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Calculating Infinite Series Using Parseval's Identity

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CALCULATING INFINITE SERIES USING PARSEVAL'S IDENTITY

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ABSTRACT: Parseval's identity is an equality from Fourier analysis that relates an infinite series over the integers to an integral over an interval, which can be used to evaluate the exact value of some classes of infinite series. We compute the exact value of the Riemann zeta function at the positive even integers using the identity, and then we use it to compute the exact value of an infinite series whose summand is a rational function summable over the integers.

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CONTENTS

1. INTRODUCTION

It is a well-known fact that the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is equal to $\frac{\pi^2}{6}$. The Basel problem, which is to find the value of this series, was initially solved by Euler in 1735, but his proof contained a logical gap. However, Euler's result was later rigorously proven by Weierstrass, as well as others [3], [10]. Of the numerous ways to solve the Basel problem, one is to use Parseval's identity, which states that for a particularly "nice" function f, it holds that

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 = \int_0^1 |f(x)|^2 \, dx,$$

where $\hat{f}(n) = \int_0^1 f(x)e^{2\pi inx}\, dx$ for every integer n is its Fourier transform. Consider $f(x) = -2\pi i \left(x - \frac{1}{2}\right)$, whose Fourier transform is $\hat{f}(n) = 1/n$ for $n \neq 0$ and $\hat{f}(0) = 0$. Observe that $1/n^2$ is an even function of n, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\sum_{n\neq 0} \frac{1}{n^2} = \frac{1}{2}\sum_{n\neq 0} \left|\hat{f}(n)\right|^2 .$$

By Parseval's identity,

$$\left|\hat{f}(0)\right|^2 + \sum_{n\neq 0} \left|\hat{f}(n)\right|^2 = \int_0^1 |f(x)|^2 \, dx,$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\int_0^1 \left|f(x)\right|^2 \, dx - \frac{1}{2}\left|\hat{f}(0)\right|^2 = \frac{1}{2}\int_0^1 \left(4\pi^2x^2 - 4\pi^2 x + \pi^2\right) \, dx - 0 = \frac{\pi^2}{6}.$$

This thesis explores the extent to which Parseval's identity can be used to evaluate other infinite series. We will show that Parseval's identity can be used to compute the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for any $k \in \mathbb{N}$, as well as series of form $\sum_{n\in\mathbb{Z}} g(n)$, where $g \in \mathbb{R}(x)$ is summable over the integers. For instance, we can use Parseval's identity to find the exact value of series like $\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1}$, $\sum_{n\in\mathbb{Z}} \frac{1}{(0.5-n)^2}$, and $\sum_{n\in\mathbb{Z}} \frac{1}{(3n+1)^3}$.

2. PARSEVAL'S IDENTITY

There is plenty of theory behind how the sum of a series over $\mathbb{Z}$ can be evaluated using an integral over $[0,1]$. After all, such a claim that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \int_0^1 |f(x)|^2\,dx$ for certain functions f is not obvious. We shall prove this identity, but we start by discussing which functions are considered "nice" for the identity to work. These functions arise from the space we call $L^2([0,1])$.

2.1. The Space of Functions $L^2([0,1])$.

Definition 2.1. We define $L^2([0,1])$ to be the space of all $\mathbb{C}$-valued measurable functions f on $[0,1]$ such that $\int_0^1 |f(x)|^2\,dx$ is finite. We identify two functions in this space if they agree almost everywhere.

The space $L^2([0,1])$ has an inner product defined to be

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

for $f, g \in L^2([0,1])$. Moreover, $L^2([0,1])$ is a normed space whose norm is given by

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 |f(x)|^2\,dx. \qquad \text{[9], p. 139}$$

With this inner product and norm, $L^2([0,1])$ becomes a complete inner product space, also known as a Hilbert space, over $\mathbb{C}$ ([6], p. 321, Theorem 5.59). Despite the elements of $L^2([0,1])$ being equivalence classes, we will still refer to a function f as an element of this space because we are not concerned with two functions that differ by a set of Lebesgue measure zero. Since $\|f\|^2 = \|g\|^2$ if any two functions $f, g \in L^2([0,1])$ differ by a set of measure zero (that is, they agree almost everywhere), we will focus only on functions that are continuous on $[0,1]$.

2.2. **Orthonormal Basis for $L^2([0,1])$.**

Definition 2.2. A nonempty subset S of a Hilbert space H is *orthonormal* if the inner product of any two elements $x, y \in S$ is given by

$$\langle x, y\rangle = \begin{cases} 0, & x \neq y \\ 1, & x = y, \end{cases}$$

and such a set S is an *orthonormal basis* if its span is dense in H.

For our purposes of establishing the theory, we will show that the set $\{e^{2\pi inx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis of the Hilbert space $L^2([0,1])$. Let us call this set B. We show that B is orthonormal using the Fundamental Theorem of Calculus, but we use the Stone-Weierstrass Theorem to show that it is an orthonormal basis.

Theorem 2.3 (Stone-Weierstrass, [4], p. 823)**.** *Suppose the set E is a vector-subspace of the space of continuous functions $C(S, \mathbb{C})$, where S is a compact, Hausdorff space. If:*

(i) for each point $x \in S$ there exists an element $f \in E$ such that $f(x) \neq 0$,

(ii) for every pair of distinct points $x, y \in S$ there exists an element $f \in E$ such that $f(x) \neq f(y)$, and

(iii) E is closed under multiplication and complex conjugation,

then E is dense in $C(S, \mathbb{C})$ under the max norm.

Proposition 2.4. *The set B is an orthonormal basis for $L^2([0,1])$; that is,*

$$\langle e^{2\pi imx}, e^{2\pi inx}\rangle = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n, \end{cases}$$

and the span of B is dense in $L^2([0,1])$.

Proof. We first show that the set B is orthonormal. If $m \neq n$, then

$$\begin{aligned}
\langle e^{2\pi imx}, e^{2\pi inx}\rangle &= \int_0^1 e^{2\pi imx}\overline{e^{2\pi inx}}\,dx \\
&= \int_0^1 e^{2\pi i(m-n)x}\,dx \\
&= \left.\frac{e^{2\pi i(m-n)x}}{2\pi i(m-n)}\right|_0^1 \\
&= \frac{1}{2\pi i(m-n)} - \frac{1}{2\pi i(m-n)} = 0.
\end{aligned}$$

If $m = n$, then the integral evaluates to $\int_0^1 e^{2\pi i(m-n)x}\,dx = \int_0^1 1\,dx = 1$. This shows that $\langle e^{2\pi imx}, e^{2\pi inx}\rangle = 0$ for each $m \neq n$, and $\left\|e^{2\pi inx}\right\| = 1$.

We next show that $\operatorname{span}(B)$ is dense in $L^2([0,1])$. Since the space of continuous complex-valued functions $C([0,1]/\{0,1\},\mathbb{C})$, where the points 0 and 1 are identified, is dense in $L^2([0,1])$ ([9], p. 153, Theorem 12), it suffices to show $\operatorname{span}(B)$ is dense in $C([0,1]/\{0,1\},\mathbb{C})$ by showing that it satisfies the hypotheses of Theorem 2.3. Observe that $\operatorname{span}(B)$ is a vector-subspace of $C([0,1]/\{0,1\},\mathbb{C})$, as well as the fact $[0,1]/\{0,1\}$ is a compact, Hausdorff space. Since the constant function $e^{2\pi i(0)x} = 1$ is in $\operatorname{span}(B)$, criterion 2.3(i) is satisfied. For 2.3(ii), consider the exponential function $e^{2\pi ix} \in \operatorname{span}(B)$. Let $x_1, x_2 \in [0,1]/\{0,1\}$ such that $x_1 \neq x_2$. Then $e^{2\pi ix_1} \neq e^{2\pi ix_2}$ by comparing real and imaginary parts, satisfying 2.3(ii). Lastly, expanding the product of two finite linear combinations of exponential functions from B shows that the product is in $\operatorname{span}(B)$, and $\operatorname{span}(B)$ is clearly closed under complex conjugation; these satisfy 2.3(iii). With all criteria of Theorem 2.3 satisfied, $\operatorname{span}(B)$ is dense in $C([0,1]/\{0,1\},\mathbb{C})$ with respect to the max norm. Since the L^2-norm is no larger than the max norm ([9], p. 142, Corollary 3), $\operatorname{span}(B)$ is dense in $C([0,1]/\{0,1\},\mathbb{C})$ with respect to the L^2-norm and is therefore dense in $L^2([0,1])$ because $C([0,1]/\{0,1\},\mathbb{C})$ is dense in $L^2([0,1])$. We conclude that B is an orthonormal basis of $L^2([0,1])$. $\square$

We lastly define the *Fourier transform* of a function $f \in L^2([0,1])$ to be

$$\begin{aligned}\hat{f}(n) &= \langle f(x), e^{2\pi i n x} \rangle \\ &= \int_0^1 f(x) e^{-2\pi i n x}\, dx\end{aligned}$$

for every $n \in \mathbb{Z}$. Now all the pieces are in place to prove Parseval's identity.

Theorem 2.5 (Parseval's Identity)**.** *If $f \in L^2([0,1])$, then*

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right|^2 = \|f\|^2 .$$

Proof. Let $f \in L^2([0,1])$, and let $(P_N f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$ be the orthogonal projection of $f(x)$ onto $\operatorname{span}\{e^{2\pi i(-N)}, ..., e^{2\pi i N}\}$. Since $\operatorname{span}(B)$ is dense in $L^2([0,1])$ by Proposition 2.4, for every $\epsilon > 0$, there exists $g \in \operatorname{span}(B)$ such that $\|g - f\| < \epsilon$. With $g \in \operatorname{span}(B)$, there exists some N such that $g \in \operatorname{span}\{e^{2\pi i(-N)}, \cdots, e^{2\pi i N}\}$. It follows that

$$\left\| P_m(f) - f \right\| \leq \|g - f\| < \epsilon$$

for all $m \geq N$. This shows that

$$f(x) = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}.$$

Then by the Pythagorean Theorem ([6], p. 576),

$$\|f\|^2 = \left\| P_N(f) \right\|^2 + \left\| f - P_N(f) \right\|^2 = \sum_{n=-N}^{N} |\hat{f}(n)|^2 + \left\| f - P_N(f) \right\|^2 .$$

Letting N approach infinity, $\left\| f - P_N(f) \right\|$ approaches 0 by what we have just shown, so

$$\|f\|^2 = \lim_{N \to \infty} \sum_{n=-N}^{N} |\hat{f}(n)|^2.$$

Since the series converges absolutely, it therefore follows that $\|f\|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$. $\square$

3. FINDING THE POSITIVE EVEN ZETA VALUES

Back in the introduction where we used Parseval's identity to calculate $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for any $s \in \mathbb{C}$ is the Riemann zeta function, observe that our choice of function $f(x) = -2\pi i \left(x - \frac{1}{2}\right)$ contains the expression $x - \frac{1}{2}$. This expression is also known as the first Bernoulli polynomial, denoted $B_1(x)$. Given how the Fourier transform of f is conveniently $1/n$ for all nonzero n, and f is a constant multiple of $B_1(x)$, we look to use any given Bernoulli polynomial $B_k(x)$ to construct a polynomial whose Fourier transform is $1/n^k$ for all nonzero n. We will then use this polynomial to evaluate $\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ (similar approaches were taken in [2] and [5]). We begin with a definition of the Bernoulli polynomials.

Definition 3.1 (cf. [8], p. 19)**.** We define the *Bernoulli polynomial* $B_k(x)$ for any integer $k \geq 0$ as the polynomial with rational coefficients resulting from the following recurrence relation:

(i) $B_0(x) = 1$,

(ii) $B_k'(x) = kB_{k-1}(x)$, for all $k \in \mathbb{N}$,

(iii) $\int_0^1 B_k(x)\, dx = 0$, for all $k \in \mathbb{N}$.

We further define $B_k \coloneqq B_k(0)$ to be the k^{th} *Bernoulli number.*

From criteria 3.1(ii) and 3.1(iii), we immediately find that $B_k(1) = B_k(0)$ for all integers $k \geq 2$:

$$\begin{aligned} 0 &= \int_0^1 B_{k-1}(x)\, dx \\ &= \int_0^1 \frac{1}{k} B_k'(x)\, dx \\ &= \frac{1}{k} B_k(1) - \frac{1}{k} B_k(0). \end{aligned}$$

By criterion 3.1(i), we further have $B_0(1) = B_0(0) = 1$. However, $B_1(1) = \frac{1}{2} = -B_1(0)$, so $B_k(1) = B_k(0)$ for all non-negative integers $k \neq 1$. Observe that criterion 3.1(iii) fails for $B_0(x)$, since $\int_0^1 B_0(x)\, dx = 1$ is nonzero.

Suppose we attempt to use $B_1(x) = x - \frac{1}{2}$ to prove $\zeta(2) = \frac{\pi^2}{6}$. Observe that the Fourier transform of $B_1(x)$ is $\frac{-1}{2\pi in}$, so we would have to account for the constant $\frac{-1}{2\pi i}$ from the Fourier transform, adding a rather inconvenient extra step to our calculation of the series. We would like to adjust any given Bernoulli polynomial $B_k(n)$ for any $k \in \mathbb{N}$ so that its Fourier transform becomes $1/n^k$ for all nonzero n and thus removing any adjustments to our calculation.

Theorem 3.2. *Define* $f_k^*(x) \coloneqq \frac{(2\pi i)^k}{-k!} B_k(x)$ *for every* $k \in \mathbb{N}$ *and* $0 \leq x \leq 1$*. Then* $\hat{f}_k^*(0) = 0$*, and* $\hat{f}_k^*(n) = 1/n^k$ *for every* $n \neq 0$*.*

Proof. The equality $\hat{f}_k^*(0) = 0$ follows from Definition 3.1(iii). We prove $\hat{f}_k^*(n) = 1/n^k$ for every $n \neq 0$ by showing the Fourier transform of $B_k(x)$ is $\frac{-k!}{(2\pi i)^k} \cdot \frac{1}{n^k}$ using induction on k. For $k = 1$, the Fourier transform of $B_1(x)$ is

$$\int_0^1 B_1(x) e^{-2\pi inx}\, dx = \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi inx}\, dx = \frac{-1}{2\pi in} = \frac{-(1)!}{(2\pi i)^1} \cdot \frac{1}{n}.$$

Now assume that $\int_0^1 B_{k-1}(x) e^{-2\pi inx}\, dx = \frac{-(k-1)!}{(2\pi i)^{k-1}} \cdot \frac{1}{n^{k-1}}$ for any $k \geq 2$. Then

$$\begin{aligned}
\int_0^1 B_k(x) e^{-2\pi inx}\, dx &= \frac{-B_k(x)}{2\pi in} e^{-2\pi inx}\Big|_0^1 + \frac{k}{2\pi in} \int_0^1 B_{k-1}(x) e^{-2\pi inx}\, dx \\
&= B_k(1)\left(\frac{-1}{2\pi in}\right) - B_k(0)\left(\frac{-1}{2\pi in}\right) - \frac{k!}{(2\pi i)^k} \cdot \frac{1}{n^k} \\
&= \frac{-k!}{(2\pi i)^k} \cdot \frac{1}{n^k}.
\end{aligned}$$

This shows $\frac{-k!}{(2\pi i)^k} \cdot \frac{1}{n^k}$ is the Fourier transform of $B_k(x)$ for every $n \neq 0$, and therefore the Fourier transform of $f_k^*(x) = \frac{(2\pi i)^k}{-k!} B_k(x)$ is $1/n^k$ for every such n. $\square$

Consider, as another example, finding the value of $\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}$. Since $\frac{1}{n^6}$ is an even function of n, $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^6}$. The summand $\frac{1}{n^6} = \left|\frac{1}{n^3}\right|^2$, so we require the function f_3^* since its Fourier transform is $1/n^3$ for every nonzero n. The third Bernoulli polynomial is $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, so $f_3^*(x) = \frac{4}{3}\pi^3 ix^3 - 2\pi^3 ix^2 + \frac{2}{3}\pi^3 ix$. Furthermore, $\hat{f}_3^*(0) = 0$ by Theorem 3.2. We now have by Parseval's identity,

$$\zeta(6) = \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^6} = \frac{1}{2} \int_0^1 \left|f_3^*(x)\right|^2 dx = \frac{1}{2} \int_0^1 \left(\frac{4}{3}\pi^3 x^3 - 2\pi^3 x^2 + \frac{2}{3}\pi^3 x\right)^2 dx = \frac{\pi^6}{945}.$$

Suppose we wish to find the value of $\zeta(2k)$ for every $k \in \mathbb{N}$. Then we can use the following corollary to Theorem 3.2.

Corollary 3.3. *For all $k \in \mathbb{N}$, the following equality holds:*

$$\zeta(2k) = \frac{1}{2}\left\| f_k^*(x) \right\|^2.$$

Proof. Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \zeta(2k) &= \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \\ &= \frac{1}{2} \sum_{n \neq 0} \left| \frac{1}{n^k} \right|^2 \\ &= \frac{1}{2} \int_0^1 \left| f_k^*(x) \right|^2 \, dx, \qquad (1) \end{aligned}$$

where line (1) is by Parseval's identity and Theorem 3.2. □

This shows Parseval's identity is capable of finding the exact value of $\zeta(s)$ for every positive even integer $s = 2k$. Furthermore, the relationship $f_k^*(x) = \frac{(2\pi i)^k}{-k!} B_k(x)$ combined with Corollary 3.3 both allow us to prove the formula for the positive even zeta values in terms of the Bernoulli numbers.

Corollary 3.4. *For all $k \in \mathbb{N}$, one has*

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2 \cdot (2k)!}.$$

Proof. Let $k \in \mathbb{N}$, and let $f_k^*(x)$ be as in Theorem 3.2. Then

$$\begin{aligned} \zeta(2k) &= \frac{1}{2} \int_0^1 \left| f_k^*(x) \right|^2 \, dx \\ &= \frac{1}{2} \int_0^1 \left| \frac{(2\pi i)^k}{-(k)!} \cdot B_k(x) \right|^2 \, dx \\ &= \frac{(2\pi)^{2k}}{2 \cdot (k!)^2} \int_0^1 B_k(x)^2 \, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{(2\pi)^{2k}}{2\cdot(k!)^2}\left(\frac{(-1)^{k-1}(k!)^2 B_{2k}}{(2k)!}\right) \qquad (2)\\
&= \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2\cdot(2k)!}. \qquad \square
\end{aligned}
$$

The identity $\int_0^1 B_k(x)^2\,dx = \frac{(-1)^{k-1}(k!)^2 B_{2k}}{(2k)!}$ on line (2) is proved in [1], p. 11, Proposition 1. As a side note, one may construct these $f_k^*(x)$ directly by the use of a recurrence relation modeled after that of the Bernoulli polynomials. That way, knowledge of the Bernoulli polynomials is not required.

Proposition 3.5. *We may construct the functions f_k^* for any $k \in \mathbb{N}$ via the following recurrence relation:*

(i) $f_1^*(x) = -2\pi i\left(x - \frac{1}{2}\right)$,
(ii) $f_{k+1}^*(x) = 2\pi i\left(\int_0^x f_k^*(t)\,dt + \int_0^1 x f_k^*(x)\,dx\right)$.

Proof. Criterion 3.5(i) follows from Theorem 3.2 for $k = 1$. For 3.5(ii), observe that Definition 3.1(ii) shows that $f_{k+1}^{*\prime}(x) = 2\pi i f_k^*(x)$ for any $k \in \mathbb{N}$, and hence $f_{k+1}^*(x) = 2\pi i\int_0^x f_k^*(t)\,dt + C$ for some constant C. Since $\int_0^1 f_k^*(x)\,dx = 0$, we may find the value of C:

$$
\begin{aligned}
0 &= \int_0^1 f_{k+1}^*(x)\,dx\\
&= \int_0^1 \left(2\pi i\int_0^x f_k^*(t)\,dt + C\right)\,dx\\
&= 2\pi i\int_0^1 \left(\int_0^x f_k^*(t)\,dt\right)\,dx + C\\
&= \underbrace{\left(2\pi i x\int_0^x f_k^*(t)\,dt\right)\Bigg|_0^1}_{=0} - 2\pi i\int_0^1 x f_k^*(x)\,dx + C,
\end{aligned}
$$

so $C = 2\pi i\int_0^1 x f_k^*(x)\,dx$. This constructs criterion 3.5(ii). $\square$

4. NON-NEGATIVE RATIONAL FUNCTIONS

Finding the exact value of the positive even zeta values utilizes ideally adjusted Bernoulli polynomials to apply Parseval's identity to. For this section, we transition from series over $\mathbb{N}$ to series over $\mathbb{Z}$ as we find another class of series we are able to calculate using Parseval's identity. To do so, we first compute the Fourier transform of a general function different from f_k^*.

4.1. Infinite Series for Non-negative Rational Functions.

We consider a function defined by the product of a monomial and an exponential: $f_{A,r}(x) = x^A e^{rx}$, where $A \geq 0$ is an integer and $r \in \mathbb{C}$. We first find the Fourier transform $\hat{f}_{A,r}(n)$, and then we rewrite the summand of a series over $\mathbb{Z}$ as the squared modulus of a linear combination of the transformed functions. That way, the sum of the series is equal to the integral of the squared modulus of a linear combination of functions $x^m e^{rx}$ over $[0,1]$. We will use M to denote the span of all functions of form $x^m e^{rx}$, where $m = 0, 1, ..., A$ and $r \in \mathbb{C}$ such that $\frac{r}{2\pi i}$ is not an integer.

Lemma 4.1. *Let $f_{A,r}(x) = x^A e^{rx}$ where $A \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{C}$ such that $\frac{r}{2\pi i}$ is not an integer. Then*

$$\hat{f}_{A,r}(n) = \frac{(-1)^{A+1} A!}{(r-2\pi i n)^{A+1}} + e^r \sum_{m=0}^{A} \frac{(-1)^m A!}{(A-m)!(r-2\pi i n)^{m+1}}$$

for all $n \in \mathbb{Z}$.

Proof. We prove Lemma 4.1 by induction. For $A = 0$, we have

$$\begin{aligned}
\hat{f}_{0,r}(n) &= \int_0^1 f_{0,r}(x) e^{-2\pi i n x}\, dx \\
&= \int_0^1 e^{rx - 2\pi i n x}\, dx \\
&= \frac{1}{r - 2\pi i n}\left(e^{r-2\pi i n} - 1\right) \\
&= \frac{(-1)^{0+1} 0!}{(r-2\pi i n)^{0+1}} + e^{r-2\pi i n} \sum_{m=0}^{0} \frac{(-1)^m 1!}{(0-m)!(r-2\pi i n)^{m+1}}.
\end{aligned}$$

For the inductive step, assume

$$\int_0^1 x^{A-1}e^{(r-2\pi in)x}\,dx = \frac{(-1)^A(A-1)!}{(r-2\pi in)^A} + e^{r-2\pi in}\sum_{m=0}^{A-1}\frac{(-1)^m(A-1)!}{(A-m-1)!(r-2\pi in)^{m+1}}.$$

Then

$$\begin{aligned}
\hat{f}_{A,r}(n) &= \int_0^1 x^A e^{(r-2\pi in)x}\,dx \\
&= \frac{x^A e^{(r-2\pi in)x}}{r-2\pi in}\bigg|_0^1 - \frac{A}{r-2\pi in}\int_0^1 x^{A-1}e^{(r-2\pi in)x}\,dx \\
&= \frac{e^{r-2\pi in}}{r-2\pi in} + \frac{(-1)^{A+1}A!}{(r-2\pi in)^{A+1}} + e^{r-2\pi in}\sum_{m=0}^{A-1}\frac{(-1)^{m+1}A!}{(A-m-1)!(r-2\pi in)^{m+2}} \\
&= \frac{(-1)^{A+1}A!}{(r-2\pi in)^{A+1}} + \frac{e^{r-2\pi in}}{r-2\pi in} + e^{r-2\pi in}\sum_{m=1}^{A}\frac{(-1)^m A!}{(A-m)!(r-2\pi in)^{m+1}} \\
&= \frac{(-1)^{A+1}A!}{(r-2\pi in)^{A+1}} + e^{r-2\pi in}\sum_{m=0}^{A}\frac{(-1)^m A!}{(A-m)!(r-2\pi in)^{m+1}}.
\end{aligned}$$

□

The Fourier transform of $f_{A,r}$ is a finite series whose sum includes terms that are rational functions of n. The structure of these rational functions of n is a fraction with a coefficient (depending on A and r) in the numerator, and the expression $(r-2\pi in)^{m+1}$ in the denominator. When $m = A$, we reach the largest degree denominator, so we next look to find a function in M that depends on A, t, where $t = \frac{r}{2\pi i}$ is not an integer, whose Fourier transform is $\frac{1}{(n-t)^{A+1}}$.

Lemma 4.2. *For all $t \in \mathbb{C} - \mathbb{Z}$ and $A \in \mathbb{N} \cup \{0\}$, there exists a function $F_{A,t} \in M$, where $r = 2\pi it$, such that $\hat{F}_{A,t}(n) = \frac{1}{(n-t)^{A+1}}$.*

Proof. Let us multiply $\frac{1}{(n-t)^{A+1}}$ by the quantity $\frac{(-2\pi i)^{A+1}}{(-2\pi i)^{A+1}}$ to obtain the equivalent expression $(-2\pi i)^{A+1} \cdot \frac{1}{(2\pi it-2\pi in)^{A+1}}$. Assign $r = 2\pi it$, and we have an expression that resembles a summand of the finite series in Lemma 4.1. Our goal is to find a function of form

$$\begin{aligned}
F_{A,t}(x) &= (-2\pi i)^{A+1}\sum_{m=0}^{A} a_m x^m e^{2\pi itx} \\
&= (-2\pi i)^{A+1}\sum_{m=0}^{A} a_m x^m e^{rx}
\end{aligned}$$

such that $\hat{F}_{A,t}(n) = \frac{1}{(n-t)^{A+1}}$. We make use of the fact that the finite series in said lemma includes the term $1/(r-2\pi in)^{A+1}$ along with the other $1/(r-2\pi in)^m$ terms from $m=1$ to $m=A$. The following $(A+1)\times(A+2)$ augmented coefficient matrix represents the linear system

$$\hat{F}_{A,t}(n) = (-2\pi i)^{A+1}\sum_{m=0}^{A} a_m \hat{f}_{m,r}(n) = \frac{1}{(n-t)^{A+1}},$$

where the $\hat{f}_{m,r}(n)$ are as in Lemma 4.1:

$$\left[\begin{array}{ccccccc|c}
e^r-1 & e^r & e^r & \cdots & e^r & e^r & e^r & 0 \\
0 & e^r-1 & 2e^r & \cdots & (A-2)e^r & (A-1)e^r & Ae^r & 0 \\
0 & 0 & 2(e^r-1) & \cdots & (A-2)(A-3)e^r & (A-1)(A-2)e^r & A(A-1)e^r & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (A-2)!(e^r-1) & (A-1)!e^r & \frac{A!}{2}e^r & 0 \\
0 & 0 & 0 & \cdots & 0 & (A-1)!(e^r-1) & A!e^r & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & (-1)^A A!(e^r-1) & 1
\end{array}\right].$$

Row-reducing this upper-triangular matrix gives us the column vector of coefficients $[a_m]_{m=0}^{A}$ necessary to construct $F_{A,t}$. $\square$

In general, a rational function $h(n)$ with complex coefficients is not of form $\frac{1}{(n-t)^{A+1}}$. Although, by using partial fraction decomposition, h can be written as a linear combination of these $\frac{1}{(n-t)^{A+1}}$.

Lemma 4.3. *For all $h \in \mathbb{C}(x)$ in which $\lim_{n\to\infty} h(n) = \lim_{n\to-\infty} h(n) = 0$, there exists a function $f \in M$ such that $\hat{f}(n) = h(n)$.*

Proof. We decompose such $h(n)$ into its partial fractions

$$h(n) = \sum_{j=1}^{L} \frac{C_j}{(n-t_j)^{A_j+1}},$$

where $C_j, t_j \in \mathbb{C}$, $A_j \in \mathbb{N}\cup\{0\}$, and L is the number of partial fractions in the decomposition. By Lemma 4.2, there exists some function $F_{A,t} \in M$ whose Fourier transform is $\frac{1}{(n-t_j)^{A_j+1}}$.

We now use linearity to construct the function

$$f(x) = \sum_{j=1}^{L} C_j F_{A_j,t_j}(x)$$

whose Fourier transform is

$$\hat{f}(n) = \sum_{j=1}^{L} C_j \hat{F}_{A_j,t_j}(n) = h(n).$$ □

In order to use Parseval's identity, we require functions of form $|h(n)|^2$, which is certainly non-negative. We therefore show that any non-negative rational function of real coefficients has this form for some h as in Lemma 4.3.

Lemma 4.4. *For all non-negative $g \in \mathbb{R}(x)$, there exists $h \in \mathbb{C}(x)$ such that $g(n) = |h(n)|^2$.*

Proof. Since g is non-negative, g does not have real zeros or real poles of odd order. Thus, by the Fundamental Theorem of Algebra, $g(n)$ factors as

$$\begin{aligned}\frac{\alpha(n-r_1)(n-\overline{r_1})(n-r_2)(n-\overline{r_2})\cdots(n-r_p)(n-\overline{r_p})}{(n-s_1)(n-\overline{s_1})(n-s_2)(n-\overline{s_2})\cdots(n-s_q)(n-\overline{s_q})} &= \frac{\alpha|n-r_1|^2|n-r_2|^2\cdots|n-r_p|^2}{|n-s_1|^2|n-s_2|^2\cdots|n-s_q|^2} \\ &= \left|\frac{\sqrt{\alpha}(n-r_1)(n-r_2)\cdots(n-r_p)}{(n-s_1)(n-s_2)\cdots(n-s_q)}\right|^2\end{aligned}$$

where $\alpha \geq 0$ is a constant, and each $r_j, s_k \in \mathbb{C}$ with $j = 1, 2, ..., p$ and $k = 1, 2, ..., q$. Observe that the real zeros and real poles of even order are listed as $\overline{r_j} = r_j$ and $\overline{s_k} = s_k$, respectively. Define $h(n) = \frac{\sqrt{\alpha}(n-r_1)(n-r_2)\cdots(n-r_p)}{(n-s_1)(n-s_2)\cdots(n-s_q)}$. □

All the pieces are in place to establish the following result.

Theorem 4.5. *Given $g \in \mathbb{R}(x)$ everywhere non-negative and vanishing at infinity, there exists a function $f \in M$ such that $g(n) = |\hat{f}(n)|^2$.*

Remark. If g is summable over the integers, then combining Theorem 4.5 with Parseval's identity shows that

$$\sum_{n \in \mathbb{Z}} g(n) = \int_0^1 |f(x)|^2 \, dx.$$

Hence, this theorem gives us a way to find the exact sum of the series using an integral over $[0,1]$.

Proof. Since g is non-negative, there exists a function $h \in \mathbb{C}(x)$ such that $g(n) = |h(n)|^2$ by Lemma 4.4. By Lemma 4.3, there exists a function $f \in M$ such that $\hat{f}(n) = h(n)$, so $g(n) = |\hat{f}(n)|^2$. $\square$

Suppose we would like to determine the sum of $\sum_{n\in\mathbb{Z}} g(n)$ for a given non-negative rational g summable over $\mathbb{Z}$. By Theorem 4.5, we can find a function $f(x) = \sum_{n=1}^{L} C_n x^{A_n} e^{r_n x}$ such that $g(n) = |\hat{f}(n)|^2$. Then the value of the infinite series is given by

$$\begin{aligned}
\int_0^1 |f(x)|^2\,dx &= \int_0^1 f(x)\overline{f(x)}\,dx \\
&= \sum_{n=1}^{L}\sum_{m=1}^{L}\left(C_n\overline{C_m}\int_0^1 x^{A_n+A_m}e^{x(r_n+\overline{r_m})}\,dx\right) \\
&= \sum_{n=1}^{L}\sum_{m=1}^{L} C_n\overline{C_m}\cdot \hat{f}_{A_n+A_m,r_n+\overline{r_m}}(0). \qquad (3)
\end{aligned}$$

The value $\hat{f}_{A_n+A_m,r_n+\overline{r_m}}(0)$ is computed in Lemma 4.1.

If we are given such a function g, then the latter three lemmas and Theorem 4.5 provide us with a method to find f such that $g(n) = |\hat{f}(n)|^2$:

1) Express $g(n)$ as $|h(n)|^2$ with $h \in \mathbb{C}(x)$ by factoring g.
2) Decompose $h(n)$ into its partial fractions.
3) Construct $F_{A_j,t_j}(x)$ as seen in Lemma 4.2 for the j^{th} partial fraction by row-reducing the corresponding matrix from said lemma.
4) Further construct $f(x) = \sum_{j=1}^{L} C_j F_{A_j,t_j}(x)$, where L is the number of partial fractions and C_j are coefficients. This gives us $\hat{f}(n) = h(n)$.
5) By Parseval's identity, $\sum_{n\in\mathbb{Z}} g(n) = \int_0^1 |f(x)|^2\,dx$.

Due to the length of time it takes to find the required function of x, as well as computing the integral $\int_0^1 |f(x)|^2\,dx$, the value of these infinite series are best found using software. As a result of the research, we have developed a code package for the mathematics software system SageMath [11]. The package, called "ParsevalSum," inputs a rational function of n and outputs the exact sum of the series. The value of $\int_0^1 |f(x)|^2\,dx$ given by the finite series in equation (3) is the operation used to evaluate the integral. ParsevalSum also computes the even zeta values, and it determines the function $f \in M$ such that $g(n) = |\hat{f}(n)|^2$. The software package can be accessed via GitHub:

`https://github.com/JamesRPoulin/ParsevalSum`.

4.2. **Example: Sum of $\frac{2}{n^2+1}$ Over $\mathbb{Z}$.** For a simple example, consider the series

$$\sum_{n\in\mathbb{Z}} \frac{2}{n^2+1}.$$

One may take the constant 2 outside the summation, but we incorporate it into the example to demonstrate that the following works nicely with constants. Observe that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{2}{n^2+1} &= \sum_{n\in\mathbb{Z}} \frac{(\sqrt{2})^2}{(n-i)(n+i)} \\ &= \sum_{n\in\mathbb{Z}} \left|\frac{\sqrt{2}}{n+i}\right|^2,\end{aligned}$$

so we will look for a function f such that $\hat{f}(n) = \frac{\sqrt{2}}{n+i}$. The denominator has form $(n-t)^{A+1}$ from Lemma 4.2, where $t = -i$ and $A = 0$. Hence, our $r = 2\pi i t = 2\pi$ and $A = 0$, so we row-reduce the matrix

$$\left[\begin{array}{c|c} e^{2\pi}-1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{c|c} 1 & \frac{1}{e^{2\pi}-1} \end{array}\right].$$

Thus, our $\hat{f}(n)$ is equivalent to

$$\hat{f}(n) = (-2\pi i)\left(\frac{\sqrt{2}}{e^{2\pi}-1}\right)\hat{f}_{0,2\pi}(n).$$

Therefore, by linearity, our required function f is

$$f(x) = \left(\frac{-2\pi i\sqrt{2}}{e^{2\pi}-1}\right)e^{2\pi x}.$$

Finally, the exact sum of the infinite series is therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{2}{n^2+1} &= \int_0^1 |f(x)|^2\,dx \\ &= \frac{2\pi(e^{2\pi}+1)}{e^{2\pi}-1} \approx 6.3067.\end{aligned}$$

4.3. **Example: Sum of $\frac{289}{(n^2+4)(n-1/2)^2}$ Over $\mathbb{Z}$.** For an example that requires partial fractions, consider

$$\sum_{n\in\mathbb{Z}} \frac{289}{(n^2+4)(n-1/2)^2}.$$

Again, the constant 289 can be taken out, but the partial fraction decomposition is more convenient with the constant. Observe that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{289}{(n^2+4)(n-1/2)^2} &= \sum_{n\in\mathbb{Z}} \frac{17^2}{(n+2i)(n-2i)(n-1/2)^2} \\ &= \sum_{n\in\mathbb{Z}} \left|\frac{17}{(n+2i)(n-1/2)}\right|^2,\end{aligned}$$

so we will look for a function f such that $\hat{f}(n) = \frac{17}{(n+2i)(n-1/2)}$. Using partial fraction decomposition, we get

$$\hat{f}(n) = \frac{-2+8i}{n+2i} + \frac{2-8i}{n-1/2}.$$

The denominator of each partial fraction has form $(n-t)^{A+1}$ from Lemma 4.2. The quantity $\frac{1}{n+2i}$ shows that the corresponding $A=0$ and $t=-2i$, hence $r = 2\pi i t = 4\pi$. Thus, we row-reduce the matrix

$$\left[\begin{array}{c|c} e^{4\pi}-1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{c|c} 1 & \frac{1}{e^{4\pi}-1} \end{array}\right].$$

Similarly, the quantity $\frac{1}{n-1/2}$ shows that the corresponding $A=0$ and $t=1/2$, hence $r = 2\pi i t = \pi i$. Thus, we row-reduce the matrix

$$\left[\begin{array}{c|c} e^{\pi i}-1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{c|c} 1 & -\frac{1}{2} \end{array}\right].$$

Thus, our $\hat{f}(n)$ is equivalent to

$$\hat{f}(n) = (-2\pi i)\left(\frac{-2+8i}{e^{4\pi}-1}\right)\hat{f}_{0,4\pi}(n) + (-2\pi i)(-1+4i)\hat{f}_{0,\pi i}(n).$$

Therefore, by linearity, our required function f is

$$\begin{aligned} f(x) &= \left(\frac{16\pi+4\pi i}{e^{4\pi}-1}\right) f_{0,4\pi}(x) + (8\pi+2\pi i) f_{0,\pi i}(x) \\ &= \left(\frac{16\pi+4\pi i}{e^{4\pi}-1}\right) e^{4\pi x} + (8\pi+2\pi i)e^{\pi i x}. \end{aligned}$$

Finally, the exact sum of the infinite series is therefore

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \frac{289}{(n^2+4)(n-1/2)^2} &= \int_0^1 |f(x)|^2\, dx \\ &= 68\pi^2 - \frac{30\pi(e^{4\pi}+1)}{e^{4\pi}-1} \approx 576.8847. \end{aligned}$$

4.4. **Example: Sum of $\frac{1}{(n^2+1)(n^2+4)^2}$ Over $\mathbb{Z}$.** For an example that uses a larger matrix, consider

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+1)(n^2+4)^2}.$$

Observe that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \frac{1}{(n^2+1)(n^2+4)^2} &= \sum_{n\in\mathbb{Z}} \frac{1^2}{(n+i)(n-i)(n+2i)^2(n-2i)^2} \\ &= \sum_{n\in\mathbb{Z}} \left|\frac{1}{(n+i)(n+2i)^2}\right|^2, \end{aligned}$$

so we will look for a function f such that $\hat{f}(n) = \frac{1}{(n+i)(n+2i)^2}$. Using partial fraction decomposition, we get

$$\hat{f}(n) = -\frac{1}{n+i} + \frac{1}{n+2i} + \frac{i}{(n+2i)^2}.$$

The denominator of each partial fraction has form $(n-t)^{A+1}$ from Lemma 4.2. The first quantity $\frac{1}{n+i}$ shows that the corresponding $A=0$ and $t=-i$, and hence our $r = 2\pi i t = 2\pi$.

Thus, we row-reduce the matrix

$$\left[\begin{array}{c|c} e^{2\pi}-1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{c|c} 1 & \frac{1}{e^{2\pi}-1} \end{array}\right].$$

The second quantity $\frac{1}{n+2i}$ shows that the corresponding $A=0$ and $t=-2i$, hence $r=2\pi it = 4\pi$. We then row-reduce the matrix

$$\left[\begin{array}{c|c} e^{4\pi}-1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{c|c} 1 & \frac{1}{e^{4\pi}-1} \end{array}\right].$$

The third quantity $\frac{1}{(n+2i)^2}$ shows that the corresponding $A=1$ and $r=4\pi$. We then row-reduce the 2×3 matrix

$$\left[\begin{array}{cc|c} e^{4\pi}-1 & e^{4\pi} & 0 \\ 0 & -(e^{4\pi}-1) & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{cc|c} 1 & 0 & \frac{e^{4\pi}}{(e^{4\pi}-1)^2} \\ 0 & 1 & -\frac{1}{e^{4\pi}-1} \end{array}\right].$$

Thus, our $\hat{f}(n)$ is equivalent to

$$\begin{aligned}\hat{f}(n) &= (-2\pi i)\left(-\frac{1}{e^{2\pi}-1}\right)\hat{f}_{0,2\pi}(n) + (-2\pi i)\left(\frac{1}{e^{4\pi}-1}\right)\hat{f}_{0,4\pi}(n) \\ &\quad + (-2\pi i)^2\left(\left(-\frac{i}{e^{4\pi}-1}\right)\hat{f}_{1,4\pi}(n) + \left(\frac{ie^{4\pi}}{(e^{4\pi}-1)^2}\right)\hat{f}_{0,4\pi}(n)\right).\end{aligned}$$

Therefore, by linearity, our required function f is

$$f(x) = \frac{2\pi i e^{2\pi x}}{e^{2\pi}-1} - \frac{2\pi i e^{4\pi x}}{e^{4\pi}-1} + \frac{4\pi^2 ixe^{4\pi x}}{e^{4\pi}-1} - \frac{4\pi^2 e^{4\pi} ie^{4\pi x}}{(e^{4\pi}-1)^2}.$$

Finally, the exact sum of the infinite series is therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+1)(n^2+4)^2} &= \int_0^1 |f(x)|^2\,dx \\ &= \frac{5\pi e^{8\pi} + 32\pi e^{6\pi} - 24\pi^2 e^{4\pi} - 32\pi e^{2\pi} - 5\pi}{144e^{8\pi} - 288e^{4\pi} + 144} \approx 0.1104.\end{aligned}$$

Note that there are multiple choices of $\hat{f}$ in each of these examples, especially this one. We could have chosen, for instance,

$$\hat{f}(n) = \frac{1}{(n+i)(n-2i)(n+2i)}$$

for this example. With this choice, our corresponding function f is

$$f(x) = \frac{\pi i e^{4\pi x}}{2(e^{4\pi}-1)} - \frac{2\pi i e^{2\pi x}}{3(e^{2\pi}-1)} + \frac{\pi i e^{-4\pi x}}{6(e^{-4\pi}-1)}.$$

Nonetheless, the integral $\int_0^1 |f(x)|^2\, dx$ has the same value as the previous integral. This shows that some choices of $\hat{f}$ may be better than others. In this case, the second choice of $\hat{f}$ only requires row-reducing augmented 1×2 matrices within Theorem 4.5, and integration by parts or computing a triple sum in equation (3) are both not required when evaluating the integral.

4.5. **More Interesting Identities.** Theorem 4.5 provides us with some interesting identities.

Proposition 4.6. *For all $t \in \mathbb{R} - \mathbb{Z}$, one has*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-t)^2} = \pi^2 \csc^2(\pi t).$$

Proof. Suppose $t \in \mathbb{R} - \mathbb{Z}$. With $\frac{1}{(n-t)^2} = \left|\frac{1}{n-t}\right|^2$, our choice of $\hat{f}(n)$ is $\frac{1}{n-t}$, which means our $A = 0$ and $r = 2\pi i t$. We thus find that $f(x) = \frac{-2\pi i \cdot e^{2\pi i t x}}{e^{2\pi i t}-1}$. Now we apply Parseval's identity:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{(n-t)^2} &= \int_0^1 \left|\frac{-2\pi i \cdot e^{2\pi i t x}}{e^{2\pi i t}-1}\right|^2 dx \\
&= \int_0^1 \left(\frac{2\pi}{|e^{2\pi i t}-1|}\right)^2 dx \\
&= \frac{4\pi^2}{2-2\cos(2\pi t)} \\
&= \pi^2 \csc^2(\pi t),
\end{aligned}$$

with the last equality holding by the half-angle identity of $\sin^2(\pi t)$. □

This identity makes sense since adding 1 (or any integer) to n in the summand does not change the value of the series. Let us next consider the same series, but for t a complex number.

Proposition 4.7. *More generally, for all $t \in \mathbb{C}$ such that $\mathrm{Im}(t) \neq 0$, one has*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|n-t|^2} = \frac{\pi - \pi e^{-4\pi \mathrm{Im}(t)}}{\mathrm{Im}(t)(e^{-4\pi\mathrm{Im}(t)} - 2e^{-2\pi\mathrm{Im}(t)}\cos(2\pi\mathrm{Re}(t)) + 1)}.$$

Proof. With $f(x) = \frac{-2\pi i \cdot e^{2\pi i t x}}{e^{2\pi i t}-1}$ from before, now our integral becomes

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{|n-t|^2} &= \int_0^1 \left| \frac{-2\pi i \cdot e^{2\pi i t x}}{e^{2\pi i t} - 1} \right|^2 dx \\
&= \int_0^1 \left| \frac{-2\pi i \cdot e^{2\pi i x(\mathrm{Re(t)} + i\mathrm{Im}(t))}}{e^{2\pi i(\mathrm{Re(t)} + i\mathrm{Im}(t))} - 1} \right|^2 dx \\
&= \int_0^1 \frac{4\pi^2 e^{-4\pi\mathrm{Im}(t)x}}{|e^{-2\pi\mathrm{Im}(t)}\cos(2\pi\mathrm{Re}(t)) - 1 + ie^{-2\pi\mathrm{Im}(t)}\sin(2\pi\mathrm{Re}(t))|^2}\, dx \\
&= \int_0^1 \frac{4\pi^2 e^{-4\pi\mathrm{Im}(t)x}}{e^{-4\pi\mathrm{Im}(t) - 2e^{-2\pi\mathrm{Im}(t)}}\cos(2\pi\mathrm{Re}(t))+1}\, dx \\
&= \frac{\pi - \pi e^{-4\pi\mathrm{Im}(t)}}{\mathrm{Im}(t)(e^{-4\pi\mathrm{Im}(t)} - 2e^{-2\pi\mathrm{Im}(t)}\cos(2\pi\mathrm{Re}(t)) + 1)}.
\end{aligned}$$

□

Proposition 4.8. *For all $k \in \mathbb{R}^\times$, one has*

$$\sum_{n\in\mathbb{Z}} \frac{k^2}{n^2+k^2} = \pi k \cdot \coth(\pi k).$$

Proof. Let k be any nonzero real number. Observe that the summand factors as $\frac{k^2}{(n+ik)(n-ik)}$, so

$$\sum_{n\in\mathbb{Z}} \frac{k^2}{n^2+k^2} = \sum_{n\in\mathbb{Z}} \left| \frac{k}{n+ik} \right|^2.$$

Thus, our choice of $\hat{f}(n)$ is $\frac{k}{n+ik}$. We see that our $A = 0$ and $t = -ik$, so $r = 2\pi k$. We then find that $f(x) = \frac{-2\pi i k \cdot e^{2\pi k x}}{e^{2\pi k}-1}$.

By Parseval's identity,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{k^2}{n^2+k^2} &= \int_0^1 \left| \frac{-2\pi ik \cdot e^{2\pi kx}}{e^{2\pi k}-1} \right|^2 dx \\
&= \frac{4\pi^2 k^2}{(e^{2\pi k}-1)^2} \int_0^1 e^{4\pi kx}\, dx \\
&= \frac{\pi k(e^{4\pi k}-1)}{(e^{2\pi k}-1)^2} \\
&= \frac{\pi k(e^{2\pi k}+1)}{e^{2\pi k}-1} \\
&= \pi k \cdot \coth(\pi k).
\end{aligned}$$

□

This identity, in particular, is very interesting. The sum of $\frac{k^2}{n^2+k^2}$ as n ranges through all the integers is analogous to the integral

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{k^2}{x^2+k^2}\, dx &= k \cdot \arctan\left(\frac{x}{k}\right)\Big|_{-\infty}^{\infty} \\
&= \pi|k|
\end{aligned}$$

where x ranges through all real numbers.

5. GENERAL RATIONAL FUNCTIONS

Theorem 4.5 provides a way to find the exact sum of a non-negative rational function over $\mathbb{Z}$, but most rational functions are not non-negative. Of course, we can handle non-positive rational functions by simply factoring out -1 from the summand, which results in a non-negative summand, but how would we handle rational functions that are neither non-negative nor non-positive? Such an expression cannot be written in the form $|h(n)|^2$, but we may instead consider a difference of such expressions. Such a difference can be determined for any rational function by using the following lemma, and thus extending Theorem 4.5 even further.

5.1. Infinite Series for General Rational Functions.

Lemma 5.1. *Any rational function with real coefficients is a difference of two non-negative rational functions.*

Proof. Let $\frac{p(n)}{q(n)}$ be a rational function of n with polynomials p and q of real coefficients. Multiplying both the numerator and denominator by $q(n)$ yields $\frac{p(n)q(n)}{(q(n))^2}$, for which the denominator is a non-negative polynomial. Considering the numerator, it is a polynomial of form $\sum_{k=0}^{\deg(pq)} a_k n^k$ where $a_k \in \mathbb{R}$ is the coefficient of the n^k term. For k even, $\frac{n^k}{(q(n))^2}$ is a non-negative rational function. For k odd, observe that

$$\frac{n^k}{(q(n))^2} = \frac{n^{k+1} + n^k + n^{k-1}}{(q(n))^2} - \frac{n^{k+1}}{(q(n))^2} - \frac{n^{k-1}}{(q(n))^2}$$

is a linear combination of non-negative rational functions since $k+1$ and $k-1$ are even, and $n^{k+1} + n^k + n^{k-1} = n^{k-1}(n^2+n+1) \geq 0$ for all real n. Lastly, add the non-negative rational functions together, and then add the non-positive rational functions together. Factoring out -1 from the non-positive quantity, we are left with a difference of two non-negative rational functions. $\square$

We now have an immediate corollary to Theorem 4.5 that gives us a means to calculate the exact sum of any summable rational function over the integers using Parseval's Identity.

Corollary 5.2. *Suppose $g \in \mathbb{R}(x)$. There exist functions $u, v \in M$ such that*

$$g(n) = \left|\hat{u}(n)\right|^2 - \left|\hat{v}(n)\right|^2.$$

Remark. Recall that M is the span of all functions of form $x^m e^{rx}$, where $m = 0, 1, ..., A$ with non-negative integer A, and $r \in \mathbb{C}$ such that $\frac{r}{2\pi i}$ is not an integer. If g is summable over the integers, then the exact value of $\sum_{n\in\mathbb{Z}} g(n)$ is that of the difference of integrals

$$\int_0^1 |u(x)|^2 \, dx - \int_0^1 |v(x)|^2 \, dx.$$

Proof. By Lemma 5.1, $g(n)$ is a difference of two non-negative rational functions of n, each of which we know has form $|\hat{f}(n)|^2$ for some function $f \in M$ by Theorem 4.5. Denote this difference as $g(n) = |\hat{u}(n)|^2 - |\hat{v}(n)|^2$ for functions $u, v \in M$. $\square$

Remark. It should be noted that a more standard method for finding the sum of an infinite series for a general rational function over $\mathbb{Z}$ involves contour integration. By taking a contour integral of the function $g(z)\cot(\pi z)$, where z is a complex variable and g is a rational function summable over $\mathbb{Z}$, we can use the residue theorem to calculate the contour integral and write the resulting value in terms of $\sum_{n\in\mathbb{Z}} g(n)$. Our method uses Parseval's identity instead of contour integration, and thus provides flexibility to how we may find the exact value of $\sum_{n\in\mathbb{Z}} g(n)$ for such g.

5.2. **Application: Dirichlet L-Functions.** A useful application to Corollary 5.2 is finding certain special values of Dirichlet L-functions.

Definition 5.3. A *Dirichlet character χ of modulus k*, where $k \in \mathbb{N}$, is a group homomorphism $\chi : (\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times$. By defining

$$\chi(n) := \begin{cases} \chi(n \bmod k), & \gcd(n,k) = 1 \\ 0, & \gcd(n,k) > 1, \end{cases}$$

the domain of χ is extended to $\mathbb{Z}$. Furthermore, we define the *Dirichlet L-function* as the infinite series

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

for any $s\in\mathbb{C}$. Lastly, we say that χ is *odd* if $\chi(-1)=-1$, and we say it is *even* if $\chi(-1)=1$.

Remark. Since χ is a group homomorphism, we automatically find that $\chi(1)=1$.

Consider the sum $S(A;q,k)$ defined to be

$$S(A;q,k):=\sum_{\substack{n\in\mathbb{Z}\\ n\equiv q \bmod k}}\frac{1}{n^A}=\sum_{n\in\mathbb{Z}}\frac{1}{(kn+q)^A},$$

where $q,k\geq 1$ and $A\geq 2$ are integers. Observe that Theorem 4.5 allows us to compute this sum for even A, and Corollary 5.2 allows us to compute this sum for odd A. If A has the same parity as χ, meaning that either both A and χ are even or they are both odd, then we have the following proposition to relate S with L:

Proposition 5.4. *Let $k\geq 3$ be an integer, χ a Dirichlet character of modulus k, and $A\geq 2$ an integer with the same parity as χ. Then*

$$L(A,\chi)=\sum_{q=1}^{\lfloor k/2\rfloor}\chi(q)S(A;q,k).$$

Proof. With such k, 2, and χ, the right-hand side of the equation is

$$\begin{aligned}\sum_{q=1}^{\lfloor k/2\rfloor}\chi(q)\sum_{\substack{n\in\mathbb{Z}\\ n\equiv q \bmod k}}\frac{1}{n^A}&=\sum_{q=1}^{\lfloor k/2\rfloor}\left(\sum_{\substack{n=1\\ n\equiv q \bmod k}}^{\infty}\frac{\chi(q)}{n^A}+\sum_{\substack{n=-\infty\\ n\equiv q \bmod k}}^{-1}\frac{\chi(q)}{n^A}\right)\\ &=\sum_{q=1}^{\lfloor k/2\rfloor}\left(\sum_{\substack{n=1\\ n\equiv q \bmod k}}^{\infty}\frac{\chi(n)}{n^A}+\sum_{\substack{n=1\\ n\equiv -q \bmod k}}^{\infty}\frac{\chi(-n)}{(-n)^A}\right)\\ &=\sum_{q=1}^{\lfloor k/2\rfloor}\left(\sum_{\substack{n=1\\ n\equiv q \bmod k}}^{\infty}\frac{\chi(n)}{n^A}+\sum_{\substack{n=1\\ n\equiv -q \bmod k}}^{\infty}\frac{\chi(-1)}{(-1)^A}\cdot\frac{\chi(n)}{n^A}\right)\end{aligned}\tag{4}$$

$$= \sum_{q=1}^{\lfloor k/2 \rfloor} \left(\sum_{\substack{n=1 \\ n \equiv q \bmod k}}^{\infty} \frac{\chi(n)}{n^A} + \sum_{\substack{n=1 \\ n \equiv -q \bmod k}}^{\infty} \frac{\chi(n)}{n^A} \right) \tag{5}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^A}$$

$$= L(A, \chi).$$

Line (4) uses the fact that A and χ have the same parity, thus $\frac{\chi(-1)}{(-1)^A} = 1$. Line (5) requires that $\gcd(k, k/2) > 1$ to avoid repeated terms $q = k/2$ whenever k is even, which is true for $k \geq 3$. □

Proposition 5.4 fails for $k = 2$, but since

$$\chi(n) = \begin{cases} 1, & \gcd(n, 2) = 1 \\ 0, & \gcd(n, 2) > 1 \end{cases}$$

is the only Dirichlet character of modulus 2, and $L(s, \chi) = (1 - 2^{-s})\zeta(s)$, nothing is new for $k = 2$. We may show that $L(s, \chi) = (1 - 2^{-s})\zeta(s)$ since

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \tag{6}$$

$$= \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots$$

$$= \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \right) - \frac{1}{2^s} \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \right) \tag{7}$$

$$= \zeta(s) - 2^{-s}\zeta(s)$$

$$= (1 - 2^{-s})\zeta(s).$$

Line (6) uses the fact that χ of modulus 2 sends positive even integers to zero and sends positive odd integers to one. Line (7) adds and subsequently subtracts the reciprocals of positive even integers raised to the power s.

An example of using Parseval's identity to calculate a Dirichlet L-Function is finding the value of $L(3,\chi)$, where χ is an odd Dirichlet character of modulus 3. Observe that

$$\chi(n) = \begin{cases} 1, & n \equiv 1 \bmod 3 \\ -1, & n \equiv 2 \bmod 3 \\ 0, & n \equiv 0 \bmod 3 \end{cases}$$

is the only odd Dirichlet character of modulus 3. Then by Proposition 5.4, we have that

$$L(3,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^3} = \sum_{n\in\mathbb{Z}} \frac{1}{(3n+1)^3}.$$

Using Corollary 5.2 on $\sum_{n\in\mathbb{Z}} \frac{1}{(3n+1)^3}$, we find that $L(3,\chi) = \frac{4}{243}\sqrt{3}\pi^3$.

6. POSITIVE EVEN ZETA VALUES REVISITED

A similar method to finding the required function f given a summable rational function of n applies to finding such f given a rational function $1/n^{2A}$ for any $A \in \mathbb{N}$. This method does not depend on knowing f_1^* from Proposition 3.5 nor the Bernoulli polynomials beforehand and instead is solely dependent on the summand. Thus, we show an alternative method to Theorem 3.2 by first finding the Fourier transform of just the monomial $f_A(x) = x^A$ for any $A \in \mathbb{N}$ and then writing $1/n^{2A}$ as a linear combination of these $\hat{f}_A(n)$. That way, we find a polynomial function f such that $\hat{f}(n) = 1/n^{2A}$ for all $n \neq 0$. Extra adjustments are necessary to account for the fact that $1/n^{2A}$ is undefined at $n = 0$, and thus the rational function is not summable over the integers.

6.1. The Alternate Method.

Lemma 6.1. *Let $f_A(x) = x^A$ for any $A \in \mathbb{N}$. Then $\hat{f}_A(0) = \frac{1}{A+1}$, and for all $n \neq 0$, one has*

$$\hat{f}_A(n) = \sum_{m=1}^{A} \frac{-A!}{(A-m+1)!(2\pi i)^m n^m}.$$

Proof. If $n = 0$, then

$$\int_0^1 f_A(x) e^{-2\pi i(0)x}\,dx = \int_0^1 x^A\,dx = \frac{1}{A+1}.$$

Suppose $n \neq 0$ and proceed by induction. For $A = 1$, we have

$$\hat{f}_1(n) = \int_0^1 x e^{-2\pi i n x}\,dx = -\frac{1}{2\pi i n} = \frac{-(1)!}{(1-1+1)!(2\pi i)^1 n^1}.$$

For the inductive step, assume $\hat{f}_{A-1}(n) = \sum_{m=1}^{A-1} \frac{-(A-1)!}{(A-m)!(2\pi i)^m n^m}$. Then

$$\begin{aligned}
\hat{f}_A(n) &= \int_0^1 x^A e^{-2\pi i n x}\,dx \\
&= -\frac{x^A e^{-2\pi i n x}}{2\pi i n}\bigg|_0^1 + \frac{A}{2\pi i n}\int_0^1 x^{A-1} e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n} + \frac{A}{2\pi i n}\sum_{m=1}^{A-1} \frac{-(A-1)!}{(A-m)!(2\pi i)^m n^m}
\end{aligned}$$

$$= -\frac{1}{2\pi i n} + \sum_{m=2}^{A} \frac{-A!}{(A-m+1)!(2\pi i)^m n^m}$$

$$= \sum_{m=1}^{A} \frac{-A!}{(A-m+1)!(2\pi i)^m n^m}. \qquad \square$$

We proceed to the next lemma similarly to how we proceeded from Lemma 4.1 to Lemma 4.2.

Lemma 6.2. *For all $A \in \mathbb{N}$, there exists a polynomial function f such that $\hat{f}(n) = 1/n^A$ for all $n \neq 0$.*

Proof. We look to find a function of form $f(x) = \sum_{m=1}^{A} a_m x^m$ such that $\hat{f}(n) = 1/n^A$ for all $n \neq 0$. We make use of the fact that the finite series in Lemma 6.1 includes the term $1/n^A$ along with the other $1/n^m$ terms from $m = 1$ to $m = A - 1$. The following $A \times (A+1)$ augmented coefficient matrix represents the linear system

$$\hat{f}(n) = \sum_{m=1}^{A} a_m \hat{f}_m(n),$$

where the $\hat{f}_m(n)$ are as in Lemma 6.1:

$$\left[\begin{array}{cccccccc|c}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 2 & 3 & 4 & \cdots & A-2 & A-1 & A & 0 \\
0 & 0 & 6 & 12 & \cdots & (A-2)(A-3) & (A-1)(A-2) & A(A-1) & 0 \\
0 & 0 & 0 & 24 & \cdots & (A-2)(A-3)(A-4) & (A-1)(A-2)(A-3) & A(A-1)(A-2) & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & (A-2)! & \frac{(A-1)!}{2} & \frac{A!}{3} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & (A-1)! & \frac{A!}{2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & A! & -(2\pi i)^A
\end{array}\right].$$

Row-reducing this matrix gives us the column vector of coefficients $[a_m]_{m=1}^{A}$ necessary to construct f. $\square$

We now have the following result.

Theorem 6.3. *For any $A \in \mathbb{N}$, there exists a polynomial function f such that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2A}} = \frac{1}{2}\int_0^1 \left|f(x)\right|^2 dx - \frac{1}{2}\left|\hat{f}(0)\right|^2 .$$

Proof. Observe that $\sum_{n=1}^{\infty} \frac{1}{n^{2A}}$ is equivalent to $\sum_{n=1}^{\infty} \left|\frac{1}{n^A}\right|^2$. By Lemma 6.2, there exists a polynomial function f such that $\hat{f}(n) = 1/n^A$ for all $n \neq 0$. We can thus find the exact sum of the series by instead finding the value of $\int_0^1 |f(x)|^2\, dx$ and $\hat{f}(0)$. The equality by Parseval's identity is

$$\left|\hat{f}(0)\right|^2 + \sum_{n\neq 0}\left|\hat{f}(n)\right|^2 = \int_0^1 \left|f(x)\right|^2 dx.$$

Since $1/n^{2A}$ is an even function of n, we have $\frac{1}{2}\sum_{n\neq 0}\frac{1}{n^{2A}} = \sum_{n=1}^{\infty}\frac{1}{n^{2A}}$, and therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2A}} = \frac{1}{2}\int_0^1 \left|f(x)\right|^2 dx - \frac{1}{2}\left|\hat{f}(0)\right|^2 . \qquad \square$$

We may evaluate the integral $\int_0^1 \left|f(x)\right|^2 dx$ directly to find

$$\int_0^1 \left|f(x)\right|^2 dx = \sum_{p=1}^{A}\sum_{q=1}^{A} \frac{a_p\overline{a_q}}{p+q+1}, \tag{8}$$

where this f is as in the statement and proof of Lemma 6.3, and a_m is the m^{th} coefficient of f. Furthermore, we know $\hat{f}(0) = \frac{1}{A+1}$ by Lemma 6.1, so we end up with

$$\sum_{n=1}^{\infty} \frac{1}{n^{2A}} = -\frac{1}{2(A+1)^2} + \frac{1}{2}\sum_{p=1}^{A}\sum_{q=1}^{A} \frac{a_p\overline{a_q}}{p+q+1}$$

which is no longer in terms of integrals.

This alternate method is how we programmed ParsevalSum to compute the $(2A)^{\text{th}}$ zeta value, as well as find a polynomial function f such that $\zeta(2A) = \frac{1}{2}\int_0^1 |f(x)|^2\, dx - \frac{1}{2}|\hat{f}(0)|^2$. ParsevalSum uses equation (8) to evaluate the integral.

6.2. **How We Found $f(x)$ for $\zeta(2)$.** This provides some extra insight for how we came up with $f(x) = -2\pi i\left(x - \frac{1}{2}\right)$ for our choice of function when calculating the value of $\zeta(2)$.

However, this method instead yields $f(x) = -2\pi i x$ because this method constructs the required function f with a zero constant term. Since we need to find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we first will look for the function f whose Fourier Transform is $\hat{f}(n) = 1/n$ for all $n \neq 0$. Note that we will use the function $\hat{f}_A(n)$ as the Fourier transform of $f_A(x) = x^A$ for such n. Following Theorem 6.3,

$$\begin{aligned}\frac{1}{n} &= a_1 \hat{f}_1(n) \\ &= a_1 \left(\frac{-1}{2\pi i n} \right)\end{aligned}$$

for all $n \neq 0$, so $a_1 = -2\pi i$. Therefore, by linearity, $f(x) = -2\pi i x$. With this choice of f, we find that $\frac{1}{2}\int_0^1 |f(x)|^2\,dx = \frac{1}{2}\int_0^1 |-2\pi i x|^2\,dx = \frac{2\pi^2}{3}$ and $\frac{1}{2}|\hat{f}(0)|^2 = \frac{1}{2}\left|\int_0^1 -2\pi i x\,dx\right|^2 = \frac{\pi^2}{2}$. We then get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\int_0^1 \left|-2\pi i x\right|^2\,dx - \frac{1}{2}\left|\int_0^1 -2\pi i x\,dx\right|^2 = \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{\pi^2}{6}.$$

6.3. **Example:** $\zeta(4)$**.** Consider $\zeta(4)$. To compute $\sum_{n=1}^{\infty} \frac{1}{n^4}$, we will look for a function f such that $\hat{f}(n) = 1/n^2$ for all $n \neq 0$. We next row-reduce the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & -(2\pi i)^2 \end{array}\right] \text{ to obtain } \left[\begin{array}{cc|c} 1 & 0 & -2\pi^2 \\ 0 & 1 & 2\pi^2 \end{array}\right].$$

This shows that

$$\begin{aligned}\hat{f}(n) &= \frac{1}{n^2} \\ &= 2\pi^2 \cdot \hat{f}_2(n) - 2\pi^2 \cdot \hat{f}_1(n),\end{aligned}$$

and so our desired function is $f(x) = 2\pi^2 x^2 - 2\pi^2 x$ by linearity. The sum of the series is therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2}\int_0^1 \left|2\pi^2 x^2 - 2\pi^2 x\right|^2\,dx - \frac{1}{2}\left|\int_0^1 (2\pi^2 x^2 - 2\pi^2 x)\,dx\right|^2 = \frac{\pi^4}{90}.$$

6.4. **Example:** $\zeta(10)$**.** For a more complicated example, consider $\zeta(10)$. In order to compute $\sum_{n=1}^{\infty} \frac{1}{n^{10}}$, we will look for a function f such that $\hat{f}(n) = 1/n^5$ for all $n \neq 0$. We next row-reduce the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 6 & 12 & 20 & 0 \\ 0 & 0 & 0 & 24 & 60 & 0 \\ 0 & 0 & 0 & 0 & 120 & -(2\pi i)^5 \end{array}\right] \text{ to obtain } \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{2}{45}\pi^5 i \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{9}\pi^5 i \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}\pi^5 i \\ 0 & 0 & 0 & 0 & 1 & -\frac{4}{15}\pi^5 i \end{array}\right].$$

This shows that

$$\begin{aligned} \hat{f}(n) &= \frac{1}{n^5} \\ &= -\frac{4}{15}\pi^5 i \cdot \hat{f}_5(n) + \frac{2}{3}\pi^5 i \cdot \hat{f}_4(n) - \frac{4}{9}\pi^5 i \cdot \hat{f}_3(n) + \frac{2}{45}\pi^5 i \cdot \hat{f}_1(n), \end{aligned}$$

and so our desired function is $f(x) = -\frac{4}{15}\pi^5 i x^5 + \frac{2}{3}\pi^5 i x^4 - \frac{4}{9}\pi^5 i x^3 + \frac{2}{45}\pi^5 i x$ by linearity. The sum of the series is therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{10}} &= \frac{1}{2}\int_0^1 \left|f(x)\right|^2 \, dx - \frac{1}{2}\left|\int_0^1 f(x)\, dx\right|^2 \\ &= \frac{1}{2}\left(\frac{2\pi^{10}}{93555}\right) - \frac{1}{2}\left|0\right|^2 \\ &= \frac{\pi^4}{93555}. \end{aligned}$$

7. EXTENSIONS

Over the course of this paper, we have established many results that allow us to use Parseval's identity to calculate the exact value of an infinite series, as well as some useful applications. There are minor extensions that take the idea of our results to introduce other methods to finding the exact sum of an infinite series.

7.1. Exact Sums Over $\mathbb{N}$.

While Parseval's identity finds the exact sum of series over $\mathbb{Z}$, we are often interested in finding the exact sum of series over $\mathbb{N}$ instead. Calculus already includes a couple classes of functions for which we can compute exact sums over $\mathbb{N}$, but Theorems 3.2 and 4.5 introduce two more classes of functions. We may thus find the exact sum of a linear combination of the following list of functions over the natural numbers:

- telescoping summands
- geometric summands
- summands of $1/n^{2k}$ for any $k \in \mathbb{N}$
- even rational summands over $\mathbb{Z}$. If g is an even rational function summable over the integers, then we may find its exact sum over the natural numbers by using $\sum_{n=1}^{\infty} g(n) = \frac{1}{2}\sum_{n\in\mathbb{Z}} g(n) - \frac{1}{2}g(0)$ and applying Parseval's identity to $\sum_{n\in\mathbb{Z}} g(n)$ using either Theorem 4.5 or Corollary 5.2.

For example, one can find the value of $\sum_{n=1}^{\infty} \frac{2n^2+n+2}{n^4+n^3+2n^2+2n}$ by decomposing the summand and using linearity to obtain $\sum_{n=1}^{\infty} \frac{1}{n^2+2} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$. The first sum may be found by instead using Theorem 4.5 to compute $\frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+2} - \frac{1}{4}$, and the second sum evaluates to 1 as it is a telescoping series. Likewise, we can find $\sum_{n=1}^{\infty} \frac{n^4+3^n}{n^4\cdot 3^n}$ by rewriting the series as $\sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{1}{3^n}$. Theorem 3.2 allows us to find the value of the first series, and the second series is a geometric series starting at $n = 1$.

7.2. Exact Sums Using Fourier Series.

Another way to compute the exact sum of a series over $\mathbb{Z}$ is to use the Fourier series expansion of f. That is, if f is a continuously differentiable function on $(0, 1)$ with the endpoints identified, and the right-derivative exists at $x = 0$ and the left-derivative exists at $x = 1$, then it is equal pointwise to its Fourier series as

$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\cos(2\pi nx) + i\sum_{n\in\mathbb{Z}} \hat{f}(n)\sin(2\pi nx)$. If we let $x = 0$ and $x = 1$, then $\frac{f(0)+f(1)}{2} = \sum_{n\in\mathbb{Z}} \hat{f}(n)$ provided that $\hat{f}(n)$ is summable ([7], p. 37, Theorem 2.3.4). If instead $x = 1/2$, then we get the sum of the alternating series $f(1/2) = \sum_{n\in\mathbb{Z}}(-1)^n\hat{f}(n)$ provided it is summable. The former gives us a way to compute the exact sum of a series over $\mathbb{Z}$ whose summand is a rational function in $\mathbb{C}$. If $\sum_{n\in\mathbb{Z}}(-1)^n\hat{f}(n)$ diverges because either the real or imaginary part of $(-1)^n\hat{f}(n)$ is not summable, but the other part is summable, then we may instead compute only the sum of the summable part. The same thing applies to $\sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Consider, for example, $\mathrm{Im}\left(\sum_{n\in\mathbb{Z}} \frac{1}{n+i}\right)$. The real part of the summand $\frac{n}{n^2+1}$ is not summable over $\mathbb{Z}$, but its imaginary part $-\frac{1}{n^2+1}$ is. Take $\hat{f}(n) = \frac{1}{n+i}$, and apply Theorem 4.5 to $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{n^2+1}$ using our initial choice of $\hat{f}(n)$. We then get the resulting $f(x) = \frac{-2\pi ie^{2\pi x}}{e^{2\pi}-1}$ we need to find that $\mathrm{Im}\left(\sum_{n\in\mathbb{Z}} \frac{1}{n+i}\right) = \mathrm{Im}\left(\frac{f(0)+f(1)}{2}\right) = -\pi\coth(\pi)$, which is what we would expect from Proposition 4.8. If we instead use $x = 1/2$, then the alternating series $\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n+i}$ is summable in both the real and imaginary parts and has exact value $f(1/2) = -\frac{2\pi e^{\pi}}{e^{2\pi}-1}i$. The real part evaluates to 0 because the real part of the summand $\frac{(-1)^n \cdot n}{n^2+1}$ is a summable odd function of n by the Alternating Series Test.

REFERENCES

[1] Takashi Agoh and Karl Dilcher. Integrals of Products of Bernoulli Polynomials. *Journal of Mathematical Analysis and Applications*, 381(1):10–16, 2011.

[2] Krishnaswami Alladi and Colin Defant. Revisiting the Riemann Zeta Function at Positive Even Integers. *International Journal of Number Theory*, 14(07):1849–1856, 2018.

[3] Oscar Ciaurri. Euler's Product Expansion for the Sine: An Elementary Proof. *The American Mathematical Monthly*, 122(7):693–695, 2015.

[4] Louis De Branges. The Stone-Weierstrass Theorem. *Proceedings of the American Mathematical Society*, 10(5):822–824, 1959.

[5] Asghar Ghorbanpour and Michelle Hatzel. Parseval's Identity and Values of Zeta Function at Even Integers. *arXiv preprint arXiv:1709.09326*, 2017.

[6] Anthony W Knapp. *Basic Real Analysis*. Springer Science & Business Media, 2005.

[7] J Korevaar. Fourier analysis and related topics. *Amsterdam, Spring*, 2011.

[8] M Ram Murty. *Problems in Analytic Number Theory*, volume 206. Springer Science & Business Media, 2008.

[9] Halsey Lawrence Royden and Patrick Fitzpatrick. *Real Analysis*, volume 32. Macmillan New York, 1988.

[10] David Salwinski. Euler's Sine Product Formula: An Elementary Proof. *The College Mathematics Journal*, 49(2):126–135, 2018.

[11] The Sage Developers. *SageMath 9.0, the Sage Mathematics Software System (Version 0.5.2)*, 2020. `https://www.sagemath.org`.

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