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Investigation of Student Understanding of Implicit Differentiation

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INVESTIGATION OF STUDENT UNDERSTANDING
OF IMPLICIT DIFFERENTIATION

By

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B.A. University of Maine, 2015

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Challenges that students face in first semester calculus have been found to be a factor in high attrition rates of students from science, technology, engineering, and mathematics (STEM) majors. With an increase in the demand for STEM graduates, an attempt must be made to remedy this issue. Research has shown that students have difficulties with many topics in the realm of calculus. Of these, students have been found to struggle with the concept of derivative and ideas related to it. However, some derivative topics have not been examined as thoroughly as others. Implicit differentiation, a technique that allows us to differentiate equations that are not explicit functions, is one such topic. The goal of the study was to examine student understanding of and ability to carry out implicit differentiation and to identify whether student work on such problems is influenced by the same factors as other derivative topics or if there are additional challenges that arise for students in the context of implicit differentiation.

Written surveys were collected from 136 first semester calculus students. Clinical interviews were then conducted with five calculus students. For the surveys, students were tasked with explaining what they believed implicit differentiation to be and asked to solve
problems using this technique. The interviews were similar in nature, but students were additionally asked to explain their reasoning and thought processes.

Findings suggest that implicit differentiation is challenging for students. Approximately 50% of the survey responses to implicit differentiation problems were correct. The interviews suggested that some students had a strong understanding of implicit differentiation and others did not. Students who have a strong understanding of the idea of implicit differentiation appear to be more successful in carrying out the procedures. For other calculus topics, researchers have found that students can show skill with procedures without an understanding of the ideas. This appears to be less so with implicit differentiation. The interviews also suggested that students do not have a strong understanding of what the symbols of differentiation represent.

Student difficulties with implicit differentiation appeared to be influenced by what they knew about function and derivative. Prior research has identified that understanding of function influences students’ abilities to recognize when and how to use derivative rules. In the case of the chain rule, students need a solid understanding of composition of functions. For the product rule, students need to be fluent in function notation and identifying the product of two functions. Difficulties identified in student work on implicit differentiation problems are similar to those found in other areas of calculus. Though, in addition to these areas of difficulty, there appear to be other ideas about function that influence performance. In particular, function equality and the idea of applying operators to both sides of an equation may be sources of difficulty. These ideas have been examined in algebra, but not in the context of calculus. Implications for educators, as well as opportunities for future research, are proposed to address these points of discussion.
ACKNOWLEDGMENTS

I would like to thank all my fans!
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CHAPTER

1. INTRODUCTION

1.1. Importance of calculus for STEM majors

Approximately half of students who choose to pursue science, technology, engineering, and mathematics (STEM) majors either change to different majors or are unsuccessful in finishing their degree (e.g., Holdren & Lander, 2012). Studies have identified that student attrition from these STEM majors often occurs in the first or second year of college, and that student experiences in introductory STEM courses are key factors in making the decision to leave the field (e.g., Watkins & Mazur, 2013). Consequently, calculus, one of the first courses encountered by incoming STEM majors, serves as one of the first hurdles for these students. This is of importance as most students taking calculus are there because some of the ideas and techniques presented within it are required for applications in their own respective majors. As such, a student’s experiences with and performance in calculus play a large part in shaping the decision as to whether or not they would like to stay in the STEM field.

1.2. Calculus and derivatives are difficult

Having discussed the importance of calculus for these students, attention should now be directed to how well students perform in first semester calculus courses. It has been found that such courses have high failure rates and, as such, are indeed steep hurdles for prospective STEM majors to overcome (e.g., Anderson & Loftsgaarden, 1987; Bressoud, Mesa, & Rasmussen, 2015). Of the topics in calculus, differentiation in general has been found to be difficult for students (e.g., Orton, 1983). Differentiation is broken down into multiple sub-topics, with various applications resulting from them, which are taught in a typical calculus course. One such
sub-topic, implicit differentiation, is a specific method of differentiation that relies on an assortment of skills related to the derivative and is needed for various applications. Yet, little is known about what causes students to struggle with it. As such, student understanding of this topic is the focus of this study.

1.3. Available literature

As of right now, there is a sizable amount of literature on student understanding of general calculus concepts such as the derivative. This is not surprising given their importance, and the degree to which students struggle with them in introductory calculus courses. However, much less literature is available about research on student understanding of specific aspects and applications of the derivative. There is some research on the chain rule (e.g., Horvath, 2008), a process that allows you to take the derivative of composite functions which is needed to take derivatives when using implicit differentiation. There is also some research on related rates (e.g., Martin, 2000), an application of derivatives that requires one to use implicit differentiation. These studies do not spend much time discussing student understanding of implicit differentiation, and often just state or make general comments on how students fared when using this technique. That said, while there is some research on topics surrounding implicit differentiation, there is very little literature available with implicit differentiation itself as the focal point. One study (Mirin & Zazkis, 2019) focuses on the mathematical justification of implicit differentiation as a procedure and the case for its importance, though there is not much, if any, literature on the difficulties students have when solving problems using this technique. As such, it could be, and has been, argued that there is a need for research on implicit differentiation to supplement the existing literature on the derivative and chain rule (Speer & Kung, 2016).
1.4. Motivation for researching implicit differentiation

Having served as a teaching assistant in an introductory calculus course on multiple occasions, I have experienced firsthand the difficulties undergraduate students have with implicit differentiation. Likewise, I have also struggled when trying to find ways to help remedy their issues with this topic. This is worrisome, as this is an important topic with applications later in calculus that require students to be proficient with it.

As mentioned in the previous section, there is currently a lack of literature addressing student understanding of implicit differentiation. Thus, it would be beneficial to explore how students think about implicit differentiation and the areas where they struggle when working with it. This would help us understand how student thinking on related topics, such as differentiation and the chain rule, could be used to explain difficulties they have with implicit differentiation as well as if there are any unique difficulties that stem from implicit differentiation itself. Knowing these things could help to identify where student understanding of this topic, and related ones, could be improved. This serves as my primary motivation for looking into this topic.

1.5. What is implicit differentiation?

When students are first taught how to take the derivative of an equation, they are given functions with \( f(x) \) or \( y \) explicitly expressed in terms of a variable \( x \). An example of such an equation would be \( y = x^2 - 2x + 5 \). When taking the derivative of such an equation, students are told that the derivative of \( y \) is \( \frac{dy}{dx} \) or \( f'(x) \). The derivative operator is then applied to the right side of the equation which is written in terms of one variable, usually \( x \). The derivative of each term is then taken using the appropriate basic derivatives rules and then the resulting terms are
combined and simplified as needed. An example of this process, that I will refer to as explicit differentiation, can be seen in Figure 1.1.

<table>
<thead>
<tr>
<th>Explicit differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 - 2x + 5 )</td>
</tr>
<tr>
<td>( y' = \frac{d}{dx}(x^2 - 2x + 5) )</td>
</tr>
<tr>
<td>( y' = 2x - 2 + 0 )</td>
</tr>
<tr>
<td>( y' = 2x - 2 )</td>
</tr>
</tbody>
</table>

Figure 1.1. Explicit differentiation example

However, not all equations are functions. The equation \( x^2 + y = 2y^2 + 1 \), for example, cannot be manipulated in any way to make it a function of \( y \) in terms of \( x \). As such, it is not possible to use explicit differentiation to take the derivative of such an equation. This is where implicit differentiation comes into play. Implicit differentiation is a technique in which you differentiate an equation with respect to a specific variable, usually \( x \) at the time of instruction, and treat all other variables as functions of that variable. When using implicit differentiation, the derivative operator is applied to both sides (left-hand and right-hand) of an equation at the same time. No rearranging of the equation is required prior to applying this technique. As such, it allows us to take the derivative of equations that are not functions. For equations such as these, the derivative depends on one or more variables which introduces a new element to be considered when taking derivatives. Here, variables need to be treated as functions of the variable the equation is being differentiated with respect to. Thus, for example, the derivative of \( y \), when differentiating with respect to \( x \), is \( \frac{dy}{dx} \). An example of the process of implicit differentiation is shown in Figure 1.2.
Implicit differentiation

\[ x^2 + y = 2y^2 + 1 \]

Original equation

\[ \frac{d}{dx}(x^2 + y) = \frac{d}{dx}(2y^2 + 1) \]

Apply the derivative operator to BOTH sides

\[ 2x + \frac{dy}{dx} = 4y \frac{dy}{dx} \]

Take the derivative of BOTH sides

\[ \frac{dy}{dx} - 4y \frac{dy}{dx} = -2x \]

Isolate terms with the derivative you are solving for

\[ \frac{dy}{dx} = \frac{-2x}{1-4y} \]

Factor out and isolate the derivative you are solving for

Figure 1.2. Implicit differentiation example

Implicit differentiation is important as it allows us to take the derivative of an equation with multiple variables without having to worry about whether the equation is a function. Since multiple variables come into play, we can do things such as examine how different changing quantities (such as the velocities of two vehicles moving away from each other over time) are related to one another in applications such as related rates.

1.6. Research questions

As mentioned earlier, while there is plenty of literature available on derivatives, and there is some on applications of the derivative, there is currently a lack of literature on implicit differentiation. Thus, the primary purpose of this study is to help build the literature base by examining how students think about implicit differentiation and identifying reasons as to why they struggle with it. As such, my research questions are as follows:

1. How do students perform when solving implicit differentiation problems?

2. What do students know about implicit differentiation and related topics?
One goal of the first research question is to see how successful undergraduate calculus students are when solving implicit differentiation problems, derivative computation problems involving equations that are not given in the form \( y = f(x) \). Implicit differentiation is the most efficient way to solve these problems, although other methods sometimes work. When looking at how successful students are when solving such problems, the focus is simply on whether their final answers are correct or incorrect. Although these kinds of problems have sometimes been among the set used in other research on differentiation or applications of the derivative, performance specifically on implicit differentiation problems has not been widely reported. Having such a set of questions and student performance data on them would benefit the mathematics community by providing another sample set of data to see just how difficult, or perhaps not, these problems are for undergraduate calculus students. The second goal of the first research question is to examine what types of errors students make, in a general sense, when working on this type of problem. That is, errors will be examined to determine if they appeared to result from difficulties with algebra or with calculus. This would help to identify the extent to which algebra and calculus appear to serve as roadblocks when working on implicit differentiation problems.

One goal of the second research question is to examine how students think about implicit differentiation. Here, their ability to identify what they believe implicit differentiation to be alongside what they say when trying to explain it are of interest. Knowing this would help to reveal what students know conceptually about this topic, which may help to identify if there is a need to strengthen understanding of it during instruction. The second goal of the second research question is to examine if students know when they need to use implicit differentiation. This would give educators an idea of the features of equations that signal students to use implicit
differentiation. Examining these features can help to further reveal if students understand why implicit differentiation is useful. The final goal of the second research question is to examine how student knowledge of related topics affects their understanding of, and ability to use, implicit differentiation. This would help to identify possible areas where students need more understanding in order to improve their success when dealing with implicit differentiation. It would also help the field understand if theory and findings from studies on these related topics fully account for the difficulties students appear to have with implicit differentiation, or if additional theory development is required.
CHAPTER

2. STUDENT UNDERSTANDING OF IMPLICIT DIFFERENTIATION AND RELATED TOPICS

Implicit differentiation, a topic within the first semester of a typical college calculus course, is a source of difficulty for students (e.g., Clark et al., 1997; Martin, 2000). It builds upon one of the most difficult topics and introduces manipulations that are complex and may be confusing to students. Having said that, knowledge of and skill with this technique allows students to solve interesting problems that are applicable in the real world, such as relating rates of varying quantities to each other. As such, it is evident that, in the realm of calculus, implicit differentiation plays an important role. However, little is known about what makes implicit differentiation difficult for students and what understanding is needed to be successful at using it. In this chapter, we examine some of what is known from existing literature about implicit differentiation and related topics.

2.1 Knowledge of related topics

Before we look at implicit differentiation, let us first examine what has been found on two primary interrelated concepts. These two concepts are the derivative and the chain rule. Because implicit differentiation depends on these concepts, we examine what others have found to be potential reasons for student difficulty in regards to each of them.

2.1.1 The derivative

The derivative is of interest in relation to implicit differentiation, as the implicit differentiation process involves applying the derivative operator to both sides of an equation at
the same time to solve for a derivative. Since it is a technique for finding derivatives, it follows that derivative rules and other ideas surrounding the derivative are relevant to this study. The derivative, one of the essential concepts in college calculus, is a source of difficulty for many students. It is the instantaneous rate of change of a function \( f(x) \) at a specific value of \( x \). It is the slope of the tangent line to the graph of this function at a specific value of \( x \). Definitions for the derivative such as these are typical in textbooks used in introductory calculus courses. More formally, by the limit definition, the derivative \( (f'(x)) \) can be expressed as

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]

In other words, as the change in \( x \) becomes arbitrarily close to 0, the change in \( y \) divided by the change in \( x \) approaches the derivative. This can be interpreted as the rate of change of \( y \) with respect to the change in \( x \). For example, if \( y \) had units of meters and \( x \) had units of seconds, the derivative could be describing a velocity in meters per second. Many reasons as to why students struggle with the derivative concept have been identified.

Zandieh (2000) states that there are three layers of the derivative concept. In order for students to fully understand the derivative concept, they must be able to identify these three layers. They are the ratio or difference quotient layer, the limit layer, and function layer. In other words, students must first understand ratios, limits, and functions in order to fully understand the derivative concept. So why do students need to be able to identify and understand these layers? Briefly, students need to know what a ratio is in order to interpret the derivative as a rate of change. They need to know how to relate the change in one variable to the change in another variable and how to convey the meaning into words. It would be hard, and unlikely, for students to be able to understand derivatives without this prerequisite knowledge. Students need to know what a limit is since the derivative, as seen earlier, is a limit. Students need to know what a
function is since it is what they are operating on when they take a derivative. Furthermore, they need to distinguish between the product of functions and a composition of more than one function, as each situation requires a different rule when taking the derivative. The limit is one of the first concepts to be taught/reviewed in a typical college course, so students have adequate exposure to this topic. Unfortunately, coverage of ratios and functions as topics is not typically built into calculus books and courses. As such, if students do not come into a calculus course with solid understanding of these ideas, they may not get a chance to strengthen them prior to being expected to use them in a calculus context.

Zandieh (2000) also mentions that the language used to describe derivatives can cause difficulties for students. Depending on a student’s understanding of the terms being used, whether consciously or unconsciously, key terms can be omitted when said student tries to memorize definitions. For example, a student could mistakenly state that the derivative is a tangent line or simply a change when it is really the slope of the tangent line and a rate of change respectively. Students could also simply say that the derivative is a slope, without knowing that it is the slope of the tangent line. Thus, there are many related ideas for students to understand as they formulate their own definitions and understanding of these concepts.

Orton (1983) brings up other issues that students encounter when working with the derivative concept. He states that, sometimes differentiation is introduced to students as nothing more than an applicable rule. In other words, it is introduced as an “abstract-apart” concept without any form of context. While this may not interfere with the computation of derivatives and the application of the rules for differentiation, it does, unsurprisingly, bring up numerous issues. For one, there are almost always students who wish to understand what they learn either deeper and/or contextually. Even students who do not care to develop conceptual knowledge
may wish to have a reason as to why some rules only work in some situations and why they have
to use them when they do. Whether or not they are comfortable with ratios, this also prevents
students from being able to interpret the derivative as a rate of change. Without units and
knowing that they are actually comparing the change in one variable to the change in another
variable, students may draw a blank when asked to describe what a contextualized derivative is
telling them.

Orton (1983) also states that algebra may confuse students and take away from their
understanding of the derivative. Students come into calculus courses with a wide variety of
algebra skillsets. Orton claims that there was evidence from his study that, when introducing
aspects of calculus, the usage of algebra should be kept to a minimum as it may distract from the
learning of calculus ideas. In other words, if a student struggles with algebra when working on
tasks involving freshly introduced calculus ideas, their focus may be shifted from these ideas to
their difficulties with the algebra. As such, Orton suggests that the amount of algebra required to
solve problems should be taken into consideration when having students practice new calculus
skills.

Finally, Orton (1983) also brings up that short-term methods to find solutions can take
away from student understanding. He claims that it may be the case that the short-term methods
students are initially taught may not be replaced by the more appropriate, general methods when
they are introduced. That is, there are several rules and shortcuts that instructors will teach
students that are only application in certain situations. When students are first taught derivatives,
they may be told, for example, that, when differentiating with respect to \( x \), the derivative of \( x \) is
\[
\frac{dx}{dx} = 1
\]
They are then told, in some way or another, that \( \frac{dx}{dx} \) is equal to 1, and need not be written,
which may lead students to forget about it. This may cause students to be unfamiliar with this
notation, which may lead to confusion when terms such as \( \frac{dy}{dx} \) appear within equations as a result of taking derivatives later on when dealing with implicit differentiation. As such, it may be the case that, if short-term methods are taught students may not have opportunities to learn why they work, where they come from, and their limitations, which may cause difficulties for students when the more general methods are introduced.

Once again, as implicit differentiation is a type of differentiation, student thinking on and difficulties with the derivative are of relevance. It is possible that these ways of thinking and areas of difficulty may resurface when students work on implicit differentiation problems. Such difficulties include, but are not limited to, recognizing functions, knowing the derivative rules and knowing when they need to be applied, and a lack of understanding of what derivatives are. It is also possible that other issues in relation to the derivative may come up when working on implicit differentiation problems.

2.1.2 Function and the chain rule

The idea of a mathematical function and the skills for working with functions come up throughout calculus. Researchers who examine calculus students’ understanding of function, as well as those who examine student understanding of calculus, often find that student understanding of function impacts their understanding and learning of calculus ideas (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Cottrill et al., 1996; Monk, 1994; Thompson, 1994; Zandieh, 2000). The issues students have with function range from underdeveloped understandings of variable, challenges in understanding the idea of equality indicated by the equal sign, and difficulties making sense of how quantities/variables co-vary in equations and other representations.
In differential calculus, ideas related to function are also relevant for the topic of the chain rule. The chain rule is of interest in relation to implicit differentiation, as the chain rule is required to successfully find derivatives when using this technique. The chain rule, a method to take the derivative of composite functions, is arguably one of the most important concepts that students learn in the first semester of college calculus. In fact, the chain rule and its applications (including implicit differentiation) take up approximately one half of the chapter on differentiation in an average calculus textbook (Horvath, 2008). It remains a key concept from the time that a student learns it until the end of the semester and beyond, if that student decides to continue their journey into the realm of mathematics. It is also one of the most difficult concepts in calculus to present to students in a meaningful manner. Gordon (2005), for example, states that the chain rule is awkward to express, both in words and using symbols, making it hard for students to remember and apply. As such, even if a student can solve routine problems involving the chain rule, they may not have the same success with non-routine ones (e.g., Horvath, 2008).

The chain rule is typically introduced in terms of function composition as \((f \circ g)'(x) = f'(g(x)) \cdot g'(x)\), or in Leibniz notation as \(\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\). While easy to write, both notations may be quite intimidating and hard to understand for students. Researchers have found that there may be specific aspects of each of the notations that are conceptually confusing for students.

The Leibniz notation for the chain rule, \(\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\), can be confusing to students in a number of ways. It is possible that students do not yet recognize the difference between \(\frac{dy}{dx}\) and \(\frac{d}{dx}\). That is, they may not understand that \(\frac{dy}{dx}\) is a derivative and that \(\frac{d}{dx}\) is an operator. Tall (1993), for example, also notes that \(\frac{dy}{dx}\) can be especially confusing to students insofar that they may not
understand whether it is a fraction or a standalone term. Students who think of it as a fraction, or even as a standalone term for that matter, may wonder if the \( du \) in the denominator of \( \frac{dy}{du} \) and in the numerator of \( \frac{du}{dx} \) can cancel each other out when the terms are being multiplied.

Likewise, the composite function notation for chain rule,

\[
(f \circ g)'(x) = f'(g(x)) \cdot g'(x),
\]

can also be confusing to students. The main difficulties with this notation often seem to stem from a student’s lack of understanding regarding the composition, and decomposition, of functions (e.g., Clark et al., 1997). That is, it is not unusual for students to have a hard time recognizing whether or not a function is a composition of functions. Even if a student thinks or recognizes that a function is composite, they may not be able to identify the components of the function. It was also found that students have a tendency to mistakenly use function multiplication in place of function composition (e.g., Horvath, 2008). In the case of the chain rule, given that \( (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \), students will mistakenly interpret \( f'(g(x)) \) as \( f'(x) \cdot g(x) \) and compute it as such.

While we are more interested in what is difficult for students and why, it should be noted that suggestions have been made to potentially make the chain rule more accessible to students. Uygur and Ozdas (2007) conducted a study in which they used arrow diagrams in an attempt to improve student proficiency at the chain rule. In their study, arrow diagrams are used to show, from left to right, what function/variable each other function/variable depends on. For example, for the composite function \( (f \circ g)(x) \), you would have a diagram \( f \rightarrow g \rightarrow x \). From here, you can see that the function \( f \) depends on the function \( g \), which depends on the variable \( x \). They found that the implementation of arrow diagrams showed significant improvement in student understanding of the chain rule over traditional means. Thus, they make the case that the chain rule should be taught using these diagrams or something similar. Also, given that students still
struggle with the topic of function composition and decomposition as they are learning the chain rule, Clark et al., for example, suggest that more emphasis should be placed on this topic while teaching the chain rule.

In the context of implicit differentiation, students need to treat variables as functions. As such, when differentiating a term including $y$ with respect to $x$, the resulting derivative will contain $\frac{dy}{dx}$. This is something students may not understand, and they may not know that it is often a consequence of the chain rule. As mentioned earlier in this section, it was found that there is confusion around $\frac{dy}{dx}$ and what it represents. This confusion may cause difficulties for students as they try to solve implicit differentiation problems. Also, because the usage of implicit differentiation often requires students to use the chain rule, it is possible that the areas of difficulty students have with the chain rule may resurface when they solve implicit differentiation problems. Such difficulties include, but are not limited to, recognizing composite functions and being able to decompose them and understanding what the chain rule is and when it is needed. It is also possible that other issues in relation to the chain rule may come up when students work on implicit differentiation problems.

2.2 What we know about implicit differentiation

While there is currently not an extensive amount of literature available with implicit differentiation as the focal point, some studies do touch upon certain aspects of this topic. One study addressed the more conceptual aspects of implicit differentiation. Mirin and Zazkis (2019) examined what makes implicit differentiation work (the reasoning behind why we can take the derivative of both sides of an equation), analyzing what one would need to know to fully
understand it. They claim the reasoning as to why this process, for differentiating an equation defined on an interval in \( x \), works can be broken down into and understood via four steps:

1. Defining \( f \).
2. Viewing both sides of the equation as functions (of \( x \)).
3. Recognizing that the functions defined by the left hand side and the right hand side are equal on the relevant interval.
4. Recognizing that, since the functions are equal on an interval, the respective derivatives of the functions are also equal on that interval.

For clarification, the first step here, when dealing with \( x \) and \( y \), involves treating \( y \) as a function of \( x \) on a specific interval dependent on the equation. These steps, Mirin and Zazkis claim, show the validity of the process of taking the derivative of both sides of an equation at the same time, which stems back to the idea of function equality. Studies have shown that understanding the ideas of equality and of what the equal sign represents pose significant challenges for younger students (e.g., Kieran, 1981; Knuth et al., 2006). This idea of function equality is something they say that instructors often do not touch upon and is needed to conceptualize the process of implicit differentiation.

Other studies provide data on student success when finding derivatives using implicit differentiation (although implicit differentiation is not be the focus and is not discussed in detail). In a study on student performance on geometric related rates problems, Martin (2000) examined student performance \((n = 58)\) on a test given during a calculus course as part of their data collection, breaking the results down for each topic it covered. Some of the problems on this test asked or required students to find derivatives using implicit differentiation. It was found that the mean success score was 44% for these problems. In a study on student understanding of the
chain rule, Clark et. al (1997) provided participants \( n = 41 \) with tasks to help reveal what they knew about this topic. For one of these tasks, students were asked to take the derivative of the equation \( x\sqrt{y} + y\sqrt{x} = A \), where \( A \) is some real number. In order to do so, implicit differentiation was required. It was found that only 39% of the participants were successful in completing this task. While we do not know how comparable the tasks are for these different studies, the success rates provide evidence that implicit differentiation is a challenging topic for students.

2.3 Goals of this study

When considering student understanding of implicit differentiation, the knowledge and difficulties students have with related topics may very well influence their understanding of and performance when using this technique. That is, students who do not have a robust understanding of these concepts may have the same difficulties when dealing with implicit differentiation. At the same time, there may be other, different aspects of students’ understanding about these topics that are further brought to light when investigating what they know about implicit differentiation. That said, due to the lack of literature on implicit differentiation, there are still areas about how students think about this technique and about how they perform when using it that have not yet been examined. As such, the research questions of this study were designed to investigate some of these gaps in the literature. As a reminder, the research questions are as follows:

1. How do students perform when solving implicit differentiation problems?
2. What do students know about implicit differentiation and related topics?

As stated in the previous section, there is some literature that documents student performance on implicit differentiation tasks. However, these studies do not go any further. That
is, they do not investigate why students performed as they did. The first research question aims to help address this. In addition to examining overall success rates on implicit differentiation problems as other studies have, the goal of this research question is to see what type of errors (algebra or calculus) appear to be more prominent when working on these problems. While the specifics of these errors may not be examined, it is useful to know, when considering ways to improve the instruction of implicit differentiation, whether algebra ideas or calculus ideas seem to be contributing more to difficulties for students as they attempt to solve these problems.

Also stated in the previous section, there is a study that provides an analysis on what it means to understand why implicit differentiation works. For the purposes of this study, the focus is not on whether students understand why implicit differentiation works, but rather what they seem to know about it in general. The second research question aims to investigate this in terms of how students explain what they believe it to be, and when they feel it is needed. This is useful to know, as there is currently little data on what students understand about this topic.

Answers to these research questions contribute to the literature base on student understanding of derivatives in general and, in particular, strengthen the mathematics education research community's understanding of student thinking about, and challenges with, implicit differentiation. These findings can help focus attention on specific challenges which can then inform the design of instruction to improve student understanding of, and ability with, implicit differentiation.
CHAPTER

3. RESEARCH DESIGN

In order to investigate why undergraduate introductory calculus students struggle with implicit differentiation, data from both written surveys and clinical interviews were collected. Written surveys were distributed and collected to amass a substantial amount of quantitative data and to help formulate ideas for interview prompts. Upon doing a brief analysis of the initial survey data, it was decided that more data might prove beneficial, so an additional wave of slightly modified surveys was distributed the next year with an invitation to partake in an interview. Following this, students were contacted, and clinical interviews were conducted to collect in-depth, qualitative data involving students’ thoughts and reasoning on implicit differentiation and the processes they employ to work through problems that require its use.

3.1 Theoretical framework

For this study, a cognitive theoretical framework (e.g., Siegler, 2003) was used. This framework focuses on how individuals think about, and explain, material using their prior experiences as the foundation of their rationale. It has been used many times by others studying student understanding of topics in the undergraduate field (e.g., Horvath, 2008; Orton, 1983; Tall, 1993) and was appropriate for this study because the goal was to understand individual students’ thinking about implicit differentiation.

At the time of writing of this thesis, there were not many reports of research available with implicit differentiation as its focus. As such, the main goal of this study was to begin building the literature base in this area by collecting data on how students think about and explain implicit differentiation, as well as how they perform on tasks requiring them to use this
specific method of differentiation. To achieve this, survey instruments were used to collect data on these aspects in writing. Interviews were later conducted to get additional data on how students explain and reason through problems on and about implicit differentiation. Through this, insight into student thinking on this topic was gathered via analysis using this theoretical framework.

### 3.2 Setting and participants

This study was conducted at a large, public northeastern university between the mid fall of 2016 and the early spring of 2018. Participants in this study were students in the university’s primary introductory calculus course (until recently the university had offered one calculus course for all majors, but there was now another small course for non-physical science and engineering students). In order to take the calculus course, students either needed to have completed pre-calculus at another institution, passed the university pre-calculus course with a C or higher, or they needed to score above a certain baseline on the university’s mathematics placement exam. Students in this course were from a variety of disciplines and majors and approximately half of them had taken some form of calculus previously. Students in this course attended a large lecture led by a faculty member three times a week (50 minutes per meeting) and a smaller recitation led by a graduate student teaching assistant (TA) two times a week (50 minutes per meeting).

### 3.3 Written survey data collection

The first set of surveys was collected in the mid-late fall of 2016. For this sample, 40 surveys were collected from one section of the course and 47 were from a different section (each
section had a different faculty instructor and TA pair). The survey instruments used in this set were designed to probe for both conceptual knowledge and computational skills in regards to implicit differentiation. That is, some questions on the instrument were designed to see how students think about and explain what it is they believe implicit differentiation to be and how it should be used. Other questions were designed to investigate what methods students apply, and the type of errors they make, when solving implicit differentiation problems. As there is little, if any, prior research explicitly on this topic, the problems on this survey drew upon inspiration from generic problems, similar to those seen in the required textbook for the course.

Three different versions of the survey instrument were distributed in the fall of 2016. The questions probing conceptual knowledge of implicit differentiation were the same on all versions. The computational questions, however, differed to some degree on each version. In particular, each version had two (out of four) of the implicit differentiation problems from the problem bank. This was done to get data from a variety of problems without making each version of the survey take an unreasonable amount of time to complete.

The questions shown in Figure 3.1 are some of the sources of survey data for the findings in this thesis. For the different versions of the surveys, see appendices A through D.
1. A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell your friend to help them understand what implicit differentiation is?

2. Solve for $\frac{dy}{dx}$.

   a) $x + y^2 = x^3 - 5$
   
   b) $y - 2x = (4y)^2$
   
   c) $x^2 = 3xy - 5y$
   
   d) $x^3 - 7 = 2y$

Figure 3.1. Survey tasks, Fall 2016

The first question was designed to probe for student understanding of implicit differentiation. It was made to see how students describe implicit differentiation either in general, or in a way that they believe would help a friend understand what it is and how it can be applied. The terminology that participants used in their responses to this question was examined to check for key words and any productive and/or unproductive ways of thinking they may have in relation to this topic. It should be noted that while this question appeared on these surveys, it was not used for data as it was decided it would serve better as an interview task.

The second problem was designed to investigate how students approach solving implicit differentiation problems. That is, it was made to see what methods participants employ to solve these problems and to examine the different types of errors they make when doing so. While it is possible to algebraically manipulate some of the parts of this question in such a way that a student would not have to use implicit differentiation (they could isolate $y$ and then differentiate as done before instruction of this topic), others do require its usage. Assuming students actually do use implicit differentiation, all parts of this question were made to test whether or not they
recognize $y$ as a function of $x$ and, as such, apply the chain rule (as needed) when taking the derivative of terms involving $y$ when differentiating with respect to $x$. That said, each part of this question was also intended to test for different calculus and algebra skills (save for part $d$ which is similar in nature to part $a$). Part $a$ is a basic problem aiming to probe whether students can employ the power rule and can take the derivative of a constant. Part $b$ aims to test if students can use the chain rule or if they can compute the square of a term using exponent rules. Part $c$ aims to test if students can use the product rule or if they can recognize that they can factor a term out of the right side of the equation, which could then lead to the usage of the quotient rule.

In the fall of 2017, it was decided that it would be useful to have more data on student performance when solving implicit differentiation problems. This was done to generate a larger sample size to try and verify patterns from previous data. Therefore, another wave of surveys was collected. A total of 49 surveys were collected, all from one section of the course (the students had the same instructor and TA).

These surveys did not have the conceptual question from the 2016 instrument. This is because the conceptual question were deemed to be better suited for interviews as it was difficult to extract meaning from the previous written responses. As such, the instrument consisted of one computational problem with four parts (as displayed in Figure 3.2), three of which were recycled from the 2016 instrument.
Solve for \( \frac{dy}{dx} \).

a) \( x + y^2 = x^3 - 5 \)

b) \( y - 2x = (4y)^2 \)

c) \( x^2 = 3xy - 5y \)

d) \( 9y^2 + 5x = y \)

Figure 3.2. Survey tasks, Fall 2017

The only new problem in this instrument is part \( d \) as seen in Figure 3.2. It was found that, when analyzing the surveys from the fall semester of 2016, many students struggled with, or did not answer, part \( b \). It was conjectured that the term \((4y)^2\) is what students had difficulty with. To verify this, part \( d \) was created. These two parts are very similar, but, rather than having a term in parentheses raised to the second power, a simplified term was presented instead. That is, \( 9y^2 \) was used in place of \((3y)^2\).

3.4 Responses of an ideal knower

While there are many possible answers that can be viewed as correct to some degree or another, there are some key aspects and solution methods that the investigator would consider necessary for an ideal answer. This section demonstrates these aspects. We examine the concept of implicit differentiation as well as the solution process for solving an implicit differentiation problem. Note that the solution process to only one of the four implicit differentiation tasks is displayed and discussed in this section. That said, example “ideal” solutions to questions 1 and 2b are as follows:
1. A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell your friend to help them understand what implicit differentiation is?

I would tell my friend that implicit differentiation is a method of differentiation in which we simultaneously take the derivative of both sides (left-hand and right-hand) of the equation. To do this, no rearranging of the equation is required. We are simply taking the derivative of each term, so it does not matter which side of the equation a specific term is on (we do not need to get y by itself on one side of the equation). This is important, as this allows us to take the derivative of expressions that are not functions (such as an equation for a circle). This method of differentiation also allows us to take the derivative of equations with multiple variables. The only added difficulty from this process is that we have to treat variables as functions of the variable the equation is being differentiated with respect to.

2b) \( y - 2x = (4y)^2 \)

\[
\frac{dy}{dx} - 2 = 2(4y) \left( 4 \frac{dy}{dx} \right) \quad \text{(Take the derivative of each individual term)}
\]

\[
\frac{dy}{dx} - 2 = 32y \frac{dy}{dx} \quad \text{(Simplify the right side of the equation)}
\]

\[
\frac{dy}{dx} - 32y \frac{dy}{dx} = 2 \quad \text{(Get all of the terms with } \frac{dy}{dx} \text{ onto one side of the equation)}
\]

\[
\frac{dy}{dx} (1 - 32y) = 2 \quad \text{(Factor } \frac{dy}{dx} \text{ out on the left side of equation)}
\]

\[
\frac{dy}{dx} = \frac{2}{1 - 32y} \quad \text{(Isolate } \frac{dy}{dx} \text{ giving you the solution)}
\]
To solve for \( \frac{dy}{dx} \), we first want to apply implicit differentiation. That is, we start by taking the derivative of both sides of the equation simultaneously. This allows us to take the derivative of each individual term. When doing this, it is important to apply the necessary derivative rules (power rule, product rule, etc.). In this case, one of the terms was the composition of two functions. As such, we applied the chain rule when taking its derivative. Once we have taken the derivative of both sides of the equation, we want to simplify if possible. In this case, we had three terms being multiplied together on one side of the equation, so we computed this and combined like terms. Following this, because we are solving for \( \frac{dy}{dx} \), we moved all terms with \( \frac{dy}{dx} \) to one side of the equation. We then factor \( \frac{dy}{dx} \) out from these terms and isolate it by dividing by what was left over after doing this.

### 3.5 Coding of survey data

Coding for the 2016 surveys as well as the 2017 surveys involved initially sorting through and marking them as either correct or incorrect. This was done to give a baseline idea of how many of the total responses were correct. It was then determined that more data could be drawn from the survey results, so codes were made to represent different types of correct and incorrect answers. Because there were no codes made for tasks on this topic before, the techniques of grounded theory (Strauss & Corbin, 1990) were used to create and refine these codes.
While refining these codes, the above flow chart was developed. First, each response was examined to see whether it was blank. From there, they were examined based upon conditions such as techniques used and whether an error was made. Below, the resulting codes are discussed based upon whether they described correct or incorrect responses.

For the correct answers, three different categories were made. A “C” was used to represent responses that were correct and implicit differentiation was used. A “U” was used to represent responses that were correct and implicit differentiation was used, but the final answer was not fully simplified. An “R” was used to represent responses that were correct, but implicit differentiation was not used to arrive at the answers. For example, some problems were made in
such a way that the dependent variable could be isolated and explicit differentiation could be used as students had learned when they had first started derivatives.

For the incorrect answers, four different categories were made. An “I” was used to represent responses where implicit differentiation was used, but a calculus error was made along the way. For example, a student may have made a mistake using one of the derivative rules such as the product or chain rule. An “A” was used to represent responses where implicit differentiation was used, but an algebra or some other type of error was made. For example, a student may have made an addition error or may have forgotten a term when going from one step to another. A “W” was used to represent responses where implicit differentiation was not used, and the answer was incorrect. Finally, a “B” was used to represent responses that were left blank.

A group of mathematics education researchers were asked to apply these codes to data samples in the fall of 2018 to test their reliability. In nearly all cases, the researchers agreed upon, and were able to match, the intended codes to the samples. While there were a few discrepancies, they were all resolved after a brief discussion.

3.6 Analysis of survey data

After the survey responses were coded, responses were counted by code, and groups of codes (based on whether they were codes for correct or incorrect responses). Using the initial coding scheme (which responses were correct, and which were incorrect), percentages were calculated to determine how many responses were correct versus incorrect overall. This coding scheme was also used to break down the percentages of how many responses were correct and incorrect per each individual question. Similarly, using the coding from what types of errors were made (“I”, “A”, “W”, and “B”), percentages were calculated to determine how many of
each error type was made overall as well as per each individual question. Tables were then made to see if there were any patterns or anything else of interest. Further details of this process are presented along with findings in the survey analysis chapter.

3.7 Interviews

To get more in-depth qualitative data, in addition to the quantitative data collected from the surveys, clinical interviews were conducted. Four interviews were done in the fall of 2017 and one was completed early in the spring semester of 2018 (these students had previously taken the survey in the fall of 2017). The instrument used for these interviews was designed using the survey results as a framework. Upon doing a brief analysis of the survey results, questions (from the survey) were recycled to be used in this instrument based upon their potential usefulness in a qualitative interview where students are asked to explain their thinking and processes. Additionally, a new problem was added on to the instrument from a pre-existing study. The questions used on this instrument are shown in Figure 3.4.
1. A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell your friend to help them understand what implicit differentiation is?

2. Examine the following table. For each of the equations, circle Y/N as to whether or not you could use implicit differentiation to find $\frac{dy}{dx}$. Then circle Y/N as to whether or not you MUST use implicit differentiation to find $\frac{dy}{dx}$.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Can you use it?</th>
<th>Do you need to?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $x^2 + 5 = x - 3y$</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>b) $y = 4xy - 8$</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>c) $y^2 = 8x^3 - 2x + 1$</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>d) $5x - y^3 = 9y - 24$</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
</tbody>
</table>

3. Solve for $\frac{dy}{dx}$.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solve for $\frac{dy}{dx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $x + y^2 = x^3 - 5$</td>
<td>$y = \frac{3x^2 - 2y}{2y}$</td>
</tr>
<tr>
<td>b) $y - 2x = (4y)^2$</td>
<td>$y = \frac{2x}{16y}$</td>
</tr>
<tr>
<td>c) $x^2 = 3xy - 5y$</td>
<td>$y = \frac{x^2 + 5y}{3x}$</td>
</tr>
<tr>
<td>d) $9y^2 + 5x = y$</td>
<td>$y = \frac{-5x}{18y}$</td>
</tr>
</tbody>
</table>

4. Given the formula $z = ps + rs^2 + \frac{pr}{s}$, calculate $\frac{dz}{ds}$ and explain how you did it.

Figure 3.4. Interview tasks (Fall 2017, Spring 2018)
The first and third question of the interview instrument, as shown in Figure 3.4, were recycled from the questions used for survey data and, as such, have been previously discussed. They were included as a part of this instrument since it was decided that more data could be extracted from these problems in a clinical interview, since students could be asked to explain and verify (based on their own understanding) what they said/did and why.

The second question was designed to probe whether students understand that they are always able to use implicit differentiation. It also has them assess if the usage of implicit differentiation is necessary to compute a specific derivative (or if they could do so explicitly). To achieve this, four different problems were given to observe what a student was thinking and the reasoning behind their thoughts as they worked on them. Each of these problems was designed to be unique in one way or another from each other, similar to how the parts of the computational question from the survey (question 3 in this interview prompt) were designed as discussed earlier. The reasoning behind this was to see if there are any features of a given equation that lead students to believe whether or not implicit differentiation could be used and if it was needed. As an aside, it should be noted that this problem was a part of the survey instrument in the fall of 2016 but was not included in the data for those surveys as it was not possible to extract meaning from the responses.

The fourth question is an interview task that was taken from a previous study by Jones (2017) examining student understanding of derivatives in the real world. The purpose of this task was to test if participants could take a derivative with multiple symbols, and to test whether they considered $s$, $p$, and $r$ to be constants or implicit functions. In that study it was found that, because they were asked to find $\frac{dz}{ds}$, all participants (six in total) identified $s$ as a variable. Only one participant chose to treat $p$ and $r$ as implicit functions, and only two of them considered the
possibility of $p$ and $r$ being implicit functions. Regardless, the participants all calculated a correct derivative based on what they interpreted the symbols to be. That said, the question is being used in this study because the way a student identifies a symbol is important when applying implicit differentiation.

### 3.8 Analysis of interview data

After all the interviews had been conducted, the audio recordings for each interview were analyzed alongside the accompanying written work to identify how each participant thought about and explained implicit differentiation. To do this, the recordings were listened to multiple times, and notes, with timestamps to signify potentially revealing insights into the participant’s knowledge of implicit differentiation, were taken. Following this, the notes were then examined to learn where similar topics and trains of thoughts were covered and were organized as such to get a better idea of the students’ understanding of implicit differentiation and other relevant topics. In particular, a greater understanding of what students were thinking, when solving the tasks to arrive at their answers, whether they were correct or incorrect, was desired. This would help to get some understanding of the reasoning behind the answers that fit the categories identified in the survey data analysis. Additional information about this process and the findings from doing this can be found in the interview analysis chapter.
CHAPTER

4. SURVEY ANALYSIS

Using the codes and methods described in the previous chapter, responses to the tasks on the surveys from both 2016 and 2017 were analyzed. The results from this analysis are presented below in two parts: student performance on the tasks and student errors when working on them. It was found that the success rate of participants on the tasks varied from student to student but was quite low overall, and that incorrect responses seemed to result more from calculus errors than from algebra errors.

4.1 Student performance on implicit differentiation problems

In this section we examine student performance on the tasks from the surveys. To do this, responses that were coded as correct using the guidelines described in the previous chapter were compiled. A response was considered correct if it fell into one of the three codes as described in Figure 4.1.

<table>
<thead>
<tr>
<th></th>
<th>The response was correct and implicit differentiation was used.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>The response was correct and implicit differentiation was used, but the final response was not fully simplified. That is, the student took the derivative correctly, but did not solve for ( \frac{dy}{dx} ).</td>
</tr>
<tr>
<td>R</td>
<td>The response was correct, but implicit differentiation was not used. That is, the student first isolated ( y ) and then took the derivative.</td>
</tr>
</tbody>
</table>

Figure 4.1. Codes for correct responses
To get a better idea of the types of responses that would fit into each of these categories, student samples from the surveys are presented in Figure 4.2, Figure 4.3, and Figure 4.4.

Figure 4.2. Student sample of a response coded as “C”

In this sample response, the student immediately applied implicit differentiation which means that they took the derivative of both sides of the equation simultaneously. Following this, they isolated $\frac{dy}{dx}$ to arrive at the correct answer. Thus, since implicit differentiation was used and the answer was correct, this response was coded as “C.”

\[ x + y^2 = x^3 - 5 \]
\[
\begin{align*}
1 + 2y \frac{dy}{dx} &= 3x^2 \\
\frac{dy}{dx} &= \frac{3x^2 - 1}{2y}
\end{align*}
\]

Figure 4.2. Student sample of a response coded as “C”

b) \[ y - 2x = (4y)^2 \]

\[
\begin{align*}
\frac{dy}{dx} - 2 &= 8y \frac{dy}{dx} \\
\frac{dy}{dx} - 2 &= 32y \frac{dy}{dx} \\
\frac{dy}{dx} &= 32y \left(\frac{dy}{dx}\right) + 2
\end{align*}
\]

Figure 4.3. Student sample of a response coded as “U”

In this sample response, the student also took the derivative of both sides of the equation to get started. They then isolated $\frac{dy}{dx}$ on one side of the equation, but still had a $\frac{dy}{dx}$ term on the
other side of the equation as well. Thus, although the response was not fully simplified since \( \frac{dy}{dx} \) was not fully solved for, implicit differentiation was used and each derivative term was correct, so it was coded as a “U.”

\[
a) \quad x^2 = 3xy - 5y \\
  x^2 = y(3x - 5) \\
  y = \frac{x^2}{3x - 5}
\]

Figure 4.4. Student sample of a response coded as “R”

In this sample response, the student first factored \( y \) out from the right side of the equation and proceeded to reorganize the equation to be explicit. They then correctly took the derivative by applying the quotient rule. Thus, since implicit differentiation was not used (the equation was made explicit) and the answer was correct, this response was coded as an “R.”

Having examined sample responses for each correct code, it was found that very few students solved tasks correctly without simplifying their response, or without using implicit differentiation. In 2016, 7% of correct responses were coded as a “U,” and 10% were coded as an “R.” In 2017, 2% of correct responses were coded as a “U,” and no responses were coded as an “R.” Due to the small number of responses that were coded as either “U” or “R,” a decision was made to consolidate the correct codes (“C,” “U,” and “R”) and to treat them all as “C.” Here, it is not important whether an answer was simplified, or if implicit differentiation appeared to be used for that matter, since we are examining success on these types of problems in general, regardless of the methods used. Treating all correct answers the same also allows us to compare success rates on implicit differentiation problems with those found in other studies.
Figure 4.5. Percentage of correct, incorrect, and blank responses overall per year

To get an idea of how successful students were on the tasks, the overall percentage of correct, as well as incorrect and blank, responses for both the 2016 and 2017 data were compiled and can be seen in Figure 4.5. It should be noted that, while a blank response could be counted as an incorrect response, they were included separately here since far more students left problems blank in 2016 than in 2017.

That being said, approximately 40% of all responses to the survey tasks were correct in 2016 as opposed to the 55% correct in 2017. Though both percentages are far from ideal, there is a notable difference in the percentage of correct responses when comparing the two years. While there are several factors that could play into this, it is important to note that the 2016 surveys were given closer to the time of instruction on implicit differentiation whereas the 2017 surveys were distributed when students were reviewing the topic to prepare for an assessment. As such, it is possible that more responses were correct in 2017 since the students had more practice solving implicit differentiation problems.
To get an idea of how individual students performed on the tasks, the number of responses each student got correct was examined as can be seen in Figure 4.6. As mentioned in the previous chapter, students who took the 2016 survey only had two tasks to solve as opposed to the four tasks participants in the 2017 survey were provided with. It should also be noted that blank responses are being counted as incorrect here.

Roughly 26% of students who took the 2016 survey got both questions correct, with another 26% of students getting one of the two tasks correct. This means that approximately 52% of participants in 2016 were able to get at least one task correct. This also means that 47% of students did not get either of the tasks correct (or left them blank).

Roughly 22% of students who took the 2017 surveys got all four of the tasks correct, with 16% getting three correct, 27% getting two correct, and another 27% getting one correct. This means that approximately 92% of participants in 2017 were able to get at least one task correct. This also means that 8% of students did not get any of the tasks correct.
Comparing the data from both years, many fewer participants (8% vs. 47%) got all the questions incorrect in 2017. At the same time, the number of students who got all the questions correct was comparable for 2016 and 2017 (26% vs. 22% respectively). When considering the percentages of students who got all questions incorrect, as mentioned earlier, it should be noted that students who took the 2017 surveys had more time and practice with these kinds of tasks. It should also again be noted that students in 2016 only had to complete two tasks as opposed to the four students had to complete in 2017 so it may not be appropriate to compare the two sets of survey data.

<table>
<thead>
<tr>
<th>Question</th>
<th>2016 # Correct</th>
<th>2016 % Correct</th>
<th>2017 # Correct</th>
<th>2017 % Correct</th>
<th>Combined # Correct</th>
<th>Combined % Correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: (x^2 = 3xy - 5y)</td>
<td>19/45 (19/39)</td>
<td>42.22% (48.72%)</td>
<td>26/49 (26/48)</td>
<td>53.06% (54.17%)</td>
<td>45/94 (45/87)</td>
<td>47.87% (51.72%)</td>
</tr>
<tr>
<td>2: (x^3 - 7 = 2y)</td>
<td>14/20 (14/19)</td>
<td>70% (73.68%)</td>
<td>N/A</td>
<td>N/A</td>
<td>14/20 (14/19)</td>
<td>70% (73.68%)</td>
</tr>
<tr>
<td>3: (y - 2x = (4y)^2)</td>
<td>22/67 (22/53)</td>
<td>32.84% (41.51%)</td>
<td>14/49 (14/47)</td>
<td>28.57% (29.79%)</td>
<td>36/116 (36/100)</td>
<td>31.03% (36%)</td>
</tr>
<tr>
<td>4: (x + y^2 = x^3 - 5)</td>
<td>14/42 (14/38)</td>
<td>33.33% (36.84%)</td>
<td>40/49 (40/49)</td>
<td>81.63% (81.63%)</td>
<td>54/91 (54/87)</td>
<td>59.34% (62.07%)</td>
</tr>
<tr>
<td>5: (9y^2 + 5x = y)</td>
<td>N/A</td>
<td>N/A</td>
<td>27/49 (27/48)</td>
<td>55.10% (56.25%)</td>
<td>27/49 (27/48)</td>
<td>55.10% (56.25%)</td>
</tr>
<tr>
<td>Overall</td>
<td>69/174 (69/149)</td>
<td>39.66% (46.31%)</td>
<td>107/196 (107/192)</td>
<td>54.59% (55.73%)</td>
<td>176/370 (176/341)</td>
<td>47.57% (51.61%)</td>
</tr>
</tbody>
</table>

Figure 4.7. Number and percent of problems correct per problem and overall per year

*Number in parentheses is the percentage correct without counting blank responses

Having looked at overall success and student performance, it now makes sense to explore the success rate for each individual task. In Figure 4.7, the percentage of responses that were correct for each question are displayed. Here, it should be noted that a total of four different versions of the survey were distributed with two out of four tasks in 2016. As such, each task was given a different number of times, as opposed to 2017 where each task had the same number
of respondents. It should also be noted that Question 2 was only used on the 2016 surveys and Question 5 was only used in the 2017 surveys.

Question 1 was designed to be one of the more challenging problems, and this was evident from the percentage of correct responses. In 2016, about 42% (49% if you don’t include blank responses) of the responses were correct and about 53% (54% without blanks) were correct in 2017. This suggests that students struggle with implicit differentiation problems that require them to employ the product rule of derivatives.

As mentioned above, Question 2 was only used on the 2016 surveys. This is because it was a rather basic problem and, as such, had a fairly high success rate of 70% (74% without blanks). Thus, this problem was pulled from the surveys as it made more sense to get more data on problems that gave students a bit more of a challenge.

The task that, perhaps, students struggled most with for both years was Question 3. In 2016, about 33% (42% without blanks) of the responses were correct and about 29% (30% without blanks) were correct in 2017. This is most likely because the chain rule must be applied if implicit differentiation is attempted without first simplifying the right side of equation.

Question 4 is interesting as the success rates between the two years is vastly different. In 2016, only 33% (37% without blanks) of the responses were correct as opposed to the 82% (82% without blanks) in 2017. Given that the participants in 2017 had more experience with implicit differentiation, it is possible that participants in 2016 tried using a method that would not be used if they had more practice.

As mentioned above, Question 5 was only used on the 2017 surveys. This question was designed to be similar to Question 3 to try and get an idea of why the success rate was so low. Students performed better on this task with about 55% (56% if blank responses are not included)
as opposed to the 33% (37% without blanks) who got Question 3 correct in 2016 and the 29% (30% without blanks) who got it correct in 2017. This suggests that there is something different enough between these two problems that alter their success rates.

4.1.1 Summary of student performance on implicit differentiation problems

Given that 40% of responses in 2016 and 55% of responses in 2017 to the tasks were correct, the combined success rate for the two years was below 50%. Although the students did slightly better in 2017, this is perhaps to be expected given that they had more experience and practice with tasks involving this technique. That said, there was also a higher percentage of correct responses for each question in 2017 than 2016 except for Question 3 (ignoring Question 2 which was not included in 2017). This may suggest that there is something about Question 3 that makes it particularly difficult, even for those with more experience dealing with these types of problems.

The percentage of students who got at least one of the tasks correct was quite different between the years, with 53% of students in 2016 and 92% in 2017. While it is a bit hard to compare these figures since the 2016 students only had two, as opposed to four, tasks to complete, we also cannot assume they would have done better if they had more tasks. Regardless, this may imply, at the very least, that students are more likely to use implicit differentiation, as opposed to first trying to reorganize an equation, as they gain more experience using it. That being said, just because a student got one task correct did not mean it was likely they would get all tasks correct. This suggests that there is something about the different problems that causes students to struggle on them to varying degrees. As such, the different types
of errors, as mentioned in the previous chapter, and the percentage of times they occurred for each question and overall are examined.

### 4.2 Student errors when working on implicit differentiation problems

In this section we examine student errors when working on the tasks on the surveys. To do this, responses that were coded as incorrect using the guidelines described in the previous chapter were compiled. A response was considered incorrect if it fell into one of the four codes as described in Figure 4.8.

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Implicit differentiation was used, but the response was incorrect due to a calculus error (derivative rules, etc.).</td>
</tr>
<tr>
<td>A</td>
<td>Implicit differentiation was used, but the response was incorrect due to an algebra, or some other type of error. (usually more correct than when a calculus error (I) is made)</td>
</tr>
<tr>
<td>W</td>
<td>The response was incorrect, and implicit differentiation was not used.</td>
</tr>
<tr>
<td>B</td>
<td>The response was blank, the student simply rewrote the equation, or nothing of significance with written.</td>
</tr>
</tbody>
</table>

Figure 4.8. Codes for incorrect responses

To get a better idea of the types of responses that would fit into each of these categories, student samples from the surveys are presented in Figure 4.9, Figure 4.10, and Figure 4.11. Take note that a student sample is not given for “B,” as it would simply be blank or display nothing of interest.
Figure 4.9. Student sample of a response coded as “I”

In this sample response, the student starts out by using implicit differentiation. However, when they took the derivative of the $3xy$ term, whether they recognized that two functions were being multiplied together or not, they did not use the product rule for derivatives. Thus, since implicit differentiation was used, but a calculus error was made this response was coded as an “I.”

\[
a) \quad x^2 = 3xy - 5y \\
\frac{dx}{dy} : 2x = 3 \frac{dy}{dx} - 5 \frac{dy}{dx} \\
\frac{2x}{-2} = (3 - 5) \frac{dy}{dx} \\
-x = \frac{dy}{dx}
\]

Figure 4.10. Student sample of a response coded as “A”

In this sample response, the student starts out by reorganizing the equation to place terms with different variables on opposite sides of the equation. They then make an algebra mistake

\[
b) \quad y - 2x = (4y)^2 \\
8y^2 - y = -2x \\
16y \frac{dy}{dx} + \frac{dy}{dx} = -2 \\
\frac{dy}{dx} (16y - 1) = -2 \\
\frac{dy}{dx} = \frac{-2}{16y - 1}
\]
here by incorrectly simplifying \((4y)^2\) as \(8y^2\). However, they go on to use implicit differentiation and follow the steps to arrive at an answer that would have been correct had they not made the algebra error. Thus, since an algebra mistake was made but implicit differentiation was used, this response was coded as an “A.”

\[
\begin{align*}
  x + y^2 &= x^3 - 5 \\
  y^2 &= x^3 - x - 5 \\
  y &= \sqrt{x^3 - x - 5} \\
  \frac{dy}{dx} &= \frac{1}{2} \left(x^3 - x - 5\right)^{-\frac{1}{2}} \cdot \left(3x^2 - 1\right) \\
  y' &= \frac{3x^2 - 1}{2 \sqrt{x^3 - x - 5}}
\end{align*}
\]

Figure 4.11. Student sample of a response coded as “W”

In this sample response, the student first made the equation explicit by isolating \(y\) on one side of the equation. However, when they took the square root, they forgot that the resulting expression could be either positive or negative. Following this, they went on to take the derivative. Thus, since implicit differentiation was not used (they first made the equation explicit) and the answer was incorrect, this response was coded as a “W.”

It should be noted that these categories were purposely made to be broad in order to test the waters and get a feel for what causes student errors when solving implicit differentiation problems, rather than to pinpoint something more specific. As such, the types of algebra and calculus errors are not distinguished here. It should also be noted that while a “W” involves an algebra or calculus error, implicit differentiation was not used, which is what we care about here.
Furthermore, blanks responses will not be examined or counted in the following statistics as they reveal nothing about student errors. That being said, let us now examine how often the different error types led students to incorrect responses.

Figure 4.12. Percentage of all errors per error type per year for Task 1

Looking at the data for Task 1, as seen in Figure 4.12, the percentage of errors types between the years is nearly identical. About 80% or incorrect responses for both years were coded as “I,” with the rest being “A,” except for in 2016 where a sprinkle of these were coded as “W.” This was not surprising, as this equation had two functions being multiplied together. If students did not realize this, they would not have used the product rule. Even if they did realize this, it is possible they did not remember the product rule, or they did not remember it correctly. Considering students who forgot to multiply by \(\frac{dy}{dx}\) when taking the derivative of terms with \(y\) and general mistakes with derivative rules, it makes sense that most errors were calculus-based here.
Looking at the data for Task 2, as seen in Figure 4.13, there was nothing of interest. Most of the incorrect responses here were coded as “I,” meaning they resulted from calculus errors. However, it should be noted that very few students got this question incorrect (only five). As such, as mentioned previously, this task was not included on future versions of the survey as it did not generate useful data in terms of what types of errors students made while doing implicit differentiation problems.
Looking at the data for Task 3, as seen in Figure 4.14, about 60% of incorrect responses for both years were coded as “I,” meaning they resulted from calculus-based errors. This was to be expected, since if a student did not simplify the \((4y)^2\) term they would need to use the chain rule, which has been found to be difficult for introductory calculus students. This left about 40% of incorrect responses in 2017 to have algebra errors. This could possibly result from students incorrectly simplifying \((4y)^2\) in some way or another (misapplying the product rule of exponents) before taking the derivative.
Looking at the data for Task 4, as seen in Figure 4.15, the percentage of error types were quite different for each year. In 2016, the majority of incorrect responses (about 54%) were coded as “W,” meaning most errors from this year involved students not using implicit differentiation. In 2017, most of the incorrect responses (about 78%) were coded as “I,” meaning they resulted from calculus errors. One possible explanation for this difference could be that this task only had one $y$ term in the equation, which may have led the less experienced students who took the 2016 survey to try and make the equation explicit, whereas the students in 2017 were more used to employing implicit differentiation whenever they took derivatives.
Figure 4.16. Percentage of all errors per error type per year for Task 5

Looking at the data for Task 5, as seen in Figure 4.16, the results are intriguing. 43% of incorrect responses were coded as “I” here, with 57% being coded as “A.” This is interesting, as this is the only question where there was a higher percentage of “A” codes than “I” codes. That is, more students made algebra errors than they did calculus errors on this task. Furthermore, as stated previously, this task was essentially the same as Task 3 (the equation was simplified to make taking the derivative easier) where most incorrect responses resulted from calculus errors. While it makes sense for fewer calculus errors to be made on an easier problem, the algebra on this task was not difficult, which makes this a bit unusual.
To get an idea of which error type was most prominent, all incorrect responses were compiled by error type per year as seen in Figure 4.17. Again, it should be noted that these percentages are out of responses that were coded as incorrect by year exclusively.

Roughly 56% of incorrect responses were coded as “I” in 2016, with 16% coded as “A” and 28% coded as “W.” This means that calculus errors seem to be responsible for over half of all incorrect responses in 2016. This also means that implicit differentiation was not used for roughly 30% of the incorrect responses.

Roughly 62% of incorrect responses were coded as “I” in 2017, with 38% were coded as “A” and no responses were coded as “W.” This means that calculus errors were also the leading mistake for over half of the incorrect responses in 2017. Also, since none of the responses were coded as “W,” this means that implicit differentiation was used for every response in 2017.

Comparing the data from both years, over half of incorrect responses for each year was coded as an “I,” meaning that calculus errors were the leading cause of incorrect responses. One point of interest is that none of the incorrect responses were coded as a “W” in 2017 as opposed
to the 28% of responses coded as “W” in 2016. This suggests that, as students gain more experience using implicit differentiation, it may be unlikely for them to first try to make the equation explicit.

### 4.2.1 Summary of student errors

As mentioned above, the majority (over 50%) of incorrect responses for both the 2016 and 2017 surveys resulted from calculus errors. In fact, this was the case in three of the five tasks. Of the other two, Task 4 was a bit of an outlier due to the number of students who did not apply implicit differentiation. It is also unclear as to why more incorrect responses resulted from algebra errors than from calculus errors on Task 5. That being said, it should again be noted that the codes for incorrect responses were quite broad. In fact, without having a student explain their thought process, it can be rather ambiguous whether an incorrect response was algebra or calculus based. For example, if a student did not apply the product rule, it was coded as a calculus error. However, if a student did not apply the product rule because they did not realize they were dealing with a product of two functions, it could be argued that it could also be counted as an algebra error. A similar argument could be made in the case of the chain rule. This left something to be desired from the survey data as there was no way to distinguish whether or not students knew they should be applying a specific derivative rule.

One thing of note is that not a single student in 2017 attempted to make an equation explicit (no responses were coded as “W”) as opposed to the 28% of incorrect responses in 2016 fitting this code. This means that every student who attempted the tasks in 2017 used implicit differentiation where only about 70% used it in 2016. Again, a potential reason for this is that the 2016 surveys were collected shortly after students had learned the technique of implicit
differentiation whereas the 2017 surveys were collected while students were reviewing this technique. As such, the 2016 students were likely still at the point where it was routine for them to first try and isolate $y$, which would explain why they did not immediately take the derivative of both sides of the equation simultaneously. This suggests that such a mistake may be less likely to show up after students have had more experience using implicit differentiation since this technique eliminates the need to first manipulate the equation.

4.3 Summary of survey results

Some results of interest came from the survey data. Student performance on the tasks averaged at just about 50% correct responses out of all responses overall between 2016 and 2017. When examining student performance for each individual task, it was found that they struggled most on the Task 3, which involved the chain rule (36% average correct for 2016 and 2017 combined) and Task 1, which involved the product rule (52% correct average for both years combined). This is not ideal as there are applications in calculus that require the usage of implicit differentiation, such as related rates.

It was found that most of the incorrect responses to the tasks resulted from calculus-based errors (56% of incorrect responses in 2016 and 62% in 2017). This perhaps suggests that a greater effort should be placed on examining calculus instruction, rather than focusing on algebra. However, as mentioned previously, it is possible that some of the incorrect responses that were coded as calculus errors should also have been coded as algebra errors. Analysis of written work can suggest if a student attempted to apply a derivative rule. However, there is no way to tell whether a student knew they needed to apply a specific rule based upon the algebra. At the same time, it is also possible a student understood the algebra aspect of a task but did not
remember how to deal with the calculus portion. To discern things such as these some sort of explanation would be needed, leaving something to be desired from the written survey data.

It was also found that students who were less experienced with implicit differentiation were more likely to try and make an equation explicit before attempting to take the derivative. As mentioned previously, participants who took the 2016 surveys had less experience and practice with implicit differentiation than the participants who took the 2017 surveys. That said, students attempted to make the equation explicit, rather than using implicit differentiation, on approximately 30% of the incorrect responses in 2016. On the other hand, not a single student attempted to make an equation explicit in 2017. This suggests that students supersede the usage of explicit differentiation, or are far less likely to use it, with implicit differentiation once they have gotten a better hang of it.

In summary, the survey results showed that students struggle with implicit differentiation problems. It appears that they have more difficulty with some types of these problems than others (such as ones involving product rule and chain rule) and it appears that most errors were calculus-based, but it was ambiguous in some cases as to where these difficulties stem from when examining written work alone.
CHAPTER

5. INTERVIEW ANALYSIS

While the written survey data provided useful insight regarding how students describe implicit differentiation and the types of errors they make when completing implicit differentiation tasks, there were areas of ambiguity related to student understanding, and the steps they took, that could not be understood from written work alone. As such, interviews were conducted to help shed light on these areas lacking clarity. As a reminder, the interview instrument, discussed in Section 3.7, had tasks that were designed to probe for student understanding of implicit differentiation, as well as tasks to examine their ability to take derivatives using this technique. After going through and analyzing each of the interviews, it appeared that each of the participants could be placed into one of four categories comparing their knowledge of implicit differentiation and related topics, with their ability to solve implicit differentiation problems. The framework for this categorization is presented below, followed by explanations of where each participant fit into the framework, along with a comparison between what participant knew from category to category.

5.1. Framework for comparing interview participants

When going through the interviews, it was found that the participants had differing levels of understanding of implicit differentiation and differing levels of ability to employ it. Given this, it seems there may be a certain level of understanding and ability needed to solve tasks using this technique. That is, there may be some amount of understanding and ability of some combination of implicit differentiation and related topics that a student needs to enable them to
solve these tasks. As such, a framework was developed to see if there is a potential relationship between understanding the ideas and success in solving the problems.

<table>
<thead>
<tr>
<th>Has at least some knowledge of related topics</th>
<th>Can solve implicit differentiation problems</th>
<th>Cannot solve implicit differentiation problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has limited knowledge of related topics</td>
<td>Limited knowledge, can solve</td>
<td>Limited knowledge, cannot solve</td>
</tr>
</tbody>
</table>

Figure 5.1. Framework comparing student knowledge with their ability to solve problems

The two-by-two table displayed in Figure 5.1 shows the four possible categories resulting from this comparison. This table matches a participant’s knowledge of implicit differentiation and related topics with their success at solving implicit differentiation problems. For example, if an interview participant appeared to have at least some knowledge and could solve implicit differentiation problems, they would represent the category “Some knowledge, can solve.” It should be noted here that a participant would be placed in the “can solve implicit differentiation problems” column if they had a correct response for any of the implicit differentiation problems. That said, interview participants fit into three out of the four categories. The following sections contain discussion of where each interview participant was placed in the framework.

5.2 Margaret: “Some knowledge, cannot solve”

The first interview participant was one of four female participants and will be referred to as “Margaret” here. Margaret admitted that implicit differentiation was one of the topics she did not really understand, but, despite this, she knew a surprising amount of information related to it. Her knowledge of implicit differentiation and related topics, as well as her success at solving implicit differentiation problems are examined below to show why she fits the category of “Some knowledge, cannot solve” from the framework.
5.2.1 Margaret’s knowledge of implicit differentiation

When asked to explain what implicit differentiation is, Margaret said the following:

*Margaret*: I know how to do it with circles, and that’s about it.

*Interviewer*: So, something with circles?

*Margaret*: Yeah, so circles, basically $x^2 + y^2$ and a lot of times you get a radius of 16, when you graph that, you don’t get, um, you have to split it in half so you have like the top half and the bottom half, because, um, otherwise you can’t get the, um, limit of the function. And then you have to essentially solve for $y$ or treat it like $y$ isn’t there. It’s actually one of the topics I don’t really know.

*Interviewer*: What do you mean by treat it like $y$ is not even there?

*Margaret*: Like, um, so when you’re getting the top half and bottom half of the function, because otherwise you’re going to get, um, no real line through it that way (it would not pass vertical line test). You have to, like, if you square rooted $y$, so if you had $y^2 = 16 - x^2$ and you have to square root both sides, $y = \pm \sqrt{16 - x^2}$, so then from there you would have to find, like the limit of, like, you can find the limit of $y$, or the derivative of $y$, otherwise you can’t do anything else. But as long as you don’t have a $y$ on this side, then normally you are good. But when you have a $y = x^3 + xy$, you have to get the $y$ out of there to be able to anything, but when you are doing implicit differentiation you are supposed to sort of ignore it and just keep going, which is part of the reason why I don’t get it.

Although Margaret was not completely sure of what implicit differentiation was, she was able to explain, to some degree, what it allows us to do and why it is important. She brings up, and partially explains, the common example to motivate implicit differentiation by examining the equation of a circle and noting that, since its graph has both a top half and a bottom half, it
does not pass the vertical line test. She also states that if you do not treat a circle in such a way, you would not be able to find the derivative. She stops the discussion of circles there, and goes on to say that, if there is not a $y$ on the right side of an equation, you are normally good. This gets at the idea that if $y$ is isolated, you will be able to take the derivative normally (explicitly). She then comes up with an equation with $y$ on both sides where she believed that $y$ could not be isolated and states that this is where implicit differentiation comes into play. This is a good start, as it gets at the idea the implicit differentiation allows us to take the derivative of an equation where the variables cannot be separated.

While it is not entirely clear what Margaret meant by when she said to treat $y$ like it was not there (in an equation), one possible explanation is as follows. As she mentioned, if you are unable to isolate $y$ in an equation, you would not be able to use normal (explicit) differentiation. However, when you are using implicit differentiation, it does not matter if $y$ is isolated, as you take the derivative of both sides of the equation simultaneously. Given that she could not remember how to use implicit differentiation, it is possible that she was simply referring to the idea that with implicit differentiation, it may seem as though we are ignoring $y$ because we do not isolate it when using this method. Again, this is just one possible explanation of what she was referring to by saying we could ignore $y$ with implicit differentiation.

Following this, Margaret then went on to try and take the derivative of $y = x^3 + xy$, and said the following:

*Margaret:* It’s just weird, because when you’re… doing stuff with… normally when you’re doing an equation, anything with equations you’re changing $x$, $y$ is the dependent variable, so you don’t normally mess around with $y$ because $y$ is the value of the function. But when you have it like over here into the problem, like getting it over is not going to be an option.
Here, Margaret gets at the idea that, before implicit differentiation was introduced, equations were structured in such a way that you would have a dependent variable, usually \( y \), and an independent variable, usually \( x \). But, in the equation that she came up with, and others one will regularly see after implicit differentiation is introduced, that is no longer the case. She noted that such an equation is different, as getting \( y \) by itself would not be possible. This, once again, alludes to the fact that implicit differentiation allows one, and is needed, to take the derivative of equations in which the variables cannot be separated.

When examining equations for one of the interview problems, Margaret was able to identify criteria for when implicit differentiation could be used. When asked if implicit differentiation could be used on \( x^2 + 5 = x - 3y \), Margaret said the following:

**Margaret:** If you really wanted to you probably could use it, but you don’t need to. So, honestly I’m going to say no for can you… well you could, but do you need to, no.

**Interviewer:** So?

**Margaret:** Well there is no \( y \) over here, so that's, so I definitely would not use implicit differentiation for it, because you could easily move the \( x \) over, divide by \( -3 \), and then just do \( y' \) and all the fun stuff that happens after that. So you really don’t need it, you just need to do quotient rule.

**Interviewer:** Okay, so you wouldn’t need to use it, but could you use it?

**Margaret:** Yeah, you can always use a rule, doesn’t mean it's the right way to do the problem though.

Here, Margaret addresses the idea that implicit differentiation is not always necessary to take the derivative of an equation. For this problem, Margaret realized that she could get \( y \) by itself and decided that implicit differentiation was not necessary to take the derivative since she
could take the derivative the normal way (explicitly). She follows the same reasoning she
employed when she was asked how she would define implicit differentiation. As such, it appears
that Margaret believed implicit differentiation would not be necessary to find the derivative of
equations in which $y$ could be isolated.

During this dialogue, Margaret also addresses the idea of when one could use implicit
differentiation. Upon starting this problem, she states that one could probably use implicit
differentiation, but that it would not be necessary (because she could find the derivative
explicitly). When asked to confirm whether or not she could use implicit differentiation, she
responded more confidently, saying that you could. She followed this up by saying that a rule
can always be applied, but that it would not necessarily be the best way to solve a problem. As
such, it appears that Margaret believed implicit differentiation could always be applied.

Margaret was also able to identify criteria as to when she believed implicit differentiation
would be needed to take the derivative of an equation. When asked if implicit differentiation
could be used, and if it was necessary, to take the derivative of $y = 4xy - 8$, Margaret said the
following after trying to manipulate the equation to isolate $y$:

_Margaret:_ The only real problem that would come in is that, um, $\frac{y + y}{y}$. So if I was solving it, I
would most likely use implicit differentiation looking at it now, because it might be easier, but...

_Interviewer:_ Do you think you could do it without using it?

_Margaret:_ You... could, it would probably take longer though. Um, so then... how would you do
it?

Margaret then went on to try to manipulate the equation further but did not seem to be
making any progress. She was then asked the following:

_Interviewer:_ So I guess let’s say that it was not possible to get $y$ by itself over here.
**Margaret:** Okay.

**Interviewer:** Would you have to use implicit?

**Margaret:** Yes, because it is the only other way to solve it because when you can’t do it any other way, implicit is the fallback.

In this problem, Margaret found herself in a situation in which she did not think that she would be able to isolate \( y \). While she could not think of a way to isolate \( y \), it appeared that she still believed it would be possible to do so. Though, when she was asked to assume that it would not be possible, she said that implicit differentiation would be needed to find the derivative. More notably, she stated that, if the derivative could not be taken any other way, implicit differentiation would be necessary. As such, it appears Margaret believed implicit differentiation would be needed for equations in which \( y \) could not be isolated, or for any other case in which you could not take the derivative explicitly.

### 5.2.2 Margaret’s knowledge of related topics

Margaret appeared to have a relatively strong foundation when it came to derivative rules. She could apply the power and the product rules, and mostly remembered the quotient rule. Whether or not the algebra that followed was correct, she knew how the rules worked. Unfortunately, even though she was able to name the chain rule, arguably the most important rule needed for implicit differentiation, she did not seem to remember it. When she was taking the derivative of \( ps \) (she is treating \( p \) as a constant and \( s \) as a variable here), the following dialogue occurred:

**Interviewer:** So how did you go from \( ps \) to \( s \)?
Margaret: Um, oh, wait, no that’s wrong. It’s just p. Should be just p, because the s comes out. Because right now that’s s^1 and when you’re doing... I think it’s chain rule, you are subtracting one from the power and bringing the constant down. So 1 times p is p, and then, s^{1-1} would be 0 so that makes that a 1.

This is the only time during the interview when Margaret ever mentioned the chain rule. Though, from what she said, Margaret was using and describing the power rule, not the chain rule. This played a part in her being unable to solve implicit differentiation tasks, as will be discussed later, since one needs to have some amount of knowledge regarding the chain rule to do so.

Margaret was unable to successfully take the derivative of terms involving y, unless the equation was explicit, and she did not fully understand how to deal with them. Part of this confusion rooted from the fact that, when asked what she believed y was, she said y was the equation (this is not surprising as y usually represents a function before the instruction of implicit differentiation). She said that x was the input and y was the output. Such a conceptualization no longer applies when y is mixed in with other terms in an equation and cannot be isolated. This led her to believe that some of the y terms in an equation either needed to be ignored or treated differently. In fact, she even attempted to estimate the derivative of an equation by incorrectly drawing upon ideas from limits towards infinity to eliminate terms with y that she believed would be less significant since y became larger and larger. That said, at the very least, it appears she believed y was special in some way and had to be treated differently.

Margaret seemed to have some understanding of constants and variables, and treated terms accordingly (except for y) when dealing with equations involving x and y. She also seemed to have some understanding of independent and dependent variables since she referred to
$x$ as the input and $y$ as the output. However, it appears that she did not recognize that any unspecified terms could be either a variable or a constant. When examining the equation $z = ps + rs^2 + \frac{pr}{s}$ she correctly said that $z$ and $s$ were variables since she was asked to find $\frac{dz}{ds}$.

However, when asked to identify what $p$ and $r$ could be, the following dialogue occurred:

**Interviewer:** How are you classifying your $p$ and your $r$?

**Margaret:** Constants. Because they... I mean $r$ repeats, but it doesn't gain or lose any powers in it... so it's just $r$. If it was like, $r^2$ somewhere in that instead and there was still a regular $r$ then there would be two variables and then you'd probably want to use implicit differentiation to find what $\frac{dz}{ds}$ is because it would be a lot harder to separate everything out.

**Interviewer:** Okay, so same thing, because there is no power or anything like that with your $p$, that makes that a constant?

**Margaret:** Yeah. It's kind of like when you use the constant $a$ to explain everything else.

Here, she said that she believed they were constants since they were not raised to a power but would have considered them to be variables if that were the case. This is not correct, since they could be either because they were not identified, and it is possible to have a constant to a power in an equation (although this is atypical in introductory calculus examples). She then went on to explain that her reasoning was based on previous experience where letters such as $a$ were often used to describe constants. While this is a logical train of thought, it is clearly stated in such examples, or should have been, that the letter(s) in question was a constant. Thus, such a transfer of knowledge is not correct in this situation and, if one does not understand the context of a problem, could cause difficulties when dealing with applications of implicit differentiation such as related rates.
Margaret also seemed to have some trouble when dealing with exponents. When simplifying the term $(4y)^2$, she correctly applied the product rule of exponents to get $16y^2$. However, she incorrectly tried to apply the same rule when faced with the term $\sqrt{8x^3 - 2x + 1}$ since she ended with $8x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 1$ upon simplifying. From this, it appears she knew that the square root of an expression is the same as that expression raised to the one-half power. Where she went wrong is that she tried to distribute the power, which is not valid in this case. This type of error is fairly common when dealing with the chain rule where there is some inside function and the outside function is a power. Such errors affect a student’s ability to successfully solve some implicit differentiation tasks.

5.2.3 Margaret’s success with solving implicit differentiation problems

Margaret did not successfully use implicit differentiation to solve any of the interview tasks. Whenever she examined an equation, she immediately started to think about how she could isolate $y$ and would manipulate it in an attempt to do so. That is, she tried to first make an equation explicit, rather than taking the derivative of both sides of the equation simultaneously. However, as mentioned earlier, she recognized that implicit differentiation would be needed if she was unable to do so. As such, at the very least, while she could not use implicit differentiation, she was able to identify when it was needed.

Although Margaret was unable to solve implicit differentiation problems, she had some understanding of why and when implicit differentiation was needed, but she did not know how to use the process itself. Thus, since she did not recall how to use implicit differentiation and did not know how to take the derivative of $y$, she made no progress on the interview tasks. However,
something of interest came about while Margaret was examining one of the interview tasks.

When examining the equation \(5x - y^3 = 9y - 24\), the following dialogue occurred:

**Margaret:** Honestly for this one I would want to take the derivative of \(x\), because it’s easier.

*Instead of like, doing \(y'\).*

**Interviewer:** Oh, so like if you were doing find \(x'\) ?

**Margaret:** Yeah, if you were doing find \(x'\), this would be the perfect problem for it, and you don’t need implicit differentiation and... you could use it if you really wanted to, but it would be one way to go about it where you wouldn’t have to use implicit because if you... although you might not need to for the \(y\), but, for the... if you were doing \(x'\), all you would have to do is add \(y^3\) to the other side and then divide by 5.

This was interesting, since Margaret brought up the idea of taking the derivative with respect to another variable without the interviewer having mentioned doing so. While this is something that she likely had seen in class before, it is surprising that she was able to see that doing so would have resulted in an easier problem for her. This may indicate that she understands the idea that, while \(y\) is usually considered a function of \(x\), it is also the case that \(x\) can be considered a function of \(y\).

### 5.2.4 Summary of what Margaret knew and could do

In summary, Margaret did not appear to know how to use implicit differentiation, but she was able to partially explain the motivation behind why it is needed. She was also able to identify that one would need implicit differentiation to take the derivative of an equation if it could not be made explicit and recognized that it could be used even if an equation could be made explicit. Margaret was confused by the abundance of \(y\) terms and did not understand how
to treat them so she was unable to solve implicit differentiation problems. That said, she appeared to have a substantial amount of knowledge on related topics. She seemed to know how to use the basic derivative rules apart from the chain rule and seemed to understand the difference between variables and constants. Her algebra also appeared to be fairly solid, although she ran into some trouble when dealing with exponents. Thus, while Margaret appeared to have a grasp on quite a bit of knowledge related to implicit differentiation, she seemed to be missing some amount of knowledge which appeared to prevent her from being able to solve implicit differentiation problems. As such, Margaret would be a case of someone who fits in the category “Some knowledge, cannot solve.”

5.3. Geoffrey: “Limited knowledge, cannot solve”

The second interview participant happened to be the only male participant and is referred to as “Geoffrey” here. Geoffrey had the most limited understanding out of the five participants. When asked directly about ideas related to implicit differentiation, he did not provide any meaningful answers, or he said that he did not know. He also had a habit of rambling or cutting himself off before finishing his train of thought so there was less substantive dialogue to examine (in comparison to the other interviews). However, bits and pieces of knowledge were revealed as he described what he was doing. That said, his knowledge of implicit differentiation and related topics, as well as his success at solving implicit differentiation problems are examined below to show why he fits the category of “Limited knowledge, cannot solve” from the framework.
5.3.1 Geoffrey’s knowledge of implicit differentiation

When asked to explain what implicit differentiation is, Geoffrey could not come up with an answer. Whenever he was prompted to explain what he thought it was, he would try to say something, but would then abruptly stop. Nothing he ended up saying was coherent, as he kept cutting himself off, and, eventually, he gave up and said that he did not think that he would be able to come up with an explanation.

While Geoffrey could not come up with an explanation of what implicit differentiation is, he was able to provide criteria for when he believed it could be used, and when it would be needed, to take the derivative of an equation. On one of his first attempts to do so, he said the following:

Geoffrey: Implicit differentiation, you would actually need that for, say a factor that you could not get derivatives of just the regular $\frac{d}{dx}$, correct? You would actually have to use implicit differentiation so that way you could expand it out.

Here, he mentions that it would be needed for “factors” that could not be differentiated the normal way and then goes on to say that implicit differentiation would expand it out. Unfortunately, he did not elaborate on what he meant by “factors” and what he was referring to when mentioning some sort of expansion. Perhaps by “factors,” he was referring to terms, which would make sense since there are terms that cannot be differentiated explicitly. When referring to expansion, it is possible he meant simplification.

Clearer criteria for when implicit differentiation could be used and would be needed were discovered as Geoffrey examined equations and said whether or not it could be used on them.

When taking the derivative of $y^2 = 8x^3 - 2x + 1$, the following dialogue occurred:

Interviewer: Do you feel that you could, or would need to use implicit differentiation?

Geoffrey: (after a long pause) I would say that you could use it.
Interviewer: And then why would you say that? What about that equation makes you think that you could use it.

Geoffrey: Because with most differentiation, when you’re working with it, it only has constants and numbers going up to a square. When you are actually doing implicit differentiation, that’s when you have one that’s going to a higher power that you need to be able to simplify more.

Interviewer: So then you’re thinking that because you have $x^3$ is why?

Geoffrey: Yes.

Here, Geoffrey provided concrete criteria for when he thought that implicit differentiation could be used. He stated that, with normal (explicit) differentiation, terms would be squared at most. That is, he believed that equations with terms raised to the second power or lower could be differentiated without the use of implicit differentiation. He then went on to say that implicit differentiation would be used when there are higher powers involved. In this case, because there was a term raised to the third power, it would be possible to use implicit differentiation. Though, he later stated that, for this equation, it would not be needed since he was able to take the derivative normally. However, the following dialogue, which occurred after he tried to take the derivative of $5x - y^3 = 9y - 24$ and said that implicit differentiation could be used, reveals when he believed it would be necessary:

Interviewer: And then do you think that you would have to?

Geoffrey: I would say that you would because, just using regular differentiation, it still leaves the $y^2$ there so from that you wouldn’t actually be able to find out what the actual $y$ is equivalent to, if you were solving out the problem.

Interviewer: So it seems like we are ultimately trying to figure out what $y$ is?

Geoffrey: Yeah.
Here, it appears that Geoffrey believed that the goal of these tasks was to find \( y \). As such, because he thought that the derivative of \( y^3 \) was \( 3y^2 \), he thought that implicit differentiation would be needed to somehow further break down the equation to get a \( y \) by itself. Given this, it appears that he thought implicit differentiation could be used if there were any terms raised to the third power or higher, and that it would be needed if there was a term with \( y \) raised to the third power or higher. While this is correct in some cases, it is not always true and does not capture all scenarios when implicit differentiation is needed.

### 5.3.2 Geoffrey’s knowledge of related topics

Geoffrey had limited proficiency with derivative rules (he also stated this during the interview). He showed no evidence of knowing the product and chain rule, or when they would be needed, and both were needed to successfully complete a few of the implicit differentiation problems. He also did not know how to take the derivative of a constant (he believed it stayed the same). The only derivative rule he could use was the power rule. Though, even then, he believed that a standalone variable, such as \( x \) or \( y \), would not change when taking the derivative of it. On that note, he did not know that the derivative of \( y \) is \( \frac{dy}{dx} \).

Geoffrey did recall that there was something special about \( y \) and thought he had to do something different when taking the derivative of \( y^2 \). When taking the derivative of \( x + y^2 = x^3 - 5 \) the following exchange occurred:

**Interviewer:** So you said the derivative is \( x + y + 2y = 3x^2 - 5 \). I’m just not quite sure where the \( y \) between the \( x \) and \( 2y \) came from.

**Geoffrey:** I was under the impression that when you were doing derivatives, would you not in some cases have the actual derivative and then you have the actual second piece of it, or is that
just when you are doing it with pieces that are combined in their own separate kind of chunk on one side of the equation and you have another piece later on? Or does it have anything to do with ones that are directly interacting with one another?

**Interviewer:** I’m not 100% sure of...

**Geoffrey:** So in this one where you have the \( x + y^2 \), I’m under the impression that there would have to be, or at least from what I remember how I was taught, something else that would be produced by this...

**Interviewer:** \( y^2 \)?

**Geoffrey:** Yeah.

Here, it appears that Geoffrey may have been referring to the product or the chain rule when talking about how certain scenarios could produce extra items when taking the derivative, but it is unclear given the language he used. Putting that aside, he believed that the derivative of \( y^2 \) would be \( y + 2y \) since he recalled that something else happened when he had seen or been taught examples like this. So, while he did not know why something extra was produced, he did recognize that there is something special about \( y \), or \( y^2 \) at the very least. It should be noted though that this was only situational based upon what he remembered seeing, as he did not produce extra terms when taking the derivatives of \( 9y^2 \) and \( y^3 \).

Geoffrey also seemed to have limited knowledge of constants and variables. When he was first asked what he believed \( x \) and \( y \) to be, the following dialogue occurred:

**Interviewer:** How would you classify \( x \) and \( y \)? Would they be constants, would they be...

**Geoffrey:** Constants.

**Interviewer:** So you think \( x \) and \( y \) are constants?

**Geoffrey:** Yeah.
Interviewer: And then, what would the derivative of a constant be?

Geoffrey: I do not know.

Here, Geoffrey cut off the interviewer and stated that he believed $x$ and $y$ were “constants.” To check to see what he meant by “constant,” he was asked what the derivative of one would be, but he simply said he did not know. It was found, to some degree, what he meant by this later in the interview. That said, because he was being asked to find $\frac{dy}{dx}$, it is problematic that he did not label them as variables. By the end of the interview, rather than defining terms as variables or constants, Geoffrey labeled them as “constants” or “numbers.” While he did not give concrete definitions for these terms, he said enough during the interview to infer small pieces as to what he meant.

Geoffrey believed a standalone term, such as $x$ or 5, should be labeled as a “constant.” Here, it should be noted that his term “constant” is different from what one would normally refer to as a constant. Rather, he called a term a “constant” if it remained the same after taking the derivative. Since he believed that the derivative of $x$ was $x$, it was a “constant” because it did not change. A “number,” on the other hand, appeared to be something that would change when taking the derivative. Given the expression $rs^2$ (it was not specified whether these terms were variables or constants), he decided that both $r$ and $s$ were “numbers” with his reasoning being that after he took the derivative of this expression, he got $2rs$. He said that $r$ was a “number” because it changed to $2r$. He said $s$ was a “number” because it changed to $s$ (from $s^2$). Having said that, there was some inconsistency in how he used these categorizations to label terms, so it is not possible to be certain of his meaning.

That is not to say that Geoffrey did not have any knowledge of variables. While he did not label anything as a variable in the interview, he did use some language that could have
referred to independent and dependent variables. At the end of the interview, he was given the equation \( z = ps + rs^2 + \frac{pr}{s} \) and was asked how he would label each of the different letters. Here, he said that \( z \) would be a “number.” This was surprising, as he had previously defined standalone terms as “constants” (this is one of the inconsistencies referred to earlier). Putting that aside, he then went on to explain that he thought \( z \) was a number because it would change based on what was put into \( p, s, \) and \( r \). Given this explanation, it would appear as though he thought, whether or not he knew the proper terminology, that \( z \) was a dependent variable and that \( p, s, \) and \( r \) were independent variables. As such, it is evident that he knew at least something about the concept of independent and dependent variables even though he did not use those terms.

5.3.3 Geoffrey’s success with solving implicit differentiation problems

When it came down to applying implicit differentiation, Geoffrey repeatedly said he did not know how it would work. But, while he did not think that he knew how to use implicit differentiation, he used some aspects of it on every task. When he took the derivative, he did not first try to rearrange an equation to make it explicit. He took what he believed to be the derivative of both sides of an equation simultaneously, so he was, in fact, using part of the process of implicit differentiation. However, it should be noted that he did not know this and believed that implicit differentiation was something else that he did not know how to apply.

While Geoffrey was able to apply a part of the process of implicit differentiation without recognizing it as such, he was unable to solve the interview tasks. Failure to do so can be attributed to his lack of knowledge of related topics. Almost all the derivatives he found were incorrect due to his lack of knowledge of the derivative rules. Even if he did have a better grasp of at least some of the derivative rules, it is unlikely that he would have arrived at a correct
answer. This is because he did not know how to take the derivative of terms involving \( y \). He did not recognize that \( y \) needed to be treated as a function of \( x \), but he did seem to remember that there was something special about \( y \) from what he was taught. However, because he did not treat it as a function, he did not know that its derivative was \( \frac{dy}{dx} \). As such, \( \frac{dy}{dx} \) never appeared as he differentiated different equations. It does not help that, as mentioned previously, it seems he believed the end goal was to solve for \( y \), even though it was written, and he was asked, to find \( \frac{dy}{dx} \). Thus, it is not surprising that he was not able to solve any of the implicit differentiation problems.

### 5.3.4 Summary of what Geoffrey knew and could do

In summary, Geoffrey did not know how to explain implicit differentiation, but he did know some aspect of how to use the technique, although he did not recognize he was using it. That is, he took the derivative of both sides of an equation simultaneously, which is part of the process of implicit differentiation, but thought he was using regular differentiation and believed that implicit differentiation was something else he could not remember how to do. He was able to come up with some criteria for whether it was necessary to use implicit differentiation to differentiate a given equation or not, but they were either unclear or not entirely correct. He believed that implicit differentiation was necessary when equations had terms that were raised to the third power or higher, which might be consistent with examples he had seen, but does not encapsulate all situations where this technique is needed. Having said that, he also could not solve implicit differentiation problems as he did not know how to take the derivative of terms with \( y \) (though he did seem to recall that it needed to be treated differently). His mastery of knowledge of related topics was also quite limited. The only derivative rule he showed some
proficiency with was the power rule which he could not even use correctly in all applicable situations, and while he used language that alluded to knowledge of variables, he never actually called anything a variable explicitly during the interview. Thus, it appears that Geoffrey had a grasp on just a small amount of knowledge related to implicit differentiation and he was unable to solve any of the implicit differentiation problems. As such, Geoffrey would be a case of someone who fits in the category “Limited knowledge, cannot solve.”

5.4 Emily, Rena, and Yolanda: “Some knowledge, can solve”

After going through each of the interviews, it was found that three of the five participants appeared to fall under the category of “Some knowledge, can solve.” That is, each of them appeared to have a strong grasp of knowledge regarding implicit differentiation and related topics. They were also each able to successfully solve at least one of the implicit differentiation problems. While participants in this category did have some difficulties, which will be discussed, with these things, it should not take away from the fact they appeared to have robust knowledge. Having said that, these students appeared to share a lot regarding what they knew and could do. As such, the focus for this section will be on a student we will call “Yolanda” and what she knew about these topics and her success at solving implicit differentiation problems. The other two participants, “Emily” and “Rena,” will be referenced based on knowledge they seemed to share with Yolanda and if there was anything they knew, or did not know, that appeared to be of relevance to this study. Difficulties these students had are also briefly discussed.
5.4.1 Their knowledge of implicit differentiation

The three participants in this category were able to describe and explain implicit differentiation to varying degrees. Yolanda stated she did not remember what it was at first but was able to say a little bit later in her interview. When she was asked how implicit differentiation was different from normal (explicit) differentiation, she said the following:

_Yolanda_: How is it different? Well, there’s two things changing, because, so like, if there’s a graph I suppose, it’s showing two different changes happening, I don’t know how to depict that, but... If the line was like here, the tangent line would be here, which is the regular “\(dx\),” you would be accounting for like multiple factors I suppose?

Here, Yolanda says that, with implicit differentiation, there are two things that are changing (as opposed to one). She then tries to add to her argument by drawing a curve with a tangent line but is not entirely confident in her depiction or explanation of it. That said, she again says that there are multiple, or more than one, factors at play. As such, it seems as if Yolanda is getting at the idea that multiple variables come into play when implicit differentiation is introduced. She also seems to recognize that each of these variables affect the outcome, the derivative (as opposed to explicit differentiation, where the resulting derivative is written in terms of only one variable, usually \(x\)).

Emily and Rena, on the other hand, were able to say a bit more and touched upon the same ideas as each other. When Emily was asked what implicit differentiation was and how she would explain it to a friend, she said the following:

_Emilys_: So that’s when you have an equation and you have \(x\)’s and \(y\)’s mixed in together and you need to get things in terms of one variable.

_Interviewer_: So how would you go about explaining that to your friend?
Emily: I would say that you take the derivative of the entire equation with respect to that variable and then you kind of organize the equation in terms of “one.”

Here, like Yolanda, Emily notes that multiple variables come into play when implicit differentiation is introduced. She then goes a step further and mentions that it involves taking the derivative of the entire equation with respect to a variable (which is the actual process of implicit differentiation). After this, she says you get the equation in terms of “one,” which she later clarifies as \( \frac{dy}{dx} \), which describes what solving an implicit differentiation problem involves. A bit later on in the interview, Emily also mentions that if you take the derivative of an \( x \) and a \( y \) you would add a \( \frac{dy}{dx} \) to the end of it. While this does not say anything about implicit differentiation, it is important as this is first dealt with explicitly when using implicit differentiation.

All three of the participants were able to identify when implicit differentiation would not be needed. Both Emily and Rena believed that implicit differentiation was not needed if it was possible to get \( y \) by itself. This is true, since this would mean you could make the equation explicit and use explicit differentiation to find the derivative. Surprisingly, Emily initially also thought that it would not be possible to use implicit differentiation if this was the case. She changed her mind on this later in her interview when she used implicit differentiation on an equation that could have been made explicit and decided it would always be possible to use it.

Yolanda had a bit more trouble in identifying when implicit differentiation was not needed but was able to arrive at a similar conclusion to both Emily and Rena. When she was asked to construct an equation where she would not need to use implicit differentiation when taking the derivative, she came up with \( x^2 + 4x + 17 = f(x) \). In the excerpt below, she is asked why this is the case.

Interviewer: Why would you not need to use it (implicit differentiation) there?
Yolanda: Because it only has one variable and so you wouldn’t be solving the derivative of $y$ with respect to $x$. It would just be the derivative of $x$, with respect to itself.

In a way, this is true since the derivative term would already be solved for, although she is incorrect when she says it would just be the derivative of $x$, with respect to itself. That said, it was later brought to her attention that her equation had two variables, since $f(x)$ is equal to $y$ and she modified her criteria as the following:

Yolanda: Well this is like, there is only one $y$. In the other equations where we had like $y = 4xy - 8$, so it’s like just one $y$. So I guess... it only has one $y$ ($f(x)$) um... so you don’t need to solve for $\frac{dy}{dx}$ like derivative of $y$ in respect to... $x$ because it’s already like... and this would be a line, no, it would be a... parabola, and it would still pass the vertical line test.

Here, Yolanda states that implicit differentiation is not necessary because the equation she came up with only has one $y$ and because it would pass the vertical line test. Both criteria are valid and get at the same idea that the variables can be isolated, meaning the equation represents a function. That is, if there is only one $y$ in an equation, it often represents a function of $x$ (there are cases where this is not true, but they are atypical in these types of problems), which can be differentiated using explicit differentiation.

All three of them were also able to identify criteria for when implicit differentiation would be needed. Each of them referenced that it was needed when there were multiple variables, but some were more correct than the others. When Yolanda was asked to construct an equation where she would need to use implicit differentiation when taking the derivative, she came up with $x^2 + y^2 = 36$. In the excerpt below, she was asked why this was the case.

Interviewer: Could you explain why you would have to use it (implicit differentiation) there?
**Yolanda:** Because this is an equation of a circle and it wouldn’t pass the horizontal line test, or vertical line test. Like for example, if I just drew it like... like it hits the (graph) multiple times.

**Interviewer:** And then, just by looking at the function, is there something that would tell you that you would need to use it?

**Yolanda:** Yeah, and it has two variables.

Here, Yolanda states that implicit differentiation is necessary if the graph of an equation does not pass the vertical line test. This is only situationally correct, as some equations can technically be differentiated using explicit differentiation even if they do not pass the vertical line test (equations of circles for example). She also mentioned it would be needed if an equation had two variables (without specifying anything about these variables). Emily also believed that implicit differentiation would be needed if an equation had two variables but went a step further and said that the variables needed to be mixed in together or they needed to be attached to each other. This is only situationally correct, since such equations can sometimes be made explicit (meaning implicit differentiation would not be needed). Rena, however, was able to come up with the most complete criteria in saying implicit differentiation would be needed if there was more than one \( y \) and you could not combine them. That is, she appeared to be saying that it would be needed if it was not possible to make the equation explicit. This is correct and follows from the idea that implicit differentiation would not be needed if an equation could be made explicit.

### 5.4.2 Their knowledge of related topics

All three participants in this category appeared to have mastery of the basic derivative rules (power, product, and quotient) and used them all successfully when working through the
implicit differentiation problems. They also each referenced the chain rule and were able to correctly apply it when necessary. However, when taking the derivative with respect to $x$ of terms with $y$, it appeared they had varying degrees of understanding. They all knew that they needed to include $\frac{dy}{dx}$ when taking the derivative of $y$ and seemed to know that it sometimes resulted from the chain rule, but they had different ways of explaining why this was the case. Emily simply said it was because of the chain rule without providing any further details. The other two, however, seemed to have more to say.

When taking the derivative of $-(4y)^2$, Yolanda went through her process and said the following:

*Yolanda*: And then you would do the chain rule on this so it’s like... and power rule, so it’s minus, or it’s 2 out front and then it’s the 4y, and then you carry it out, and then it would be 4, because the $y$ would drop, but it would still be 4 and then you would still keep the $y'$ because that’s what we’re looking for.

While the language Yolanda used to describe her process was unclear, it is evident that she used the chain rule and she was able to produce the correct derivative for this term. When taking the derivative of the inside function $(4y)$, she said it would be 4, that the $y$ would drop, and that $y'$ remained because it is what was being solved for. This makes it seem like she knew $y'$ resulted from taking the derivative of $y$ (she did not say it just needed to be added on or something along those lines). Though, saying that it remained because it is what was being solved for is not a proper way of explaining why it is there. As such, it appears Yolanda knew that $\frac{dy}{dx}$ resulted from the chain rule.
Upon taking the derivative of $y^2$ and ending up with $2y \frac{dy}{dx}$, Rena was asked where the $\frac{dy}{dx}$ came from and said the following:

**Rena:** I’m not sure I totally remember how it technically arrives in the equation, but I know that you have to signify that you can’t just take the derivative of $y$ and say that everything is fine. You have to show, because you are taking it with respect to $x$, you need to have some form of derivative that is in respect to $x$.

Here, Rena mentions that she is not completely sure where it comes from. Though, she does seem to know it came from taking the derivative of $y$, which results from the chain rule. She then says that there needed to be some sort of derivative in respect to $x$ because she was differentiating with respect to $x$. While this is true, it is not enough to say for sure that she realized that this was because $y$ needs to be treated as a function of $x$. As such, it appears Rena knew that $\frac{dy}{dx}$ resulted from the chain rule and had some understanding as to why it did.

All three participants in this category seemed to have no trouble working with constants and variables when dealing with $x$ and $y$, though there were some differences in thought among them when working with unknown letters and determining if they should be treated as constants or variables. Rena and Emily appeared to be on the same page. When dealing with the equation $z = ps + rs^2 + \frac{pr}{s}$ and asked to find $\frac{dz}{ds}$, they both immediately realized that $z$ and $s$ were variables based upon the derivative they were solving for. Rena then decided to treat the other letters as variables since it was not specified that they were constants. Emily, on the other hand, decided to treat them as constants since the equation already had two variables (which is the usual number of variables one deals with in equations until after learning implicit differentiation). That said, they both stated that it was possible for these letters to be either constants or variables.
Yolanda however, first asked if the other letters were variables, but then decided that they had to be constants as seen in the following dialogue:

*Interviewer*: Could $p$ and $r$ be variables?

*Yolanda*: I would say no, just because they are not listed, like, it would say like given respect to, like, given respect to like different variables like $ds$ or like $dp$ or $dr$. Like I think it would ask you to solve for more, or... I don't know. Just, no, because it's not listed here.

Here, Yolanda decided that these letters would have to be constants since there was nothing to indicate that they were variables (for example, she was not told to find the derivative with respect to one of those letters). While this is similar to the reasoning Rena had as to why she did not treat the letters as constants, this is problematic for answering these questions because taking the derivative while treating the letters as constants would result in an incorrect solution if they actually were variables. Though, this was for an equation without context, so nothing can be said about what would have happened if context was provided.

5.4.3. Their difficulties with knowledge on related topics

Having discussed what these students knew and any issues in relation to their knowledge of topics related to implicit differentiation, there are some other issues that came up. These issues will be mentioned briefly here.

One issue that came up was confusion around terms for referencing the derivative. Two of the students in this category used terms to reference derivatives that were not actually derivatives. That is, instead of using correct terms such as $\frac{dy}{dx}$ or $y'$, students referenced $dx$, $dy$, and $\frac{d}{dx}$ in their place. This is problematic, as $dx$ and $dy$ are differentials and $\frac{d}{dx}$ is a derivative operator. In other words, none of them are correct terms for a derivative.
Another issue that came up involved student conceptions of how something should turn out or look. One student knew that $\frac{dx}{dx} = 1$, but did not think the derivative term should simplify to one. Because of this, they then thought that it may be incorrect and considered what else it could have been. They later decided it was fine because they thought their final answer looked, based on appearance, correct. Another student was bothered that the derivative term they solved for was written in terms of $x$ and $y$. This led them to try and investigate ways to rewrite their derivative in terms of only one variable even though what they had was actually correct.

The final issues of note involved algebra mistakes. One mistake involved failure to apply an operation to both sides of an equation to maintain equality. When attempting to divide equations by a term, one student failed to divide the entire side of equation by that term on a few occasions. The other mistake involved a student incorrectly canceling out terms in the numerator and denominator of a rational expression. When simplifying $\frac{2}{32y-1}$, they arrived at $\frac{1}{16y-1}$. Here, they divided both the 2 in the numerator and the 32$y$ in the denominator by 2, but not the 1. This is incorrect as the 1 is being subtracted from (not multiplied by) 32$y$ in the denominator, so it would also need to be divided by 2.

5.4.4 Their success with solving implicit differentiation problems

All three participants in this category were able to use implicit differentiation. In other words, they knew how to take the derivative of both sides of the equation at the same time, treating variables properly. When it came to solving tasks using this technique, Rena and Yolanda did not seem to have any issues. Emily, however, struggled a bit at first when working on these tasks due to algebra errors and confusion regarding the derivative of $x$. But, after talking through some of her issues, she too was able to solve some of these tasks.
When working on these tasks, Emily and Rena followed the same steps to solve for their derivative term. They would first check to see if there was anything that could be simplified, and would then use implicit differentiation, simultaneously taking the derivative of both sides of the equation. Following this, they would move any \( \frac{dy}{dx} \) terms to one side of the equation and would then solve for \( \frac{dy}{dx} \).

Yolanda had a slightly different approach to solving these tasks. While she knew how to use implicit differentiation, rather than taking the derivative of each side of the equation immediately, she would first try to separate her variables by moving all terms with \( y \) onto one side of the equation. She would then see if she could isolate \( y \), in which case she would differentiate explicitly. Otherwise, she would then use implicit differentiation.

When asked to solve for \( \frac{dx}{dy} \), rather than \( \frac{dy}{dx} \), they were all able to do so without any issues. Though, it should be noted that Emily stated she was not sure why one would want to do this. That said, this means that they were all able to use implicit differentiation to take the derivative with respect to a different variable. This is something that is required for related rates, which comes shortly after implicit differentiation in calculus instruction.

5.4.5 Summary of what they knew and could do

In summary, Emily, Rena, and Yolanda were all able to provide some explanation of what they thought implicit differentiation was. They each touched upon the idea that multiple variables are involved with implicit differentiation. Emily and Rena were then able to correctly say that implicit differentiation involves taking the derivative of both sides of an equation simultaneously. All three of them were able to provide criteria for when implicit differentiation would not be needed, indicating, in some way or other, that it would not be needed if an equation
was explicit or could be made explicit. They each were also able to provide criteria for when it
would be needed. Emily and Yolanda said that it would be needed if there were multiple
variables, which is situationally correct. Rena said that it would be needed if there were multiple
\(y\) terms that could not be combined. This is more correct, as it gets at the idea that implicit
differentiation is needed for equations that cannot be made explicit. They all were able to take
derivatives without any issues and were able to use the chain rule. They also seemed to know
that \(\frac{dy}{dx}\) resulted from taking the derivative of \(y\). They had no issues dealing with constants and
variables and took correct derivatives based upon whether they believed a term was a constant or
a variable. Finally, they were all able to correctly solve one or more of the implicit differentiation
problems using similar approaches to arrive at their final answers. While Emily and Yolanda had
some issues that could have, and did in some cases, affect their success at solving implicit
differentiation problems, they were able to talk through or put aside some of them, which
allowed them to correctly solve at least a few of the tasks. Thus, Emily, Rena, and Yolanda each
appeared to have enough knowledge in regard to implicit differentiation and related topics and
could each correctly solve at least one of the implicit differentiation problems. As such, they
exemplify someone who fits in the category “Some knowledge, can solve.”

5.5. Potential framework for characterizing student knowledge, and ability, of implicit
differentiation

When examining what the different interview participants knew, in relation to implicit
differentiation, there were some things that were sought out during analysis. It was checked to
see how well each participant fulfilled each of the items (described starting in Section 5.2) when
examining each participant’s knowledge of and skills with implicit differentiation and related topics. These items can be seen in Figure 5.2.

<table>
<thead>
<tr>
<th>Criteria related to implicit differentiation for analysis</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Could describe implicit differentiation</td>
<td>Could identify when it is needed</td>
</tr>
<tr>
<td>Could distinguish between variables and constants</td>
<td>Knew the derivative of terms with $y$</td>
</tr>
</tbody>
</table>

Figure 5.2. Criteria related to implicit differentiation

Four of these items had to do directly with implicit differentiation, and four involved skills from related topics. It should be noted that some of these items depend on one another. For example, to take a derivative using the chain rule, one must first know the basic derivative rules. That being said, the set of these items appear to be useful in distinguishing differences in student understanding and ability related to implicit differentiation. It seems they reveal differences in knowledge among the interview participants, as can be seen in Figure 5.3.
In Figure 5.3, you can see whether an interview participant satisfied a criterion, or if they were partially successful in doing so. From this, it is evident that the three participants who could solve implicit differentiation problems had the most robust understanding and mastery of these criteria, revealing a difference between them and the others who could not solve these problems. As such, it may be the case that these criteria play a role in a student’s ability to use and/or understand implicit differentiation. These may be something to consider when examining data involving student knowledge, and ability, of implicit differentiation and could perhaps be the beginnings of a framework for characterizing student thinking and ability in relation to this topic. Perhaps not all of these criteria are useful and there are other things not listed here that are of importance, but that is something that could be addressed in future research. At the very least,
these eight criteria appear to be useful in characterizing student thinking and work in relation to implicit differentiation and do so in more detail than previous methods (e.g. right vs. wrong).

5.6. Comparison, discussion of categories

Having examined which category each interview participant fit into based on the framework outlined at the beginning of this chapter, it was found that only three out of the four categories were filled. Here, we will examine how the knowledge of the participants who fit into the different categories compared with one another. We will then discuss the category that did not have any participant fit into, “Limited knowledge, can solve.”

First, since Geoffrey (“Limited knowledge, cannot solve”) and Margaret (“Some knowledge, cannot solve”) could not solve any of the tasks, they were found to fit in the “cannot solve implicit differentiation problems” portion of the framework. Let us now compare what these participants knew regarding implicit differentiation and related topics. While they knew $y$ needed to be treated differently when dealing with implicit differentiation, neither of them knew how to take the derivative of it. In terms of solving implicit differentiation problems, Geoffrey was able to use part of the process of implicit differentiation, even though he did not know he was doing so, whereas Margaret could not. While neither of them could explain what implicit differentiation involved, Margaret could at least talk about what it allows us to do. Margaret was also able to identify when implicit differentiation was needed while Geoffrey was only partially correct in doing so. Margaret appeared to have mastery of all basic derivative rules, other than the chain rule. Geoffrey, on the other hand, only knew the power rule and could only use it situationally. Finally, Margaret seemed to have adequate understanding of constants and variables whereas Geoffrey did not appear to know how to define them. Thus, given the
difference in knowledge of these topics, it was decided Margaret was an example of someone who fits into the category “Some knowledge, cannot solve” whereas Geoffrey was an example of someone who fits in the category “Limited knowledge, cannot solve.”

Having examined how the two participants who could not solve any of the tasks compared to one another, let us now compare the participants who fit into the category of “Some knowledge, can solve” with Margaret. Although Margaret could not solve implicit differentiation problems, much of what she seemed to know was at a similar level to Emily, Rena, and Yolanda. However, the three of them knew additional information which Margaret only mentioned or did not know. Like Margaret, they knew the basic derivative rules, but they also knew how to use the chain rule. It also appeared that they knew the derivative of $y$ was $\frac{dy}{dx}$. They were also able to provide more accurate explanations of what they believed implicit differentiation to be and were able to use it. Finally, unlike Margaret, they at least considered that unknown letters in a contextless problem could be either constants or variables. Thus, it may be the case that some of the extra information that Emily, Rena, and Yolanda knew that Margaret did not know may be needed to solve implicit differentiation problems.

It was found that none of the interview participants fit into the category of “Limited knowledge, can solve.” This means that none of them both knew very little about implicit differentiation and its related topics and were able to solve at least one of the implicit differentiation problems. This is of interest, since it had been found that there are areas in calculus, and mathematics in general, in which students are able to successfully complete procedural tasks even if they have limited understanding of the underlying concepts (e.g., Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009). Given the low student success rate at solving implicit differentiation problems, this does not seem to be the case. While there were
only five interview participants, this may suggest that, in the case of implicit differentiation, it is not possible for students to successfully solve problems without having at least some knowledge about it and related topics. That is, computational skills alone may not be enough to be successful when working on implicit differentiation problems. This may mean that it would not be possible, or highly unlikely, for someone with knowledge comparable to that of Geoffrey to solve implicit differentiation tasks. This will be discussed in more detail in the following chapter.
CHAPTER

6. CONCLUSIONS AND IMPLICATIONS

The purpose of this study was to investigate student understanding of implicit differentiation, a technique which allows one to take the derivative of equations that are not explicit functions. As there was little literature available on the topic, one goal was to get a general idea of how well undergraduate calculus students perform when solving problems using implicit differentiation as well as the type of errors that are made when doing so. Another goal was to get an idea of how they think about and explain implicit differentiation and an idea of what they appear to know about prerequisite topics.

The findings of this study, which include data on student performance on implicit differentiation problems and reveal potential areas of difficulty related to this topic, help to reveal some aspects of student’s understanding of implicit differentiation. Implications of these findings are provided in this chapter to help inform educators and researchers alike in the mathematics community of areas where the instruction of this topic could be improved. These implications could be used to give direction towards boosting students’ understanding of this technique, and their ability to use it in practice, which could thus potentially help to improve the retention rate of undergraduate STEM majors.

6.1 Results regarding the first research question

The first research question for this study was designed to investigate student success when solving implicit differentiation problems and the type of errors, in a broad sense, that appear to be prominent. As a reminder, the first research question is as follows: Research Question #1: How do students perform when solving implicit differentiation problems?
Findings that align with the first question are summarized below and are discussed in the following sections:

- Implicit differentiation problems appear to be difficult for students.

- Calculus errors were the cause of more incorrect responses than algebra errors.

These findings concur with prior research (e.g., Clark et al., 1997; Martin, 2000) that suggests implicit differentiation problems are difficult for students. These findings add to the literature by suggesting that calculus errors may cause more incorrect responses than algebra errors to implicit differentiation problems.

### 6.1.1 Implicit differentiation problems appear to be difficult for students

When examining student responses to the tasks from the survey data, it was found that overall the success on these problems was low. Fewer than 50% of all responses (2016 and 2017 data combined) were correct. While the success rate from the 2017 data is a bit higher (about 55%), it is still quite low. A breakdown of the percentage of correct, incorrect, and blank responses is displayed in Figure 6.1.

<table>
<thead>
<tr>
<th></th>
<th>Correct responses</th>
<th>Incorrect responses</th>
<th>Blank responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>2016 surveys</td>
<td>40%</td>
<td>46%</td>
<td>14%</td>
</tr>
<tr>
<td>2017 surveys</td>
<td>55%</td>
<td>43%</td>
<td>2%</td>
</tr>
<tr>
<td>Overall</td>
<td>47%</td>
<td>45%</td>
<td>8%</td>
</tr>
</tbody>
</table>

Figure 6.1. Percentage of correct, incorrect, and blank response per year and overall

This should not be surprising, as prior studies have found similar results, noting that implicit differentiation problems were a source of difficulty for students (e.g., Clark et al., 1997; Martin, 2000). Thus, this data provides further evidence suggesting that students struggle with implicit differentiation problems. It should also be noted that the success rates varied from
problem to problem. Some of these problems were found to be more difficult than others, which will be briefly mentioned in the following section.

6.1.2 Calculus errors resulted in more incorrect responses than algebra errors

When examining incorrect responses to the tasks from the survey data, it was found that calculus errors appeared to be the primary cause. Roughly 60% of all incorrect responses (2016 and 2017 data combined) seemed to result from students first making a calculus error. A breakdown of the error types for each year and overall is displayed in Figure 6.2. When looking at this data, there are two things that need to be taken into consideration. First, while a student made either an algebra or calculus error on incorrect responses where they did not use implicit differentiation, this data is kept separate and is not of interest, as we only want to know what type of error was made if implicit differentiation was used. Second, it is possible that a calculus error (such as not using a derivative) may have resulted from difficulties with algebra (e.g. not recognizing a product of two functions).

<table>
<thead>
<tr>
<th></th>
<th>% of incorrect responses resulting from calculus errors</th>
<th>% of incorrect responses resulting from algebra errors</th>
<th>% of incorrect responses in which implicit differentiation was not used</th>
</tr>
</thead>
<tbody>
<tr>
<td>2016 surveys</td>
<td>56%</td>
<td>16%</td>
<td>28%</td>
</tr>
<tr>
<td>2017 surveys</td>
<td>62%</td>
<td>38%</td>
<td>0%</td>
</tr>
<tr>
<td>Overall</td>
<td>60%</td>
<td>27%</td>
<td>13%</td>
</tr>
</tbody>
</table>

Figure 6.2. Percentage of errors per error type per year and overall

It is not surprising that there appeared to be more calculus errors than algebra errors, as these problems were designed to be simple, so the algebra was not intended to be particularly difficult. That said, two of these problems were found to be a greater source of difficulty for students. One of these problems was designed to have students use the product rule (52% correct overall), and the other one was designed to have them use the chain rule (36% correct overall).
This suggests that both the product rule and the chain rule may cause particular difficulties for students when they are needed in the context of an implicit differentiation problem. In contrast, students were generally more successful on implicit differentiation problems where they only needed to use the power rule.

This data provides evidence that calculus-related ideas and skills are a source of difficulty for students as they work on implicit differentiation problems. It appeared the chain and product rules were particularly difficult. That said, it is not known whether these students had more general difficulties with the product and chain rules, or if something in the context of implicit differentiation contributed to these difficulties.

6.2 Results regarding the second research question

The second research question for this study was designed to investigate student understanding of implicit differentiation and related topics, such as the derivative and the chain rule. As a reminder, the second research question is as follows: Research Question #2: What do students know about implicit differentiation and related topics?

Findings that align with the second research question are summarized below and are discussed in the following sections:

- Not all students could explain implicit differentiation.
- Students could identify criteria for when implicit differentiation is needed.
- Students seemed to have difficulties with derivative ideas.
- Understanding of functions may be of importance for implicit differentiation.
- Students may not be able to solve implicit differentiation tasks without at least some understanding of the underlying concepts.
These findings concur with some prior research that suggest some of the topics related to implicit differentiation are difficult for students. These findings add to the literature by providing a look into how students think about and explain implicit differentiation. They also reveal evidence of potential areas of difficulty in related topics that may arise when implicit differentiation is introduced as well as evidence that computational knowledge alone may not be enough to solve implicit differentiation problems.

6.2.1 Not all interview participants could explain what implicit differentiation was

Not all interview participants were able to recall and explain what they thought implicit differentiation was. Three of the interview participants were able to say what the method of implicit differentiation involved. The other two either did not know or could not recall it.

The three participants who were able to explain implicit differentiation each mentioned, in some way or another, that multiple variables were involved. Two of them were then able to state that it was the method of taking the derivative of both sides of equation. While the other participant did not state this at first, as they had trouble remembering it, they were able to remember and successfully use it later in the interview.

One of the participants who could not explain implicit differentiation did not appear to have any relevant knowledge. The other participant who could not explain implicit differentiation recalled hearing about it and that it had something to do with circles. This is most likely since examples involving circles are sometimes used to introduce implicit differentiation.

All three of the interview participants who could explain implicit differentiation were able to use this technique to take derivatives. One of the participants who could not describe it was able to use part of the process of implicit differentiation but did not recognize that they were
doing so. This suggests that students may not always link names with the techniques they learn, which may cause confusion when they are asked to use techniques by name.

6.2.2 Students could describe when they thought implicit differentiation was needed

All five of the interview participants were able to provide some sort of criteria for when implicit differentiation was needed. Two of the participants were able to provide correct criteria, saying in some way that it would be needed if they were given an equation that was not explicit, and could not be made explicit. The other three participants were able to provide criteria that was only correct in certain situations (they did not represent all cases in which implicit differentiation would be needed).

One of the participants who was able to provide partially correct criteria thought it would be needed if there were terms with $y$ that were raised to the third power of higher. This is correct in some cases, but there are some cases where this is false. Furthermore, this criterion fails to account for equations that have $y$ terms with lesser powers (lower than the third power) that cannot be differentiated explicitly.

The other two participants who were able to provide partially correct criteria thought it would be needed if an equation had multiple variables in it. More specifically, they both either appeared to think, or said that these variables had to be mixed into the equation (the equation was not explicit). This is also only situationally correct, as some equations that do not appear to be explicit can be made explicit through algebraic manipulations.

This suggest that students can recognize when implicit differentiation may be needed, even if they cannot recall or describe implicit differentiation itself. This may mean that students can recognize some aspects of an equation that allow them to determine whether it is explicit. It
is also possible that, if a student did not know implicit differentiation, they would say it is needed if they believed they were presented with an equation they did not know how to take the derivative of using explicit differentiation.

6.2.3 Students seemed to have difficulties involving the derivative

There were some issues interview participants had in relation to the derivative that appeared to be of particular relevance to their work on the interview tasks. These include student confusion around symbols of differentiations and student conceptions of what they expected or thought a derivative to be or look like.

Two of the interview participants were found to have poor understanding of symbols of differentiation. These students used the symbols to reference derivatives that were not representative of derivatives. Rather than using correct symbolic representations such as $\frac{dy}{dx}$ or $y'$, these students used $dx$, $dy$, and $\frac{d}{dx}$ when using and describing derivatives. This is problematic since it is important for students to be able to distinguish between the derivative operator, $\frac{d}{dx}$, and the symbols used for derivatives, such as $\frac{dy}{dx}$. This suggests that students struggle with understanding the difference between the symbols of differentiation in the context of implicit differentiation which supports claims made in prior research (e.g., Orton, 1983).

Two of the interview participants, who were able to eventually solve implicit differentiation problems, initially had some issues when starting these tasks due to conceptions they held about derivatives and derivative solutions. One student doubted that one of their derivative terms, $\frac{dx}{dx}$, was correct, as it simplified to 1 and they were expecting something more. This resulted in them rethinking their work, but they ultimately decided it was probably fine as
they believed their final answer looked, based on appearance, correct. Another student was bothered that the derivative term they had was written in terms of $x$ and $y$. This led them to try and investigate ways to rewrite their derivative in terms of only one variable, even though what they had was actually correct. These issues suggest that students may try to reason with derivative terms based on their appearance rather than an understanding of what these terms actually represent. This is an example of a “manipulation focus” that was identified in student work on other kinds on derivative problems (White & Mitchelmore, 1996).

The survey data presented evidence that some of the implicit differentiation problems were more difficult for students than others. Problems involving the usage of the product rule, and the chain rule in particular were found to have lower success rates than the others. This suggests that student difficulties with these rules may affect their performance on implicit differentiation rules. Alternatively, it could be that the context of implicit differentiation contributes in some way to students’ making these errors (while they might not make them in the context of other kinds of problems). It also suggests that, since the product and chain rules require students to have some knowledge of functions, this knowledge, which is discussed in the following section, may also affect their performance.

**6.2.4 Understanding of functions may be of importance for implicit differentiation**

Ideas related to function appear to be needed to understand derivatives and implicit differentiation. Regarding derivatives in general, some understanding of function may be needed for some of the derivative rules. Regarding implicit differentiation, students may have difficulty recognizing variables as functions of other variables and may struggle with reconceptualizing how they think about $y$. 
Analysis of the survey data revealed that students were found to struggle more on implicit differentiation problems with the chain rule or the product rule. Both rules rely quite extensively on knowledge of functions. The chain rule requires students to be able to identify composite functions and to recognize the individual functions that have been composed. The product rule requires students to identify functions and recognize if there are functions that are being multiplied together. Thus, as prior research (e.g., Horvath, 2008) has found, it appears function concepts such as these are both important and a source of difficulty for students.

A concept in relation to functions that surfaces when introducing implicit differentiation is the notion that variables should be treated as functions of the variable an equation is being differentiated with respect to. This was something that four of the five interview participants appeared to have knowledge of. Without knowing this, students may not recognize why they end up with terms such as $\frac{dy}{dx}$, which may then lead to difficulties with chain rule when taking the derivative of terms with variables.

A difficulty in relation to functions that may surface when introducing implicit differentiation is the shift from viewing $y$ strictly as a dependent variable to either an independent or dependent variable depending on the context. One interview participant stated that they believed that $y$ represented the equation, and struggled with implicit differentiation as, all of a sudden, multiple $y$ terms would appear within equations. This is not surprising, since until implicit differentiation is introduced, $y$ is almost always used to represent a dependent variable. Building on this, the notion of acting upon terms with $y$ (taking the derivative of them), which was not explicitly the case when dealing with explicit differentiation, may be a source of difficulty for students. This may parallel research that found that students struggle to make the shift from thinking about $y$ as a placeholder for something that is being calculated in early
education, to something that needs to be solved for in algebra (e.g., Knuth et al., 2006). Other research has found that students sometimes have an under-developed understanding of equality and the equals sign (e.g., Kieran, 1992) which affects their understanding of and ability to solve algebraic equations. It follows that manipulating an equation in ways that maintain the equality is challenging, resulting in part from under-developed understanding about the nature of equality. Thus, areas such as these may be worth considering when examining how students think about implicit differentiation.

6.2.5 Students with limited knowledge may not be able to solve these problems

None of the interview participants were able to solve implicit differentiation problems if they did not appear to know a fair amount about implicit differentiation and related topics. This is intriguing as prior research has found that there are calculus topics, such as integration, where students with limited understanding of the underlying concepts are successful in completing procedural tasks (e.g., Grundmeier et al., 2006; Jones, 2015; Mahir, 2009). In these situations, students can generate correct answers, but further probing reveals that they do not have strong command of the ideas that the problems are about. Although there were only five interview participants in this study, this may suggest that it is not possible, or highly unlikely, for students to solve implicit differentiation problems without at least some amount of knowledge of implicit differentiation and related topics. That is not to say that computational skills are not important, but that having them alone may not be enough. In other words, there may be some knowledge in relation to these skills or implicit differentiation that students need to have to be successful when working on these problems.
6.3 Implications for educators

This study has helped to build upon literature citing student difficulties with implicit differentiation and related topics. Findings suggest that these difficulties may stem from how students understand symbols of differentiation, from what they understand in relation to function and derivative rules, and from how implicit differentiation is explained and introduced. These findings have some instructional implications for calculus educators.

When introducing explicit differentiation and the derivative rules, more focus may need to be placed on the idea that the derivative of \( y \), when differentiating with respect to \( x \), is \( \frac{dy}{dx} \). That is, the derivative operator is also being applied to \( y \) (not just the other side of the equation), thus resulting in \( \frac{dy}{dx} \). More emphasis may also need to be placed on the fact that the derivative of \( x \) is \( \frac{dx}{dx} \), which, while equivalent to 1, is important as it represents a derivative. That is, it may be beneficial for students to understand that the derivative of \( x \) is not simply 1. This may help to familiarize students with this notation, which may prevent some confusion when dealing with terms in this form, such as \( \frac{dy}{dx} \), when implicit differentiation is introduced. This may help students recognize terms such as \( \frac{dy}{dx} \) as derivatives.

It also may be of benefit to place more emphasis on discussing the variable of differentiation. This is an important piece of information when dealing with implicit differentiation and its applications. For example, rather than saying “take the derivative,” it may be beneficial for student understanding of this concept to say, “take the derivative with respect to \( x \).” That is, if you just say, “take the derivative,” it is not made obvious to students that they are taking the derivative with respect to a specific variable. This would also potentially make it easier to use more varied examples, as you could, for example, then also ask them to take the
derivative with respect to $t$, resulting in derivative terms such as $\frac{dr}{dt}$. This would help to get students accustomed to dealing with variables other than $x$ and $y$, giving them the opportunity to generalize, which may help resolve issues such as the “$x$, $y$ syndrome” which was described previously. To this end, it is recommended that terms such as $\frac{dy}{dx}$ are used to label derivatives, as terms such as $y'$ and $f'(x)$ mask the variable an equation was differentiated with respect to. These things may help build student understanding of these ideas, as it would provide students with exposure to taking the derivative of equations with variables other than $x$, which may make them more comfortable when dealing with derivatives such as $\frac{dr}{dt}$ before they are found with implicit differentiation.

Before the instruction of implicit differentiation, it may be beneficial for instructors to test for student knowledge of related topics that were found to be possible sources of difficulty when solving implicit differentiation problems, such as the chain rule or the product rule. This would give them an opportunity to identify if students struggle with these topics in a general sense, while also providing them with the opportunity to address these issues if they were indeed found to be difficult for students. If it is then found that students still struggle with these topics when working on implicit differentiation problems, it may be the case that implicit differentiation adds a new layer of difficulty in relation to these topics.

It may be the case that using the equation of a circle as the motivating example for implicit differentiation has the potential to set students up to think about implicit differentiation in unproductive ways. When explaining how implicit differentiation allows us to take the derivative of equations like those of circles, instructors sometimes show how explicit differentiation sometimes could also be used to take the derivative. This may lead some students to believe that it may be possible to make any equation explicit. This is problematic, as this is not
true, and may explain why some students first try to make equations explicit before resorting to using implicit differentiation. Students may simply note how implicit differentiation made taking the derivative of the equation of a circle easier, which may lead to unproductive ways of describing implicit differentiation. That is, for example, a student may say that it is an easier way to take derivatives when asked to describe what it is. Such conceptualizations may inhibit the development of understanding in relation to what implicit differentiation is and what it allows us to do. There is value in using examples such as these since we want students to understand they will get the same answers using either method of differentiation in this case. However, it may be important to emphasize that equations cannot always be made explicit and what it is implicit differentiation allows us to do that is not possible with explicit differentiation alone.

6.4 Future research opportunities

While this study helped to fill some holes in the implicit differentiation literature, there is still much we do not know about how students think about this topic. This study helped to provide evidence that implicit differentiation is difficult for students and that it may be the case that knowledge of at least some of the underlying concepts is required to be successful when solving implicit differentiation problems. The survey and interview data provide some evidence of, and allows us to speculate about, some areas of difficulty related to the function and differentiation. That said, more research needs to be done to help further understand what makes implicit differentiation difficult for students.

On a small scale, it may be of interest to refine existing survey and interview instruments, or to create new ones, to examine targeted areas of student understanding in relation to implicit differentiation such as the eight criteria mentioned in Section 5.5. The instruments used for this
study helped to give a broad idea of what students struggled with. However, a few of the tasks were problematic in some ways. It was possible to manipulate some of them algebraically to make them explicit equations. There were also algebraic manipulations students could do to avoid using the product rule and the chain rule. As such, it would be best to carefully design tasks to avoid workarounds to the ideas we seek to understand.

Conducting more interviews may also be of interest to get a larger sample of data on how students think about and explain implicit differentiation. Five interviews were conducted over the course of this study and there was not much variety between the different type of responses. That is, the interview participants from this study either knew it and could solve implicit differentiation problems, or they did not know it and could not solve these problems. In addition to getting more explanations from students such as these, it would be interesting to hear from ones who could not be categorized as such. Also, because the participants in this study were asked how to explain implicit differentiation, they may not have been inclined to define it initially. Thus, it may be a good idea to ask students to first define it, and then explain it. As such, it would be beneficial to conduct additional interviews to investigate if there are other things students say about this topic when asked to explain or define it.

It may also be of interest to identify how students think about the ideas of function equality and applying function operators to both sides of an equation in the context of implicit differentiation. As mentioned in the findings, students struggle to make the transition from doing things to one side of an equation to applying function operations to both sides of an equation. Because a similar shift is made from explicit differentiation, where the derivative operator only appears to be applied to one side of an equation, to implicit differentiation, where the derivative operator is applied to both sides of an equation, difficulties with the idea of function equality
from earlier in a student’s education may resurface when learning implicit differentiation and may contribute to their difficulties with this topic. As such, research on function equality (e.g., Knuth et al., 2006) may be of relevance when considering what about implicit differentiation makes it difficult for students. Thus, it may be beneficial to create a study examining these ideas to investigate if they factor into the learning of implicit differentiation.

On a larger scale, it may be beneficial to conduct a longitudinal study of student understanding of implicit differentiation. Through the use of carefully designed surveys and/or interviews, it may be possible to better target and identify the specific struggles students have with implicit differentiation and related topics as well as how these issues may influence their ability to solve implicit differentiation problems over the course of their study of calculus. First, student knowledge of differentiation and functions could be examined before the instruction of implicit differentiation. Thinking and ability in relation to implicit differentiation could then be examined shortly after learning it. This would then be reexamined after students had more time to practice and master this topic. Doing this would give us insight in both what appears to be hard for students as they progress through the instruction of implicit differentiation, as well as what more complete student conceptualizations of this topic look like. This data could also be used to compare how students who appeared to have a strong understanding of the related material before learning implicit differentiation fared by the end of the study in comparison to students who demonstrated poor understanding of this material at the start of the study. This would give us a better understanding of how these related topics affect, if they do, student learning of implicit differentiation. Research has found (e.g., Selden et al., 2000) that having an adequate knowledge base of related topics does not always translate to success when using it in
different settings. As such, it would be useful to know whether this is the case with implicit differentiation.

Implicit differentiation, a technique that allows us to differentiate equations that are not explicit functions, has been found to be a source of difficulty for students. Ideas around function and derivative concepts and skills have been identified as possible causes of these difficulties. Given the importance of what implicit differentiation allows us to do and the fact that applications such as related rates rely on it, this is something that needs to be addressed. Learning more about how students think about and work with this topic can help us to identify reasons as to why it is difficult, which can then be used to guide instruction to improve the teaching of implicit differentiation and related topics. If we can increase student performance and understanding in relation to this topic, students would be more successful in calculus. In a broad sense, this should then translate to an increase in STEM major retention.
REFERENCES


A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell to your friend to help them understand what implicit differentiation is?

Solve for \( \frac{dy}{dx} \):

a) \( x + y^2 = x^3 - 5 \)

b) \( y - 2x = (4y)^2 \)
Solve for \( \frac{dy}{dx} \).

a) \( x^2 = 3xy - 5y \)

b) \( x^3 - 7 = 2y \)

A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell to your friend to help them understand what implicit differentiation is?
Solve for \( \frac{dy}{dx} \).

a) \( x^2 = 3xy - 5y \)

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A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell to your friend to help them understand what implicit differentiation is?
Solve for $\frac{dy}{dx}$.

a) $x + y^2 = x^3 - 5$

b) $y - 2x = (4y)^2$

c) $x^2 = 3xy - 5y$

d) $9y^2 + 5x = y$
APPENDIX E: INTERVIEW INSTRUMENT

1. A friend of yours is taking calculus and was absent the day everyone learned about implicit differentiation. What would you tell your friend to help them understand what implicit differentiation is?

2. Examine the following table. For each of the equations, circle Y/N as to whether or not you could use implicit differentiation to find \( \frac{dy}{dx} \). Then circle Y/N as to whether or not you MUST use implicit differentiation to find \( \frac{dy}{dx} \).

<table>
<thead>
<tr>
<th>Equation</th>
<th>Can you use it?</th>
<th>Do you need to?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ( x^2 + 5 = x - 3y )</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>b) ( y = 4xy - 8 )</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>c) ( y^2 = 8x^3 - 2x + 1 )</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
<tr>
<td>d) ( 5x - y^3 = 9y - 24 )</td>
<td>Y/N</td>
<td>Y/N</td>
</tr>
</tbody>
</table>
3. Solve for $\frac{dy}{dx}$.

a) $x + y^2 = x^3 - 5$

b) $y - 2x = (4y)^2$

c) $x^2 = 3xy - 5y$

d) $9y^2 + 5x = y$

4. Given the formula $z = ps + rs^2 + \frac{pr}{s}$, calculate $\frac{dz}{ds}$ and explain how you did it.
BIOGRAPHY OF THE AUTHOR

Connor Chu was raised in Elmont, New York and later moved to Winthrop, Maine where he graduated from Winthrop High School in 2011. He attended the University of Maine and graduated magna cum laude in 2015 with a Bachelor of Arts in Mathematics. He is a candidate for the Master of Science in Teaching degree from the University of Maine in August 2019.