiplicative properties of Ind and Res. Recall that if $V \in \mathrm{Rep}(G)$ and $W \in \mathrm{Rep}(H)$, $V \otimes W \in \mathrm{Rep}(G \times H)$, with action specified in 2.1.6.

Lemma 2.2.10. *Suppose $H' \subseteq H$ and $G' \subseteq G$ are subgroups, and let $\pi' \in \mathrm{Rep}(G')$ and $\rho' \in \mathrm{Rep}(H')$. Then $\mathrm{Ind}_{G'\times H'}^{G\times H}(\pi' \otimes \rho') \cong \mathrm{Ind}_{G'}^{G} \pi' \otimes \mathrm{Ind}_{H'}^{H} \rho'$. Additionally, if $\pi \in \mathrm{Rep}(G)$ and $\rho \in \mathrm{Rep}(H)$, then $\mathrm{Res}_{G'\times H'}^{G\times H}(\pi \otimes \rho) \cong \mathrm{Res}_{G'}^{G} \pi \otimes \mathrm{Res}_{H'}^{H} \rho$.*

Proof. The case with Res is clear. For Ind, let V and W be the spaces associated with π' and ρ', respectively. Then

$$\mathrm{Ind}_{G'\times H'}^{G\times H}(V \otimes W) \cong \mathbb{C}[G \times H] \underset{\mathbb{C}[G'\times H']}{\otimes} (V \otimes W)$$

which is isomorphic to $[\mathbb{C}G \otimes \mathbb{C}H] \underset{\mathbb{C}G'\otimes\mathbb{C}H'}{\otimes} (V \otimes W)$. Now define

$$\varphi : (\mathbb{C}G \otimes \mathbb{C}H) \underset{\mathbb{C}G'\otimes\mathbb{C}H'}{\otimes} (V \otimes W) \to (\mathbb{C}G \underset{\mathbb{C}G'}{\otimes} V) \otimes (\mathbb{C}H \underset{\mathbb{C}H'}{\otimes} W)$$

by $g \otimes h \otimes v \otimes w \mapsto g \otimes v \otimes h \otimes w$, and

$$\psi : (\mathbb{C}G \underset{\mathbb{C}G'}{\otimes} V) \otimes (\mathbb{C}H \underset{\mathbb{C}H'}{\otimes} W) \to (\mathbb{C}G \otimes \mathbb{C}H) \underset{\mathbb{C}G'\otimes\mathbb{C}H'}{\otimes} (V \otimes W)$$

by $g \otimes v \otimes h \otimes w \mapsto g \otimes h \otimes v \otimes w$.

It's easy to see that if they're well-defined, φ and ψ are mutually inverse. To show that φ is well-defined, let $g' \otimes h'$ be an elementary tensor and observe: $\varphi(g \otimes h \otimes (g' \otimes h')(v \otimes w)) = \varphi(g \otimes h \otimes g'v \otimes h'w)$ by the definition of how $G' \times H'$ acts on $V \otimes W$. But applying φ yields $g \otimes g'v \otimes h \otimes h'w = gg' \otimes v \otimes hh' \otimes w$ since the two factors of the codomain of φ are balanced over $\mathbb{C}G'$ and $\mathbb{C}H'$. Again looking at the definition of φ we see that this quantity equals $\varphi(gg' \otimes hh' \otimes v \otimes w) = \varphi((g \otimes h \otimes)(g' \otimes h') \otimes v \otimes w)$, as desired. A similar argument shows that ψ is a balanced map.

To see that φ is a morphism of $\mathbb{C}G \otimes \mathbb{C}H$-modules, let $g_1 \otimes h_1 \in \mathbb{C}G \otimes CH$. Then

$$\begin{aligned}\varphi\big((g_1 \otimes h_1)(g \otimes h \otimes v \otimes w)\big) &= \varphi(g_1 g \otimes h_1 h \otimes v \otimes w)\\ &= g_1 g \otimes v \otimes h_1 h \otimes w\\ &= (g_1 \otimes h_1)\varphi(g \otimes h \otimes v \otimes w)\end{aligned}$$

as needed. □

Lemma 2.2.11. *Let $K \subseteq H \subseteq G$ be a chain of subgroups, $\pi \in \mathrm{Rep}(K)$, and $\rho \in \mathrm{Rep}(G)$. Then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H \pi \cong \mathrm{Ind}_K^G \pi$, and $\mathrm{Res}_K^H \mathrm{Res}_H^G \rho \cong \mathrm{Res}_K^G \rho$.*

Proof. The case for Res is immediate. To prove the Ind case, recall that the adjoint of a composition of functors is the composition of the adjoints in the opposite order [1, Proposition 18.5]. We proved above that Res_K^G is the adjoint functor to Ind_K^G. This proves our desired result. □

2.3 Mackey Theory

Our foray into Mackey theory will consist of studying the composition $\mathrm{Res} \circ \mathrm{Ind}$. The formula we arrive at is often used to investigate the irreducibility of induced representations. We will use it to verify the Hopf axiom in the (yet-to-be-defined) Hopf algebra $\mathrm{R}(S)$.

Let G be a finite group, and let $M, N \subseteq G$ be two subgroups, with $W \in \mathrm{Rep}(N)$. Recall that for $X \in \mathrm{Rep}(G)$, we have $\mathrm{Res}_M^G X \cong \mathbb{C}G \otimes_{\mathbb{C}M} X$ and $\mathrm{Ind}_N^G W \cong \mathbb{C}G \otimes_{\mathbb{C}N} W$. For $x \in G$, define the notation ${}^xM := xMx^{-1}$, and recall the functor $\mathrm{Ad}_x^* : \mathrm{Rep}({}^xM) \to \mathrm{Rep}(M)$ of example 2.1.4.

We will prove the following.

Theorem 2.3.1. *With the above hypotheses, we have*

$$\mathrm{Res}_M^G \circ \mathrm{Ind}_N^G W \cong \bigoplus_{MxN \in M\backslash G/N} \mathrm{Ind}_{{}^xN \cap M}^M \circ \mathrm{Ad}_{x^{-1}}^* \circ \mathrm{Res}_{N \cap {}^{x^{-1}}M}^N W.$$

The proof of this theorem will rely heavily on the tensor product formulas for Ind and Res, along with some facts about the structure of the bimodules those formulas use. An immediate consequence of 2.2.7 and 2.2.9 is that

$$\mathrm{Res}_M^G \mathrm{Ind}_N^G W \cong \mathbb{C}G \underset{\mathbb{C}G}{\otimes} \mathrm{Ind}_N^G W \cong \mathbb{C}G \underset{\mathbb{C}G}{\otimes} \mathbb{C}G \underset{\mathbb{C}N}{\otimes} W,$$

which tells us that

$$\mathrm{Res}_M^G \mathrm{Ind}_N^G W \cong \mathbb{C}G \underset{\mathbb{C}N}{\otimes} W.$$

In order to study this module, we will prove some structure lemmas about its constituents.

Lemma 2.3.2. *As $(\mathbb{C}M, \mathbb{C}N)$-bimodules, we have*

$$\mathbb{C}G = \bigoplus_{MxN \in M\backslash G/N} \mathbb{C}[MxN],$$

where

$$\mathbb{C}[MxN] = \left\{ \sum_{(m,n)\in M\times N} \alpha_{m,n} mxn \,\middle|\, \alpha_{m,n} \in \mathbb{C} \right\}$$

is the $\mathbb{C}$-subspace of $\mathbb{C}G$ with $(\mathbb{C}M, \mathbb{C}N)$-bimodule structure given by multiplication in $\mathbb{C}G$.

The $(\mathbb{C}M, \mathbb{C}N)$-bimodule structure on $\mathbb{C}[MxN]$ is given by multiplication in $\mathbb{C}G$.

Proof. The double cosets partition G. □

Next, a tool for counting the dimension of an induced representation.

Lemma 2.3.3. *For any finite group G, subgroup N thereof, and W a representation of N, we have* $\dim_{\mathbb{C}}(\mathbb{C}G \underset{\mathbb{C}N}{\otimes} W) = |G/N| \dim_{\mathbb{C}} W$.

Proof. First, we observe that for any $g \in G$, $\mathbb{C}[gN] \cong \mathbb{C}N$ as right $\mathbb{C}N$-modules. Next, $\mathbb{C}G \cong \bigoplus_{gN \in G/N} \mathbb{C}[gN]$ as right $\mathbb{C}N$-modules. Thus, as vector spaces,

$$\begin{aligned} \mathbb{C}G \underset{\mathbb{C}N}{\otimes} V &\cong \bigoplus_{gN\in G/N} \left(\mathbb{C}[gN] \otimes_{\mathbb{C}N} V\right) \cong \bigoplus_{gN\in G/N} \left(\mathbb{C}[gN] \otimes_{\mathbb{C}N} W\right) \\ &\cong \bigoplus_{gN\in G/N} (\mathbb{C}N \otimes_{\mathbb{C}N} W) = |G/N|(\mathbb{C}N \otimes_{\mathbb{C}N} W) \cong \frac{|G|}{|N|} W. \end{aligned}$$

This proves our result. □

Lemma 2.3.4. *Let $M, N \subseteq G$ be two subgroups, and $x \in G$. Then, as $(\mathbb{C}M, \mathbb{C}N)$-bimodules, we have* $\mathbb{C}[MxN] \cong \mathbb{C}M \underset{\mathbb{C}[M\cap {}^xN]}{\otimes} \mathbb{C}[xN]$

Proof. Define the map

$$\varphi : \mathbb{C}M \underset{\mathbb{C}[M\cap {}^xN]}{\otimes} \mathbb{C}[xN] \to \mathbb{C}[MxN]$$

by the formula $f \otimes g \mapsto fg$ (product in $\mathbb{C}G$). Suppose now that $l \in \mathbb{C}[M \cap {}^xN]$. Then $\varphi(fl \otimes g) = flg = \varphi(f \otimes lg)$. So φ is a balanced map by associativity of multiplication in $\mathbb{C}G$. The same reasoning tells us that φ is a map of $(\mathbb{C}M, \mathbb{C}N)$-bimodules.

Let $mxn \in \mathbb{C}[MxN]$ be a basis element. Then $m \otimes xn \mapsto mxn$, proving that φ is surjective. Now we'll prove that

$$\dim(\mathbb{C}M \underset{\mathbb{C}[M\cap {}^xN]}{\otimes} \mathbb{C}[xN]) = |MxN| = \dim(\mathbb{C}[MxN]).$$

Using 2.3.3, we must only show that $|M/(M \cap {}^xN)||N| = |MxN|$. Consider $M \times N$ acting on G via $(m,n) \cdot g := mgn^{-1}$. Then Mxn is the orbit of x. By the Orbit Stabilizer Theorem (see [8, Proposition IV.4.34]), $M \times N/\operatorname{stab}(x) \cong MxN$ as $M \times N$-sets. We may deduce that

$$|MxN| = |(M \times N)/\operatorname{stab}(x)| = \frac{|M||N|}{|\operatorname{stab}(x)|}.$$

Now note that $(m,n) \in \operatorname{stab}(x)$ if and only if $m = xnx^{-1}$, and so the map $M \cap {}^xN \to \operatorname{stab}(x)$ given by $m \mapsto (m, x^{-1}mx)$ is a bijection. The existence of such a bijection tells us that

$$|MxN| = \frac{|M|}{|M \cap {}^xN|}|N|,$$

and thus φ is an isomorphism. □

Lemma 2.3.5. *As representations of $M \cap {}^xN$, we have $\mathbb{C}[xN] \otimes_{\mathbb{C}N} W \cong \mathrm{Ad}^*_{x^{-1}} \circ \mathrm{Res}^N_{{}^{x^{-1}}M\cap N} W$.*

Proof. Define the map $\varphi : \mathbb{C}[xN] \otimes_{\mathbb{C}N} W \to Ad^*_{x^{-1}} \circ \mathrm{Res}^N_{{}^{x^{-1}}M\cap N} W$ by $xn \otimes w \mapsto nw$. First note that φ is balanced over $\mathbb{C}N$ since multiplication in $\mathbb{C}G$ is associative. Next, let $xnx^{-1} \in M \cap {}^xN$. Then, for $xn' \in xN$, we have $\varphi(xnx^{-1}(xn' \otimes w)) = \varphi(xnn' \otimes w) = nn'w = n\varphi(xn' \otimes w)$ so φ is a morphism of $M \cap {}^xN$-representations.

Suppose $w \in W$. Then $\varphi(x1 \otimes w) = w$ and thus φ is onto. To prove injectivity we will use the following three facts:

- $\mathbb{C}[xN] \cong \mathbb{C}N$ as right $\mathbb{C}N$ modules. The map $\psi : \mathbb{C}[xN] \to \mathbb{C}N$ given by $xn \mapsto n$ is easily seen to be an isomorphism.

- $\mathbb{C}N \underset{\mathbb{C}N}{\otimes} W \cong W$ as left $\mathbb{C}N$-modules (representations of N). By our tensor product formulas for the functors Ind and Res, we know this is rephrasing the statement that $\mathrm{Ind}^N_N W \cong W$ which is clear.

□ $\dim(\mathbb{C}[xN] \underset{\mathbb{C}N}{\otimes} W) = \dim(W)$. This follows immediately from the above facts.

It follows that the domain and codomain of the survective map φ have equal dimension, and so φ is injective and thus an isomorphism. □

With these few lemmas in hand, we're now in a position to prove Mackey's Theorem.

Proof of Theorem 2.3.1. As stated before, $\text{Res}^G_M \text{Ind}^G_N W \cong \mathbb{C}G \otimes_{\mathbb{C}N} W$. We may rewrite this as

$$\bigoplus_{MxN \in M\backslash G/N} \mathbb{C}[MxN] \underset{\mathbb{C}N}{\otimes} W.$$

Next, use 2.3.4 to express this module as

$$\bigoplus_{MxN \in M\backslash G/N} \mathbb{C}M \underset{\mathbb{C}[M \cap {}^x N]}{\otimes} \mathbb{C}[xN] \otimes_{\mathbb{C}N} W.$$

Finally, by 2.3.5, this is isomorphic to

$$\bigoplus_{MxN \in M\backslash G/N} \mathbb{C}M \underset{\mathbb{C}[M \cap {}^x N]}{\otimes} Ad^*_{x^{-1}} \circ \text{Res}^N_{{x^{-1}}_{M \cap N}} W.$$

finally, 2.2.9 allows us to rewrite this as

$$\bigoplus_{MxN \in M\backslash G/N} \text{Ind}^M_{M \cap {}^x N} \circ \text{Ad}^*_{x^{-1}} \circ \text{Res}^N_{x^{-1}} W.$$

□

2.4 Generalized Induction and Restriction

In this section we will see generalizations of the Ind and Res functors. Along with these functors will come formulas and a Mackey theorem similar to those above. And, like before, these formulas will be the cornerstone of the computations with these new functors. We will define the multiplication and comultiplication structures of our Hopf algebra in section 3.3. Here we follow closely the text [11, section 1.1], filling in gaps as we go.

Suppose G is a finite group, and let $M, U \subseteq G$ be subgroups of G such that M normalizes U, i.e., $mum^{-1} \in U$ for all $m \in M$ and $u \in U$. Suppose also that $M \cap U = 1$. Under these assumptions, $P := MU = \{mu | m \in M, u \in U\}$ is a subgroup of G since
$(m_1u_1)(m_2u_2) = m_1m_2(m_2^{-1}u_1m_2)u_2$.

Definition 2.4.1. Let $\rho : M \to \mathrm{GL}(V)$ be a representation of M. Extend ρ to a representation ρ' of $P := MU$ on V via $\rho'(mu) := \rho(m)$. Define the functor $i_U : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ by $\rho \mapsto \mathrm{Ind}_P^G(\rho')$.

Lemma 2.4.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and let $V^U = \{v \in V | uv = v \text{ for all } u \in U\}$. Then V^U is M-invariant.*

Proof. Indeed, let $v \in V^U$, $m \in M$, $u \in U$. Then we have $umv = mm^{-1}umv = mm^{-1}umv$. Recall that M normalizes U and v is U-invariant; this last expression is thus equal to $muv = mv$. □

Definition 2.4.3. Let $(\rho, V) \in \mathrm{Rep}(G)$. Define the functor $r_U : \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ by $(\rho, V) \mapsto (\rho|_M, V^U)$.

Remark 2.4.4. When $U = \{1\}$, the functors r_U and i_U are the usual restriction and induction functors Res_M^G and Ind_M^G. In this case, $P = M$ and $V^U = V$.

Lemma 2.4.5. *Let M, U, and G be as above. Suppose N, $V \subseteq M$ are subgroups such that N normalizes V, and $N \cap V = 1$. Then the functors $i_V : \mathrm{Rep}(N) \to \mathrm{Rep}(M)$ and $r_V : \mathrm{Rep}(M) \to \mathrm{Rep}(N)$ exist and are defined as before. The equalities $i_U \circ i_V = i_{UV}$ and $r_V \circ r_U = r_{UV}$ hold.*

Proof. Let $(\rho, W) \in \mathrm{Rep}(N)$, and set $Q := NV$. Then we define $(\rho', W) \in \mathrm{Rep}(Q)$ via $\rho'(nv) := \varphi(n)$. We know that $i_V(\rho) = \mathrm{Ind}_Q^M(\rho') \cong \mathbb{C}M \otimes_{\mathbb{C}Q} W$.

Call $\overline{\rho} = i_V(\rho)$. Then $\overline{\rho}' \in \mathrm{Rep}(P)$ is given by a definition analogous to the above. So now $i_U(\overline{\rho}) = \mathrm{Ind}_P^G(\overline{\rho}') \cong \mathbb{C}G \otimes_{\mathbb{C}P} \mathrm{Ind}_Q^M(\rho')$ which we know is isomorphic to $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}M \otimes_{\mathbb{C}Q} W$.

Now define $(\tilde{\rho}, W) \in \mathrm{Rep}(NUV)$ by the formula $\tilde{\rho}(nuv) := \rho(n)$. Then we have $i_{UV}(\rho) = \mathrm{Ind}_{NUV}^G(\tilde{\rho})$ which is isomorphic to $\mathbb{C}G \otimes_{\mathbb{C}NUV} W$. Now define the map $\psi : \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}M \otimes_{\mathbb{C}Q} W \to \mathbb{C}G \otimes_{\mathbb{C}NUV} W$ by $g \otimes m \otimes w \mapsto gm \otimes w$. This map is balanced over $\mathbb{C}P$ since multiplication is associative in $\mathbb{C}G$. It is balanced over $\mathbb{C}Q$ since its codomain is balanced over $\mathbb{C}NUV \supseteq \mathbb{C}Q$.

The map ψ is evidently surjective. Now observe that the dimension of $\mathbb{C}G \otimes_{\mathbb{C}NUV} W$ is equal to $\frac{|G|}{|N||U||V|}\dim(W)$. The dimension of $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}M \otimes_{\mathbb{C}Q} W$ is

$$\frac{|G|}{|M||U|}\frac{|M|}{|N||V|}\dim(W) = \frac{|G|}{|N||U||V|}\dim(W).$$

Thus ψ is an isomorphism, so $i_U \circ i_V = i_{UV}$.

It remains to show that $r_V \circ r_U = r_{UV}$. Observe that $r_V(r_U(W)) = (W^U)^V = W^{UV}$. □

Now the tensor product formulas for i and r. From now on $M, U \subseteq G$ are as above unless otherwise specified.

Proposition 2.4.6. *Let*

$$e_U := \frac{1}{|U|}\sum_{u \in U} u \in \mathbb{C}G.$$

Then $\mathbb{C}Ge_U$ is a $(\mathbb{C}G, \mathbb{C}M)$-bimodule, and the functor i_U is given by $i_U(V) \cong \mathbb{C}Ge_U \underset{\mathbb{C}M}{\otimes} V$.

Lemma 2.4.7. *1. For any $m \in M$, $me_U = e_U m$.*

2. For any representation V of G, the map $e_U : v \to V$ given by $v \mapsto e_U v$ is the projection of V onto V^U.

Proof. For (1), we compute directly: $me_U = m\frac{1}{|U|}\sum_u u = \frac{1}{|U|}\sum_u mu = \frac{1}{|U|}\sum_u mum^{-1}m$. Since M normalizes U, this last expression equals $\frac{1}{|U|}\sum_u um = e_U m$.

(2) follows from the observation that the map e_U is idempotent with image V^U. □

Proposition 2.4.8. *If $E \in \mathrm{Rep}(G)$, then $r_U(E) = E^U$ is isomorphic to $\mathrm{Hom}_G(\mathbb{C}Ge_U, E)$ as representations of M.*

Proof. Define $\varphi : \mathrm{Hom}_G(\mathbb{C}Ge_U, V) \to V^U$ by $T \mapsto Te_U$. First we must show this map is well-defined. That is, that $Te_U \in V^U$. Let $u \in U$. Then
$uT(e_U) = T(ue_U) = T(\frac{1}{|U|}\sum_{u_0 \in U} uu_0) = Te_U$, as desired. It is also easy to see that φ is linear.

Now we'll show that φ is a morphism of M representations. Let $m \in M$. Then $\varphi(mT) = (mT)(e_U) = T(e_U m) = T(me_U) = m(Te_U) = m(\varphi(T))$, as needed.

Now suppose that $Te_U = 0$. Then for all $k \in G$, $T(ke_U) = kTe_U = k0 = 0$. Thus $T = 0$, so φ is injective. To see that φ is surjective, suppose $v \in V^U$. Define a G-equivariant map $S : \mathbb{C}Ge_U \to V$ by $S(fe_U) := fe_Uv$. Then S is G-equivariant by the associativity of multiplication in $\mathbb{C}G$. Now note that $\varphi(S) = S(e_U) = e_Uv = v$, since v is U-invariant. Thus φ is an isomorphism of representations of M. □

Proposition 2.4.9. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then V^U is isomorphic to $[e_U\mathbb{C}G] \otimes_{\mathbb{C}G} V$ as representations of M.*

Proof. Define the map $\varphi : [e_U\mathbb{C}G] \otimes_{\mathbb{C}G} V \to V^U$ by $\varphi : e_Uf \otimes v \mapsto e_Ufv$. To see that φ is balanced over $\mathbb{C}G$, observe that $\varphi((e_Ufg) \otimes v) = (e_Ufg)v = (e_Uf)(gv)$ which is equal to $\varphi(e_Uf \otimes gv)$.

Now define $\psi : V \to [e_U\mathbb{C}G] \otimes_{\mathbb{C}G} V$ by $\psi(V) := e_U \otimes v$. Then

$$\varphi(\psi(v)) = \varphi(e_U \otimes v) = e_Uv = v,$$

and

$$\psi(\varphi(e_Uf \otimes v)) = \psi(e_Ufv) = e_U \otimes e_Ufv = {e_U}^2f \otimes v = e_Uf \otimes v.$$

Thus φ and ψ are mutually inverse isomorphisms of vector spaces. The M-equivariance of the map φ is clear. □

Proposition 2.4.10. *The functors i_U is left-adjoint to r_U; that is,*
$\mathrm{Hom}_G(i_U(V), W) \cong \mathrm{Hom}_M(V, r_U(W))$

Proof. By 2.4.6 and 2.4.8, this fact follows from the tensor-Hom adjunction (2.2.1). □

Now consider the case where $G = MU$. In this case $\mathrm{Ind}_{MU}^G = \mathrm{id}$ and $\mathrm{Res}_{MU}^G = \mathrm{id}$. What we've shown is that inflating a representation of M to one of MU by defining U to act trivially is adjoint to taking unvariants under U. In symbols, let $\mathrm{Infl}_M^{MU} : \mathrm{Rep}(M) \to \mathrm{Rep}(MU)$ be given by $\rho \mapsto \rho'$ as above. Also define $\mathrm{Inv}_M^{MU} : \mathrm{Rep}(MU) \to \mathrm{Rep}(M)$ by $V \mapsto V^U$. By restricting the above arguments to the $G = MU$ case, we've shown that Infl is a one-sided adjoint to Inv.

2.5 Another Mackey Theorem

In order to continue, we'll need a Mackey-style formula for the composition for the i and r functors defined in the previous section. This result is stated in [11, A.3.1] as a special case of [2, § 5], with a note to the reader that it may also be verified via a character computation. This character computation is what follows.

Recall that by 2.1.18 the irreducible characters of G form a basis for the complex vector space of class functions on G; i.e., the irreducible characters are linearly independent and span

$$Cl(G) = \{f : G \to \mathbb{C} | f(gxg^{-1}) = f(x) \text{ for all } x, g \in G\}.$$

By the correspondence between irreducible characters of G and irreducible representations of G, we may consider $\mathrm{R}(G) = \mathbb{Z}\,\mathrm{Irr}(G)$ as an additive subgroup of $Cl(G)$. If $F : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is a functor that commutes with direct sums, then F induces a linear map $Cl(G) \to Cl(H)$. We now compute the linear maps on class functions induced by the functors i_U and r_U.

Let $M, U, N, V \subseteq G$ be subgroups satisfying $M \cap U = \{1\}$ and $N \cap V = \{1\}$, with M and N normalizing U and V, respectively.

Proposition 2.5.1. *The map* $r_U : Cl(G) \to Cl(M)$ *induced by the functor* $r_U : \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ *is given by*

$$r_U(f)(m) = \frac{1}{|U|} \sum_{u \in U} f(mu).$$

Proof. Let $(\rho, V) \in \mathrm{Rep}(G)$. Then by part 2 of 2.4.7, e_U is a projection onto V^U. If χ_V is the character of V and χ_{V^U} is the character of V, then

$$\chi_{E^U}(m) = \mathrm{Tr}(\pi(m)\big|_{E^U}) = \mathrm{Tr}(\rho(m)\pi(e_U)) = \mathrm{Tr}\left(\pi(m)\frac{1}{|U|}\sum_u \pi(u)\right)$$
$$= \frac{1}{|U|}\sum_u \mathrm{Tr}(\pi(mu)) = \frac{1}{|U|}\sum_u \chi_E(mu).$$

□

To prove the corresponding forumla for i, we'll first need a result known as the Frobenius character formula.

Theorem 2.5.2 (Frobenius character formula)**.** *Let χ be the character of $\operatorname{Ind}_H^G V$, and χ_V the character of V. Then*

$$\chi(g) = \sum_{\substack{Hx \in H\backslash G \\ xgx^{-1} \in H}} \chi_V(xgx^{-1}).$$

Proof. For each coset Hx, define $V_x = \{f \in \operatorname{Ind}_H^G V | f(g) = 0 \text{ for each } g \notin Hx\}$. To understand these subspaces, let $y \in Hx$, so that $y = hx$ for some $h \in H$. Then for $f \in \operatorname{Ind}_H^G V$, we have $f(y) = f(hx) = hf(x)$; so values of f on particular cosets are determined by values of f on a single representative of the coset. Thus let $\{v_1, \ldots, v_m\}$ be a basis for V, and let $X \subseteq G$ be a complete set of representatives for the cosets $H\backslash G$. Define for each coset Hx the function $\delta_{x,i} \in \operatorname{Ind}_H^G V$ by

$$\delta_{x,i}(y) = \begin{cases} hv_i & \text{if } y = hx \text{ for some } h \in G \\ 0 & \text{else.} \end{cases}$$

The set $\{\delta_{x,i} | x \in X, i = 1, \ldots, m\}$ is a basis for $\operatorname{Ind}_H^G V$. It follows that $\operatorname{Ind}_H^G V = \bigoplus_{x \in X} V_x$. Hence $\chi(g) = \sum_{x \in X} \operatorname{Tr}|_{V_x}(g)$, where $\operatorname{Tr}|_{V_x}$ denotes taking the trace of g as an operator on the space V_x. Now let $g \in G$, and observe that $g \cdot \delta_{x,i}(x) = \delta_{x,i}(xg) = 0$ if $xg \notin Hx$. Hence $\operatorname{Tr}|_{V_x}(g) = 0$ in this case.

In the case that $xg \in Hx$, we have $xg = hx$ for some h. Hence $h = xgx^{-1} \in H$. Now define the function $\alpha : V_x \to V$ given by $\alpha(f) := f(x)$. Considering that the functions $\delta_{x,i}$ span V_x and are determined by their values on x, it follows that α is an isomorphism. Next, observe that

$$\alpha(g \cdot f) = g \cdot f(x) = f(xg) = f(hx) = hf(x) = h\alpha(f).$$

Since α is invertible, this says that $g \cdot f = \alpha^{-1} h \alpha(f)$. Using these facts and the property $\operatorname{Tr}(bab^{-1} = \operatorname{Tr}(a)$, we see that $\operatorname{Tr}|_{V_x}(g) = \operatorname{Tr}|_{V_x}(\alpha^{-1} h \alpha) = \operatorname{Tr}|_V(h)$. Thus we have

$$\chi(g) = \sum_{\substack{Hx \in H\backslash G \\ xgx^{-1} \in H}} \chi_V(xgx^{-1})$$

as claimed. ☐

Now recall that in the definition of the i_U functor, we pull back a representation of M to a representation of MU along the homomorphism given by projection $\mu : MU \twoheadrightarrow M$. Thus the element mu acts as the element m. So if V is a representation of M with character χ_1, and we also consider V as a representation of MU with character χ_2, we have
$\chi_2(mu) = \chi_1(m) = \chi_1(\mu(mu))$.

Proposition 2.5.3. *Let $f \in Cl(M)$. We have*

$$i_U(f)(g) = \sum_{\substack{MUx \in MU \backslash G \\ xgx^{-1} \in MU}} f(\mu(xgx^{-1}))$$

for all $g \in G$.

Proof. The functor i_U is defined as the pullpack along $\mu : MU \twoheadrightarrow M$ followed by induction from MU to G. This follows immediately from the Frobenius formula and the above comments about pulling back along the projection μ. ☐

Definition 2.5.4. Let M and U be subgroups of a group G, such that M normalizes U and $M \cap U = \{1\}$. We say a subgroup H of G is *decomposable with respect to* (M, U) if $H \cap (MU) = (H \cap M)(H \cap U)$.

Let M, N, U, and V be subgroups of a group G with M normalizing U and N normalizing V, and satisfying $N \cap V = M \cap U = \{1\}$. Define also $P := MU$ and $Q := NV$. Let $W \subseteq G$ be a complete set of double coset representatives for $Q \backslash G / P$, and assume that for each $w \in W$, the groups ${}^wP, {}^wM$, and ${}^wU$ are decomposable with respect to (N, V), and ${}^{w^{-1}}Q, {}^{w^{-1}}N$, and ${}^{w^{-1}}V$ are decomposable with respect to (M, U).

With the above data, we may define functors $i_{N \cap {}^wU} : \mathrm{Rep}({}^wM \cap N) \to \mathrm{Rep}(N)$, $\mathrm{Ad}^*_{w^{-1}} : \mathrm{Rep}(M \cap {}^{w^{-1}}N) \to \mathrm{Rep}({}^wM \cap N)$, and $r_{M \cap {}^{w^{-1}}V} : \mathrm{Rep}(M) \to \mathrm{Rep}(M \cap {}^{w^{-1}}N)$.

Theorem 2.5.5. *Let M, U, N, V, and G be as above. Then*

$$r_V \circ i_U = \bigoplus_{\substack{NVwMU \\ \in NV \backslash G / MU}} i_{N \cap {}^wU} \circ Ad^*_{w^{-1}} \circ r_{M \cap {}^{w^{-1}}V}$$

as maps $Cl(M) \to Cl(N)$.

We will again prove this theorem via a series of lemmas. This time, though, we will use the lemmas to reindex and rewrite sums to achieve our goal.

Lemma 2.5.6. *For each $w \in W$, the map*

$$\psi : Q/(Q \cap {}^wP) \to QwP/P, \;\; q[Q \cap {}^wP] \mapsto qwP$$

is a bijection.

The set QwP is not a group, but a subset of G on which P acts by right multiplication. So QwP/P denotes the set of orbits of this action.

Proof. First we must show ψ is well-defined. Indeed, suppose

$$q_1[Q \cap {}^wP] = q_2[Q \cap {}^wP].$$

Then $q_1 = q_2wpw^{-1}$ for some $p \in P$. Thus $q_1wP = q_2wpw^{-1}wP = q_2wpP = q_2wP$.

In order to prove ψ is injective, suppose now that $q_1wP = q_2wP$. Then $q_1wp = q_2w$ for some $p \in P$; i.e. $wpw^{-1} = q_1^{-1}q_2$. So, $q_1^{-1}q_2 \in Q \cap {}^wP$ and therefore $q_1[Q \cap {}^wP] = q_2[Q \cap {}^wP]$.

To see that ψ is surjective, let $qwpP = qwP \in QwP/P$. Then $q[Q \cap {}^wP] \mapsto qwP$, as desired. □

Lemma 2.5.7. *The map $\psi : N/(N \cap^w P) \times (V/(V \cap {}^wP) \to Q/(Q \cap {}^wP)$ given by $(\eta[N \cap {}^wP], v[V \cap {}^wP]) \mapsto \eta v[Q \cap {}^wP]$ is a bijection.*

Proof. We will prove that the map ψ is well-defined and injective in the same way. Suppose $\eta_1 v_1[N \cap {}^wP] = \eta_2 v_2[N \cap {}^wP]$. This is equivalent to saying $v_2^{-1}\eta_2\eta_1 v_1 \in N \cap {}^wP$, which, again, is equivalent to the statement that

$$(\eta_2^{-1}\eta_1^{-1})(\eta_2^{-1}\eta_1^{-1})^{-1}v_2^{-1}(\eta_2\eta_1)v_1 \in Q \cap {}^wP.$$

This quantity $\eta_2^{-1}\eta_1^{-1}$ is obviously in N and, since N normalizes V, the product $(\eta_2^{-1}\eta_1^{-1})^{-1}v_2^{-1}(\eta_2\eta_1)v_1$ lies in V. By our decomposability hypothesis, we know

$$Q \cap {}^wP = (N \cap {}^wP)(V \cap^w P).$$

The above equation is thus equivalent to the statement that $\eta_2^{-1}\eta_1 \in N \cap {}^wP$ and $(\eta_2^{-1}\eta_1^{-1})^{-1}v_2^{-1}(\eta_2\eta_1)v_1 \in V \cap {}^wP$. This is true if and only if $\eta_1 = \eta_2$ and $v_1[V \cap {}^wP] = v_2[v \cap {}^wP]$.

The surjectivity of this map should be obvious: if $q[Q \cap {}^wP] = nv[Q \cap {}^wP] \in Q/(Q \cap {}^wP)$, then $(n[Q \cap {}^wP], v[V \cap {}^wP]) \mapsto nv[Q \cap {}^wP]$, as desired. □

Lemma 2.5.8. *Fix $w \in W$ and let $\lambda : N \cap {}^wP \twoheadrightarrow N \cap {}^wM$ be the projection whose existence is guaranteed by the equality $N \cap {}^wP = (N \cap {}^wM)(N \cap {}^wU)$. Then $\mu(w^{-1}gw) = w^{-1}\lambda(g)w$ for all $g \in N \cap {}^wP$.*

Proof. Suppose $g = wmuw^{-1} = n$. Then $\mu(w^{-1}gw) = \mu(w^{-1}(wmuw^{-1})w) = \mu(mu) = m$. On the other hand, by definition, $\lambda((wmw^{-1})(wuw^{-1})) = wmw^{-1}$. Hence $w^{-1}\lambda(g)w = w^{-1}wmw^{-1}w = m$, as desired. □

We're now in a position to prove Theorem 2.5.5.

Proof of 2.5.5. Let $f \in Cl(M)$ and let $n \in N$. Then we may use 2.5.1 and 2.5.3 to compute

$$r_V \circ i_U(f)(n) = \frac{1}{|V|}\sum_{v \in V} i_U(f)(nv) = \frac{1}{|V|}\sum_{v \in V}\sum_{\substack{xP \in G/P: \\ x^{-1}nvx \in P}} f(\mu(x^{-1}nvx)).$$

Note that the double cosets of G by Q and P partition G; thus we may rewrite this sum as

$$\sum_{w \in W}\sum_{xP \in QwP/P}\frac{1}{|V|}\sum_{\substack{v \in V: \\ x^{-1}nvx \in P}} f(\mu(x^{-1}nvx)).$$

Fixing a single w, let us consider the corresponding summand:

$$f_w(n) = \sum_{xP \in QwP/P}\frac{1}{|V|}\sum_{\substack{v \in V: \\ x^{-1}nvx \in P}} f(\mu(x^{-1}nvx)).$$

By Lemma 2.5.6 we may rewrite $f_w(n)$ as

$$f_w(n) = \sum_{\substack{q[Q \cap {}^wP] \\ \in Q/(Q \cap {}^wP)}}\frac{1}{|V|}\sum_{\substack{v \in V: \\ w^{-1}q^{-1}nvqw \in P}} f(\mu(w^{-1}q^{-1}nvqw)).$$

We may rewrite the condition $w^{-1}q^{-1}nvqw \in P$ as $q^{-1}nvq \in Q \cap {}^wP$. Set $q = \eta y$ with $\eta[N \cap {}^wP] \in N/[N \cap {}^wP]$ and $y[V \cap {}^wP] \in V/(V \cap {}^wP)$. We may now rewrite $f_w(n)$ using Lemma 2.5.7 to replace the first sum with two sums:

$$f_w(n) = \frac{1}{|V|} \sum_{\eta \in N/(N \cap {}^wP)} \sum_{y \in V/(V \cap {}^wP)} \sum_{\substack{v \in V: \\ y^{-1}\eta^{-1}nv\eta y \in Q \cap {}^wP}} f(\mu(w^{-1}y^{-1}\eta^{-1}nv\eta yw)).$$

We will now make the following change of variables:

$$v_{\text{new}} = [(\eta^{-1}n\eta)^{-1}y^{-1}(\eta^{-1}n\eta)] \cdot \eta^{-1}v_{\text{old}}\eta \cdot y$$

(This is a valid change of variables: that is, the mapping $v_{\text{old}} \mapsto v_{\text{new}}$ is a permutation of V. Indeed, recall that N normalizes V, and observe that $\eta^{-1}n\eta \in N$; thus $(\eta^{-1}n\eta)^{-1}y^{-1}(\eta^{-1}n\eta) \in V$. Note also that $\eta^{-1}v_{old}\eta \in V$). Now

$$\eta^{-1}n^{-1}\eta v_{\text{new}} = y^{-1}(\eta^{-1}n\eta) \cdot \eta^{-1}v_{\text{old}}\eta \cdot y$$

Therefore with this substitution the argument of μ becomes $w^{-1}\eta^{-1}n\eta vw$, which is constant in y. This allows us to replace the $\sum_y$ with $|V/(V \cap {}^wP)| = \frac{|V|}{|V \cap {}^wP|}$. Additionally, the condition $y^{-1}\eta^{-1}nv\eta y \in Q \cap {}^wP$ becomes $\eta^{-1}n\eta v \in N \cap {}^wP$. Our formula for $f_w(n)$ thus simplifies to

$$f_w(n) = \frac{1}{|V \cap^w P|} \sum_{\eta \in N/(N \cap {}^wP)} \sum_{\substack{v \in V: \\ \eta^{-1}n\eta v \in Q \cap {}^wP}} f(\mu(w^{-1}\eta^{-1}n\eta vw)).$$

Recalling our decomposability assumption, we know that $\eta^{-1}n\eta v \in Q \cap {}^wP$ if and only if $\eta^{-1}n\eta \in N \cap {}^wP$ and $v \in V \cap {}^wP$. Thus we may write

$$f_w(n) = \frac{1}{|V \cap {}^wP|} \sum_{\eta \in N/(N \cap {}^wP)} \sum_{\substack{v \in V \cap {}^wP \\ \eta^{-1}n\eta \in N \cap {}^wP}} f(\mu(w^{-1}\eta^{-1}n\eta vw)).$$

Now, since μ is a homomorphism, we know that

$$\mu(w^{-1}\eta^{-1}n\eta vw) = \mu(w^{-1}\eta^{-1}n\eta w \cdot w^{-1}vw) = \mu(w^{-1}\eta^{-1}n\eta w)\mu(w^{-1}vw).$$

Again by the decomposition hypothesis we may write $v \in V \cap {}^wP$ as xy with $x \in V \cap {}^wM$ and $y \in V \cap {}^wU$. Thus we have $\mu(w^{-1}vw) = \mu(w^{-1}xw \cdot w^{-1}yw) = w^{-1}xw$, by the definition of μ.

So the $V \cap {}^wU$ part of our sum is constant. We may therefore write

$$f_w(n) = \frac{1}{|V \cap {}^wM|} \sum_{\substack{\eta \in N/(N \cap {}^wP) \\ \eta^{-1}n\eta \in N \cap {}^wP}} \sum_{v \in V \cap {}^wM} f(\mu(w^{-1}\eta^{-1}n\eta w) \cdot \mu(w^{-1}vw)).$$

Now recall that $v \in V \cap {}^wM$. Call $x := w^{-1}vw \in {}^{w^{-1}}V \cap M$. Let $\lambda : N \cap {}^wP \twoheadrightarrow N \cap {}^wM$ be the projection as in Lemma 2.5.8. According to that lemma, we may rewrite $f_w(n)$ once again as

$$f_w(n) = \sum_{\eta \in N/(N \cap {}^wP)} \frac{1}{|M \cap {}^{w^{-1}}V|} \sum_{\substack{x \in M \cap {}^{w^{-1}}V \\ \eta^{-1}n\eta \in N \cap {}^wP}} f(w^{-1}\lambda(\eta^{-1}n\eta)w \cdot x)). \tag{2.5.9}$$

We will now calculate the image of f under the composition

$$Cl(M) \xrightarrow{r_{M \cap w^{-1}V}} Cl(M \cap {}^{w^{-1}}N) \xrightarrow{Ad_w} Cl(N \cap^w M) \xrightarrow{i_{N \cap {}^wU}} Cl(N).$$

Observe that 2.5.3 gives

$$i_{N \cap {}^wU} \circ Ad_w \circ r_{M \cap {}^{w^{-1}}V}(f)(n) = \sum_{\substack{\eta \in N/(N \cap {}^wP) \\ \eta^{-1}n\eta \in N \cap {}^wP}} Ad_w \circ r_{M \cap {}^{w^{-1}}V}(f)(\lambda(\eta^{-1}n\eta)).$$

We may rewrite this sum first as

$$\sum_{\substack{\eta \in N/(N \cap {}^wP) \\ \eta^{-1}n\eta \in N \cap {}^wP}} r_{M \cap {}^{w^{-1}}V}(f)(w^{-1}\lambda(\eta^{-1}n\eta)w),$$

and finally, using 2.5.1, as

$$\sum_{\substack{\eta \in N/(N \cap {}^wP) \\ \eta^{-1}n\eta \in N \cap {}^wP}} \frac{1}{|M \cap {}^{w^{-1}}V|} \sum_{x \in M \cap {}^{w^{-1}}V} f(w^{-1}\lambda(\eta^{-1}n\eta)w \cdot x).$$

Comparing this expression to 2.5.9 then proves our desired result; that is

$$r_V \circ i_U = \bigoplus_w i_{N \cap {}^wU} \circ Ad^*_{w^{-1}} \circ r_{M \cap {}^{w^{-1}}V}. \qquad \square$$

CHAPTER 3

THE HOPF ALGEBRA $R(S)$

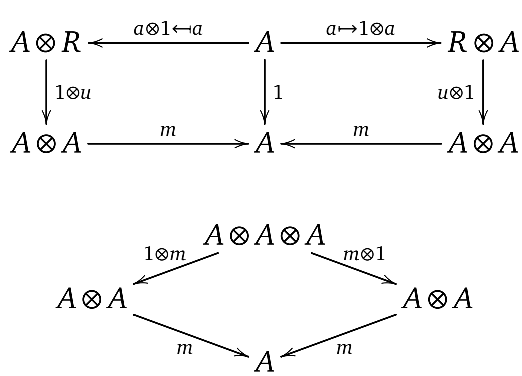
3.1 Overview

In the present chapter we begin to explore the representation theory of the symmetric groups, following the work of Zelevinsky in [11]. We will define the algebra and coalgebra R(S) of virtual representations of the family S_n. The construction of the structures will use the language of the functors r and i for convenience, but in this case what we're really using are the functors Res and Ind.

3.2 Algebras and Coalgebras

We will define (co)algebras in terms of diagrams, not elements. All rings will be assumed to have a unit.

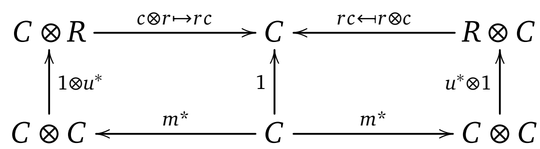
Definition 3.2.1. An *algebra* over a commutative ring R consists of an R-module A along with maps $u : R \to A$ and $m : A \otimes_R A \xrightarrow{a\otimes b \mapsto ab} R$ such that the following diagrams commute:

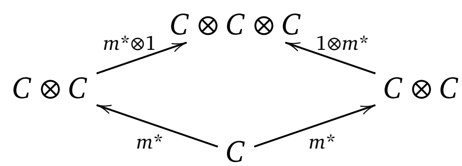


We call the element $u(1) \in A$ the *unit* of A.

In terms of elements, these diagrams express the axioms $1a = a1 = a$ and $(ab)c = a(bc)$, respectively. The reason for expressing these properties with diagrams instead of elements is so that we may more easily define the dual structure by reversing arrows as follows.

Definition 3.2.2. A *coalgebra* over a commutative ring R is an R-module C along with maps $u^* : C \to R$ and $m^* : C \to C \otimes C$ such that the following diagrams commute:





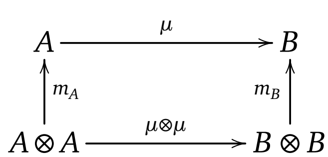
Remark 3.2.3. Based on the above diagrams, algebras and coalgebras deserve their names, relative to one another. The diagrams are opposites of one another.

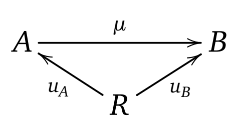
Remark 3.2.4. The lower diagram in the (co)algebra definition will be referred to as the (co)associativity condition.

Example 3.2.5. Let A be an algebra. Then we may endow $A \otimes A$ with the structure of an algebra by defining multiplication componentwise. That is, define

$$m_{A\otimes A} : A \otimes A \otimes A \otimes A \xrightarrow{a\otimes b\otimes c\otimes d \mapsto a\otimes c\otimes b\otimes d} A \otimes A \otimes A \otimes A \xrightarrow{m\otimes m} A \otimes A.$$

Definition 3.2.6. In this new context, a *morphism* of algebras $\mu : A \to B$ is a function $A \to B$ making the following diagrams commute:

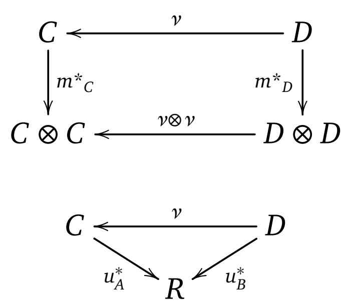




Remark 3.2.7. This definition of a morphism reduces to the usual definition when we consider specific elements. That is, if $\mu : A \to B$ is a morphism of algebras, then

$\mu(a_1a_2)=\mu(a_1)\mu(a_2)$, and μ preserves the unit. A morphism of coalgebras must make the opposite diagrams commute.

Definition 3.2.8. A *morphism of coalgebras* is a map $\nu : C \to D$ which makes the following diagrams commute:



The algebras and coalgebra we will study are *graded*.

Definition 3.2.9. 1. An algebra A over R is *graded* if for each positive integer n there is an R-submodule A_n, such that $A=\oplus_{n\geq 0}A_n$; further, for $a\in A_k$ and $a'\in A_l$, it must be that $aa'\in A_{k+l}$. For a particular n, elements of A_n are called *homogeneous* elements of degree n. Another way to express this fact is to say $m(A_k\otimes A_l)\subseteq A_{k+l}$.

2. A coalgebra C is *graded* if for each positive integer n there is an R-submodule C_n, such that $C=\oplus_{n\geq 0}C_n$; additionally, for $c\in C_n$, c must satisfy $m^*(c)\in\bigoplus_{n=k+l}C_k\otimes C_l$.

3.3 Algebra and Coalgebra Structures on R(S)

For each positive integer n, denote by S_n the symmetric group on n letters; that is, the group of permutations of $\{1,\ldots,n\}$. Consider the free abelian group with a basis consisting of isomorphism classes of irreducible representations of S_n; that is, define $\mathrm{R}(S_n):=\mathbb{Z}\,\mathrm{Irr}(S_n)$ (by convention, say S_0 is the trivial group). This is the $\mathbb{Z}$-module of virtual representations of S_n. We call nonnegative integral combinations of irreducibles positive elements of this module, and identify the collection thereof with the set of isomorphism classes in the category

$\mathrm{Rep}(S_n)$. An isomorphism class $[\rho] \in \mathrm{Irr}(S_n)$ is identified with the basis element $\rho \in \mathrm{R}(S_n)$. Define the abelian group

$$\mathrm{R}(S) := \bigoplus_{n \geq 0} \mathrm{R}(S_n)$$

as at the outset of chapter 3. We will prove that the functors Ind and Res give graded algebra and coalgebra structures for this group.

Suppose $k + l = n$. The stabilizer of $\{1, \ldots, k\} \subseteq \{1, \ldots, n\}$ is easily seen to be isomorphic to $S_k \times S_l$ via the canonical isomorphism

$$(\sigma, \tau) \mapsto \left(i \mapsto \begin{cases} \sigma(i) & \text{if } i \leq k \\ k + \tau(i - k) & \text{if } k < i \end{cases} \right).$$

We may thus regard $S_k \times S_l$ as a subgroup of S_n. Recall the bijection

$$\mathrm{Irr}(G) \times \mathrm{Irr}(H) \to \mathrm{Irr}(G \times H)$$

given by $(V, W) \mapsto V \otimes_{\mathbb{C}} W$ of Proposition 2.1.12. This allows us to make the identification $\mathrm{R}(S_k \times S_l) \cong \mathrm{R}(S_k) \otimes \mathrm{R}(S_l)$ via the bijection of basis elements $V \otimes_{\mathbb{C}} W \mapsto V \otimes_{\mathbb{Z}} W$ which sends tensor products of representations to elementary tensors in $\mathrm{R}(S_k) \otimes \mathrm{R}(S_l)$. The following definitions generalize this idea.

Definition 3.3.1. Let n be a positive integer. A *composition* of n is a tuple $\beta = (b_1, \ldots, b_s)$ satisfying $b_1, \ldots, b_s > 0$ and $\sum_i b_i = n$. Denote the set of compositions of n by $\mathscr{C}_n$.

Next, let $\gamma = (\gamma_i)_{i \in I}$ be a set-theoretic partition of $\{1, \ldots, n\}$. That is, $\gamma_i \subseteq \{1, \ldots, n\}$, $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$, and $\sqcup_i \gamma_i = \{1, \ldots, n\}$; the subsets γ_i are called the *blocks* of γ. We call γ a *set partition* of n. Say γ_1 and γ_2 are two set partitions of n such that each block of γ_1 is contained in some block of γ_2. Then we say $\gamma_1 \leq \gamma_2$. Denote by $\mathscr{P}_n$ the collection of all set partitions of n.

Example 3.3.2. There is a canonical injection $\Theta : \mathscr{C}_n \hookrightarrow \mathscr{P}_n$ given by sending a composition $(b_1, \ldots, b_s)$ to the set partition with blocks $\{1, \ldots, b_1\}, \{b_1 + 1, \ldots, b_1 + b_2\}, \ldots, \{n - b_s + 1, \ldots, n\}$. This allows us to consider compositions as set partitions.

On the collection $\mathscr{P}_n$ define the operation $\wedge$ by declaring $\alpha \wedge \beta$ to be the set partition whose blocks are the nonempty intersections $\alpha_i \cap \beta_j$. The group S_n acts on $\mathscr{P}_n$ in a natural way: the permutation σ sends $\gamma = (\gamma_i)$ to the set partition $\sigma\gamma = (\sigma(\gamma_i))$.

Set partitions interact with the symmetric groups in a predictable way, as the next proposition shows. For a set partition α of n, let S_α be the subgroup of S_n containing all permutations which restrict to bijections of each block of α. So $\sigma \in S_\alpha$ if and only if $\sigma|_{\alpha_i}$ is a permutation of α_i.

Proposition 3.3.3. *1. For any set partitions α and β of n, $S_\alpha \cap S_\beta = S_{\alpha\wedge\beta}$*

2. For each $w \in S_n$, $\alpha \leq \beta$ if and only if $w\alpha \leq w\beta$

3. For $w \in S_n$, $w(\alpha \wedge \beta) = w\alpha \wedge w\beta$

4. For $w \in S_n$, ${}^{w}S_\alpha = S_{w\alpha}$

5. For any set partition α of n, $S_\alpha \cong S_{|\alpha_1|} \times \cdots S_{|\alpha_r|}$

6. For set partitions $\alpha, \beta \in \mathscr{P}_n$, if $\alpha \leq \beta$, then S_α is a subgroup of S_β.

Proof. For (1), suppose $w \in S_n$ preserves all the α_i and the β_j. Then it preserves the $\alpha_i \cap \beta_j$. Hence $w \in S_{\alpha\wedge\beta}$. On the other hand, if w' preserves each set $\alpha_i \cap \beta_j$, then it evidently preserves each α_i and β_j individually, since $\alpha_i = \sqcup_j(\alpha_i \cap \beta_j)$ and $\beta_j = \sqcup_i(\alpha_i \cap \beta_j$.

For (2), suppose $\alpha \leq \beta$ and consider two blocks $\alpha_i \subseteq \beta_j$. Then $w(\alpha_i) \subseteq w(\beta_j)$, so $w\alpha \leq w\beta$. To prove the converse, use the same argument with w^{-1}.

For (3), consider one block $w(\alpha_i) \cap w(\beta_j)$ of $w\alpha \wedge w\beta$. We know that since w is a permutation, we have $w(\alpha_i) \cap w(\beta_j) = w(\alpha_i \cap \beta_j)$. So every block of $w\alpha \wedge w\beta$ is the image of some block of $\alpha \wedge \beta$. A similar argument using w^{-1} shows the converse.

To prove (4), let $w \in S_\alpha$. Then for $k \in w(\alpha_i)$, we have $w\sigma w^{-1}(k) \in w(\alpha_i)$ as well. Hence ${}^{w}S_\alpha \subseteq S_{w\alpha}$. Now suppose $\rho \in S_{w\alpha}$ and $l \in \alpha_j$. Then $w^{-1}\rho w(l) \in \alpha_j$, so $w^{-1}\rho w \in S_\alpha$, as desired.

To see why (5) is true, we begin by considering the set partition defined by the composition (k,l) the map $\varphi : S_k \times S_l \to S_{(k,l)}$ given by

$$(\sigma,\tau) \mapsto \left(i \mapsto \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq k \\ \tau(i-k)+k & \text{if } k+1 \leq i \leq k+l \end{cases} \right)$$

is easily seen to be an isomorphism. By induction this process is valid for all compositions. The case for set partitions is similar.

Finally, for (6), suppose $\sigma \in S_\alpha$. Then σ preserves the blocks of α, which are all subsets of blocks of β. Hence σ preserves the blocks of β as well. □

Definition 3.3.4. For any set partitions $\alpha \leq \beta$ of n, we define functors

$$i_\alpha^\beta : \mathrm{Rep}(S_\alpha) \to \mathrm{Rep}(S_\beta)$$

and

$$r_\alpha^\beta : \mathrm{Rep}(S_\beta) \to \mathrm{Rep}(S_\alpha)$$

by $i_\alpha^\beta = \mathrm{Ind}_{S_\alpha}^{S_\beta}$ and $r_\alpha^\beta = \mathrm{Res}_{S_\alpha}^{S_\beta}$. These functors induce $\mathbb{Z}$-linear maps

$$i_\alpha^\beta : \mathrm{R}(S_\alpha) \to \mathrm{R}(S_\beta)$$

defined on basis elements by $\rho \mapsto i_\alpha^\beta(\rho)$ and

$$r_\alpha^\beta : \mathrm{R}(S_\beta) \to \mathrm{R}(S_\alpha)$$

defined on basis elements by $\pi \mapsto r_\alpha^\beta(\pi)$.

Let $\pi \in \mathrm{Rep}(S_n)$ be some positive element of $\mathrm{R}(S)$. We define comultiplication on $\mathrm{R}(S)$ by

$$m^*(\pi) := \sum_{k+l=n} r_{k,l}^n(\pi).$$

Note that

$$m^*(\pi) \in \bigoplus_{n=k+l} \mathrm{R}(S_k) \otimes \mathrm{R}(S_l) \subseteq \mathrm{R}(S) \otimes \mathrm{R}(S).$$

To prove coassociativity, we need to show that $(m^* \otimes 1) \circ m^* = (1 \otimes m^*) \circ m^*$ as maps $\mathrm{R}(S) \to \mathrm{R}(S) \otimes \mathrm{R}(S) \otimes \mathrm{R}(S)$. Indeed, let π be a representation of S_n (i.e., a positive element of $\mathrm{R}(S_n)$. Then we may use the associativity of restriction (Lemma 2.2.11) and of tensor products to deduce that

$$\begin{aligned}(m^* \otimes 1) \circ m^*(\pi) &= \sum_{k+l=n} r^n_{k,l}(\pi) = \sum_{k+l=n} \sum_{a+b=k} r^n_{a,b,l}(\pi) \\ &\cong \sum_{k+l=n} \sum_{a+b=k} r^n_{l,a,b}(\pi) = (1 \otimes m^*) \circ m^*(\pi).\end{aligned}$$

The isomorphism is an equality when we move to the level of Grothendieck groups $\mathrm{R}(S_n)$.

Example 3.3.5. We will compute the comultiplication of the representation $\rho : S_3 \to \mathrm{GL}_2(\mathbb{C})$ of example 2.1.16. For each n, define triv_n to be the trivial representation of S_n, and sgn_n to be the sign representation of S_n; write $\mathrm{triv}_0 = 1$. According to the definition,

$$m^*(\rho) = \sum_{3=k+l} r^3_{k,l}(\rho) = r^3_{0,3}(\rho) + r^3_{1,2}(\rho) + r^3_{2,1}(\rho) + r^3_{3,0}(\rho).$$

Now $r^3_{1,2}(\rho) = \left((23) \mapsto \rho((12)(123) = \begin{pmatrix} 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 \end{pmatrix} \right)$, which is a 2-dimensional representation of $S_{1,2} \cong S_2$. Then the subspace $\{(a,b) | b = e^{2\pi i/3} a\}$ is a subrepresentation isomorphic to triv_2, and the subspace $\{(a,b) | b = -e^{2\pi i/3} a\}$ is a subrepresentation isomorphic to sgn_2.

On the other hand, now consider $r^3_{2,1}(\rho)$. This representation is given by

$$(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This representation has $\mathbb{C}(e_1 + e_2)$ and $\{(a,b) | a = -b\}$ as subrepresentations isomorphic to triv_2 and sgn_2, respectively (e_1 and e_2 are the standard basis vectors for $\mathbb{C}^2$).

We may conclude that

$$m^*(\rho) = 1 \otimes \rho + \mathrm{triv}_1 \otimes [\mathrm{triv}_2 + \mathrm{sgn}_2] + [\mathrm{triv}_2 + \mathrm{sgn}_2] \otimes \mathrm{triv}_1 + \rho \otimes 1.$$

Now we'll define $m : \mathrm{R}(S) \otimes_{\mathbb{Z}} \mathrm{R}(S) \to \mathrm{R}(S)$ in order to give $\mathrm{R}(S)$ an algebra structure. Let $\pi \in \mathrm{R}(S_k), \rho \in \mathrm{R}(S_l)$, and suppose $k + l = n$. Then $\pi \otimes_{\mathbb{C}} \rho \in \mathrm{R}(S_k \times S_l)$. We define multiplication in $\mathrm{R}(S)$ by $\pi\rho = m(\pi \otimes_{\mathbb{Z}} \rho) := i^n_{k,l}$. To check associativity, suppose $\pi_l \in \mathrm{R}(S_l), \pi_m \in \mathrm{R}(S_m)$, and $\pi_n \in \mathrm{R}(S_n)$. We need to show that $m \circ (1 \otimes m)(\pi_l \otimes \pi_m \otimes \pi_n) = m \circ (m \otimes 1)(\pi_l \otimes \pi_m \otimes \pi_n)$. It will be sufficient to show that

$$i^{l+m+n}_{l,m+n}(\pi_l \otimes i^{m+n}_{m,n}(\pi_m \otimes \pi_n)) \cong i^{l+m+n}_{l+m,n}(i^{m+m}_{l,m}(\pi_l \otimes \pi_m) \otimes \pi_n)$$

as representations of S_{l+m+n}. But we may rewrite the left hand side of this equivalence as $i^{l+m+n}_{l,m+n}(i^{l,m+n}_{l,m,n}(\pi_l \otimes \pi_m \otimes \pi_n))$ by definition of the functors being used. This is equivalent to $i^{l+m+n}_{l,m,n}(\pi_l \otimes \pi_m \otimes \pi_n)$ by the transitivity of Ind (Lemma 2.2.11). Similarly, the right hand side of equation (3.3) is equivalent to $i^{l+m+n}_{l,m,n}(\pi_l \otimes \pi_m \otimes \pi_n)$. This proves the associativity of multiplication in $\mathrm{R}(S)$.

Next we define the unit and counit. For a unit, define $u : \mathbb{Z} \to \mathrm{R}(S)$ by sending $1 \in \mathbb{Z}$ to the trivial representation of S_0; denote $u(1)$ by I. What we need to do amounts to proving that if $\rho \in \mathrm{Rep}(S_n)$, then $m(I \otimes \rho) \cong \rho$. We only need to show isomorphism because $\mathrm{R}(S_n)$ is defined in terms of equivalence classes of representations. But it's evident that $m(I \otimes \rho) = i^n_{0,n}(\rho) \cong \rho$.

For a counit, define u^* on basis elements. Say

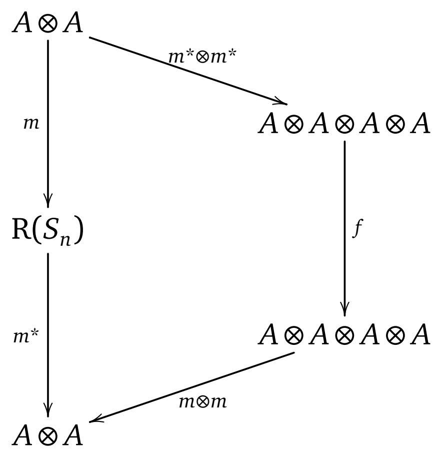
$$u^*(\rho) := \begin{cases} 1 & \rho \in \mathrm{Irr}(S_0) \\ 0 & \text{else} \end{cases}$$

The condition to be satisfied amounts to checking that if $\pi \in \mathrm{Rep}(S_n)$, then $(u^* \otimes 1)(m^*(\pi)) \cong \pi$ as representations. Following definitions, we arrive at

$$(u^* \otimes 1)(m^*(\pi)) = (u^* \otimes 1)(\sum_{n=k+l} r^n_{k,l}(\pi)) = r^n_{0,n}(\pi) = \pi.$$

The other calculation is carried out similarly.

Definition 3.3.6. Let $A = \bigoplus_{n \geq 0} A_n$ be a graded algebra over and coalgebra over $\mathbb{Z}$. Then A is *connected* if $u : \mathbb{Z} \to A$ and $u^*|_{A_0} : A_0 \to \mathbb{Z}$ are mutually inverse isomorphisms. If A is connected, then A is a *Hopf algebra* if m^* is a morphism of algebras; that is, if the following diagram commutes:



where all tensor products are over $\mathbb{Z}$.

Remarks 3.3.7. The definition of a Hopf algebra usually doesn't include the graded and connected conditions, but instead includes another piece of data called the *antipode* of the Hopf algebra (see [5, section 1.4] for details). However, every algebra and coalgebra A satisfying the above definition admits a unique antipode; see [5, Theorem 1.4.14]. Since only connected graded Hopf algebras are within the scope of this thesis, definition 3.3.6 will suffice.

Example 3.3.8. Let G be a finite group. The group ring $\mathbb{C}G$ has the structure of an algebra and a coalgebra. Multiplication is defined by $g \cdot h := gh$ on basis elements and extended by linearity; comultiplication is defined by $m^*(g) := g \otimes g$. With these definitions, one may easily verify $m^*(gh) = m^*(g)m^*(h)$. Let V and W be representations of G. Then we already know that $V \otimes W$ is a representation of G via $g \cdot (v \otimes w) := gv \otimes gw$. This definition tells us that G acts on $V \otimes W$ through m^*. That is, $g \cdot (v \otimes w) = m^*(g)(v \otimes w)$. Thus the fact that comultiplication is a map of algebras gives us the necessary associativity that assures this is a group action.

Remarks 3.3.9. It's immediate from the definitions of u and u^* that $\mathrm{R}(S)$ is connected. We give a grading on $\mathrm{R}(S)$ by declaring $\mathrm{R}(S)_n := \mathrm{R}(S_n)$. Thus our definitions of m and m^* immediately imply that $\mathrm{R}(S)$ is a graded algebra and coalgebra.

The following theorem appears as [11, section 6.1].

Theorem 3.3.10. $\mathrm{R}(S)$ *is a Hopf algebra.*

Proving Theorem 3.3.10 will require a bit of setup.

Say $\pi \in \mathrm{R}(S_{k'})$ and $\rho \in \mathrm{R}(S_{l'})$ with $k' + l' = n$. Then we must compute the composition

$$\mathrm{R}(S_{k'}) \otimes \mathrm{R}(S_{l'}) \xrightarrow{m} \mathrm{R}(S_n) \xrightarrow{m^*} \mathrm{R}(S) \otimes \mathrm{R}(S).$$

By the definition of m^*, the image is in

$$\bigoplus_{k+l=n} \mathrm{R}(S_k) \otimes \mathrm{R}(S_l).$$

Fixing one summand, we would like to calculate

$$\mathrm{R}(S_{k'}) \otimes \mathrm{R}(S_{l'}) \xrightarrow{m} \mathrm{R}(S_n) \xrightarrow{{m^*}_{k,l}} \mathrm{R}(S_k) \otimes \mathrm{R}(S_l).$$

By the identifications made earlier, this amounts to the composition

$$\mathrm{R}(S_{k',l'}) \xrightarrow{\mathrm{Ind}} \mathrm{R}(S_n) \xrightarrow{\mathrm{Res}} \mathrm{R}(S_{k,l}).$$

The Mackey Theorem (2.3.1) describes this composition. In order to utilize the theorem, we must choose a suitable set of double coset representatives for

$$S_{k,l} \backslash S_n / S_{k',l'}.$$

3.3.1 Double Cosets $S_\beta \backslash S_n / S_\alpha$

Let α and β be compositions of n. In this section we will give an explicit description of a canonical set of representatives for the double cosets $S_\beta \backslash S_n / S_\alpha$. This will be accomplished by parameterizing such representatives by matrices with nonnegative integer entries that sum to

n. To each such matrix we will associate a representative which will give us a twisting map as in example 3.2.5.

We'll separate the problem of paramterizing $S_\beta \backslash S_n / S_\alpha$ by first understanding S_n/S_α. Next, we'll use associativity of multiplication to realize $S_\beta \backslash S_n / S_\alpha$ as the set of orbits for the action of S_β on S_n/S_α by multiplication.

Suppose $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ and $\beta = (\beta_1, \beta_2, ..., \beta_s)$ are compositions of n. Define

$$X_\alpha = \{f : \{1, \ldots, n\} \to \{1, \ldots, r\} | f^{-1}(i)| = \alpha_i \forall i = 1, \ldots, r\}.$$

We can think of X_α as the set of colorings of n numbered balls in at most r colors, with α_i balls having the color i. There is an action of S_n on X_α via the formula $w \cdot f(k) := f(w^{-1}(k))$. In our colored ball analogy, this amounts to renumbering the already colored balls according to the permutation w.

Now let $I_1, ..., I_r$ and $J_1, ..., J_s$ be the blocks of α and β, respectively; define the function $\mathbb{X} \in X_\alpha$ by $k \mapsto i$ where $k \in I_i$. Let stab($\mathbb{X}$) denote the stabilizer of $\mathbb{X}$ in S_n. This subgroup consists of all permutations which preserve $I_1, ..., I_r$; i.e. $\text{stab}(\mathbb{X}) = S_\alpha$. Recall the Orbit Stabilizer Theorem:

Lemma 3.3.11. *Let A be a set with a transitive action of the group G, and let $H := \text{stab}(x)$ for $x \in A$. Then the map $G/H \to A$ given by $gH \mapsto g(x)$ is an isomorphism of G-sets.*

Proof. To see that the map is G-equivariant, let $k, g \in G$. Then $k(gH) = kgH \mapsto (kg)(x) = k(g(x))$. The surjectivity of the map follows form the transisitivity of the G-action. Finally, if $g_1(x) = g_2(x)$, then $x = g_2^{-1} g_1(x)$, so $g_2^{-1} g_1 \in H$. □

Proposition 3.3.12. *The group S_n acts transitively on X_α.*

Proof. Let $f, g \in X_\alpha$, and define $w \in S_n$ as follows. Define

$$k_1 := \min(\{m = 1, ..., n | f(m) = g(1)\}),$$

and let w^{-1} send $1 \mapsto k_1$. Similarly, define

$$k_2 := \min(\{m = 1, ..., n | f(m) = g(2)\} - \{k_1\}),$$

and and let w^{-1} send $2 \mapsto k_2$. Continue inductively, defining

$$k_l := \min(\{m = 1, ..., n | f(m) = g(l)\} - \{k_1, k_2, ..., k_{l-1}\})$$

and letting w^{-1} send $l \mapsto k_l$. Then w is a permutation by the pigeonhole principle, and $w \cdot f(i) = f(w^{-1}(i))$. By the definition of w^{-1}, $f(w^{-1}(i)) = g(i)$. Thus $w \cdot f = g$, so S_n acts transitively on X_α. □

It follows, then, that the map $S_n/S_\alpha \to X_\alpha$ given by $wS_\alpha \mapsto w \cdot \mathbb{X}$ is an isomorphism of S_n-sets. We'll now study the orbits in X_α under S_β to understand $S_\beta \backslash S_n / S_\alpha$. To do this, we'll place s boxes around our n balls which are colored using r colors.

To do this, define for each $f \in X_\alpha$ the $r \times s$ matrix $[f]$ over $\mathbb{Z}_{\geq 0}$ by $[f]_{ij} = |f^{-1}(i) \cap J_j|$. So the matrix $[f]$ answers the question, "How many balls of color i are there in box j?" The answer is $[f]_{ij}$. With this interpretation in mind, it's clear that $\sum_i [f]_{ij} = |J_j| = \beta_j$ (the total number of balls in box j) and $\sum_j [f]_{ij} = |f^{-1}(i)| = \alpha_i$ (the total number of balls of color i).

Definition 3.3.13. For each pair of compositions α and β of n, let $M_{\beta\alpha}$ be the collection of matrices

$$\left\{ [m]_{ij} \in M_{r\times s}(\mathbb{Z}_{\geq 0}) \,\middle|\, \sum_i m_{ij} = \beta_j \text{ and } \sum_j m_{ij} = \alpha_i \right\}$$

Proposition 3.3.14. *The map $\Theta : S_\beta \backslash X_\alpha \to M_{\beta\alpha}$ given by $S_\beta f \mapsto [f]$ is a bijection.*

Proof. To show that Θ is well-defined, let $f \in X_\alpha$ and let $w \in S_\beta$. Since $w \cdot f := f \circ w^{-1}$, we know that $(w \cdot f)^{-1}(i) = (f \circ w^{-1})^{-1}(i) = w(f^{-1}(i))$. But w sends elements of J_j to elements of J_j since $w \in S_\beta$, so $|w(f^{-1}(i)) \cap J_j| = |f^{-1}(i) \cap J_j|$. It follows that $[w \cdot f] = [f]$, and thus the map Θ is well-defined.

To show injectivity, first suppose that f and g are in X_α and are nondecreasing on each block of β (that is, suppose $f(i) \leq f(j)$ whenever $i \leq j$ and both lie in the same block of β, and similarly for g). Furthermore, suppose $[f] = [g]$. We'll show $f = g$. Fix $j \in \{1, ..., s\}$. Then for each color i, there are $[f]_{ij} = [g]_{ij}$ balls colored i in block J_j. By the nondecreasing hypothesis, the first $[f]_{1j} = [g]_{1j}$ of these balls are colored 1, the next $[f]_{2j} = [g]_{2j}$ are colored 2, and so on. So $f = g$ on J_j, which was chosen arbitrarily. Hence $f = g$ everywhere.

Now by transitivity, for each $f \in X_\alpha$ there is some $w \in S_\beta$ such that $\cdot f$ is nondecreasing on each block of β. So suppose now that $[f] = [g]$ for some $f, g \in X_\alpha$. Then there are $w, v \in S_\beta$ such that $w \cdot f$ and $v \cdot g$ are nondecreasing on each block of β. But then $[w \cdot f] = [f] = [g] = [v \cdot g]$. But by the previous argument, it follows that $w \cdot f = v \cdot g$. Thus $S_\beta w \cdot f = S_\beta v \cdot g$ so $S_\beta f = S_\beta g$.

To show Θ is surjective, let $[m] \in M_{\beta\alpha}$. By the definition of these matrices, it's obvious we may define some element $f \in X_\alpha$ to be nondecreasing on each block of β satisfying $|f^{-1}(i) \cap J_j| = [m]_{ij}$. It follows readily that $[f] = [m]$. □

Corollary 3.3.15. *For any two compositions α and β of n, the map*

$$S_\beta \backslash S_n / S_\alpha \to M_{\beta\alpha}, \;\; S_\beta w S_\alpha \mapsto [w\mathbb{X}]$$

is a bijection.

Proof. The bijections of 3.3.11 and 3.3.14 combine to give this bijection. □

The case which will be useful to us is when $r = s = 2$. We give a more explicit description of this case below.

3.4 The Hopf Axiom for $\mathrm{R}(S)$

We now demonstrate that the canonical representatives constructed in **??** do indeed give us the twisting map we need.

Lemma 3.4.1. *Let k, l, k', l' be nonnegative integers satisfying $k + l = k' + l' = n$. For each nonnegative integer matrix*

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a+b=k', c+d=l', a+c=k$, and $b+d=l$, we define the permutation $w=w_K\in S_n$ defined by

$$w(i)=\begin{cases} i & i\in\{1,\dots,a\} \\ i+c & i\in\{a+1,\dots,a+b\} \\ i-b & i\in\{a+b+1,\dots,a+b+c\} \\ i & i\in\{a+b+c+1,\dots,a+b+c+d\}\end{cases}.$$

Then the bijection $S_{(k,l)}\backslash S_n/S_{(k',l')}\to M_{(k,l)(k',l')}$ of 3.3.15 sends the double coset of w_K to the matrix K.

Proof. We may compute

$$\begin{aligned} w\cdot\mathbb{X}(m)=\mathbb{X}(w^{-1}(m)) &= \begin{cases} 1 & w^{-1}(m)\in\{1,\dots,k'\} \\ 2 & w^{-1}(m)\in\{k'+1,\dots,n\}\end{cases} \\ &= \begin{cases} 1 & m\in\{1,\dots,a\}\cup\{a+c+1,\dots,a+c+b\} \\ 2 & m\in\{a+1,\dots,a+c\}\cup\{a+c+b,\dots,n\}.\end{cases}\end{aligned}$$

Therefore since $[w\cdot\mathbb{X}]_{i,j}=|(w\cdot\mathbb{X})^{-1}(i)\cap J_j|$, we conclude that

$$[w\cdot\mathbb{X}]=\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

□

Recall that we set $G=S_n$, $M=S_{k',l'}$, and $N=S_{k,l}$. Fix $K\in M_{(k,l)(k',l')}$ for now and let $w=w_K$. Then we have:

Proposition 3.4.2. *$M'=M\cap {}^{w^{-1}}N=S_{a,b,c,d}$ and $N'={}^{w}M\cap N=S_{a,c,b,d}$.*

Proof. Using 3.3.3, we may compute

$$M'=S_{(k',l')}\cap S_{w^{-1}(k,l)}=S_{(k',l')\wedge w(k,l)}=S_{a,b,c,d}.$$

The computation for N' is similar. □

Recall also that, following example 2.1.4, we denote by Ad_w^* the functor $\mathrm{Rep}(M') \to \mathrm{Rep}(N')$ given by $(\rho, Z) \mapsto (\rho \circ w^{-1}, Z)$, where the symbol w^{-1} denotes the isomorphism given by conjugating by w^{-1}. We must find a formula for the action of the functor Ad_w^*.

Lemma 3.4.3. *Let $w \in S_n$ be the canonical double coset representative associated to the matrx $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose definition is given in 3.4.1. Then ${}^{w}S_{a,b,c,d} = S_{a,c,b,d}$.*

Proof. Suppose $\sigma \in S_{a,b,c,d}$, and $1 \le i \le n$. If $1 \le i \le a$ or $n - d \le i \le n$, it's clear that $w\sigma w^{-1}(i)$ is where it should be. So suppose $a + 1 \le i \le a + c$. Then we have $a + b + 1 \le w^{-1}(i) \le a + b + c$, $a + b + 1 \le \sigma \circ w^{-1}(i) \le a + b + c$, and finally $a + 1 \le w \circ \sigma \circ w^{-1}(i) \le a + c$, as needed. The calculation for when $a + c + 1 \le i \le a + c + b$ is carried out similarly. □

Next, define

$$f : S_a \times S_b \times S_c \times S_d \xrightarrow{(\alpha,\beta,\gamma,\delta)\mapsto(\alpha,\gamma,\beta,\delta)} S_a \times S_c \times S_b \times S_d$$

to be the map that flips the two inner factors. Define

$$\varphi : S_a \times S_b \times S_c \times S_d \xrightarrow{\cong} S_{a,b,c,d}$$

and

$$\psi : S_a \times S_c \times S_b \times S_d \xrightarrow{\cong} S_{a,c,b,d}$$

to be the canonical isomorphisms defined in the proof of 3.3.3.

Proposition 3.4.4. $\mathrm{Ad}_w \circ \varphi = \psi \circ f$ *as maps $S_a \times S_b \times S_c \times S_d \to S_{a,c,b,d}$. In diagram form, we have*

$$\begin{array}{ccc} S_{a,b,c,d} & \xleftarrow{\mathrm{Ad}_{w^{-1}}} & S_{a,c,b,d} \\ \varphi \uparrow & & \uparrow \psi \\ S_a \times S_b \times S_c \times S_d & \xleftarrow{f} & S_a \times S_c \times S_b \times S_d \end{array}$$

Proof. If $g \in S_a$ or $g \in S_d$, then the two images of g are obviously equal. So let $g \in S_c$, and let $i \in \{1, \dots, n\}$. Then $\varphi \circ f(1, g, 1, 1) = \varphi(1, 1, g, 1)$. Now

$$\varphi(1,1,g,1)(i) = \begin{cases} g(i-a-b)+a+b & a+b+1 \le i \le a+b+c \\ i & \text{else} \end{cases}.$$

On the other hand, $\mathrm{Ad}_{w^{-1}} \circ \psi(1, g, 1, 1) = w^{-1} \circ \psi(1, g, 1, 1) \circ w$, which gives

$$w^{-1} \circ \psi(1,g,1,1) \circ w(i) = \begin{cases} g(i-a-b)+a+b & a+c+1 \le i \le a+c+b \\ i & \text{else} \end{cases},$$

as needed. A similar argument holds for the S_b factor. □

Remarks 3.4.5. Pulling back along all these isomorphisms, we arrive at the following commutative diagram:

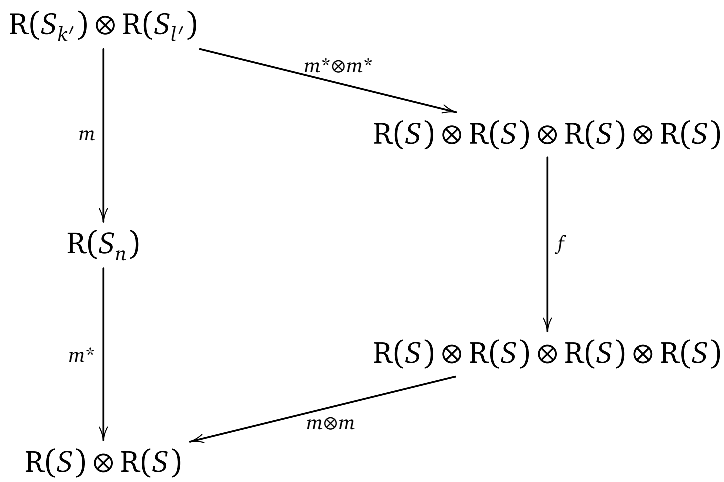
$$\begin{array}{ccc} \mathrm{Rep}(S_a \times S_b \times S_c \times S_d) & \xrightarrow{f^*} & \mathrm{Rep}(S_a \times S_c \times S_b \times S_d) \\ \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} \\ \mathrm{Rep}(S_{a,b,c,d}) & \xrightarrow[\mathrm{Ad}^*_{w^{-1}}]{} & \mathrm{Rep}(S_{a,c,b,d}) \end{array} \tag{3.4.6}$$

The bijection in 2.1.12 tells us that f^* induces an isomorphism

$$\mathrm{R}(S_a) \otimes_{\mathbb{Z}} \mathrm{R}(S_b) \otimes_{\mathbb{Z}} \mathrm{R}(S_c) \otimes_{\mathbb{Z}} \mathrm{R}(S_d) \xrightarrow{\alpha\otimes\beta\otimes\gamma\otimes\delta \mapsto \alpha\otimes\gamma\otimes\beta\otimes\delta} \mathrm{R}(S_a) \otimes_{\mathbb{Z}} \mathrm{R}(S_c) \otimes_{\mathbb{Z}} \mathrm{R}(S_b) \otimes \mathrm{R}(S_d).$$

We're now in a position to verify the Hopf axiom for $\mathrm{R}(S)$.

Proof of 3.3.10. Let $\pi \in \mathrm{R}(S_{k'})$ and $\rho \in \mathrm{R}({}_S l')$ be positive elements, and set $n = k' + l'$. Our goal is to prove that $m^*(\pi\rho) = m^*(\pi)m^*(\rho)$; that is, we would like to show that the diagram



(3.4.7)

commutes, where f is the twisting map of example 3.2.5. It was noted in remark 3.4.5 that the action of the functor Ad^* on representations of $S_{a,b,c,d}$ is given by flipping the two inner factors (by 3.3.3 and 2.1.12, representations of $S_{a,b,c,d}$ are tensor products of representations of S_a, S_b, S_c, and S_d).

We now compute:

$$m^*(\pi\rho) = \sum_{n=k+l} r^n_{k,l}(\pi\rho) = \sum_{n=k+l} r^n_{k,l} \circ i^n_{k',l'}(\pi \otimes \rho). \tag{3.4.8}$$

By the Mackey Theorem (Theorem 2.3.1) and Proposition 3.4.2, the quantity in 3.4.8 may be rewritten as

$$\sum_{n=k+l} \sum_{\substack{S_{k,l} w S_{k',l'} \\ \in S_{k,l} \backslash S_n / S_{k',l'}}} i^{k,l}_{a,c,b,d} \circ \mathrm{Ad}^*_{w^{-1}} \circ r^{k',l'}_{a,b,c,d}(\pi \otimes \rho). \tag{3.4.9}$$

We now use 2.2.10 and 3.4.1 to express 3.4.9 as

$$\sum_{n=k+l} \sum_{\substack{a,b,c,d \\ a+b=k',c+d=l' \\ a+c=k,b+d=l}} [i^k_{a,c} \otimes i^l_{b,d}] \circ \mathrm{Ad}^*_{w^{-1}} \circ [r^{k'}_{a,b} \otimes r^{l'}_{c,d}](\pi \otimes \rho). \tag{3.4.10}$$

Now we notice that, for fixed a, b, c, d, the composition $[i^k_{a,c} \otimes i^l_{b,d}] \circ \mathrm{Ad}^*_{w^{-1}}$ is exactly the definition of multiplication in the algebra $\mathrm{R}(S) \otimes \mathrm{R}(S) \otimes \mathrm{R}(S) \otimes \mathrm{R}(S)$. This, along with a reindexing of the sums to delete redundancies, allows us to write the quantity in 3.4.10 as

$$\sum_{\substack{a,b,c,d\\ a+b=k',\\ c+d=l'}} r^{k'}_{a,b}(\pi) r^{l'}_{c,d}(\rho). \tag{3.4.11}$$

On the other hand,

$$\begin{aligned} m^*(\pi)m^*(\rho) &= \sum_{k'=a+b} r^{k'}_{a,b}(\pi) \sum_{l'=c+d} r^{l'}_{c,d}(\rho) \\ &= \sum_{\substack{a,b,c,d\\ k'=a+b,\\ l'=c+d}} r^{k'}_{a,b}(\pi) r^{l'}_{c,d}(\rho). \end{aligned} \tag{3.4.12}$$

The quantities in 3.4.11 and 3.4.12 are equal, so m^* is a morphism of algebras.

By remarks 3.3.9, we may now conclude that $\mathrm{R}(S)$ is a Hopf algebra.

□

3.5 Further Properties of $\mathrm{R}(S)$

This section explores further properties of the Hopf algebra $\mathrm{R}(S)$. Namely, we will prove that $\mathrm{R}(S)$ is actually a *positive, self-adjoint* Hopf algebra.

Definition 3.5.1. Let R be a Hopf algebra over $\mathbb{Z}$. We say R is *self-adjoint* if there exists a positive definite, bilinear form on R with values in $\mathbb{Z}$, such that multiplication and comultiplication are adjoint maps.

Suppose R has a distinguished basis $\Omega = \Omega(R)$ consisting of homogeneous elements. If the maps m and u are positive maps in the sense that they send nonnegative integral combinations of elements of Ω to nonnegative integral combinations, then R is said to be *positive*.

Theorem 3.5.2. *The Hopf algebra* $\mathrm{R}(S)$ *is a PSH algebra.*

In order to prove 3.5.2, we must first define a positive definite, bilinear form on $\mathrm{R}(S)$.

Definition 3.5.3. We define $\langle\cdot,\cdot\rangle : \mathrm{R}(S) \times \mathrm{R}(S) \to \mathbb{Z}$ by declaring that the basis $\bigsqcup_{n\geq 0} \mathrm{Irr}(G_n)$ for $\mathrm{R}(S)$ forms an orthonormal set.

Remarks 3.5.4. Let π_1, π_2 be representations of S_n. Schur's Lemma (2.1.8) implies that $\langle \pi_1, \pi_2 \rangle = \dim \mathrm{Hom}_{S_n}(\pi_1, \pi_2)$, according to 2.1.11. If π_1 is a representation of S_m and π_2 a representation of S_n for $m \neq n$, then $\langle \pi_1, \pi_2 \rangle = 0$. So the definition of the inner product on $\mathrm{R}(S)$ is representation theoretic.

Lemma 3.5.5. *The maps m and m^* are adjoint maps.*

Proof. Recall Frobenius reciprocity (2.2.6) tells us that if H is a subgroup of G, $V \in \mathrm{Rep}(G)$, and $W \in \mathrm{Rep}(H)$, then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$. We apply this directly to conclude self-adjointness. Indeed, let π, ρ, φ be representations of S_l, S_m,, and S_n, respectively. Then we have

$$\langle \pi \otimes \rho, m^*(\varphi) \rangle = \left\langle \pi \otimes \rho, \sum_{n=k_1+k_2} r^n_{k_1,k_2} \varphi \right\rangle = \sum_{n=k_1+k_2} \langle \pi \otimes \rho, r^n_{k_1,k_2} \varphi \rangle.$$

The only term of this final sum which is nonzero is the case when $n = l + m$. In this case we arrive at $\langle \pi \otimes \rho, m^*(\varphi) \rangle = \langle \pi \otimes \rho, r^n_{l,m} \varphi \rangle = \langle i^{l+m}_{l,m}(\pi \otimes \rho), \varphi \rangle = \langle m(\pi \otimes \rho), \varphi \rangle$. This proves that m and m^* are adjoint maps. □

Remarks 3.5.6. Our distinguished basis for $\mathrm{R}(S)$ is $\Omega = \sqcup_{n \geq 0} \mathrm{Irr}(S_n)$. The positivity of m comes from the fact that Ind takes representations to representations. Since $u(1)$ is defined to be the trivial representation of S_0 (in particular, $u(1)$ is positive), u is also a positive map. Hence $\mathrm{R}(S)$ is positive.

Proof of 3.5.2. Theorem 3.3.10 tells us that $\mathrm{R}(S)$ is a Hopf algebra. Lemma 3.5.5 says that $\mathrm{R}(S)$ is self-adjoint, and remarks 3.5.6 says that $\mathrm{R}(S)$ is positive. Hence $\mathrm{R}(S)$ is a PSH algebra. □

What we've shown is that $\mathrm{R}(S)$ is a positive, self-adjoint Hopf algebra, with the structure given by the functors Ind and Res. Almost all of the properties except the Hopf axiom were proven directly from definitions. The key to demonstrating that the Hopf axiom holds is the Mackey formula (Theorem 2.3.1). In the next chapter we will use the generalized Mackey formula (Theorem 2.5.5) to exhibit a PSH structure on another family of groups.

CHAPTER 4

THE HOPF ALGEBRA $\mathrm{R}(G)$

Throughout this chapter we will generalize all of the work from chapter 3. A new family of combinatorial groups will be defined, and properties analogous to those of S_n will be demonstrated.

4.1 Permutations Modulo p

In this section we define and characterize the groups on which our new PSH algebra will be based. For convenience and ease of notation we will often use the language of *wreath products* which will be defined below; however this isn't strictly necessary. Wreath products are special semidirect products, which we now define.

Definition 4.1.1. Given two finite groups W and N, and a homomorphism $\varphi : W \to \mathrm{Aut}(N)$, the *semidirect product* of W and N is the group $W \ltimes_{\varphi} N$, whose underlying set is $W \times N$, and whose operation is given by $(w_1, n_1) \cdot (w_2, n_2) := (w_1 w_2, \varphi(w_1^{-1})(n_1) n_2)$. We will denote this group by $W \ltimes N$ when φ is understood.

We make two immediate observations. First, note that $w \mapsto (w, 1)$ and $n \mapsto (1, n)$ embed W and N naturally as subgroups of $W \ltimes N$. Second, we note also that $(w, 1)(1, n)(w^{-1}, 1) = (1, \varphi(w)(n))$ so N is a normal subgroup of $W \ltimes N$.

We now define wreath products, which are semidirect products coming from an action of W on some set X. We will always take X to be finite. Given such an action, if H is any other group we make $H^X = \{f : X \to H\}$ into a group via the formula $f_1 f_2(x) := f_1(x) f_2(x)$. If we make the observation that $H^X \cong \Pi_{x \in X} H$, we see that the group operation for H^X which we just defined corresponds to the usual multiplication in direct products. The group H^X is also given a W action by translation. That is, $(w \cdot f)(x) := f(w^{-1} x)$.

Definition 4.1.2. The *wreath product* of W with H over X is defined to be $W \ltimes H^X$.

We will now define the family of groups that the remainder of this thesis studies. Fix a prime p, and define the group

$$G_n := \{g \in M_n(\mathbb{Z}/p^2\mathbb{Z}) | \text{g is a permutation matrix mod p}\}.$$

Recall that a permutation matrix is a matrix whose rows and columns each contain a single 1, and 0s otherwise. For $m \in M_n(\mathbb{Z}/p^2\mathbb{Z})$ to be a permutation matrix mod p means that when the homomorphism $\mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\text{reduce mod } p} \mathbb{Z}/p\mathbb{Z}$ is applied to each entry of m, the result is a permutation matrix.

We will realize this group as the semidirect product $S_n \ltimes M_n(\mathbb{Z}/p\mathbb{Z})$, where $M_n(\mathbb{Z}/p\mathbb{Z})$ denotes the additive group of $n \times n$ matrices over $\mathbb{Z}/p\mathbb{Z}$. We will often write M_n. To do so, we must first define how S_n acts on $M_n(\mathbb{Z}/p\mathbb{Z})$. Consider a matrix $(m) \in M_n$; let $\sigma \in S_n$ and define

$$(\sigma \cdot (m))_{i,j} := m_{\sigma^{-1}i, \sigma^{-1}j}. \tag{4.1.3}$$

This is an action since we can consider a square matrix as a function $\{1, \ldots, n\}^2 \to \mathbb{Z}/p\mathbb{Z}$; in this case, σ acts by translating the argument. It's clear that S_n is isomorphic to the subgroup of G_n consisting of actual permutation matrices of $M_n(\mathbb{Z}/p^2\mathbb{Z})$.

Remarks 4.1.4. This last observation that we may view an $n \times n$ matrix over $\mathbb{Z}/p\mathbb{Z}$ as a function $\{1, \ldots, n\}^2 \to \mathbb{Z}/p\mathbb{Z}$ is where the wreath product perspective will often come in handy. As we proceed we will readily switch between these two viewpoints. That is, we will identify without specification the groups $S_n \ltimes M_n(\mathbb{Z}/p\mathbb{Z})$ and $S_n \ltimes (\mathbb{Z}/p\mathbb{Z})^{\{1,\ldots,n\}^2}$.

Lemma 4.1.5. *Let N, Q, G be three groups with homomorphisms*

$$N \xrightarrow{i} G \xrightarrow{q} Q$$

such that $\operatorname{image} i = \ker q$*, i is injective, and q is surjective. Suppose also that there exists a group homomorphism $s : Q \to G$ satisfying $qs = \mathrm{id}_Q$. Then $G \cong Q \ltimes N$, where the action of Q on N is given by $x \cdot n := s(x)i(n)s(x)^{-1}$.*

Proof. Since q is onto, s is injective; identify $s(Q) = Q$. Suppose $g \in N \cap Q$. Then $q(g) = 1$ and $g = s(x)$ for some $x \in Q$. But then $1 = q(g) = q(s(x)) = x$. So $g = s(1) = 1$, and therefore $N \cap Q = 1$. Next, note that since $N = \ker q$, N is a normal subgroup of G and hence Q normalizes N.

Finally, let $g \in G$ be arbitrary. Then $g = sq(g)[sq(g)]^{-1}g$. This implies that $G = QN$, since $sq(g) \in \operatorname{image} s = Q$. Also, $q([sq(g)]^{-1}g) = q([sq(g)])^{-1}q(g) = q(g)^{-1}q(g) = 1$, which tells us that $[sq(g)]^{-1}g \in N$. This tells us that $G \cong Q \ltimes N$ with Q acting on N by conjugation in G. □

Proposition 4.1.6. *The group G_n is characterized by $G_n \cong S_n \ltimes M_n(\mathbb{Z}/p\mathbb{Z})$.*

Proof. Define $i : M_n(\mathbb{Z}/p\mathbb{Z}) \to G$ by $m \mapsto 1 + pm$ (this means we first consider the entries of $m \in M_n(\mathbb{Z}/p\mathbb{Z})$ as elements of $\mathbb{Z}/p^2\mathbb{Z}$ by choosing the smallest positive coset representatives, then we multiply them by $p \in \mathbb{Z}/p^2\mathbb{Z}$), and $q : G_n \to S_n$ by $m \mapsto m(\text{mod} p)$. Define also $s : S_n \to G_n$ to be the identity (identify a permutation with its permutation matrix). Then i, q, s satisfy the conditions of 4.1.5, and therefore $G_n \cong S_n \ltimes M_n(\mathbb{Z}/p\mathbb{Z})$ for some action of S_n on $M_n(\mathbb{Z}/p\mathbb{Z})$. Since conjugation by a permutation matrix has the effect of permuting rows and columns, this action is that given by the definition in 4.1.3. □

4.2 Properties, Subgroups, and Functors

Recall from definition 3.3.1 that a set partition of n is a set theoretic partition $\gamma = (\gamma_i)$ of the finite set $\{1, \ldots, n\}$. We denote the collection of all such set partitions $\mathscr{P}_n$.

For each $\alpha \in \mathscr{P}_n$, define M_α to be the subgroup of $M_n = M_n(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\{1,\ldots,n\}^2}$ consisting of those functions whose support is contained in the set

$$\bigcup_{i=1}^{r} \alpha_i^2.$$

Equivalently, M_α is the set of matrices m where $m_{i,j} = 0$ whenever i and j are not in the same block of α. Recall that the group S_n acts on M_n via the following formula: if $m \in M_n$ and $w \in S_n$, then $(wm)_{i,j} = m_{w^{-1}(i), w^{-1}(j)}$.

Definition 4.2.1. For a set partition α of n, we will denote by X_α the set of ordered pairs (i,j) with i and j in the same block of α.

Proposition 4.2.2. 1. *For all $\alpha,\beta \in_n$, $X_\alpha \cap X_\beta = X_{\alpha\wedge\beta}$*

2. *For all $\alpha \in \mathscr{P}_n$ and all $w \in S_n$, we have $w(X_\alpha) = X_{w\alpha}$.*

Proof. By definition, $(i,j) \in X_\alpha \cap X_\beta$ if and only if $i,j \in \alpha_k$ and $i,j \in \beta_l$ for some k and l. This is equivalent to saying $i,j \in \alpha_k \cap \beta_l$, which tells us i and j are in the same block of $\alpha \wedge \beta$. This proves (1).

Part (2) is an immediate consequence of the definition of the action of S_n on $\{1,\dots,n\}^2$. □

The groups M_α are compatible with intersections and conjugation similarly to how the groups S_α are:

Proposition 4.2.3. 1. *For all $\alpha,\beta \in \mathscr{P}_n$, we have $M_\alpha \cap M_\beta = M_{\alpha\wedge\beta}$*

2. *For any $\alpha \in \mathscr{P}_n$ $w \in S_n$ $w(M_\alpha) = M_{w\alpha}$.*

Proof. For (1), recall that i and j are in the same block of $\alpha \wedge \beta$ if and only if they are in the same block of α and the same block of β. So $X \in M_\alpha \cap M_\beta$ if and only if $X_{i,j}$ is only possibly nonzero when i and j are in the same block of α and the same block of β. But this is true if and only if i and j are in the same block of $\alpha \wedge \beta$, which is true if and only if $X \in M_{\alpha\wedge\beta}$.

Now onto (2). By definition, $m \in M_\alpha$ if and only if $(m)_{w^{-1}(i),w^{-1}(j)} = 0$ whenever $w^{-1}(i)$ and $w^{-1}(j)$ are in different blocks of α. Applying w, we see that this is equivalent to saying that i and j are in different blocks of $w\alpha$. This proves our result, since $(wm)_{i,j} = (m)_{w^{-1}(i),w^{-1}(j)}$. □

Note that if $\alpha \le \beta$, then by part 2 of Proposition 4.2.3, M_β is stable under S_α. Let $\alpha \le \beta$. Let us make the following definitions:

- $G_\alpha := S_\alpha \ltimes M_\alpha$
- $P_\alpha^\beta := S_\alpha \ltimes M_\beta$

$\square$ $U_\alpha^\beta := \{m \in M_n | m \text{ is supported inside } \cup_i \beta_i^2 - \cup_j \alpha_j^2\}$

Remarks 4.2.4. Define $H := \mathbb{Z}/p\mathbb{Z}$. Then $U_\alpha^\beta = H^{X_\beta - X_\alpha}$ (we abuse notation and declare that for a subset $Y \subseteq \{1, \ldots, n\}^2$, H^Y denotes the subgroup of functions supported inside Y).

These are all subgroups of G_n. The previous two propositions combine to give a useful fact about our new groups.

Lemma 4.2.5. *1. For all $\alpha \in \mathscr{P}_n$ and all $w \in S_n$, ${}^wG_\alpha = G_{w\alpha}$*

2. For all $\alpha, \beta \in \mathscr{P}_n$ with $\alpha \leq \beta$, $G_\alpha U_\alpha^\beta = P_\alpha^\beta$

Proof. (1) follows from Propositions 3.3.3 and 4.2.3. For (2) we'll show that the map $G_\alpha \times U_\alpha^\beta \xrightarrow{(g,u)\mapsto gu} P_\alpha^\beta$ is a bijection, and that G_α normalizes U_α^β. Let $(\sigma, x) \in G_\alpha$, and $(1, y) \in U_\alpha^\beta$. Then

$$(\sigma, x)(1, y)(\sigma^{-1}, -\sigma \cdot x) = (\sigma, x + y)(\sigma^{-1}, -\sigma \cdot x) = (1, \sigma \cdot x + \sigma \cdot y - \sigma \cdot x) = (1, \sigma \cdot y).$$

Consider y as a function $\{1, \ldots, n\}^2 \to \mathbb{Z}/p\mathbb{Z}$ supported inside $(\beta_1^2 \cup \cdots \cup \beta_s^2) - (\alpha_1^2 \cup \cdots \cup \alpha_r^2)$. Recall that both $\beta_1^2 \cup \cdots \cup \beta_s^2$ and $\alpha_1^2 \cup \cdots \cup \alpha_r^2$ are S_α-invariant subsets of $\{1, \ldots, n\}^2$. Since σ is an element of S_α, it follows that $(i, j) \in (\beta_1^2 \cup \cdots \cup \beta_s^2) - (\alpha_1^2 \cup \cdots \cup \alpha_r^2)$ if and only if $\sigma(i, j) \in (\beta_1^2 \cup \cdots \cup \beta_s^2) - (\alpha_1^2 \cup \cdots \cup \alpha_r^2)$. Hence $(1, \sigma \cdot y) \in U_\alpha^\beta$, which tells us that G_α normalizes U_α^β.

To prove that this multiplication map $G_\alpha \times U_\alpha^\beta \to P_\alpha^\beta$ is bijective, first suppose that $(\sigma_1, x_1)(1, y_1) = (\sigma_2, x_2)(1, y_2)$. This tells us that $\sigma_1 = \sigma_2$ immediately; at the same time this tells us that $x_1 + y_1 = x_2 + y_2$. Rearranging, we get that $x_1 - x_2 = y_2 - y_1$. Since $(y_1)_{i,j} = (y_2)_{i,j} = 0$ when i and j are in the same α block, we conclude that $x_1 = x_2$. It thus follows that $y_2 - y_1 = 0$, so $y_1 = y_2$. This implies injectivity of the multiplication map.

Lastly, suppose $(\tau, z) \in P_\alpha^\beta = S_\alpha \ltimes M_\beta$. Set $x_{i,j} = z_{i,j}$ when i and j are in the same block of β but not the same block of α, and 0 otherwise. Also define $(1, y) \in U_\alpha^\beta$ to be such that $y_{i,j} = z_{i,j}$ when i and j are in the same block of α, and 0 otherwise. Then we have $(\tau, x)(1, y) = (\tau, x + y) = (\tau, z)$, as desired. $\square$

Proposition 4.2.6. *Let $k, l \geq 0$, and denote by (k,l) the set partition $\Theta((k,l))$ associated with the composition (k,l) of $k+l$ defined in example 3.3.2 (we will often write $G_{k,l}$ for $G_{(k,l)}$). Then $G_k \times G_l \cong G_{(k,l)}$.*

Proof. Recall that $G_{(k,l)} = S_{(k,l)} \ltimes M_{(k,l)}$, and that the map $\varphi : S_k \times S_l \to S_{(k,l)}$ given by

$$(\sigma, \tau) \mapsto \left(i \mapsto \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq k \\ \tau(i-k)+k & \text{if } k+1 \leq i \leq k+l \end{cases} \right)$$

is an isomorphism. Now define $\psi : M_k \times M_l \to M_{(k,l)}$ to be given by

$$(m,n) \mapsto \left((i,j) \mapsto \begin{cases} m_{i,j} & \text{if } i,j \leq k \\ n_{i-k,j-k} & \text{if } k+1 \leq i,j \leq k+l \\ 0 & \text{else} \end{cases} \right).$$

We will also denote $\psi(m,n)$ by $m \oplus n$.

We'll now show that the map $\varphi \times \psi : G_k \times G_l \to G_{(k,l)}$ defined by

$$\varphi \times \psi \left(((\sigma, m), (\sigma', m')) \right) := (\varphi(\sigma, \sigma'), \psi(m, m'))$$

is an isomorphism. To show it's a homomorphism, we'll prove that for every $(w_1, w_2) \in S_k \times S_l$ and every $(m_1, m_2) \in M_k \times M_l$, $\varphi(w_1, w_2)\psi(m_1, m_2) = \psi(w_1 \cdot m_1, w_2 \cdot m_2)$. To verify this, note that

$$\varphi(w_1, w_2)\psi(m_1, m_2)_{i,j} = \begin{cases} (m_1)_{W_1^{-1}(i), w_1^{-1}(j)} & \text{if } i,j \leq k \\ (m_2)_{w_2^{-1}(i-k), w_2^{-1}(j-k)} & \text{if } k+1 \leq i,j \leq k+l \\ 0 & \text{else} \end{cases}$$

which is by definition equal to $\psi(w_1 \cdot m_1, w_2 \cdot m_2)$. This proves that the map is a homomorphism. Both φ and ψ are bijections, so $\varphi \times \psi$ is as well. □

The decomposability assumption 2.5.4 was crucial in the proof of the Mackey Theorem (Theorem 2.5.5). Here we will prove that same condition holds with our new groups.

Lemma 4.2.7. *For all $\alpha, \beta \in \mathscr{P}_n$, we have*

1. $P_\alpha^n \cap G_\beta = (G_\alpha \cap G_\beta)(U_\alpha^n \cap G_\beta)$

2. $P_\alpha^n \cap U_\beta^n = (G_\alpha \cap U_\beta^n)(U_\alpha^n \cap U_\beta^n)$

3. $P_\alpha^n \cap P_\beta^n = (G_\alpha \cap P_\beta^n)(U_\alpha^n \cap P_\beta^n)$

Proof. Throughout this proof, we will view $n \times n$ matrices over $\mathbb{Z}/p\mathbb{Z}$ as functions $\{1, \ldots, n\}^2 \to \mathbb{Z}/p\mathbb{Z}$. Set $H := \mathbb{Z}/p\mathbb{Z}$.

For (1) first note that by 3.3.3 and 4.2.3, we have $G_\alpha \cap G_\beta = G_{\alpha \wedge \beta}$. Next, observe that $G_\beta \cap U_\alpha^n = U_{\alpha \wedge \beta}^\beta$. Indeed,

$$\begin{aligned} G_\beta \cap U_\alpha^n &= (S_\beta \ltimes M_\beta) \cap H^{X_n - X_\alpha} = (S_\beta \ltimes H^{X_\beta}) \cap H^{X_n - X_\alpha} \\ &= H^{X_\beta} \cap H^{X_n - X_\alpha} = H^{X_\beta - (X_\alpha \cap X_\beta)} \\ &= H^{X_\beta - X_{\alpha \wedge \beta}} = U_{\alpha \wedge \beta}^\beta. \end{aligned}$$

So what remains to show is that $P_\alpha^n \cap G_\beta = G_{\alpha \wedge \beta} U_{\alpha \wedge \beta}^\beta$. This second group is equal to $P_{\alpha \wedge \beta}^\beta$ by Lemma 4.2.5.

Now observe that $P_\alpha^n \cap G_\beta = (S_\alpha \ltimes M_n) \cap (S_\beta \ltimes M_\beta) = S_{\alpha \wedge \beta} \ltimes M_\beta = P_{\alpha \wedge \beta}^\beta$, as desired.

For (2) first note that $U_\alpha^n \cap U_\beta^n = H^{X_n - X_\alpha} \cap H^{X_n - X_\beta} = H^{X_n - (X_\alpha \cup X_\beta)}$. Now since $G_\alpha \cap U_\beta^n = U_{\alpha \wedge \beta}^\alpha = H^{X_\alpha - (X_\alpha \cap X_\beta)}$ and $P_\alpha^n \cap U_\beta^n = (S_\alpha \ltimes H^{X_n}) \cap (1 \ltimes H^{X_n - X_\beta}) = U_\beta^n = H^{X_n - X_\beta}$, we'd like to show that

$$H^{X_n - X_\beta} = H^{X_\alpha - (X_\alpha \cap X_\beta)} H^{X_n - (X_\alpha \cup X_\beta)}.$$

To do this it will suffice to note that as sets,

$$X_n - X_\beta = [X_n - (X_\alpha \cap X_\beta)] \sqcup [X_n - (X_\alpha \cup X_\beta)].$$

To prove (3) we'll use previous results to rewrite all three terms. First,

$$P_\alpha^n \cap P_\beta^n = (S_\alpha \ltimes H^{X_n}) \cap (S_\beta \ltimes H^{X_n}) = S_{\alpha \wedge \beta} \ltimes H^{X_n} = P_{\alpha \wedge \beta}^n$$

by 3.3.3. Next, we have

$$G_\alpha \cap P^n_\beta = (S_\alpha \ltimes H^{X_\alpha}) \cap (S_\beta \ltimes H^{X_n}) = S_{\alpha\wedge\beta} \ltimes H^{X_\alpha} = P^\alpha_{\alpha\wedge\beta}$$

by 3.3.3 and 4.2.2. Lastly,

$$U^n_\alpha \cap P^n_\beta = (1 \ltimes H^{X_n - X_\alpha}) \cap (S_\beta \ltimes H^{X_n}) = 1 \ltimes H^{X_n - X_\beta}.$$

Thus what we'd like to show is that $P^n_{\alpha\wedge\beta} = P^\alpha_{\alpha\wedge\beta} U^n_\alpha$. We show this directly:

$$\begin{aligned} P^n_{\alpha\wedge\beta} &= S_{\alpha\wedge\beta} \ltimes H^{X_n} = S_{\alpha\wedge\beta} \ltimes H^{X_\alpha \sqcup (X_n - X_\alpha)} \\ &= S_{\alpha\wedge\beta} \ltimes [H^{X_\alpha} H^{X_n - X_\alpha}] \\ &= (S_{\alpha\wedge\beta} \ltimes H^{X_\alpha})(1 \ltimes H^{X_n - X_\alpha}) \\ &= P^\alpha_{\alpha\wedge\beta} U^n_\alpha, \end{aligned}$$

as needed.

□

We now define induction and restriction functors between the groups G_α, and exhibit them as special cases of the i_U and r_U functors of section 2.4. The decomposability condition established in Lemma 4.2.7 allows us to apply the generalized Mackey formula (Theorem 2.5.5) to these functors.

Definition 4.2.8. Suppose $\alpha, \beta \in \mathscr{P}_n$ are such that $\alpha \leq \beta$. Define the functors $i^\beta_\alpha : \mathrm{Rep}(G_\alpha) \to \mathrm{Rep}(G_\beta)$ and $r^\beta_\alpha : \mathrm{Rep}(G_\beta) \to \mathrm{Rep}(G_\alpha)$ as follows. Set

$$i^\beta_\alpha = i_{U^\beta_\alpha} = \mathrm{Rep}(G_\alpha) \xrightarrow{\text{pullback}} \mathrm{Rep}(P^\beta_\alpha) \xrightarrow{\text{induce}} \mathrm{Rep}(G_\beta).$$

For each $W \in \mathrm{Rep}(G_\beta)$, define $r^\beta_\alpha(W) \in \mathrm{Rep}(G_\alpha)$ to be given by

$$r^\beta_\alpha(W) = r_{U^\beta_\alpha}(W) = W^{U^\beta_\alpha},$$

where $W^{U^\beta_\alpha}$ is the restriction to the representation of G_α on the set of U^β_α-fixed vectors.

Now let (k', l') and (k, l) be compositions of n. We will apply the generalized Mackey formula (Theorem 2.5.5) to compute the composition

$$r^n_{k,l} \circ i_{k',l'} = r_{U^n_{k,l}} \circ i_{U^n_{k',l'}}.$$

In order to proceed, we must understand the double cosets $G_{k,l} \backslash G_n / G_{k',l'}$ as before. The following proposition tells us that this presents no added challenge.

Proposition 4.2.9. *Let $\alpha, \beta \in \mathscr{P}_n$. The map $\varphi : S_\beta \backslash S_n / S_\alpha \to P^n_\beta \backslash G_n / P^n_\alpha$ given by $S_\beta w S_\alpha \mapsto P^n_\beta (w, 0) P^n_\alpha$ is a bijection.*

Proof. First we must verify that this assignment doesn't depend on some choice of double coset representative. Suppose $\sigma \in S_\beta$ and $\sigma' \in S_\alpha$. Then observe that

$$\varphi(S_\beta \sigma w \sigma' S_\alpha) = P^n_\beta (\sigma w \sigma', 0) P^n_\alpha = P^n_\beta (\sigma, 0)(w, 0)(\sigma', 0) P^n_\alpha = P^n_\beta (w, 0) P^n_\alpha,$$

since $S_\alpha \subseteq P^n_\alpha$ and $S_\beta \subseteq P^n_\beta$. Hence this map is well-defined.

Next up, we will check that this function is one-to-one. Suppose that $\varphi(S_\beta w S_\alpha) = \varphi(S_\beta w' S_\alpha)$. As a consequence, we know that $P^n_\beta (w, 0) P^n_\alpha = P^n_\beta (w', 0) P^n_\alpha$. It then follows from the definition of double cosets, that $(w', 0) = (\tau, m)(w, 0)(\tau', m')$. But by definition of multiplication in the semidirect product $S_n \ltimes M_n$, this means

$$(w', 0) = (\tau w \tau', \tau'^{-1} \cdot w^{-1} \cdot m' + m).$$

The immediate implication of the above equality is that $w' = \tau w \tau'$ and $0 = \tau'^{-1} \cdot w^{-1} \cdot m' + m$. We may therefore conclude that $S_\beta w' S_\alpha = S_\beta \tau w \tau' S_\alpha = S_\beta w S_\alpha$, and so φ is injective.

This map is also onto. Indeed, let $P^n_\beta (w, m) P^n_\alpha \in P^n_\beta \backslash G_n / P^n_\alpha$. Observe that $\varphi(S_\beta w S_\alpha) = P^n_\beta (w, 0) P^n_\alpha = P^n_\beta (w, 0)(1, m) P^n_\alpha = P^n_{(k,l)} (w, m) P^n_\alpha$, as desired. This map φ has thus been shown to be a bijection. □

Set $\alpha = (k', l')$ and $\beta = (k, l)$ with $k + l = k' + l' = n$. Recall from **??** that $M_{\beta,\alpha}$ is the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + b = k'$, $c + d = l'$, $a + c = k$, and $b + d = l$, and to each such matrix $K \in M_{\beta,\alpha}$ we associate a permutation $w_K \in S_n$ defined by

$$w_K(i) = \begin{cases} i & i \in \{1, \ldots, a\} \\ i + c & i \in \{a+1, \ldots, a+b\} \\ i - b & i \in \{a+b+1, \ldots, a+b+c\} \\ i & i \in \{a+b+c+1, \ldots, a+b+c+d\} \end{cases}.$$

Corollary 4.2.10. *The map $M_{\beta,\alpha} \to P^n_\beta \backslash G_n / P^n_\alpha$ given by $K \mapsto P^n_\beta (w_K, 0) P^n_\alpha$ is a bijection.*

Proof. Lemmas 3.3.15 and 4.2.9 combine to yield the desired result. □

Corollary 4.2.11. *Letting $\alpha = (k', l')$ and $\beta = (k, l)$ with $k + l = k' + l' = n$ as above, we have*

$$r^n_\beta \circ i^n_\alpha = \bigoplus_{K \in M_{\beta,\alpha}} i_{w_K \alpha \wedge \beta} \circ \mathrm{Ad}_{w_K^{-1}} \circ r_{\alpha \wedge w_K^{-1} \beta}$$

Proof. This follows from the new Mackey Theorem, Theorem 2.5.5, when we make the following definitions:

- □ $G := G_n$
- □ $M := G_\alpha$
- □ $U := U^n_\alpha$
- □ $P := P^n_\alpha = MU$
- □ $N := G_\beta$
- □ $V := U^n_\beta$
- □ $Q := P^n_\beta = NV$.

By corollary 4.2.10, the double cosets over which we're summing is the set $\{(w_K, 0) \in S_n | K \in M_{\beta,\alpha}\}$, and thanks to Lemma 4.2.7, the decomposability condition is satisfied. For each w_K, we have $U' = N \cap {}^{w_K}U = G_\beta \cap U^n_{w_K \alpha} = U^\beta_{\beta \wedge w_K \alpha}$ and $V' = M \cap {}^{w_K{}^{-1}}V = U^\alpha_{\alpha \wedge w_K{}^{-1}\beta}$. □

4.3 Algebra and Coalgebra Structure

The construction of the PSH algebra structure on representation theory of the family of groups G_n will be directly analogous to the construction for the family S_n which was explained in chapter 3.

Definition 4.3.1. Define the graded algebra $\mathrm{R}(G) := \bigoplus_{n\geq 0} \mathrm{R}(G_n)$ with the homogenous components given by $\mathrm{R}(G_n) := \mathbb{Z}\,\mathrm{Irr}(G_n)$.

First we'll put an algebra structure on the abelian group $\mathrm{R}(G)$. Let $\pi \in \mathrm{Rep}(G_k)$, and $\rho \in \mathrm{Rep}(G_l)$. Then $\pi \otimes_{\mathbb{C}} \rho \in \mathrm{Rep}(G_k \times G_l)$. By the isomorphism $G_k \times G_l \cong G_{(k,l)}$ of 4.2.6, we may identify $\pi \otimes_{\mathbb{C}} \rho \in \mathrm{Rep}(G_{(k,l)})$. We are now in a position to define $m(\pi \otimes_{\mathbb{Z}} \rho) := i^{k+l}_{(k,l)}(\pi \otimes \rho)$. We again take the trivial representation of the trivial group G_0 as our unit.

Proposition 4.3.2. *With the aforementioned grading, multiplication, and unit,* $\mathrm{R}(G)$ *is an associative graded algebra.*

Proof. The checks that this defines an algebra structure are exactly the same as they were for $\mathrm{R}(S)$, and follow from the transitivity of the i functor, which was proved in 2.4.5. The multiplication is graded by definition. □

Now for a coalgebra structure. Suppose $\pi \in \mathrm{Rep}(G_n)$. We define a coproduct m^* as follows:

$$m^*(\pi) := \sum_{k+l=n} r^n_{(k,l)}(\pi).$$

Strictly speaking, as we defined it our coproduct takes values in

$$\bigoplus_{k+l=n} \mathrm{R}(G_{(k,l)}).$$

But by the isomorphism $G_k \times G_l \cong G_{(k,l)}$ of 4.2.6, this is isomorphic to

$$\bigoplus_{k+l=m} \mathrm{R}(G_k \times G_l) \cong \bigoplus_{k+l=n} \mathrm{R}(G_k) \otimes \mathrm{R}(G_l).$$

Hence the above formula gives a well-defined map $\mathrm{R}(G) \to \mathrm{R}(G) \otimes \mathrm{R}(G)$. For a counit, we define $u^* : \mathrm{R}(G) \to \mathbb{Z}$ in the same way as in our earlier construction. That is, define u^* on basis elements by

$$u^*(\pi) := \begin{cases} 1 & \pi \in \mathrm{Irr}(G_0) \\ 0 & \text{else} \end{cases}.$$

Again, the verification that this gives a well-defined coalgebra structure is exactly similar to that for $\mathrm{R}(S)$:

Proposition 4.3.3. *The above definitions for comultiplication and counit make* $\mathrm{R}(G)$ *into a coassociative graded coalgebra.* □

We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{R}(G)$ by defining the basis $\bigsqcup_{n \geq 0} \mathrm{Irr}(G_n)$ to be orthonormal.

Proposition 4.3.4. *The maps m and m^* are adjoint. Moreover, all of the structure maps m, u, m^*, and u^* are positive maps in the sense of Definition 3.5.1.*

Proof. The maps u and u^* are positive by definition; m and m^* are positive since i and r both take representations to representations.

To verify the self-adjointness, suppose π, ρ, and φ are representations of G_l, G_m, and G_n, respectively. Now observe that

$$\langle \pi \otimes \rho, m^*(\varphi) \rangle = \langle \pi \otimes \rho, \sum_{a+b=n} r^n_{(a,b)}(\varphi) \rangle = \langle \pi \otimes \rho, r^n_{(l,m)}(\varphi) \rangle.$$

By the adjunction of the i and r functors, this is equal to

$$\langle i^n_{(l,m)}(\pi \otimes \rho), \varphi \rangle = \langle m(\pi \otimes \rho), \varphi \rangle$$

as desired. □

We will now prove that multiplication and comultiplication in $\mathrm{R}(G)$ are compatible in the sense of definition 3.3.6. We will prove that:

Theorem 4.3.5. *The algebra and coalgebra* $\mathrm{R}(G)$ *is a PSH algebra.*

4.4 The Hopf Axiom for R(G)

To complete the proof that R(G) is a PSH algebra, we'll once again verify the Hopf axiom. That is, $m^*(\pi\rho) = m^*(\pi)m^*(\rho)$. Let $\pi \in \mathrm{R}(G_{k'})$ and $\rho \in \mathrm{R}(G_{l'})$ be positive elements (i.e. representations) with $k' + l' = n$. What we need to compute is the composition

$$\mathrm{R}(G_{k'}) \otimes \mathrm{R}(G_{l'}) \xrightarrow{m} \mathrm{R}(G_n) \xrightarrow{m^*} \mathrm{R}(G) \otimes \mathrm{R}(G).$$

In our case, this reduces to

$$\mathrm{R}(G_{k'}) \otimes \mathrm{R}(G_{l'}) \xrightarrow{m} \mathrm{R}(G_n) \xrightarrow{m^*} \bigoplus_{k+l=n} \mathrm{R}(G_k) \otimes \mathrm{R}(G_l).$$

Fix one summand and consider the composition

$$\mathrm{R}(G_{k'}) \otimes \mathrm{R}(G_{l'}) \xrightarrow{m} \mathrm{R}(G_n) \xrightarrow{m^*_{k,l}} \mathrm{R}(G_k) \otimes \mathrm{R}(G_l),$$

where we consider only one component of the coproduct. Using the identifications we've proved, it will suffice to compute

$$\mathrm{Rep}(G_{k'} \times G_{l'}) \xrightarrow{i^n_{(k',l')}} \mathrm{Rep}(G_n) \xrightarrow{r^n_{(k,l)}} \mathrm{Rep}(G_k \times G_l).$$

Make the following definitions:

- ☐ $G := G_n$
- ☐ $M := G_{(k',l')}$
- ☐ $U := U^n_{(k',l')}$
- ☐ $P := P^n_{(k',l')} = MU$
- ☐ $N := G_{(k,l)}$
- ☐ $V := U^n_{(k,l)}$
- ☐ $Q := P^n_{(k,l)} = NV$

To each matrix $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of nonnegative integers, we associate the permutation $w = w_K$ described earlier in 3.4.1. Using Lemma 4.2.5, we have the following:

- ☐ $M' = M \cap {}^{w^{-1}}N = G_{(k',l')} \cap G_{w^{-1}(k,l)} = G_{(a,b,c,d)}$
- ☐ $N' = {}^{w}M \cap N = G_{w(k',l')} \cap G_{(k,l)} = G_{(a,c,b,d)}$
- ☐ $U' = N \cap {}^{w}U = G_{(k,l)} \cap U^{n}_{w(k',l')} = U^{(k,l)}_{(a,c,b,d)}$
- ☐ $V' = M \cap {}^{w^{-1}}V = G_{(k',l')} \cap U^{n}_{w^{-1}(k,l)} = U^{(k',l')}_{(a,b,c,d)}$

Next, we'll consider the action of w on representations. To do so, consider the following.

Proposition 4.4.1. *The diagram*

$$\begin{array}{ccc} G_{a,b,c,d} & \xrightarrow{\mathrm{Ad}_w} & G_{a,c,b,d} \\ \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \psi} \\ G_a \times G_b \times G_c \times G_d & \xrightarrow{f = flip} & G_a \times G_c \times G_b \times G_d \end{array}$$

commutes, where φ and ψ are the canonical isomorphisms.

Proof. It's immediate that $\psi \circ f = \mathrm{Ad}_w \circ \varphi$ on G_a and G_d. So let $(\sigma, X) \in G_b$. Then

$$(1, (\sigma, X), 1, 1) \mapsto (1, 1, (\sigma, X), 1) \mapsto ((id \sqcup id \sqcup \sigma \sqcup id), (0 \oplus 0 \oplus X \oplus 0))$$

under $\psi \circ f$. On the other hand, we have

$$\varphi : (1, (\sigma, X), 1, 1) \mapsto ((id \sqcup \sigma \sqcup id \sqcup id), (0 \oplus X \oplus 0 \oplus 0))$$

which maps to $((id \sqcup id \sqcup \sigma \sqcup id), w \cdot (0 \oplus X \oplus 0 \oplus 0))$ under Ad_w. Now observe that

$$w \cdot (0 \oplus X \oplus 0 \oplus 0)_{i,j} = (0 \oplus X \oplus 0 \oplus 0)_{w^{-1}(i), w^{-1}(j)}$$

which, by definition, is equal to

$$\begin{cases} X_{w^{-1}(i)-a, w^{-1}(j)-a} & \text{if } a + 1 \leq w^{-1}(i), w^{-1}(j) \leq a + b \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} X_{w^{-1}(i)-a, w^{-1}(j)-a} & \text{if } a+c+1 \leq i, j \leq a+c+b \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} X_{i-a-c, j-a-c} & \text{if } a+c+1 \leq i, j \leq a+c+b \\ 0 & \text{else} \end{cases}$$

but this equals $0 \oplus 0 \oplus X \oplus 0$ as desired. A similar argument works for G_c. □

Remarks 4.4.2. If we replace w with w^{-1} and pull back along the diagram in 4.4.1, we arrive at the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}(G_a \times G_b \times G_c \times G_d) & \xrightarrow{f^*} & \mathrm{Rep}(G_a \times G_c \times G_b \times G_d) \\ \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} \\ \mathrm{Rep}(G_{a,b,c,d}) & \xrightarrow{\mathrm{Ad}^*_{w^{-1}}} & \mathrm{Rep}(G_{a,c,b,d}) \end{array}$$

and so it follows that the functor $\mathrm{Ad}^*_{w^{-1}}$ acts on representations via the formula $\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4 \mapsto \pi_1 \otimes \pi_3 \otimes \pi_2 \otimes \pi_4$.

By Theorem 2.5.5, we know that

$$r_V \circ i_U = \bigoplus_{QwP \in Q\backslash G/P} i_{U'} \circ \mathrm{Ad}_{w^{-1}} \circ r_{V'}.$$

In this specific context, this formula amounts to the statement

$$r^n_{(k,l)} \circ i^n_{(k',l')} = \bigoplus_{\substack{w_K \\ K \in M_{(k',l'),(k,l)}}} i^{(k,l)}_{(a,c,b,d)} \circ \mathrm{Ad}_{w^{-1}} \circ r^{(k',l')}_{(a,b,c,d)},$$

where $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{(k',l'),(k,l)}$ is the matrix associated to the double coset representative w_K as in 3.4.1.

For now we'll consider a single summand functor

$$\Phi_w := i^{(k,l)}_{(a,c,b,d)} \circ \mathrm{Ad}_{w^{-1}} \circ r^{(k',l')}_{(a,b,c,d)} : \mathrm{Rep}(G_{(k',l')}) \to \mathrm{Rep}(G_{(k,l)}).$$

Lemma 4.4.3. *If $V' \in \mathrm{Rep}(G_{k'})$ and $W' \in \mathrm{Rep}(G_{l'})$, then*

$$r^{(k',l')}_{(a,b,c,d)}(V' \otimes W') \cong r^{k'}_{(a,b)}(V') \otimes r^{l'}_{(c,d)}(W')$$

as representations of $G_{(a,b,c,d)}$.

Proof. By the tensor product formula for i (2.4.6), we have

- $r^{(k',l')}_{(a,b,c,d)}(V' \otimes W') \cong e_{U^{(k',l')}_{(a,b,c,d)}} \mathbb{C}G_{(k',l')} \underset{\mathbb{C}G_{(a,b,c,d)}}{\otimes} (V' \otimes W')$
- $r^{k'}_{(a,b)}(V') \cong e_{U^{k'}_{(a,b)}} \mathbb{C}G_{k'} \underset{\mathbb{C}G_{(a,b)}}{\otimes} V'$
- $r^{l'}_{(c,d)}(W') \cong e_{U^{l'}_{(c,d)}} \mathbb{C}G_{l'} \underset{\mathbb{C}G_{(c,d)}}{\otimes} W'$.

Define a map

$$\alpha : e_{U^{(k',l')}_{(a,b,c,d)}} \mathbb{C}G_{k',l'} \underset{\mathbb{C}G_{(a,b,c,d)}}{\otimes} (V' \otimes W') \to e_{U^{k'}_{(a,b)}} \mathbb{C}G_{k'} \underset{\mathbb{C}G_{(a,b)}}{\otimes} V' \otimes e_{U^{l'}_{(c,d)}} \mathbb{C}G_{l'} \underset{\mathbb{C}G_{(c,d)}}{\otimes} W'$$

by

$$e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'} \otimes g_{l'}) \otimes v \otimes w \mapsto e^{k'}_{(a,b)} g_{k'} \otimes v \otimes e^{l'}_{(c,d)} g_{l'} \otimes w.$$

Define a map β in the opposite direction by

$$e_{U^{k'}_{(a,b)}} g_{k'} \otimes v \otimes e_{U^{l'}_{(c,d)}} g_{l'} \otimes w \mapsto e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'} \otimes g_{l'}) \otimes v \otimes w$$

(these formulas are using the canonical identification $G_k \times G_l \cong G_{k,l}$ of 4.2.6).

We will show that α and β are mutually inverse isomorphisms. First we'll show that α is balanced over $\mathbb{C}G_{(a,b,c,d)}$. Let $(\sigma_a, \sigma_b, \sigma_c, \sigma_d) \in \mathbb{C}G_{(a,b,c,d)}$ be a basis element. It acts on $V' \otimes W'$ and $\mathbb{C}G_{k'} \otimes \mathbb{C}G_{l'}$ via the element $(\sigma_a, \sigma_b) \otimes (\sigma_c, \sigma_d) \in \mathbb{C}G_{(a,b)} \otimes \mathbb{C}G_{(c,d)}$. Observe:

$$\begin{aligned}
&\alpha\left(e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'} \otimes g_{l'}) \otimes (\sigma_a, \sigma_b, \sigma_c, \sigma_d)(v \otimes w)\right) \\
&= \alpha\left(e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'} \otimes g_{l'}) \otimes ((\sigma_a, \sigma_b)v \otimes (\sigma_c, \sigma_d)w)\right) \\
&= e_{U^{k'}_{(a,b)}} g_{k'} \otimes (\sigma_a, \sigma_b)v \otimes e_{U^{l'}_{(c,d)}} g_{l'} \otimes (\sigma_c, \sigma_d)w \\
&= e^{k'}_{(a,b)} g_{k'}(\sigma_a, \sigma_b) \otimes v \otimes e_{U^{l'}_{(c,d)}} g_{l'}(\sigma_c, \sigma_d) \otimes w \\
&= \alpha(e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'}(\sigma_a, \sigma_b) \otimes g_{l'}(\sigma_c, \sigma_d)) \otimes v \otimes w \\
&= \alpha\left(e_{U^{(k',l')}_{(a,b,c,d)}}(g_{k'} \otimes g_{l'})(\sigma_a, \sigma_b, \sigma_c, \sigma_d) \otimes (v \otimes w)\right).
\end{aligned}$$

Hence α is well-defined, and a similar computation shows that β is well-defined. It's clear by the definitions that $\beta = \alpha^{-1}$, so to finish the proof it will suffice to demonstrate that either α or β is a map of $\mathbb{C}G_{(a,b,c,d)}$-modules. We'll check β.

Recall from Lemma 2.4.7 that $e_{U^{(k',l')}_{(a,b,c,d)}}$ commutes with $G_{(a,b,c,d)}$. Let $(\sigma_a, \sigma_b, \sigma_c, \sigma_d) \in G_{(a,b,c,d)}$ be as above. Then we have the following:

$$
\begin{aligned}
&\beta\left((\sigma_a,\sigma_b,\sigma_c,\sigma_d)[e_{U^{k'}_{(a,b)}} g_{k'} \otimes v \otimes e_{U^{l'}_{(c,d)}} g_{l'} \otimes w]\right) \\
&= \beta\left((\sigma_a,\sigma_b) e_{U^{k'}_{(a,b)}} g_{k'} \otimes v \otimes (\sigma_c,\sigma_d) e_{U^{l'}_{(c,d)}} g_{l'} \otimes w\right) \\
&= \beta\left(\left[e_{U^{k'}_{(a,b)}} (\sigma_a,\sigma_b) g_{k'} \otimes v\right] \otimes \left[e_{U^{l'}_{(c,d)}} (\sigma_c,\sigma_d) g_{l'} \otimes w\right]\right) \\
&= e_{U^{(k',l')}_{(a,b,c,d)}} [(\sigma_a,\sigma_b) g_{k'} \otimes (\sigma_c,\sigma_d) g_{l'}] \otimes v \otimes w \\
&= e_{U^{(k',l')}_{(a,b,c,d)}} (\sigma_a,\sigma_b,\sigma_c,\sigma_d)(g_{k'} \otimes g_{l'}) \otimes v \otimes w \\
&= (\sigma_a,\sigma_b,\sigma_c,\sigma_d) e_{U^{(k',l')}_{(a,b,c,d)}} (g_{k'} \otimes g_{l'}) \otimes v \otimes w \\
&= (\sigma_a,\sigma_b,\sigma_c,\sigma_d) \beta\left(e_{U^{k'}_{(a,b)}} g_{k'} \otimes v \otimes e_{U^{l'}_{(c,d)}} g_{l'} \otimes w\right).
\end{aligned}
$$

This proves that β is a map of $\mathbb{C}G_{(a,b,c,d)}$-modules, as needed. □

What we've just seen is that the diagram

$$
\begin{array}{ccc}
\mathrm{R}(G_k) \otimes \mathrm{R}(G_l) & \xrightarrow{\ \psi\ } & \mathrm{R}(G_{(k,l)}) \\
\Big\downarrow {\scriptstyle r^k_{(a,b)} \otimes r^l_{(c,d)}} & & \Big\downarrow {\scriptstyle r^{(k,l)}_{(a,b,c,d)}} \\
\mathrm{R}(G_{(a,b)}) \underset{\mathbb{Z}}{\otimes} \mathrm{R}(G_{(c,d)}) & \xrightarrow[\ \varphi\ ]{} & \mathrm{R}(G_{(a,b,c,d)})
\end{array}
$$

commutes, where φ and ψ are the canonical isomorphisms. We will now prove the corresponding fact for the i functors.

Proposition 4.4.4. *The diagram*

$$
\begin{array}{ccc}
\mathrm{R}(G_k) \otimes \mathrm{R}(G_l) & \xrightarrow{\ \psi\ } & \mathrm{R}(G_{(k,l)}) \\
\Big\uparrow {\scriptstyle i^k_{(a,b)} \otimes i^l_{(c,d)}} & & \Big\uparrow {\scriptstyle i^{(k,l)}_{(a,b,c,d)}} \\
\mathrm{R}(G_{(a,b)}) \underset{\mathbb{Z}}{\otimes} \mathrm{R}(G_{(c,d)}) & \xrightarrow[\ \varphi\ ]{} & \mathrm{R}(G_{(a,b,c,d)})
\end{array}
$$

commutes.

Proof. To prove this, we will take advantage of the inner product on $\mathrm{R}(G)$, which is defined such that $\sqcup_{n\geq 0}\mathrm{Irr}(G_n)$ is orthonormal. Also, recall that m and m^* are adjoint maps with respect to this inner product. Additionally, φ and ψ preserve this inner product. Now suppose $V \in \mathrm{R}(G_{(a,b)}$, $W \in \mathrm{R}(G_{(c,d)}$, and let $U_1 \in \mathrm{R}(G_k)$ and $U_2 \in \mathrm{R}(G_l)$. Then

$$\begin{aligned}
&\langle \psi(i^k_{(a,b)}(V)\otimes i^l_{(c,d)}(W)), \psi(U_1\otimes U_2)\rangle \\
&= \langle i^k_{(a,b)}(V)\otimes i^l_{(c,d)}(W), U_1\otimes U_2\rangle \\
&= \langle i^k_{(a,b)}(V), U_1)(i^l_{(c,d)}(W), U_2\rangle \\
&= \langle V, r^k_{(a,b)}(U_1))(W, r^l_{(c,d)}(U_2)\rangle \\
&= \langle V\otimes W, r^k_{(a,b)}(U_1)\otimes r^l_{(c,d)}(U_2)\rangle \\
&= \langle \varphi(V\otimes W), \varphi(r^k_{(a,b)}(U_1)\otimes r^l_{(c,d)}(U_2))\rangle \\
&= \langle \varphi(V\otimes W), r^{(k,l)}_{(a,b,c,d)}\circ\psi(U_1\otimes U_2)\rangle \\
&= \langle i^{(k,l)}_{(a,b,c,d)}\circ\varphi(V\otimes W), \psi(U_1\otimes U_2)\rangle .
\end{aligned}$$

We may set the first and last lines equal to one another, and conclude that

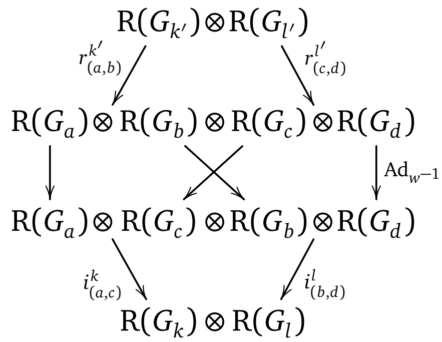
$$\langle\Big[\psi(i^k_{(a,b)}\otimes i^l_{(c,d)}) - i^{(k,l)}_{(a,b,c,d)}\circ\varphi\Big](V\otimes W), \psi(U_1\otimes U_2)\rangle = 0.$$

Since this holds for all U_1 and U_2, and since ψ is an isomorphism, we deduce that

$$\langle\Big[\psi(i^k_{(a,b)}\otimes i^l_{(c,d)}) - i^{(k,l)}_{(a,b,c,d)}\circ\varphi\Big](V\otimes W), \pi\rangle = 0$$

for every $\pi \in \mathrm{R}(G_{k,l})$. In particular, we set $\pi = \Big[\psi(i^k_{(a,b)}\otimes i^l_{(c,d)}) - i^{(k,l)}_{(a,b,c,d)}\circ\varphi\Big](V\otimes W)$ and see that we must have $\Big[\psi(i^k_{(a,b)}\otimes i^l_{(c,d)}) - i^{(k,l)}_{(a,b,c,d)}\circ\varphi\Big](V\otimes W) = 0$. So the diagram commutes. □

Proof of Theorem 4.3.5. The functor Φ_w defines a unique $\mathbb{Z}$-linear map $\Phi_w : \mathrm{R}(S_{k'}\times S_{l'}) \to \mathrm{R}(S_k\times S_l)$. According to the identification $\mathrm{R}(S_k\times S_l)\cong \mathrm{R}(S_k)\otimes\mathrm{R}(S_l)$, we may regard $\Phi_w : \mathrm{R}(S_{k'})\otimes\mathrm{R}(S_{l'}) \to \mathrm{R}(S_k)\otimes\mathrm{R}(S_l)$. In diagram form, Φ_w is the composition



The composition $(i \otimes i) \circ \mathrm{Ad}_{w^{-1}}$ turns out to be exactly the definition for multiplication in $\mathrm{R}(G) \otimes \mathrm{R}(G)$. It follows that $\Phi_w(\pi \otimes \rho) = r^{k'}_{a,b}(\pi) \cdot r^{l'}_{c,d}(\rho)$. Now, in full form,

$$m^*(\pi\rho) = \sum_{n=k+l} r^n_{k,l}(\pi\rho) = \sum_{n=k+l} r^n_{k,l} \circ i^n_{k',l'}(\pi \otimes \rho).$$

By the Mackey Theorem (Theorem 2.5.5), this sum is equal to

$$\sum_{n=k+l} \sum_{\substack{w=w_K \\ K \in M_{(k',l'),(k,l)}}} \Phi_w(\pi \otimes \rho) = \sum_{n=k+l} \sum_{\substack{a,b,c,d \\ a+b=k', c+d=l' \\ a+c=k, b+d=l}} r^{k'}_{a,b}(\pi) \cdot r^{l'}_{c,d}(\rho).$$

Reindexing, we may express this sum as

$$\sum_{\substack{a+b=k' \\ c+d=l'}} r^{k'}_{a,b}(\pi) \cdot r^{l'}_{c,d}(\rho) = \sum_{a+b=k'} r^{k'}_{a,b}(\pi) \cdot \sum_{c+d=l'} r^{l'}_{c,d}(\rho) = m^*(\pi) \cdot m^*(\rho).$$

The module $\mathrm{R}(S)$ is thus a PSH algebra, for the structure maps m and m^*. □

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