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Three Examples of Mondoromy Groups

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THREE EXAMPLES OF MONODROMY GROUPS

By

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A THESIS

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This thesis illustrates the notion of the Monodromy group of a global analytic function through three examples. One is a relatively simple finite example; the others are more complicated infinite cases, one abelian and one non-abelian, which show connections to other parts of mathematics.

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CHAPTER 1

INTRODUCTION

1.1 BRIEF HISTORY AND NOTATION

Georg Friedrich Bernhard Riemann's dissertation touched on the idea of a Riemann surface, or a surface described by the largest domain of an analytic function (for more, see Section 2.4). Skeptic, Karl Weierstrass set out to disprove such a surface existed, and thus began to develop local theory with functions whose domain is a subset of the complex plane. Eventually, Riemann was vindicated. The following is a brief description in the rise of this theory. In this thesis, we walk through complex, local theory, expand into global theory, and tie abstract algebra and topological theory to three starkly different examples (for more see Ahlfors(1966), Merzbach 2011, and Markushevich 1965).

The principal objects discussed in this thesis are the closely related notions of *holomorphic* and *analytic functions*, and the global analytic functions created through unrestricted analytic continuation. We establish definitions and some well-known theory before continuing.

Throughout the following text, we will assume the sets $\mathscr{D},\ \mathscr{D}^*,\ \mathscr{D}_1,\ \mathscr{D}_2 \ldots$ to be open. Further, we will assume $\mathscr{B}(z, R)$ to be an open disc centered at z with radius R.

Definition 1.1.1. *A function $f : \mathscr{D} \to \mathbb{C}$ is holomorphic on $\mathscr{D}$ if f is differentiable at all points $z \in \mathscr{D}$.*

Definition 1.1.2. *A function f is said to be analytic at z_0 if it can be represented by a power series*

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n$$

that converges absolutely in some open disc $\mathscr{B}(z_0, R)$ for $R > 0$.

The term "analytic" describes the holomorphic property of the power series, since the power series is infinitely differentiable by term wise differentiation.

Lemma 1.1.3. *Suppose the analytic function f is defined on $\mathscr{B}(z_0, R)$ such that $R > 0$. Further, suppose $f(z_0) = 0$. Then either f is identically zero throughout $\mathscr{D}$ or there exists a unique natural number m, called the order of zero, such that*

$$f(z) = (z - z_0)^m g(z) \tag{1.1}$$

where $g(z)$ is analytic in $\mathscr{D}$ and $g(z_0) \neq 0$.

Proof. If f is analytic in $\mathscr{D}$, then f is represented by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n$$

that converges absolutely in the disc $\mathscr{D}$. Either $A_n = 0$ for all $n \in \mathbb{N}$, or there exists $m \in \mathbb{N}$ such that A_m is nonzero. We can assume without loss of generality that m is the smallest such natural number. In the first case, if all coefficients are zero, then $f(z) = 0$ throughout the disc also. In the second case

$$\sum_{n=m}^{\infty} A_n (z - z_0)^n = (z - z_0)^m \sum_{n=0}^{\infty} A_{n+m} (z - z_0)^n$$

and the function $g(z) = \sum_{n=0}^{\infty} A_{n+m} (z - z_0)^n$ is clearly analytic since it is represented by a power series and, $g(z_0) = A_m \neq 0$. □

Theorem 1.1.4. *Suppose f is analytic in $\mathscr{D}$. If there exists a sequence of points $z_n \in \mathscr{D}$ that converges to a limit $z_0 \in \mathscr{D}$ and $f(z_n) = 0$ for all $n \in \mathbb{N}$, then f is identically zero in $\mathscr{D}$.*

Proof. By the continuity of f, $f(z_0) = 0$. By Lemma 1.1.3, if f is not identically zero in some open disc centered at z_0, then $f(z) = (z - z_0)^m g(z)$ for some natural number m, where g is analytic and $g(z_0) \neq 0$. By the continuity of g there exists a neighborhood of z_0 where g has no zero, meaning there exist no point in the neighborhood except z_0, where f vanishes. This contradicts the hypothesis that z_n tends to z_0. Hence the sequence $(f(z_n))$ is zero for all $n \in \mathbb{N}$ as it converges to $f(z_0)$. Since this argument works for all $z_0 \in \mathscr{D}$, f is identically zero where the Taylor series converges. □

CHAPTER 2

ANALYTIC CONTINUATION

2.1 ANALYTIC CONTINUATION OF A HOLOMORPHIC FUNCTION

When discussing a function $f : \mathscr{D} \to \mathbb{C}$, it is sometimes the case that $\mathscr{D}$ is the natural domain for the function, in the sense that it is the largest open set where f is defined. For example, the function $f : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ given by

$$f(z) = (1 - z)^{-1}$$

has the natural domain $\mathbb{C}\backslash\{1\}$, but its series expression,

$$f_0(z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges only on $\mathscr{B}(0, 1)$. However, were we to begin with the power series $f_0(z)$, we could sum the series to give $(1 - z)^{-1}$, which gives an expansion of the domain of f_0 to $\mathbb{C}\backslash\{1\}$. We will formalize this process of producing a function on a larger domain as the notion of analytic continuation.

Definition 2.1.1. *Let $f_0 : \mathscr{D}_0 \to \mathbb{C}$ be holomorphic. If $f_1 : \mathscr{D}_1 \to \mathbb{C}$ is holomorphic, $\mathscr{D}_0 \cap \mathscr{D}_1 \neq \emptyset$, and $f_0(z) = f_1(z)$ for all $z \in \mathscr{D}_0 \cap \mathscr{D}_1$, then f_1 is said to be an analytic continuation of f_0 to $\mathscr{D}_1$.*

Theorem 2.1.2. *If $f_0 : \mathscr{D}_0 \to \mathbb{C}$ has an analytic continuation to a simply connected domain $\mathscr{D}_1$ with $\mathscr{D}_0 \cap \mathscr{D}_1 \neq \emptyset$ then this continuation is unique.*

Proof. Suppose that $f_1 : \mathscr{D}_0 \to \mathbb{C}$ and $f_2 : \mathscr{D}_0 \to \mathbb{C}$ are two analytic continuations of f_0 to $\mathscr{D}_1$. On the intersection $\mathscr{D}_0 \cap \mathscr{D}_1$, which is open, f_1 and f_2 are both equal to f_0, so $f_1 - f_2 = 0$ on $\mathscr{D}_0 \cap \mathscr{D}_1$. By Theorem 1.1.4, $f_1 - f_2 = 0$ throughout $\mathscr{D}_1$. $\square$

Although unique when existent, finding an analytic continuation is often difficult, and often needing *ad hoc* methods. As an example, let us analytically continue the function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

First, we show Γ is uniformly convergent for all Re $z > 0$.

Lemma 2.1.3. *Let* $0 < \delta \leq \text{Re } z \leq \Delta < \infty$. *Then*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt \text{ is holomorphic for Re } z > 0.$$

Proof. Consider the integral restricted to $T < t < T'$. For $\varepsilon > 0$, we will show that there exists a $T_0 \in \mathbb{N}$, so that for all $T' > T > T_0$,

$$\left| \int_T^{T'} t^{z-1} e^{-t} dt \right| < \varepsilon.$$

Consider $T < T' \leq 1$. Since $e^{-t} \leq 1$ for $0 \leq t \leq 1$, we have
$|e^{-t} t^{z-1}| \leq |t^{z-1}| \leq t^{\delta - 1}$. Hence,

$$\left| \int_T^{T'} t^{z-1} e^{-t} dt \right| \leq \int_T^{T'} t^{\delta-1} dt = \frac{1}{\delta}(T'^{\ \delta} - T^\delta).$$

Now let $1 \leq T \leq T' < \infty$. Then for large enough values of t,

$$\left| \int_T^{T'} t^{z-1} e^{-t} dt \right| \leq \int_T^{T'} t^{\Delta-1} e^{-t} dt = \int_T^{T'} e^{-t/2} (e^{-t/2} t^{\Delta - 1}) dt.$$

The function $e^{-t/2} t^{\Delta-1}$ is convergent as $t \to \infty$, so for some $C > 0$

$$\int_T^{T'} e^{-t/2} (e^{-t/2} t^{\Delta-1}) dt \leq C \int_T^{T'} e^{-t/2} dt = -2C(e^{-T'/2} - e^{-T/2}).$$

As T and T' tend to infinity, e^{-T} and $e^{-T'}$ tend to zero. By comparison, the integral $\int\limits_{T}^{T'} t^{z-1}e^{-t}dt$ must also tend to zero for $1 \leq T < T' < \infty$. Hence $\Gamma(z)$ is uniformly convergent for all Re $z > 0$.

Then $\int_{\mathscr{C}} t^{z-1}e^{-t}dt = 0$, where $\mathscr{C}$ is any closed curve on the region Re $z > 0$. By Morera's Theorem, this implies f is holomorphic. In using integration by parts, as so:

$$\Gamma(z+1) = -t^z e^{-t}\Big|_0^{\infty} + z\int\limits_0^{\infty} t^{z-1}e^{-t}dt = z\Gamma(z)$$

we expand the defined region of the Gamma function. Hence

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}. \tag{2.1}$$

This implies that $\Gamma(z)$ is holomorphic for all Re $z > -1$, except at $z = 0$, a simple pole. Therefore, $\int_{\mathscr{C}} t^{z-1}e^{-t}dt = 0$, where $\mathscr{C}$ is any closed curve[1] on the region Re $z > -1$. Therefore, Γ can be analytically continued to the left of the imaginary axis. By iteration,

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)} = \ldots = \frac{\Gamma(z+\beta)}{z(z+1)\ldots(z+\beta-1)} \qquad \text{for } \beta \in \mathbb{N}. \tag{2.2}$$

By a similar argument to the case where Re $z > -1$, Equation (3) analytically continues Γ to a holomorphic function that defines the half plane
Re $z > -\beta$, except at $z = 0, -1, -2, \ldots, -\beta + 1$.

Thus the Gamma function can be analytically continued from an initial domain to $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$, where $0, -1, -2, \ldots$ are simple poles. Although a very elegant example, it is very specific to the function considered and does not provide a general method of analytic continuation. We can apply a more general method to a power series.

[1]Any closed curve which goes around $z = 0$ can be broken into a sum of simple curves which do not go around zero.

Definition 2.1.4. *A function element is a pair* $\{f, z_0\}$*, understood as the function*

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \tag{2.3}$$

on the open disc $\mathscr{B}(z_0, R)$*, where* $R > 0$ *is the largest value for which the series (2.3)*

convergences absolutely on $\mathscr{B}(z_0, R)$*. Note that the coefficients are given by*

$$A_n = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{f(t)dt}{(t-a)^{n+1}}, \tag{2.4}$$

where $\mathscr{C}$ *can be taken to be the positively-oriented contour* $|z - a| = R_0$*,*

$R > R_0 > 0$*.*

Definition 2.1.5. *Let* $f : \mathscr{D} \to \mathbb{C}$ *be holomorphic. If* f *can be analytically continued to some neighborhood of a point* $\zeta \in \mathscr{D}$*, then* ζ *is called a regular point of* f*; if not, then* ζ *is called a singular point of* f*.*

These definitions are of interest when ζ is a point on the boundary of $\mathscr{D}$, since every point interior of $\mathscr{D}$ is regular.

There is a method by which we can analytically continue a power series by manipulating the expansion directly, known as *direct analytic continuation*. In essence, this takes a function element $\{f_0, z_0\}$ and moves the center of the function f_0 to some point $z_1 \in \mathscr{B}(z_0, R_0)$. In expanding about z_1, we have another function element $\{f_1, z_1\}$. More precisely, let f_0 be given by Equation (4). Given $z_1 \in \mathscr{B}(z_0, R_0)$, we can apply the Binomial Theorem and change order of summations to give

$$f_0(z) = \sum_{n=0}^{\infty} A_n \sum_{j=0}^{n} \binom{n}{j} (z-z_1)^j (z_1-z_0)^{n-j} = \sum_{j=0}^{\infty} \frac{1}{j!}(z-z_1)^j \sum_{n=j}^{\infty} A_n \frac{n!}{(n-j)!}(z_1-z_0)^{n-j}$$

for any $z \in \mathscr{B}(z_0, R_0)$. The inner sum can be recognized as $f^{(j)}(z_1)$ so it converges. Hence the expression on the right is the Taylor series expansion of f near z_1. This converges absolutely in the disc $\mathscr{B}(z_1, R_1)$. It may happen that $\mathscr{B}(z_1, R_1)$ is contained in $\mathscr{B}(z_0, R_0)$, but more often $\mathscr{B}(z_1, R_1)$ extends outside $\mathscr{B}(z_0, R_0)$, and the new expansion is an analytic continuation of f_0 to $\mathscr{B}(z_1, R_1)$.

Theorem 2.1.6. *Every function given by a power series has at least one singular point on the boundary of its disc of absolute convergence.*

Proof. Let $\{f_0, z_0\}$ be a function element as above. Assume to the contrary that all boundary points of the disc $\mathscr{D}_0$ are regular. Then for each $z \in \partial\mathscr{D}_0$ there exists an open disc $\mathscr{U}_z$ containing z to which f_0 can be continued, and hence a ball $\mathscr{B}(z, R_z)$ for some $R_z > 0$ for which f can be analytically continued to $\mathscr{D}_0 \cup \mathscr{B}(z, R_z)$; furthermore, this analytic continuation is unique by Theorem 2.1.2.

By the Heine-Borel Theorem, finitely many of the sets $\mathscr{U}_z$ cover $\partial\mathscr{D}_0$, say $\mathscr{U}_1, \ldots, \mathscr{U}_n$. We will denote the continuations of f_0 to the discs $\mathscr{U}_1, \ldots, \mathscr{U}_n$ by $f_1, \ldots, f_n$ respectively. Without loss of generality, also suppose no set is contained within another and the discs are numbered counter-clockwise. Since the direct analytic continuations from f_0 to $\mathscr{U}_1$ and from f_0 to $\mathscr{U}_2$ are each unique and $f_0 = f_1 = f_2$ on $\mathscr{D}_0 \cap \mathscr{U}_1 \cap \mathscr{U}_2$, we have that $f_1 = f_2$ on $\mathscr{U}_1 \cap \mathscr{U}_2$, implying f_2 is also a unique analytic continuation of f_1. This applies to all intersections $\mathscr{U}_j \cap \mathscr{U}_{j+1}$ and $\mathscr{U}_n \cap \mathscr{U}_1$, i.e., we must get f_1 on $\mathscr{U}_1$ when we analytically continue f_n to $\mathscr{U}_1$. Thus the analytic continuation to $\mathscr{D}_0 \cup \bigcup_{j=1}^{n} \mathscr{U}_j$ is unique and single-valued.

The boundaries $\partial\mathscr{U}_j,\ \partial\mathscr{U}_{j+1}$ intersect at some point x_j for which $|x_j| = |R_0 + \varepsilon_j|$, where R_0 is the radius of convergence for $\mathscr{D}_0$ and $\varepsilon_j > 0$ (for $\partial\mathscr{U}_n,\ \partial\mathscr{U}_1$, we get ε_n). Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $\mathscr{B}(z_0, R_0 + \varepsilon) \subseteq \mathscr{D}_0 \cup \bigcup_{j=1}^{n} \mathscr{U}_j$ and $\mathscr{B}(z_0, R_0 + \varepsilon)$ is a larger disc of convergence for f_0 than $\mathscr{D}_0$, a contradiction. This implies that there must indeed exist one point of singularity on the boundary of convergence for any function element. $\square$

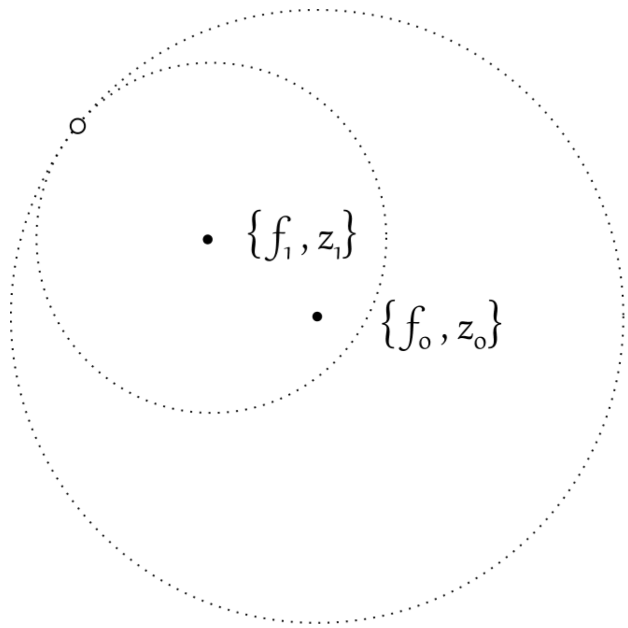

Figure 2.1. An example point of singularity on the boundary of the domain of two function elements.

Consider the case for which the region $\mathscr{D}_1$ is entirely contained in $\mathscr{D}_0$. All points of $\partial\mathscr{D}_1 \subset \mathscr{D}_0$ are regular, since f_0 is infinitely differentiable at all points interior to $\partial\mathscr{D}_0$. By the above argument f_1 must have at least one point of singularity on its boundary; therefore, the point at which $\partial\mathscr{D}_0$ and $\partial\mathscr{D}_1$ are tangent must be a singular point for both f_0 and f_1. This can be seen in Figure 2.1; in which $\mathscr{B}(z_1, R_1)$ is the disc of the absolute convergence for f_1 and therefore, has a singular point on its boundary. This must be ζ, since all other points are in $\mathscr{B}(z_0, R_0)$, and therefore regular for f_0 and f_1.

2.2 ANALYTIC CONTINUATION ALONG A CURVE

Let $\mathscr{C} \subseteq \mathbb{C}$ be a curve from α to β, and let $\{f_0, \alpha\}$ be a function element. Suppose there exists a chain of direct analytic continuations, centered at a finite sequence of points $z_0, \ldots, z_n \in \mathscr{C}$ such that $z_0 = \alpha$ and $z_n = \beta$, from $\{f_0, \alpha\}$ to $\{f_n, z_n\}$. Further, for each disc $\mathscr{B}(z_i, R_i)$ defined by f_i, let the curve between z_i

and z_{i+1} be contained within $\mathscr{B}(z_i, R_i) \cap \mathscr{B}(z_{i+1}, R_{i+1})$ for $i = 0, \ldots, n-1$. Then this method of continuing f_0 is known as *analytic continuation along a curve* $\mathscr{C}$ (see Figure 2).

Analytic continuation along curves allows us to define the largest domain to which a function can be continued. It may happen (and often does) that an analytic function f_0 may be continued to a domain $\mathscr{B}(z_n, R_n)$, so that $\mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_n, R_n) \neq \emptyset$, but $f_0 \neq f_n$ on this intersection. We return to this issue in Section 2.4. In this section we first establish that if an analytic continuation along $\mathscr{C}$ exists, any sequence of points along $\mathscr{C}$ will give rise to the same final function element.

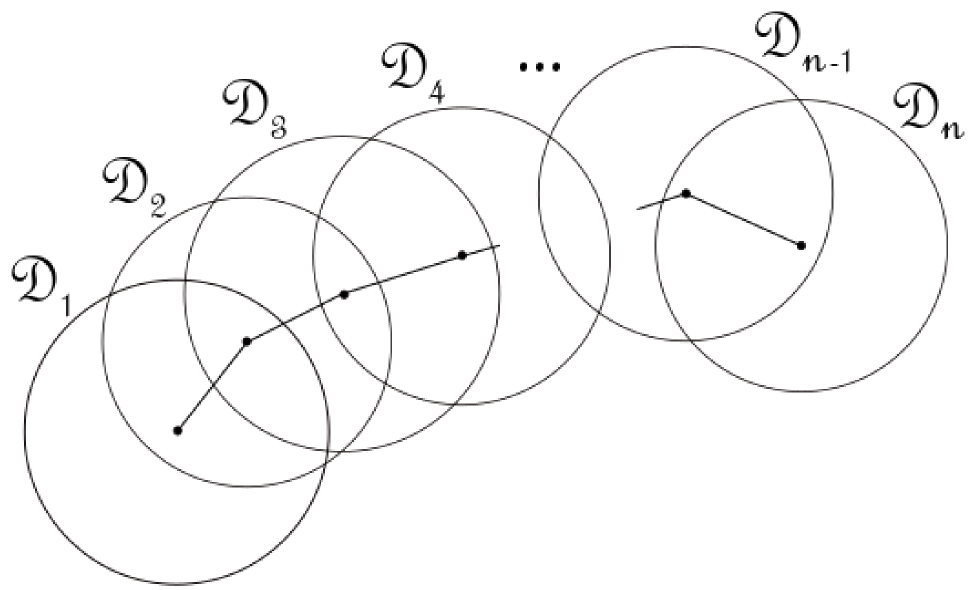

Figure 2.2. A chain of analytic continuations

Theorem 2.2.1. *Suppose $\{f_0, \alpha\}$ can be analytically continued along $\mathscr{C}$ from α to β following either $\alpha = z_0, z_1, ..., z_n = \beta$ and ending at $\{f_n, \beta\}$ or following $\alpha = z_0^*, z_1^*, ..., z_m^* = \beta$ and ending at $\{f_m^*, \beta\}$, in the same way as described above. Then $\{f_n, \beta\} = \{f_m^*, \beta\}$.*

Proof. Let $\alpha = z_0^\dagger, \ldots, z_k^\dagger = \beta$ be the finite sequence of points along $\mathscr{C}$ so that

$$\{z_0^\dagger, \ldots, z_k^\dagger\} = \{z_0, \ldots, z_n\} \cup \{z_0^*, \ldots, z_m^*\}.$$

Firstly, $z_0^\dagger = z_0 = z_0^*$ and therefore,
$\sum_{n=0}^{\infty} A_n(z - z_0^\dagger)^n = \sum_{n=0}^{\infty} A_n(z - z_0)^n = \sum_{n=0}^{\infty} A_n(z - z_0^*)^n$. By Definition 2.1.4, this implies that the radius of convergence of $f_0^\dagger$ centered at $z_0^\dagger$ is equal to the radius of convergence of f_0 centered at z_0. Similarly, this is true for f_0^* and z_0^*. Hence, $\{f^\dagger, z_0^\dagger\} = \{f_0, z_0\} = \{f_0^*, z_0^*\}$.

Next, either $z_1^\dagger = z_1$ or $z_1^\dagger = z_1^*$. Without loss of generality, suppose $z_1^\dagger = z_1$. By assumption, there exists a disc, defined by f_1, call it $\mathscr{B}(z_1, R_1)$, to which f_0 can be continued. The same is true if $z_1^\dagger = z_1^*$.

Now consider the next point, $z_2^\dagger$. There exist four cases, but for now, suppose $z_1^\dagger = z_1$ and $z_2^\dagger = z_1^*$. By assumption, there exists a continuation from f_0 to the disc defined by f_1^*, centered at z_1^*, call it $\mathscr{B}(z_1^*, R_1^*)$. By construction, the curve between z_0 and z_1^* is contained in $\mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_1^*, R_1^*)$, meaning that $z_1 \in \mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_1^*, R_1^*)$. By an earlier argument in this proof, $f_0 = f_1$ on $\mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_1, R_1)$, but also $f_0 = f_1^*$ on $\mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_1^*, R_1^*)$. Therefore, $f_0 = f_1 = f_1^*$ on $\mathscr{B}(z_0, R_0) \cap \mathscr{B}(z_1, R_1) \cap \mathscr{B}(z_1^*, R_1^*)$, meaning $f_1 = f_1^*$ on $\mathscr{B}(z_1, R_1) \cap \mathscr{B}(z_1^*, R_1^*)$. This implies that an analytic continuation from f_1 to $\mathscr{B}(z_1^*, R_1^*)$ exists, and further, f_0 remains single-valued as it continues to $\mathscr{B}(z_2^\dagger, R_2^\dagger) = \mathscr{B}(z_1^*, R_1^*)$.

Notice, if $z_1^\dagger = z_1^*$ and $z_2^\dagger = z_1$, a similar argument would work. Also, if $z_1^\dagger = z_1$ and $z_2^\dagger = z_2$ (or if $z_1^\dagger = z_1^*$ and $z_2^\dagger = z_2^*$), an analytic continuation exists by assumption. For the remaining continuations, similar arguments show f_0 remains single-valued through the chain of continuations, and hence $\{f_k^\dagger, R_k^\dagger\} = \{f_n, R_n\} = \{f_m^*, R_m^*\}$. $\square$

Theorem 2.2.2. *Suppose the function f_0 has a direct analytic continuation along the curve $\mathscr{C}$ from α to β, given by finitely many function elements. Then the union of all chains of direct analytic continuation of f_0 along $\mathscr{C}$ gives rise to $f_0(z(t))$ for $0 \leq t \leq 1$ and $\alpha \leq z(t) \leq \beta$ so that $\alpha = z(0)$ and $\beta = z(1)$.*

Proof. Assume the above premise. Lemma 2.8 gives us the existence of another finite sequence of analytic continuations, given that one exists. However, countably many exist by a repetitive argument of Lemma 2.8. Define $\mathcal{P}$ to be a sequence of finite analytic continuations from f_0 to $\mathscr{B}(z_n, R_n)$. Further, define $\mathbb{P} = \{\mathcal{P}_l | l \in \mathbb{N}\}$ to be the set of all such finite sequences of analytic continuations, where $\mathcal{P}_0$ is our original. Every point $z \in \mathscr{C}$ is in some $\mathcal{P}_l$, since every point on $\mathscr{C}$ is regular. Therefore, a parametrization exists. Below, we show the calculation of such a parameterization.

Let $f_0(z) = \sum\limits_{n=0}^{\infty} A_n(z - z_0)^n$ centered at $z_0 = \alpha$. Define $z(t)$ to be the parameterized function moving $z_0 = \alpha$ to $z_n = \beta$ along $\mathscr{C}$, as t ranges from 0 to 1. Then, by similar calculations shown in the previous proof,

$$
\begin{aligned}
f_0(z) &= \sum_{n=0}^{\infty} A_n(z - z_0)^n \\
&= \sum_{n=0}^{\infty} A_n(z - z(t) + z(t) - z_0)^n \\
&= \sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \binom{n}{k} A_n(z - z(t))^k (z(t) - z_0)^{n-k} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} A_n(z(t) - z_0)^{n-k} (z - z(t))^k \\
&= \sum_{k=0}^{\infty} A_k(t)(z - z(t))^k, \text{ for } A_k(t) = \sum_{n=k}^{\infty} \binom{n}{k} A_n(z(t) - z_0)^{n-k}
\end{aligned}
$$

For any t, $z(t) \in \mathcal{P}_l$ for some l. Therefore as t ranges from 0 to 1, $z(t)$ ranges from α to β. $\square$

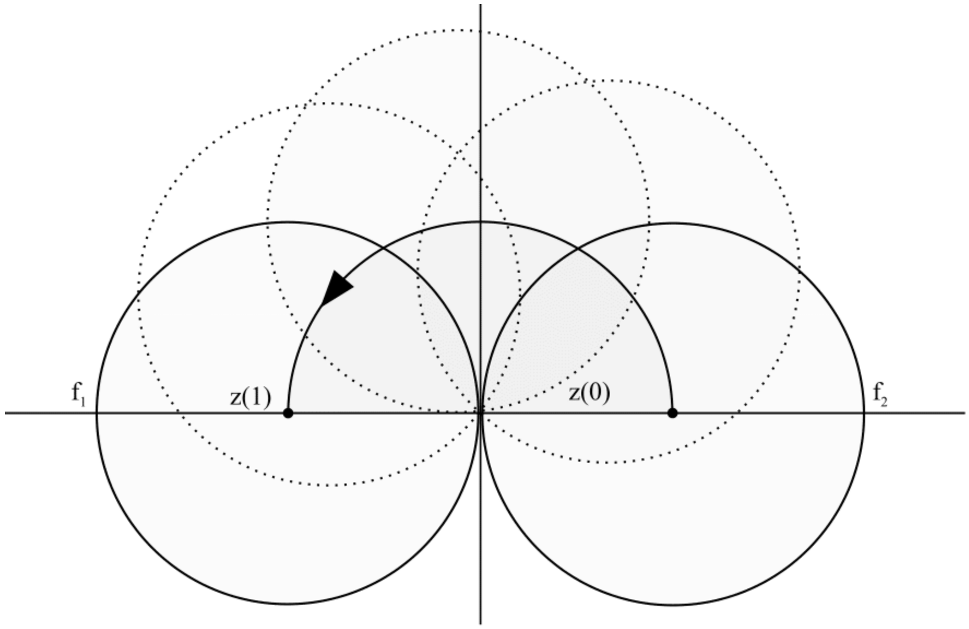

Figure 2.3. An analytic continuation of f_2 along $z(t)$, where $t \in [0, 1]$.

One useful tool in analytically continuing a function this way lies in the local to global construction of Riemann surface of a function. This construction requires that a direct analytic continuation exists, but once this is known, the parametrized version generally makes light work of the remaining computation. In the following sections, we find more efficient ways to define analytic continuations of simply connected subsets of $\mathbb{C}$.

2.3 THE SCHWARZ REFLECTION PRINCIPLE

In some situations, the Schwarz Reflection Principle provides a convenient way to analytically continue a function to a larger domain. As a simple example, suppose $\mathscr{D}$ is a set that contains the real line I, as in the left and picture of Figure 2.3.1. If f is analytic in $\mathscr{D}$, then the conjugate of $f(\overline{z})$ is analytic in $\overline{\mathscr{D}}$. If we further suppose that f is real valued on I, then $f(z)$ and the conjugate of $f(\overline{z})$ agree on I and hence by Theorem 2.1.2, they agree by uniqueness of analytic continuation on $\mathscr{D} \cap \overline{\mathscr{D}}$. A less restrictive idea is contained in the following theorem.

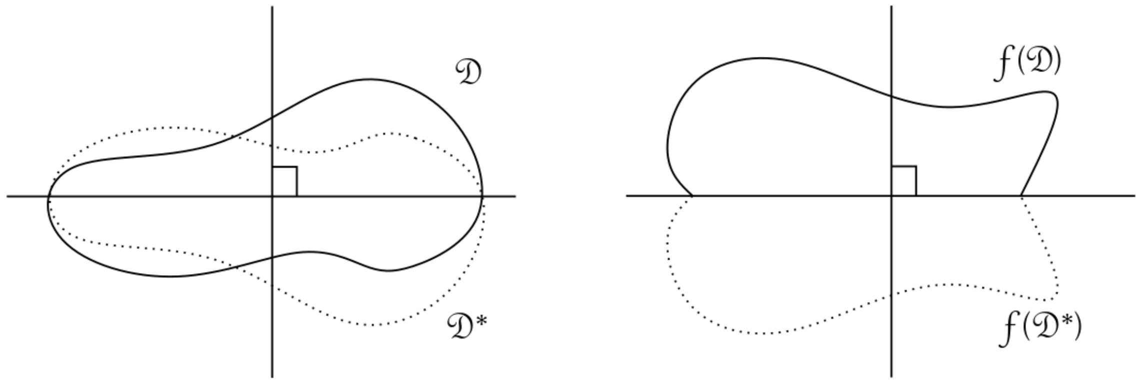

Figure 2.4. An example of the Schwarz Reflection Principle.

Theorem 2.3.1. *Define $\mathscr{H}$ as the complex upper half plane. Let $\mathscr{D} \subseteq \mathscr{H}$ be an open set with some non-empty open interval $I \in \partial\mathscr{D}$ on the real axis. If f is analytic in $\mathscr{D}$ and continuous in $\mathscr{D} \cup I$ and if f is real on I then,*

$$F(z) = \begin{cases} f(z) & z \in \mathscr{D} \\ \overline{f(\overline{z})} & z \in \overline{\mathscr{D}} \\ \lim\limits_{z \to z_0} f(z) & z \in I \end{cases}$$

is analytic in $\mathscr{D} \cup \overline{\mathscr{D}} \cup I$.

Proof. Let f be a continuous function. Let $\mathscr{C}$ be any closed, positively-oriented contour in $\mathscr{D} \cup \overline{\mathscr{D}} \cup I$. Assume $\mathscr{C} \subseteq \mathscr{D}$ (or $\mathscr{C} \subseteq \overline{\mathscr{D}}$) then

$$\int_{\mathscr{C}} f(z)dz = 0 \qquad \left(\int_{\mathscr{C}} \overline{f(\overline{z})}dz = 0 \right)$$

Now assume $\mathscr{C}$ goes through both $\mathscr{D}$ and $\overline{\mathscr{D}}$, and define B as the set contained within $\mathscr{C}$. Define α^+ as the positive traversal of I and α^- as the negative traversal of I. Further, define $\mathscr{C}_1 \subseteq \mathscr{D}$ and $\mathscr{C}_2 \subseteq \overline{\mathscr{D}}$ as subcurves of $\mathscr{C}$ so that $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$.

Then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \alpha^{\pm}$, as the sum of the integrals along α^+ and α^- is zero. Clearly, $\mathscr{C}_1 \cup \alpha^+$ and $\mathscr{C}_2 \cup \alpha^-$ are two continuous positively-oriented contours, each containing a subset of B, which is contained completely in either $\mathscr{D}$ or $\overline{\mathscr{D}}$. Now,

$$\int_{\mathscr{C}} f(z)dz = \int_{\mathscr{C}_1 \cup \alpha^+} f(z)dz + \int_{\mathscr{C}_2 \cup \alpha^-} f(z)dz = 0 + 0 = 0$$

by our first argument. Since f is defined and continuous in each region and each contour integral is exactly 0, it follows that $F(z)$ must be analytic in $\mathscr{D} \cup \overline{\mathscr{D}} \cup I$ [2] .

□

In most cases, $\partial\mathscr{D}$ may not necessarily share points with the real axis; however, with a linear transformation, we may be able to still use the Reflection Principle.

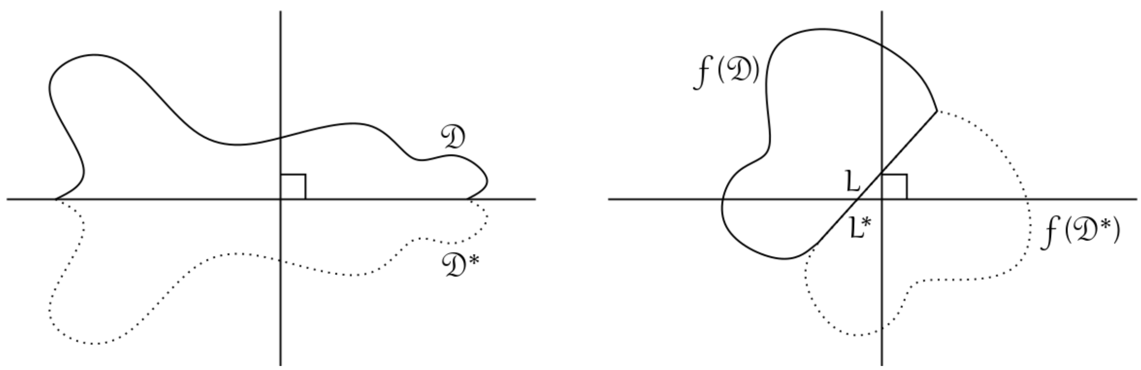

Figure 2.5. An example of the Schwarz Reflection Principle, where a subset of boundary of $\mathscr{D}$ is a subset of the real line.

Theorem 2.3.2. (The Schwarz Reflection Principle) *Let $\mathscr{D}$ be a domain whose boundaries include the linear segment L and let $\mathscr{D}^*$ be a domain whose boundary includes the linear segment L^*. If the analytic function $w = f(z)$ maps $\mathscr{D}$ to $\mathscr{D}^*$ in such a way that the segment L is transformed into the segment L^*, then $f(z)$ can be continued analytically across L.*

[2]Also known as Morera's Theorem

As a small example, take some domain $\mathscr{D}$ which intersects a subset of the real axis. Then a reflection of $\mathscr{D}$ takes all values z such that Im $z > 0$ and it maps it to $\overline{z}$ such that Im $\overline{z} < 0$ and Im $z+$ Im $\overline{z} = 0$ (see Figure 2.5).

For every function $f : \mathscr{A} \to \mathscr{A}^*$, so that $\mathscr{A}$ is simply connected, there exists a map which takes $\partial\mathscr{A}$ (or a subset of $\partial\mathscr{A}$) to the real line (or a subset of the real line). This is true, because the Riemann Mapping Theorem gives us a bijective map from any simply connected subset of $\mathbb{C}$ to the upper half plane[3] . Section 2.7 details the Elliptic Modular Function, which takes the boundary of the upper half plane, and transforms it into the hyperbolic triangle, whose vertices are on the complex unit circle. As a useful tool to that example, we briefly review how reflection in a circle works.

Take all $z \in \mathscr{B}(0,1)$, except $z = 0$. The reflection of $z \in \mathscr{B}(0,1)$ is the point z^* on the half-line starting at the origin and passing through z so that $|z||z^*| = 1$. This is the same as requiring

(i) arg $z =$ arg z^*, with respect to the origin and real axis,

(ii) and $z^* = \frac{1}{\overline{z}}$, since by definition, $z = |z|e^{i\text{arg } z}$ and $z^* = |z^*|e^{i\text{arg } z^*}$

which implies,

$$z^* = \frac{1}{|z|}e^{i\text{arg } z^*} = \frac{1}{|z|}e^{i\text{arg } z} = \frac{|z|}{|z|^2}e^{i\text{arg } z} = \frac{z}{z\overline{z}} = \frac{1}{\overline{z}}.$$

Thus, z^* is the "conjugate-reciprocal" of z. This concept is known as the reflection of a point in a circle. The mapping of z to z^* is an anti-holomorphic map, or an orientation-reversing map.

In Sections 2.7 and 2.8, we will show how the Schwarz Reflection Principle works with an elliptic integral and elliptic modular function, respectively.

[3]The Riemann Mapping Theorem says there exists a unique map from a simply connected subset of $\mathbb{C}$ and to the unit circle. Such a map can be uniquely defined and hence, bijective. So the composition of maps resulting in the transformation of the upper half plane to a simply connected set exist.

2.4 MANY-VALUED FUNCTIONS

We have thus far examined functions of power series form, which are *single-valued*. All functions $f : \mathscr{D} \to \mathbb{C}$ which send z to one and only one other element f are single-valued in the usual sense of what we mean by a function. However, if $f : \mathscr{D} \to \mathbb{C}$ sent z to elements $f_1(z), f_2(z), \ldots$, then $f(z)$ is called a many-valued (or *multiple-valued* function). Admittedly, the term is confusing because a many-valued function is not a function in the traditional sense, but for conformity, we will use the term many-valued function as defined above.

Given a many-valued function $F : \mathscr{D} \to \mathbb{C}$, it is often convenient (or necessary) to define a *branch f of F*, which is a single-valued function defined by specifying one output among the outputs of F for each z in some $\mathscr{D}^* \subseteq \mathscr{D}$.

Example 2.4.1. *The exponential function*

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C}\backslash\{0\} \\ z &\mapsto e^z \end{aligned}$$

sends any $z = x + iy$ to $e^x e^{iy}$, and hence $e^{x_1} = e^{x_2}$ if and only if $z_2 = z_1 + 2\pi n$ for some $n \in \mathbb{Z}$. Then the inverse function

$$\begin{aligned} \log : \mathbb{C}\backslash\{0\} &\to \mathbb{C} \\ \omega &\mapsto \log|\omega| + i\,arg\ \omega \end{aligned}$$

where $arg\ \omega$ is any value such that $e^{\log|\omega|}e^{i\ arg\ \omega} = \omega$, is many-valued.

Starting with a single-valued function, we could analytically continue $\omega = \log(z)$ along the unit circle to find a many-valued function with intervals larger than 2π.

Definition 2.4.2. *We say $\{f, a\}$ has unrestricted analytic continuation in an open set $\mathscr{D}$ if f can be analytically continued along any path in $\mathscr{D}$ starting at a. If F is* *the unrestricted analytic continuation of f to $\mathscr{D}$, then F is called the global analytic function of f.*

Definition 2.4.3. *Let $\{f_\gamma, a\}$ denote the function element obtained after analytically continuing f along γ.*

Lemma 2.4.4. *Suppose $\{f, a\}$ has unrestricted analytic continuation F to $\mathscr{D}$ and suppose $\mathscr{A} = \bigcup_{z\in\mathscr{A}} \mathscr{B}(z, \rho_z)$, contained in $\mathscr{D}$, is a compact set. Define $\rho = \inf_{z\in\mathscr{A}} \rho_z$. Then, $\rho > 0$.*

Proof. Suppose to the contrary that $\inf_{z\in\mathscr{A}} \rho_z = \rho$ is zero. For each $z_{j,k} \in \mathscr{A}$, there exists a function element $\{f_{\gamma_k}, z_{j,k}\}$ which defines the disc $\mathscr{B}(z_{j,k}, \rho_{z_{j,k}})$, where $\rho_{z_{j,k}}$ is the largest radius for which $f_{\gamma_k}(z_{j,k})$ converges on the disc. There must exist a sequence (z_n) in $\mathscr{A}$, for which the radius ρ_{z_n} tends to 0. Note (z_n) is bounded and therefore has a convergent subsequence. Without loss of generality, say (z_n) converges to some point z^*, say. By the closeness of $\mathscr{A}$, $z^* \in \mathscr{A}$, meaning z^* is a regular point. Each $\mathscr{B}(z_n, \rho_{z_n})$ has a singular point on its boundary of F, say ζ_n. For any $\delta > 0$, if n sufficiently large then $|z_n - z^*| < \delta/2$ and $\rho_{z_n} < \delta/2$. Therefore,

$$|\zeta_n - z^*| \leq |\zeta_n - z_n| + |z_n - z^*| = \rho_{z_n} + |z_n - z^*| < \delta/2 + \delta/2 = \delta.$$

Thus f can not be analytically continued to any neighborhood about z^*, meaning z^* is singular. However, it was shown that z^* is a regular point; hence, a contradiction. Thus, $\rho > 0$. $\square$

The following three examples will demonstrate what a global analytic function looks like.

2.5 EXAMPLE 1. A POWER SERIES (PART 1)

Define $f_0 : \mathscr{B}(1, 1) \to \mathbb{C}$ by $f_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(n+1/3)}{n!\Gamma(1/3)}(z-1)^n$ for $z \in \mathbb{C}\backslash\{0\}$. We will show that when $f_0(z)$ is analytically continued following all possible paths, the resulting global analytic function is the many-valued function $z^{-1/3}$.

Consider the series

$$f_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+1/3)}{n!\Gamma(1/3)}(x-1)^n$$

which is the Taylor series of the function $x^{-1/3}$, since

$$\left[\frac{d}{dx}\right]^n (x^{-1/3}) = (-1)^n \frac{4 \cdot 7 \cdots (1+3n)}{3^n} x^{-1/3-n}$$

for $n = 1, 2, 3, \ldots$. Note the coefficients of each term, given as $\frac{1}{3} \cdot \frac{4}{3} \cdot \ldots \frac{1+3n}{3}$, may be rewritten by the ratio of Gamma functions $\frac{\Gamma(n+1/3)}{\Gamma(1/3)}$. Since the real line $(0, 2)$ is contained in $\mathscr{B}(1, 1)$, Theorem 1.4 implies that f_0 is the complex power series of $z^{-1/3}$ throughout $\mathscr{B}(1, 1)$. This implies f_0 is single-valued, if we take a branch cut along the real interval $(-\infty, 0)$.

Let $z(t) = e^{it}$ for $t \in (0, 2\pi]$. Then for $z \in \mathscr{B}(1, 1)$, we get

$$\begin{aligned}
f_0(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+1/3)}{n!\Gamma(1/3)}(z - z(t) + z(t) - 1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+1/3)}{n!\Gamma(1/3)} \sum_{m=0}^{n} \binom{n}{m} (z - z(t))^m (z(t) - 1)^{n-m} \\
&= \sum_{m=0}^{\infty} \frac{(z - z(t))^m}{m!} \sum_{n=m}^{\infty} \frac{(-1)^n \Gamma(n+1/3)}{(n-m)!\Gamma(1/3)} (z(t) - 1)^{n-m} \\
&= \sum_{m=0}^{\infty} \frac{(z - z(t))^m}{m!} \sum_{n=0}^{\infty} \frac{(-1)^{n+m} \Gamma(n+m+1/3)}{n!\Gamma(1/3)} (z - z(t))^m (z(t) - 1)^{n-m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (z - z(t))^m}{m!} \frac{\Gamma(m+1/3)}{\Gamma(1/3)} \sum_{n=0}^{\infty} \frac{\Gamma(n+m+1/3)}{n!\Gamma(m+1/3)} (-1)^n (z(t) - 1)^n.
\end{aligned}$$

By the Binomial Theorem, $\sum\limits_{n=0}^{\infty} \frac{\Gamma(n+m+1/3)}{n!\Gamma(m+1/3)}(-1)^n(z(t)-1)^n = z(t)^{-m-1/3}$ if $z(t) \in \mathscr{B}(1,1)$. Hence

$$\begin{aligned} f_0(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m(z-z(t))^m}{m!}\frac{\Gamma(m+1/3)}{\Gamma(1/3)} z(t)^{-m-1/3} \\ &= z(t)^{-1/3}\sum_{m=0}^{\infty} \frac{(-1)^m(z-z(t))^m}{m!}\frac{\Gamma(m+1/3)}{\Gamma(1/3)} z(t)^{-m} \\ &= z(t)^{-1/3}\sum_{m=0}^{\infty} \frac{(-1)^m(z/z(t)-1)^m}{m!}\frac{\Gamma(m+1/3)}{\Gamma(1/3)} \\ &= z(t)^{-1/3} f_0(z/z(t)) \end{aligned}$$

Now consider the intersection of $\mathscr{B}(1,1)$ and $|z|=1$. At every point along this path, the function f_0 is analytic, and therefore, a continuation exists. Since $z(t) = e^{it}$ implies $z(t)^{-1/3} = e^{-it/3}$, we find that as t increases from 0 to 2π, the function $z(t)^{-1/3}$ approaches $-\frac{1}{2} - \frac{\sqrt{3}i}{2}$. Since $z(0)^{-1/3} f_0(z/z(0)) \neq z(2\pi)^{-1/3} f_0(z/z(2\pi))$ the analytic continuation of f_0 to $\mathbb{C}\backslash(0,\infty)$ is given by F_0, which is a branch defined by the restriction of $t \in (0, 2\pi]$. By expanding the range of t to $\mathbb{R}$, we obtain the global analytic function F, which is many-valued.

2.6 EXAMPLE 2. AN ELLIPTIC INTEGRAL (PART 1)

Let $\mathscr{H} := \{z |\ \mathrm{Im}\ z > 0\}$ be the upper half complex plane and let $\mathscr{R}$ be an open rectangle to be determined shortly. By the Riemann Mapping Theorem, there exists a conformal mapping $f : \mathscr{H} \to \mathscr{R}$.

We will see that one such function is given by

$$z = f(\zeta) = \int_0^{\zeta} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \tag{2.5}$$

called the *elliptic integral of the first kind* with parameter $k \in (0,1)$. We write $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}$.

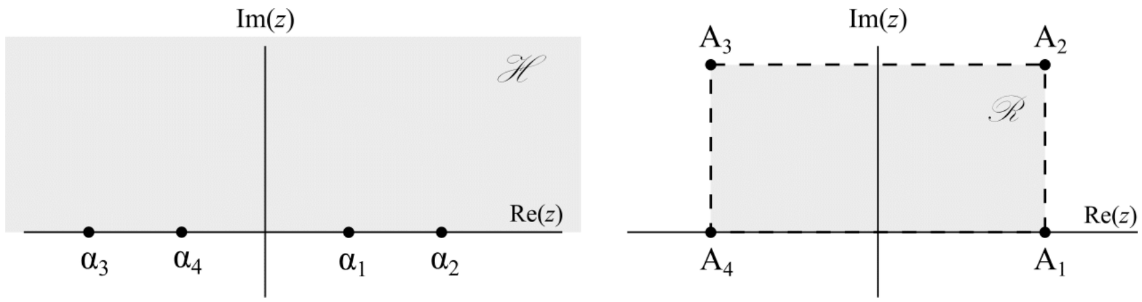

Figure 2.6. Elliptic function of the first order.

First, any closed contour in $\mathscr{H}$ which starts and ends at 0 is exactly 0, as the function is continuous and analytic in this region. Clearly, $f(0) = 0$. As ζ increases on the real line from 0 to 1, $f(\zeta)$ also increases along the positive real axis, and

$$f(1) = \int_0^1 \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} =: K, \tag{2.6}$$

As we pass $\zeta = 1$, following a semi-circle in $\mathscr{H}$, we find this does not remain true.

For each root function, take the real interval $(-\infty, 0)$ to be the branch cut, and let the argument angle vary from $-\pi$ to π. Define $w = 1 - \rho e^{i\theta}$ for fixed radius $0 < \rho < 1$ to be the small semi-circular path around 1 and θ to range from $-\pi$ to 0. Since $\theta \in (-\pi, 0)$, we can show w as it ranges from one boundary to the next with the above definition. For $\zeta = 1$, we find

$$\sqrt{1-w} = \sqrt{1-(1-\rho e^{i\theta})} = \sqrt{\rho}e^{i\theta/2}$$

When $\theta = -\pi$, $e^{i(-\pi/2)} = -i$ and $\rho = w - 1$. Thus $\sqrt{\rho}e^{i\theta/2} = -i\sqrt{\rho} = -i\sqrt{w-1}$.

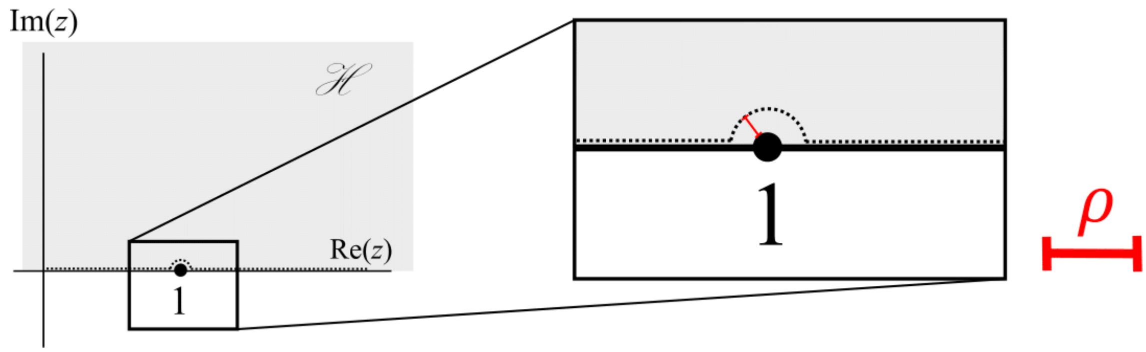

Figure 2.7. The contour of the above integral depicted by dotted lines.

Now define $\mathscr{C}_\rho := w = 1 + \rho e^{i(\pi-\theta)}$ for $0 \le \theta \le \pi$. It follows that as we pass over $\zeta = 1$, our function f changes as so:

$$\int_0^\zeta \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} = \int_0^{1-\rho} \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} + i\int_{1-\rho}^{\zeta} \frac{dw}{\sqrt{w^2-1}\sqrt{1-k^2w^2}} \tag{2.7}$$
$$+ \int_{\mathscr{C}_\rho} \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}}.$$

Define $M = \max_{z\in\mathscr{C}_\rho} \left| \frac{1}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} \right|$ and L as the length of the semi-circular arc. Notice, for $w \in \mathscr{C}_\rho$ and $\rho < \frac{|1-k|}{2k}$

$$\begin{aligned} &|1-w| = |-\rho e^{i(\pi-\theta)}| = \rho, \quad |1+w| \ge 2 - |1-w| > 1, \\ &|1-kw| \ge |1-k| - k|1-w| = |1-k| - k\rho > \frac{|1-k|}{2}, \\ &\text{and } |1+kw| = |1+k-k(1-w)| \ge |1+k| - k\rho > \frac{|1+k|}{2}. \end{aligned} \tag{2.8}$$

Therefore, $M \le \frac{2}{\rho^{1/2}(1-k^2)^{1/2}}$, and hence,

$$\left| \int_{\mathscr{C}_\rho} \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} \right| \le ML \le \left(\frac{2}{\rho^{1/2}(1-k^2)^{1/2}} \right) \pi\rho = \frac{2\pi\rho^{1/2}}{(1-k^2)^{1/2}}.$$

As $\rho \to 0^+$ this tends to 0, and Equation (2.7) becomes

$$\int_0^\zeta \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} = K + i\int_1^\zeta \frac{dw}{\sqrt{w^2-1}\sqrt{1-k^2w^2}}. \tag{2.9}$$

Since we are traversing the real line, the last term of Equation (2.9) defines a vertical path.

Similar to equation (2.6), define

$$\int_1^{1/k} \frac{dw}{\sqrt{w^2-1}\sqrt{1-k^2w^2}} =: K'.$$

We create a similar argument by choosing $w = 1/k - \rho e^{i\theta}$ for $\zeta = 1/k$ to show that,

$$f(\zeta) = K + iK' - \int_{1/k}^{\infty} \frac{dw}{\sqrt{w^2-1}\sqrt{k^2w^2-1}}.$$

We have done the calculation of $\partial\mathscr{H}$ to $\partial\mathscr{R}$ strictly for $\operatorname{Re}\zeta > 0$. Note, for all $\zeta > 0$, $f(-\zeta) = -f(\zeta)$. Naturally, the mapping of $\mathscr{H}$ to $\mathscr{R}$ strictly for $\operatorname{Re}\zeta < 0$ must then be a reflection of the above calculations over the imaginary axis of the z-plane, meaning for $\zeta < 0$

$$f(\zeta) = -K + iK' + \int_{-1/k}^{-\infty} \frac{dw}{\sqrt{w^2-1}\sqrt{k^2w^2-1}}.$$

Having shown the that $\partial\mathscr{H}$ maps to $\partial\mathscr{R}$, we now need to show that the interior of $\mathscr{H}$ maps to the interior of $\mathscr{R}$. Take $w = iT$ where $0 < T$. Then $f(\zeta)$ becomes $i\int_0^T \frac{dt}{\sqrt{1+t^2}\sqrt{1+k^2t^2}}$. The branch being the positive real quantity, i.e. having positive real roots, implies that the above integral is integrating along the imaginary. As T tends to infinity, the integral approaches K'. Thus this function does map $\mathscr{H}$ to $\mathscr{R}$.

Now we can consider the analytic continuation f. By continuity, the real boundary $\psi_1 = (-1, 1)$ maps to the real interval $\Psi_1 = (-K, K)$. Therefore, an analytic continuation exists, using the Schwarz Reflection Principle. We choose $(-\infty, -1]$ and $[1, \infty)$ as branch cuts of f. Under reflection, $\mathscr{H}$ and $\overline{\mathscr{H}}$ share the boundary ψ_1, meaning the reflection of $\mathscr{R}$ over the boundary Ψ_1 must be the codomain of $\overline{f(\overline{\zeta})}$. Let's call this reflected codomain $\mathscr{R}_1$. Notice, the branch cuts taken were necessary, because by continuity, a different cut would yield a different reflection of f. Therefore, the function

$$f_1(\zeta) = \begin{cases} f(\zeta) & \text{whenever } \zeta \in \mathscr{H} \cup \psi_1 \\ \overline{f(\overline{\zeta})} & \text{whenever } \zeta \in \overline{\mathscr{H}} \end{cases}$$

maps $\mathscr{H} \cup \psi_1 \cup \overline{\mathscr{H}}$ to $\mathscr{R} \cup \Psi_1 \cup \mathscr{R}_1$, where rectangle $\mathscr{R}_1$ has the vertices $-K, K, K - iK'$, and $-K - iK'$.

By similar reasoning, there exists three other reflections of $\mathscr{R}$, however each take a different linear transformation. Take the branch cuts $(-\infty, 1]$ and $[1/k, \infty)$. As before, by continuity $\psi_2 = (1, 1/k)$ maps to $\Psi_2 = (K, K + iK')$, and hence the reflection of $\mathscr{H}$ over ψ_2 to $\overline{\mathscr{H}}$ produces the unique analytic continuation

$$f_2(\zeta) = \begin{cases} f(\zeta) & \text{whenever } \zeta \in \mathscr{H} \cup \psi_2 \\ 2K - \overline{f(\overline{\zeta})} & \text{whenever } \zeta \in \overline{\mathscr{H}}. \end{cases}$$

This will map $\mathscr{H} \cup \psi_2 \cup \overline{\mathscr{H}}$ to $\mathscr{R} \cup \Psi_2 \cup \mathscr{R}_2$, where rectangle $\mathscr{R}_2$ has the vertices $K + iK',\ 3K + iK',\ 3K$, and K.

To simplify notation, after each reflection, let ψ_i map to $\overline{\psi_i}$ for $i = 1, 2, 3, 4$. After one reflection of $\mathscr{H}$ over ψ_i, one of two cases can occur: (1) we can reflect $\overline{\mathscr{H}}$ over the reflected boundary $\overline{\psi_i}$ or (2) over a different boundary $\overline{\psi_j}$ where $i \neq j$ and $j = 1, 2, 3, 4$. The former case is easy to see, since linear transformations are bijective. The second case we motivate with an example first, then a generalization.

Let f_2 be defined as above. Rectangle $\mathscr{R}_2$ has the boundaries $\overline{\Psi_1} = (K, 3K)$ and $\overline{\Psi_2} = (K, K + iK')$. Since the reflection of $\mathscr{R}_2$ over $\overline{\Psi_2}$ would yield $\mathscr{R}$, we chose to reflect $\mathscr{R}_2$ over $\overline{\Psi_1}$, meaning in the domain, we reflect $\overline{\mathscr{H}}$ over $\overline{\psi_1}$.

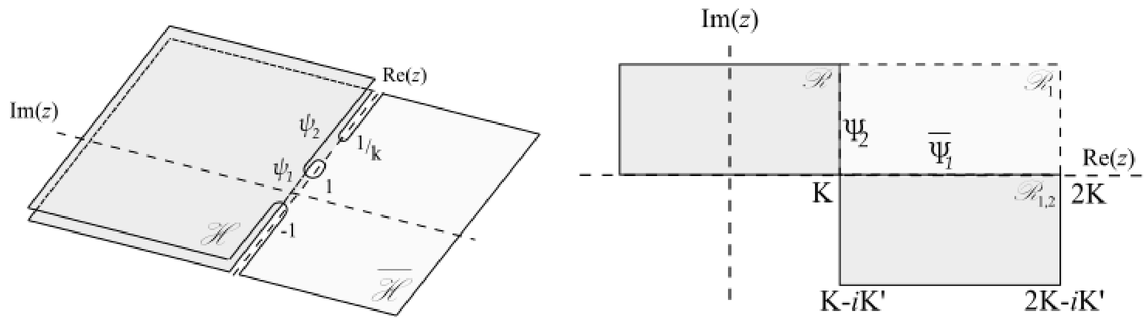

Figure 2.8. $f(\zeta)$ analytically continued to $\mathscr{H}\cup\overline{\mathscr{H}_2}\cup\mathscr{H}_{2,1}$ maps to $\mathscr{R}\cup\mathscr{R}_2\cup\mathscr{R}_{2,1}$.

Let $(-\infty,-1]$ and $[1,\infty)$ be the branch cuts of our function F_2. Then the reflection of all $\overline{\zeta}\in\overline{\mathscr{H}}$ over $\overline{\psi_1}$ gives us $\zeta\in\mathscr{H}$, as expected. This reflection is simply connected and analytic; thus by continuity and the Schwarz Reflection Principle

$$f_{2,1}(z)=\begin{cases}\overline{f(\overline{\zeta})} & \text{whenever } \zeta\in\overline{\mathscr{H}}\cup\overline{\psi_1}\\ 2K-f(\zeta) & \text{whenever } \zeta\in\mathscr{H}\end{cases}$$

takes $\mathscr{H}\cup\overline{\mathscr{H}}$ to $\mathscr{R}_2\cup\mathscr{R}_{2,1}$ (see Figure 2.8). Though similarly defined, the function $f_{1,2}(z)$ is not equivalent to $f_{2,1}(z)$, meaning

$$f_{1,2}(z)=\begin{cases}2K-\overline{f(\overline{\zeta})} & \text{whenever } \zeta\in\overline{\mathscr{H}}\cup\overline{\psi_2}\\ 2K-f(\zeta) & \text{whenever } \zeta\in\mathscr{H},\end{cases}$$

Let $g:\mathscr{D}\to\mathscr{D}^*$ be any function we can analytically continue by Theorem 2.3.1. For simpler notation, let the reflection of $\mathscr{D}$, over ψ_i, to $\overline{\mathscr{D}}$ be notated as $\mathscr{D}\overset{\psi_i}{\to}\overline{\mathscr{D}}$ for $i=1,2,3,4$. The reflective motions $\mathscr{H}\overset{\psi_2}{\to}\overline{\mathscr{H}}\overset{\overline{\psi_1}}{\to}\mathscr{H}$ resulted in a function whose codomain is $\mathscr{R}_{2,1}$, but notice that the boundaries of $\mathscr{R}_{2,1}$ are a rotation and translation of $\mathscr{R}$. In reflecting the domain of f twice, f becomes a multiple-value function, and the inverse function of f, call it $\lambda(z)$, is doubly periodic. Specific to the above example, $\lambda(z)=\lambda(-z+2K)$. We claim that other such mappings exist and are found by a unique transformation.

Proposition 2.6.1. *Let*

$$g_1 : z \mapsto z + 4K,\ \ g_2 : z \mapsto z + 2iK',\ \ \textit{and}\ h : z \mapsto -z + 2K.$$

The motions g_1, g_2, and h (and any composition of g_1, g_2, and h) take $\mathscr{R}$ to a rectangle whose inverse is $\mathscr{H}$.

Proof. Under reflections, $\mathscr{H} \overset{\psi_2}{\to} \overline{\mathscr{H}} \overset{\psi_4}{\to} \mathscr{H}$, $\mathscr{R}$ translates to $\mathscr{R}_{2,4}$. Under λ, $\mathscr{R}_{2,4}$ is mapped to $\mathscr{H}$. Notice, the translation of $\mathscr{R}$ to $\mathscr{R}_{2,4}$ is exactly g_1. A similar proof works for g_2. From earlier work, we found that $f_{2,1}$ contains the reflections which give the motions of h. Finally, since every motion, g_1, g_2, and h are derived from two sequential reflections, any composition of g_1, g_2 , h is also generated by an even amount of reflections. □

We applied these motions to all rectangles whose preimage was $\mathscr{H}$, but we can also use these motions on the set of rectangles whose preimage is $\overline{\mathscr{H}}$. In using all such compositions of $\mathscr{H}$ and $\overline{\mathscr{H}}$, we obtain the global analytic function, denoted as F, whose codomain is the complex plane, tessellated by reflections of $\mathscr{R}$ (see Figure 2.9).

2.7 EXAMPLE 3. THE ELLIPTIC MODULAR FUNCTION (PART 1)

Let the open triangular domain, $\mathscr{T}$, have the vertices 1, $\omega = e^{2\pi i/3}$, $\overline{\omega} = e^{4\pi i/3}$ and a boundary, which is orthogonal to $|z| = 1$. By the Riemann Mapping Theorem, there exists a conformal map $\chi : \mathscr{H} \to \mathscr{T}$ which maps $\infty \mapsto 1$, $0 \mapsto \omega$, and $1 \mapsto \overline{\omega}$ (Nehari, 1975) (see Figure 2.10). Denote the convex circular arcs of $\partial\mathscr{T}$, opposite to their respective vertex, as l_1, l_ω, and $l_{\overline{\omega}}$. By continuity,

$$(0,1) \to l_1, \quad (1,\infty) \to l_\omega, \quad (-\infty,0) \to l_{\overline{\omega}}. \tag{2.10}$$

By the Schwarz Reflection Principle, there exists an analytic continuation of χ to $\overline{\mathscr{H}}$ over each real interval, and hence, a corresponding reflected triangle of $\mathscr{T}$.

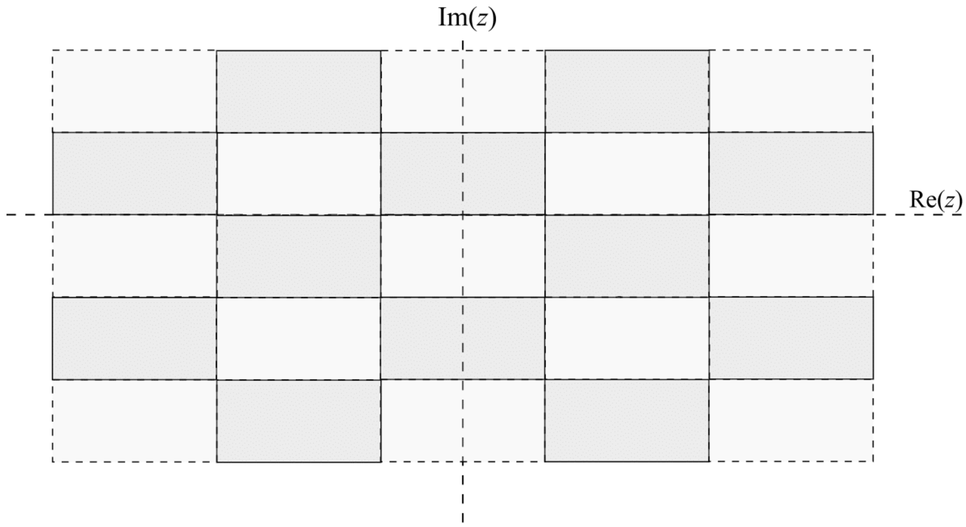

Figure 2.9. The codomain of the global analytic function $F(\zeta)$.

Suppose that $r^{(1)},\ r^{(\omega)}$, and $r^{(\overline{\omega})}$ denote a hyperbolic reflection of $l_1,\ l_\omega$, and $l_{\overline{\omega}}$, respectively.

Let

$$r^{(1)}\mathscr{T} = \mathscr{T}^1,\ r^{(\omega)}\mathscr{T} = \mathscr{T}^\omega,\ r^{(\overline{\omega})}\mathscr{T} = \mathscr{T}^{\overline{\omega}};$$

that is, the three inversions of $\mathscr{T}$ over $l_1,\ l_\omega$, and $l_{\overline{\omega}}$.

By the Schwarz Reflection Principle, the three functions

$$\chi^{(1)} : \overline{\mathscr{H}} \to \mathscr{T}^1,\ \chi^{(\omega)} : \overline{\mathscr{H}} \to \mathscr{T}^\omega,\ \chi^{(\overline{\omega})} : \overline{\mathscr{H}} \to \mathscr{T}^{\overline{\omega}}$$

which are each a composition of a complex conjugation with a Möbius transformation

$$\chi^{(i)}(z) = \frac{\alpha_i\ \chi(\overline{z}) + \beta_i}{\gamma_i\ \chi(\overline{z}) + \delta_i} \qquad \text{for } \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}, \qquad (2.11)$$

are an analytic continuation of χ.

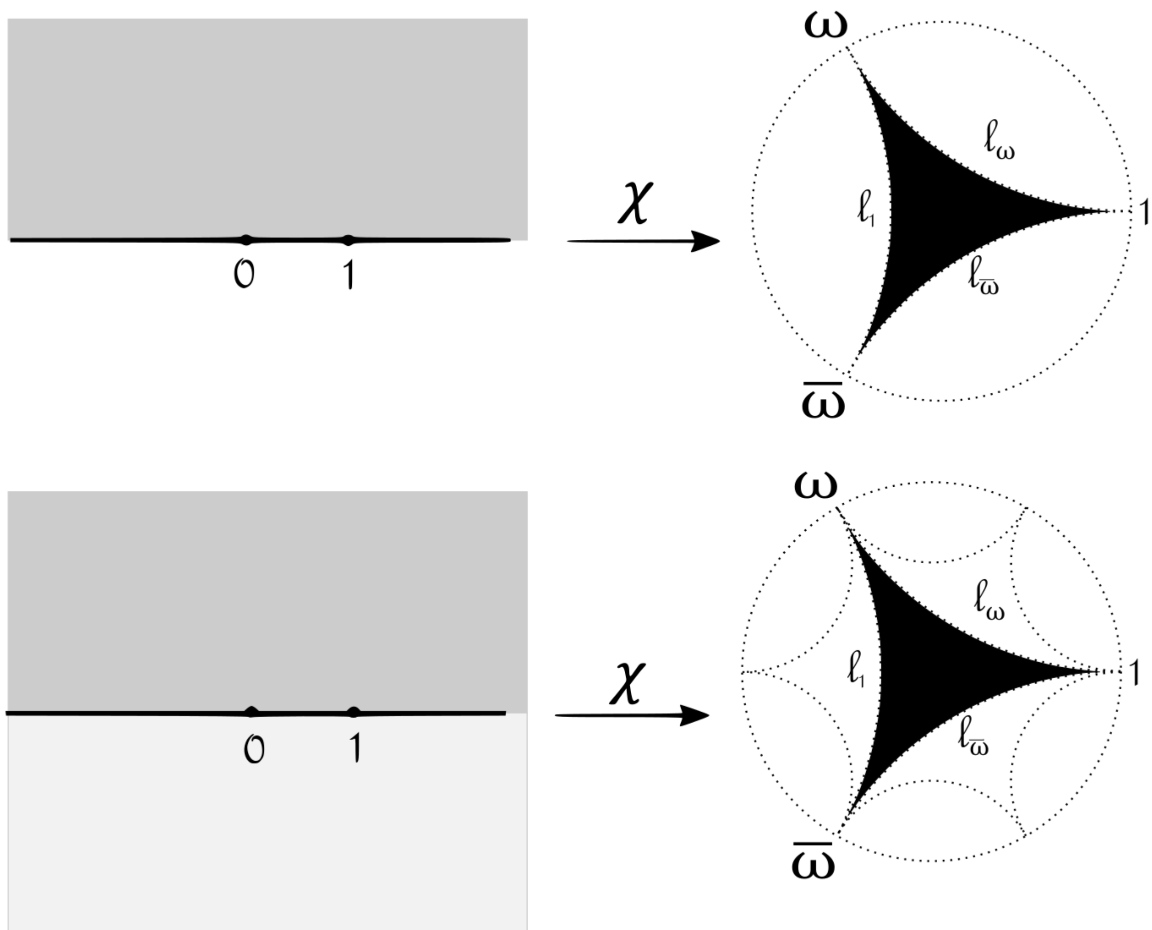

Figure 2.10. The elliptic modular function

With the appropriate branch cut and under Theorem 2.3.1, the general analytic continuation of χ to $\mathbb{C}\backslash\{0,1\}$ is given by

$$\mathrm{X}^{(i)}(z) = \begin{cases} \chi(z) & z \in \mathscr{H} \\ \chi^{(i)}(\overline{z}) & z \in \overline{\mathscr{H}} \\ \lim\limits_{z \to z_0} \chi(z) & z_0 \in (a,b). \end{cases} \tag{2.12}$$

where (a,b) is a real interval and $i = 1,\ \omega,\ \overline{\omega}$.

As a short example, we find the map $\chi^{(1)}(z)$. Take the branch cuts $(-\infty, 0) \cup (1, \infty)$ along the real line. The reflection of $\mathscr{T}$ over l_1 can be thought of as an inversion over a circle's boundary(see Section 2.3). For $\chi \in \mathscr{T}$, the function

$$\chi^{(1)} = \frac{-2\chi - 1}{\chi + 2}$$

takes $\mathscr{T}$ to $\mathscr{T}^1$, and so

$$\mathrm{X}^{(1)}(z) = \begin{cases} \chi(z) & z \in \mathscr{H} \\ \chi^{(1)}(\overline{z}) & z \in \overline{\mathscr{H}} \\ \lim\limits_{z \to z_0} \chi(z) & z_0 \in (0, 1). \end{cases} \tag{2.13}$$

is the analytic continuation of χ to $\mathbb{C} \backslash (-\infty, 0) \cup (1, \infty)$. A similar transformation exists for the other two white triangles.

We can again reflect one of the three reflected triangles, say $\mathscr{T}^1$. As before, the reflection of $\overline{\mathscr{H}}$ over one real interval versus another is unique by continuity. However under conjugation, two of the three real intervals maps to the reflection of l_ω and $l_{\overline{\omega}}$.

Let $\mathscr{T}^{\,i \circ 1} = r^{(i)}(r^{(1)}\mathscr{T})$ be the triangle obtained from first reflecting $\mathscr{T}$ over l_1, then l_i, where $i = 1, \omega, \overline{\omega}$. First, notice that $\mathscr{T}^{\,i \circ 1} \neq \mathscr{T}^{\,1 \circ i}$ unless $i = 1$. Taking the appropriate branch cuts, the continuation of $\chi^{(1)}$ to $\mathscr{H}$ is given by the analytic map

$$\chi^{(i)(1)} : \mathscr{H} \to \mathscr{T}^{\,i \circ 1}$$

which takes $\chi^{(1)}$ across one of the three real intervals $(-\infty, 0),\ (0, 1),\ (1, \infty)$ that corresponds to the side of $\mathscr{T}^1$ chosen in the second step. Any sequential reflection of $\mathscr{T}^{i \circ 1}$ works this way. Thus by the unrestricted analytic continuation of χ, we obtain the global analytic function X. The codomain of X is the unit disc $\mathbb{D}$, composed of tessellated hyperbolic triangles.

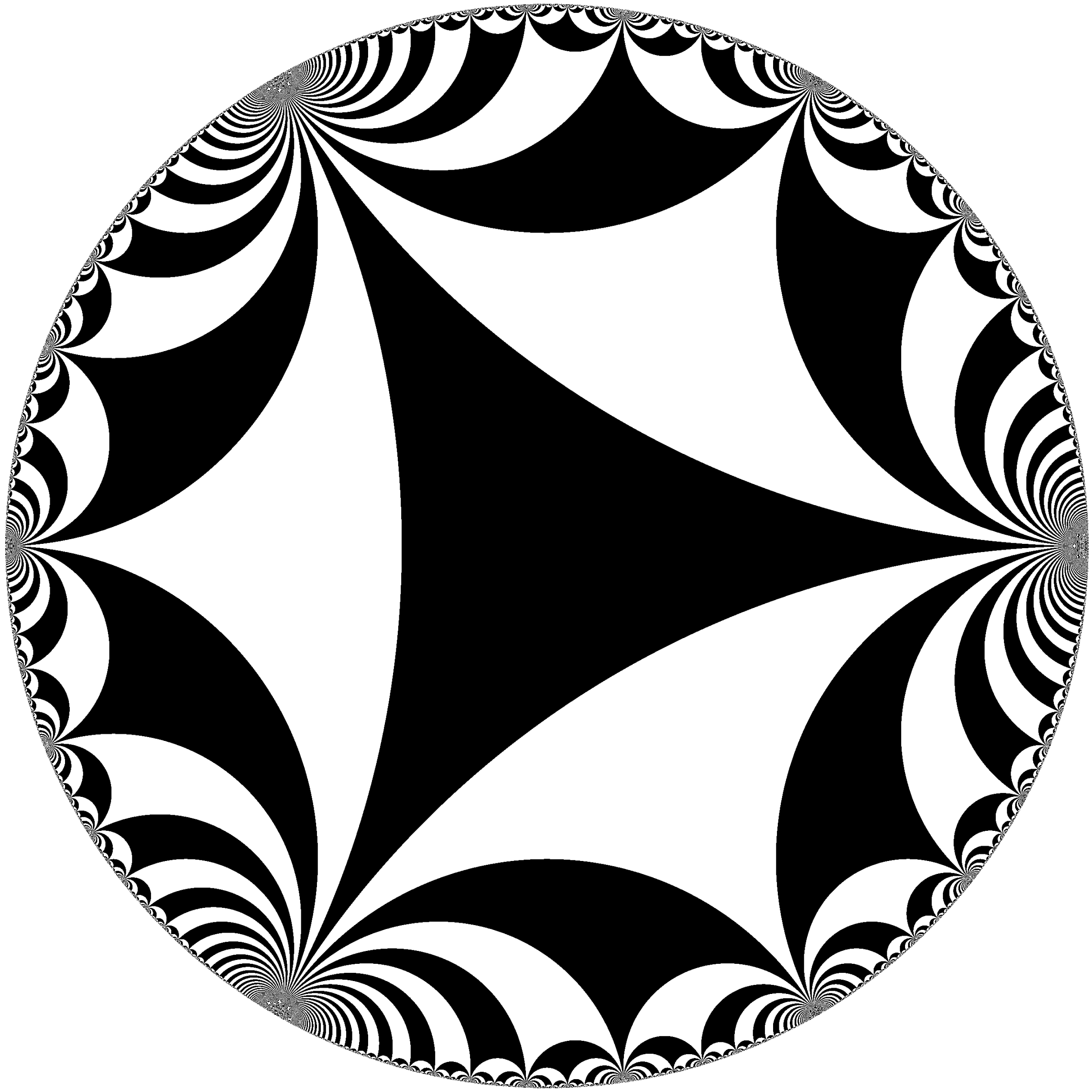

Figure 2.11. The unrestricted analytic continuation of χ to $\mathscr{B}(0,1)$; rotated image of Anton Sherwood, 2011.

Definition 2.7.1. *Let A be the set of compositions of reflective motions on $\mathscr{T}$. Rephrased, for $\alpha \in A$, $\mathscr{T}^\alpha$ is a triangle in the set of tessellations of $\mathscr{T}$.*

Theorem 2.7.2. *Let $\mathscr{T}$ be the hyperbolic triangle described above and let $\mathscr{T}^\alpha$ be a triangle in the set of tessellations of $\mathscr{T}$. Then $\mathscr{T} \cup \bigcup_{\alpha \in A} \{\mathscr{T}^\alpha\} = \mathbb{D}$, where $\mathbb{D}$ is the unit disc.*

Instead of proving this formally, we will sketch the proof found in Markushevich's *Theory of Complex Analysis* (Markushevich, 1965).

If there exists $z \in \partial\mathbb{D}$ that is not a limit of images of $1, \omega, \overline{\omega}$, then there exists an arc of $\partial\mathbb{D}$, call it σ, that contains no images of $1, \omega, \overline{\omega}$. Without loss of generality, the endpoints of σ are vertices or limits of vertices. Let τ be the hyperbolic arc linking the endpoints of σ (see Figure (2.12)).

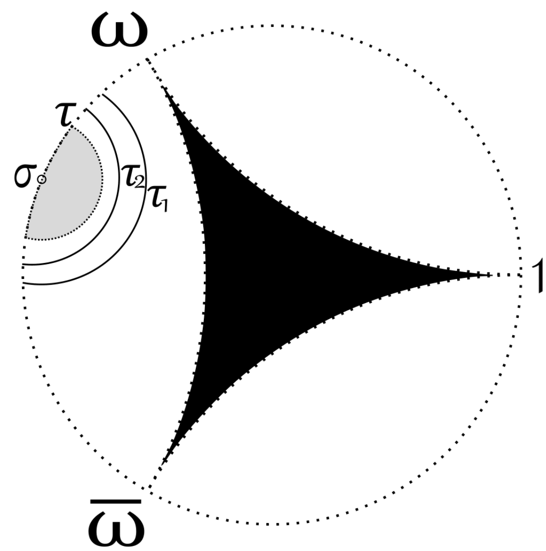

Figure 2.12. The sequence of hyperbolic boundaries $\tau_1, \tau_2, \ldots$ approach the limit τ seen as the dotted boundary between the grey and white region.

Then τ is the limit of a sequence of hyperbolic lines $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, all of which have endpoints that are images of $\mathscr{T}$. Reflecting any of the vertices of $\mathscr{T}$, say 1, in the sequence τ_n, gives a sequence of vertices, call them $\omega_1, \omega_2, \ldots, \omega_n, \ldots \notin \sigma$, which have some limit, say $\omega' \notin \sigma$. However it can be shown that ω' is the reflection of 1 in τ, and is therefore in σ, a contradiction. $\square$

Now, we turn to the motions of the hyperbolic disc, given by the product of even numbers of reflections. To the confused reader, we will wrap up the relevance of the following theory in the second part of this example (see Section 3.6).

Definition 2.7.3. *Let G be the group of the compositions of even reflection on the hyperbolic disc.*

Proposition 2.7.4. *Let $\mathscr{T}$ be the center black triangle, and let $\mathscr{T}^*$ be any other black triangle. Then the motion of the hyperbolic disc, which takes $\mathscr{T}$ to $\mathscr{T}^*$ is a unique $g \in G$ such that $\mathscr{T}^* = g\mathscr{T}$.*

Proof. Assume $\mathscr{T}^*$ is a black triangle and that there exists $g,\ g' \in G$ so that $\mathscr{T}^* = g\mathscr{T} = g'\mathscr{T}$. Then $\mathscr{T} = g^{-1}(g'\mathscr{T})$. This means that the composition $g^{-1}g' = id$, meaning $g = g'$, and there exists a unique $g \in G$ so that $\mathscr{T}^* = g\mathscr{T}$.
□

Proposition 2.7.5. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be any two black triangles. Then there exists a unique $g \in G$ so that $g\mathscr{T}_1 = \mathscr{T}_2$.*

Proof. Certainly, there is a $g_1,\ g_2 \in G$ which takes $g_1\mathscr{T} = \mathscr{T}_1$ and $g_2\mathscr{T} = \mathscr{T}_2$. Therefore $g_2(g_1^{-1}\mathscr{T}) : \mathscr{T}_1 \to \mathscr{T}_2$. Uniqueness follows from the proof above. □

CHAPTER 3

THE MONODROMY THEOREM AND MONODROMY GROUPS

3.1 HOMOTOPIC PATHS AND THE FUNDAMENTAL GROUP

Define $I = [0, 1]$ and define

$$\mathscr{C}(\mathscr{D}; a, b) = \{\gamma : I \to \mathscr{D} |\ \gamma \text{ is continuous, } \gamma(0) = a, \gamma(1) = b\}.$$

Definition 3.1.1. *Let $\gamma_0, \gamma_1 \in \mathscr{C}(\mathscr{D}; a, b)$. A homotopy from γ_0 to γ_1 is a continuous map $H : (t, s) \mapsto \mathscr{D}$ such that $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$ for all t. If such a map exists, γ_0 is said to be homotopic to γ_1.*

Proposition 3.1.2. *Homotopy defines an equivalence relation on $\mathscr{C}(\mathscr{D}; a, b)$.*

Proof. Take $\gamma_0, \gamma_1, \gamma_2 \in \mathscr{C}(\mathscr{D}; a, b)$.

(i) Define $H(t, s) = \gamma_0(t)$, which is a continuous function for all $t, s \in I$. Then $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_0(t)$. Thus $\gamma_0 \sim \gamma_0$.

(ii) Defined $H(t, s)$ to be a continuous function from γ_0 to γ_1; i.e. $H(t, 0) = \gamma_0(t)$, $H(t, 1) = \gamma_1(t)$. Then $H^*(t, s) = H(t, 1 - s)$ is also a continuous function from γ_1 to γ_0. Hence, if $\gamma_0 \sim \gamma_1$, then $\gamma_1 \sim \gamma_0$.

(iii) Let H_0 be a continuous function from γ_0 to γ_1 and let H_1 be a continuous function from γ_1 to γ_2, and define

$$H(t, s) = \begin{cases} H_0(t, 2s) & 0 \le s \le 1/2 \\ H_1(t, 2s - 1) & 1/2 \le s \le 1. \end{cases}$$

Notice that as we go from γ_1 to γ_2 that $H_0(2s, t) = H_1(t, 2s-1)$ for $s = 1/2$. Since homotopy respects equivalence relation properties, homotopy is an equivalence relation. $\square$

Definition 3.1.3. *If $\gamma : I \to \mathscr{D}$ satisfies $\gamma(0) = \gamma(1)$, then γ is called a loop. Further, let*

$$\mathscr{C}(\mathscr{D}; a) = \{\gamma : I \to \mathscr{D} | \gamma(0) = \gamma(1) = a\}$$

denote the set of all loops in $\mathscr{D}$ based at a.

Notice that homotopy is still an equivalence relation on $\mathscr{C}(\mathscr{D}; a)$ as a special case of Proposition 3.1.2. Let $[\gamma]$ denote the equivalence class of loops γ under homotopy, and by $\pi(\mathscr{D}; a) = \mathscr{C}(\mathscr{D}; a)/\sim$, the set of equivalence classes in $\mathscr{C}(\mathscr{D}; a)$. Given $\gamma_0,\ \gamma_1 \in \mathscr{C}(\mathscr{D}; a)$, we define $\gamma_0 * \gamma_1$ by

$$(\gamma_0 * \gamma_1)(t) = \begin{cases} \gamma_0(2t) & \text{for } 0 \leq t \leq 1/2 \\ \gamma_1(2t-1) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

and note that if $\gamma_0,\ \gamma_1 \in \mathscr{C}(\mathscr{D}; a)$, then $\gamma_0 * \gamma_1 \in \mathscr{C}(\mathscr{D}; a)$, so multiplication is well-defined on $\mathscr{C}(\mathscr{D}; a)$.

Proposition 3.1.4. *Let $\gamma_0, \gamma_1, \gamma_0', \gamma_1' \in \mathscr{C}(\mathscr{D}; a)$ such that $\gamma_0 \sim \gamma_0'$ and $\gamma_1 \sim \gamma_1'$. Then $\gamma_0 * \gamma_1 \sim \gamma_0' * \gamma_1'$. This multiplication is well-defined on $\pi(\mathscr{D}; a)$, meaning for $[\gamma_0] * [\gamma_1] = [\gamma_0 * \gamma_1]$.*

Proof. Let $\gamma_0, \gamma_1, \gamma_0', \gamma_1' \in \mathscr{D}$, and define H_0 to be a continuous function from γ_0 to γ_0' and H_1 to be a continuous function from γ_1 to γ_1'. Then

$$H(t, s) = \begin{cases} H_0(t, 2s) & 0 \leq s \leq 1/2 \\ H_1(t, 2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

is the homotopy which takes $\gamma_0 * \gamma_1$ to $\gamma_0' * \gamma_1'$. $\square$

Theorem 3.1.5. *The set $\pi(\mathscr{D};a)$ is a group, called the fundamental group of $\mathscr{D}$ at a.*

Proof. Proposition 3.1.4 showed that the multiplication for $[\gamma_0]$ and $[\gamma_1]$ is well-defined. Hence $[\gamma_0 * \gamma_1] \in \pi(\mathscr{D};a)$, and $\pi(\mathscr{D};a)$ is closed under multiplication. Take any $[\gamma_0], [\gamma_1], [\gamma_2] \in \pi(\mathscr{D};a)$. For $\gamma_0 \in [\gamma_0]$, $\gamma_1 \in [\gamma_1]$, and $\gamma_3 \in [\gamma_3]$, the homotopy from $\gamma_1 * (\gamma_2 * \gamma_3)$ to $(\gamma_1 * \gamma_2) * \gamma_3$ is given by

$$H(t,s) = \begin{cases} \gamma_0(4t/(2t-s)) & 0 \le t \le (2-s)/4 \\ \gamma_1(4t+s-2) & (2-s)/4 \le t \le (3-s)/4 \\ \gamma_2((4t+s-3)/(1+s)) & (3-s)/4 \le t \le 1 \end{cases}$$

Hence, under homotopy, multiplication is associative.

Suppose $id = a$ for all $z \in \mathscr{D}$. Then for any loop $\gamma \in [\gamma]$,

$$H(t,s) = \begin{cases} \gamma(2t/(s+1)) & 0 \le t \le (s+1)/2 \\ a & (s+1)/2 \le t \le 1. \end{cases}$$

Hence $\gamma * id \sim \gamma$ for all $\gamma \in [\gamma]$. The proof showing $id * \gamma \sim \gamma$ is similar, hence $id \in [id]$ and $[id] \in \pi(\mathscr{D};a)$.

Now for each $\gamma \in [\gamma]$, suppose there exists a $\gamma * \gamma^{\dagger} = id$. Then the continuous function

$$H(t,s) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2 \\ \gamma^*(2t-1) & 1/2 \le t \le 1 \end{cases}$$

defines a homotopy and it must be that $\gamma^* = \gamma^{-1}$. The proof showing $\gamma^{\dagger} * \gamma = id$ is nearly identical. □

Proposition 3.1.6. *The groups $\pi(\mathscr{D};a)$ and $\pi(\mathscr{D};b)$ are isomorphic.*

Proof. First, define $[\gamma] \in \pi(\mathscr{D};a,b)$ such that $[\overline{\gamma}]$ is the inverse equivalence class of $[\gamma]$. Further, define $\psi : \pi(\mathscr{D};a) \to \pi(\mathscr{D};b)$ so that $[\alpha] \mapsto [\overline{\gamma} * \alpha * \gamma]$. Choose $\gamma \in \mathscr{C}(\mathscr{D};a,b)$. We would like to show that ψ is a bijective homomorphism Take two curves $\alpha_0, \alpha_1 \in [\alpha]$. Then $\psi(\alpha_0 * \alpha_1) = \overline{\gamma} * \alpha_0 * \alpha_1 * \gamma = (\overline{\gamma} * \alpha_0 * \gamma) * (\overline{\gamma} * \alpha_1 * \gamma)$. Hence, ψ is a homomorphism. The inverse map $\psi^{-1} : \pi(\mathscr{D}; b) \to \pi(\mathscr{D}; a)$ so that $[\beta] \mapsto [\gamma * \beta * \overline{\gamma}]$ can be similarly shown to be a homomorphism, since $\psi^{-1} \circ \psi(\alpha) = \psi^{-1}(\gamma * \alpha * \overline{\gamma}) = \overline{\gamma} * \gamma * \alpha * \overline{\gamma} * \gamma = \alpha$. Hence ψ is an isomorphism. □

Finally, we show that the fundamental group of the punctured disc is isomorphic to the group of additive integers (adapted from Ahlfors, 1966, p. 220-21). Without loss of generality, let the punctured disc be represented by $0 < |z| < 1$ and define some point z_0 so that $|z_0| = r$ exists on a positive radius. Further, define the path of any loop $\gamma(t)$ by $z(t)$ so that $0 \leq t \leq 1$. Then there exists a central projection

$$z(t,u) = (1-u)z(t) + ur\frac{z(t)}{|z(t)|} \tag{3.1}$$

which conformally maps $z(t)$ to the fixed circle $|z| = r$, call it $\mathcal{O}$. For any t_0, there exists an open neighborhood N_0 defined as $|z(t) - z(t_0)| < r/2$. For any point $P \in \mathcal{O}$ so that $-P = P^* \in \mathcal{O}$, any neighborhood which contains P (or P^*) does not contain P^* (or P). Since such a neighborhood is possible for any t, we can define an open cover of neighborhoods $\bigcup_{\alpha \in A} N_\alpha$ of $z(t)$, and for a finite set $\{0, 1, \ldots, n\} \subset A$, a finite subcover $\bigcup_{\alpha=0}^{n} N_\alpha$ of $z(t)$ so that for any i, the intersection of N_i,N_{i+1} is exactly one point. Then, we can redefine each curve γ by the product of simple arcs $t_i t_{i+1}$; i.e., $\gamma := t_0 t_1 \ldots t_n t_0$. Here, we make no distinction that curve $t_i t_{i+1}$ is traversed clock-wise or counter clock-wise. Given this configuration, we claim that each loop is homotopic to the identity curve id_γ or it goes around the point of origin.

Without loss of generality, define $z(t_0) = P$ and redefine the arc $t_0 t_1 \ldots t_n t_0$ as $t_0 t_1 t_2 t_0 t_2 t_3 t_0 \ldots t_0 t_{n-1} t_n t_0$ so that neither $t_0 t_i$ nor $t_i t_0$ contains P^*. Each path $t_i t_0 t_i$ is certainly homotopic to id_γ, which implies that for some i, either the subcurve $t_i t_{i+1}$ contains P^* or there exists no subcurve which contains P^*. If the latter, then

$\gamma \sim id_\gamma$. If the former, then by the enumeration of $t_0t_1t_2t_0t_2t_3t_0 \ldots t_0t_{n-1}t_nt_0$, we have $\gamma \sim \mathcal{O}$ or $\gamma \sim \mathcal{O}^{-1}$. We can also conclude from the former case that every closed curve which traverses about the origin is homotopic to $\mathcal{O}$ or $\mathcal{O}^m$ for $m \in \mathbb{Z}$. The integral

$$\int_{\mathcal{O}} \frac{dz}{z} = m \cdot 2\pi i$$

demonstrates that $\mathcal{O}^m$ is homotopic to id if and only if $m = 0$. Therefore, the fundamental group of the punctured disc is isomorphic to the group of additive integers. This is to say that any analytic function which is continued along a path around a point of singularity follows this same argument. We see an application of this in Section 3.4.

3.2 MONODROMY GROUPS

Let $\mathscr{D}$ be the largest domain into which a function element $\{f, a\}$ has unrestricted analytic continuation and let $F : \mathscr{D} \to \mathbb{C}$ be the global analytic function of f.

Definition 3.2.1. *We define $\mathscr{C}_f(\mathscr{D}; a) \subseteq \mathscr{C}(\mathscr{D}; a)$ to be the subset of $\mathscr{C}(\mathscr{D}; a)$ consisting of the loops such that $f = f_\gamma$,i.e.,*

$$\mathscr{C}_f(\mathscr{D}; a) = \left\{ \gamma \in \mathscr{C}(\mathscr{D}; a) \middle| \{f, a\} = \{f_\gamma, a\} \right\}$$

Lemma 3.2.2. *If $\{f, a\}$ and $\{f^*, a\}$ are function elements of F, then $\mathscr{C}_{f^*}(\mathscr{D}; a) = \mathscr{C}_f(\mathscr{D}; a)$.*

Proof. If $\{f^*, a\}$ is a function element of F then $f^* = f_\gamma$ for some $\gamma \in \mathscr{C}(\mathscr{D}; a)$; i.e., it is the end result of analytic continuation f along some loop γ at a. This implies $f^*_{\gamma^{-1}} = f$ also. Suppose $\gamma_0 \in \mathscr{C}_f(\mathscr{D}; a)$. Then continuing f^* along $\gamma * \gamma_0 * \gamma^{-1}$ means

$$f^*_{\gamma * \gamma_0 * \gamma^{-1}} = f_{\gamma * \gamma_0} = f_\gamma = f^*.$$

Thus $\gamma_0 \in \mathscr{C}_{f^*}(\mathscr{D}; a)$ also. Showing the other direction is a similar proof to the above. $\square$

Definition 3.2.3. *Define $\pi_f(\mathscr{D}; a)$ to be the set of equivalence classes under homotopy; i.e., $\pi_f(\mathscr{D}; a) = \mathscr{C}_f(\mathscr{D}; a)/\sim$.*

Proposition 3.2.4. *The set $\pi_f(\mathscr{D}; a)$ is a group under multiplication and is defined by*

$[\gamma_0] * [\gamma_1] = [\gamma_0 * \gamma_1]$, for $[\gamma_0],\ [\gamma_1] \in \mathscr{C}_f(\mathscr{D}; a)$.

Proof. By the Monodromy Theorem, for $\gamma_0 \in [\gamma_0]$ and $\gamma_1 \in [\gamma_1]$, $\{f_{\gamma_0}, a\} = \{f_{\gamma_1}, a\}$. Thus $\pi_f(\mathscr{D}; a)$ is closed under multiplication.
Suppose that for $[\gamma_0],\ [\gamma_1], [\gamma_2] \in \pi_f(\mathscr{D}; a)$ so that $\gamma_0 \in [\gamma_0]$, $\gamma_1 \in [\gamma_1]$, and $\gamma_2 \in [\gamma_2]$. Let id be the constant loop. For all $[\gamma] \in \pi_f(\mathscr{D}; a)$, note that $id * \gamma = \gamma = \gamma * id$ for $\gamma \in [\gamma]$. Hence, $\pi_f(\mathscr{D}; a)$ has the identity element id.

Now, we need only show that $\pi_f(\mathscr{D}; a)$ has inverses. Suppose that for $[\gamma] \in \pi_f(\mathscr{D}; a)$ that $\{f, a\} = \{f_\gamma, a\}$. By definition, the inverse $(\gamma)^{-1}$ exists, and thus is a loop for which $\{f, a\} = \{f_{(\gamma)^{-1}}, a\}$. Hence $\{f_{(\gamma)^{-1}}, a\} \in \pi_f(\mathscr{D}; a)$ has inverses and $\pi_f(\mathscr{D}; a)$ is a group. $\square$

Proposition 3.2.5. $\pi_f(\mathscr{D}; a) \trianglelefteq \pi(\mathscr{D}; a)$.

Proof. Define $\psi : \pi(\mathscr{D}; a) \to \pi_f(\mathscr{D}; a)$ so that $[\gamma^\dagger] \mapsto [\gamma] * [\gamma^\dagger] * [\gamma]^{-1}$. Under ψ, we are essentially taking any $\gamma^\dagger \in \mathscr{C}(\mathscr{D}; a)$ and any $\gamma \in \mathscr{C}_f(\mathscr{D}; a)$ to show $\gamma^\dagger \mapsto \gamma * \gamma^\dagger * \gamma^{-1}$. Lemma 3.11 says that the analytic continuation of f along $\gamma^\dagger$ is the same as the analytic continuation of f along $\gamma * \gamma^\dagger * \gamma^{-1}$ for all $\gamma \in \mathscr{C}_f(\mathscr{D}; a)$ and all $\gamma^\dagger \in \mathscr{C}(\mathscr{D}; a)$, and vice versa, also. This is true for all branches of F, hence ψ is a bijective homomorphism, and $\pi_f(\mathscr{D}; a) \trianglelefteq \pi(\mathscr{D}; a)$. $\square$

This proposition allows us to define the Monodromy group as $G \cong \frac{\pi(\mathscr{D};a)}{\pi_f(\mathscr{D};a)}$. The theory provided in Section 3.1 and 3.2 proves that it does not matter where our loops are based, nor what branch of the global analytic function we take to determine $\mathscr{C}_f(\mathscr{D};a)$. Rather, it is sufficient enough to define the Monodromy group of a set using only a global analytic function.

In our next examples, we will relate G to geometric properties of their respective global analytic functions.

3.3 MONODROMY GROUPS

Let $\mathscr{D}$ be the largest domain into which a function element $\{f,a\}$ has unrestricted analytic continuation and let $F:\mathscr{D}\to\mathbb{C}$ be the global analytic function of f.

Definition 3.3.1. *We define $\mathscr{C}_f(\mathscr{D};a)\subseteq\mathscr{C}(\mathscr{D};a)$ to be the subset of $\mathscr{C}(\mathscr{D};a)$ consisting of the loops such that $f=f_\gamma$,i.e.,*

$$\mathscr{C}_f(\mathscr{D};a)=\left\{\gamma\in\mathscr{C}(\mathscr{D};a)\,\middle|\,\{f,a\}=\{f_\gamma,a\}\right\}$$

Lemma 3.3.2. *If $\{f,a\}$ and $\{f^*,a\}$ are function elements of F, then $\mathscr{C}_{f^*}(\mathscr{D};a)=\mathscr{C}_f(\mathscr{D};a)$.*

Proof. If $\{f^*,a\}$ is a function element of F then $f^*=f_\gamma$ for some $\gamma\in\mathscr{C}(\mathscr{D};a)$; i.e., it is the end result of analytic continuation f along some loop γ at a. This implies $f^*_{\gamma^{-1}}=f$ also. Suppose $\gamma_0\in\mathscr{C}_f(\mathscr{D};a)$. Then continuing f^* along $\gamma*\gamma_0*\gamma^{-1}$ means

$$f^*_{\gamma*\gamma_0*\gamma^{-1}}=f_{\gamma*\gamma_0}=f_\gamma=f^*.$$

Thus $\gamma_0\in\mathscr{C}_{f^*}(\mathscr{D};a)$ also. Showing the other direction is a similar proof to the above. □

Definition 3.3.3. *Define $\pi_f(\mathscr{D};a)$ to be the set of equivalence classes under homotopy; i.e., $\pi_f(\mathscr{D};a)=\mathscr{C}_f(\mathscr{D};a)/\sim$.*

Proposition 3.3.4. *The set* $\pi_f(\mathscr{D};a)$ *is a group under multiplication and is defined by*

$[\gamma_0] * [\gamma_1] = [\gamma_0 * \gamma_1]$*, for* $[\gamma_0],\ [\gamma_1] \in \mathscr{C}_f(\mathscr{D};a)$*.*

Proof. By the Monodromy Theorem, for $\gamma_0 \in [\gamma_0]$ and $\gamma_1 \in [\gamma_1]$, $\{f_{\gamma_0}, a\} = \{f_{\gamma_1}, a\}$. Thus $\pi_f(\mathscr{D};a)$ is closed under multiplication.
Suppose that for $[\gamma_0],\ [\gamma_1], [\gamma_2] \in \pi_f(\mathscr{D};a)$ so that $\gamma_0 \in [\gamma_0]$, $\gamma_1 \in [\gamma_1]$, and $\gamma_2 \in [\gamma_2]$. Let id be the constant loop. For all $[\gamma] \in \pi_f(\mathscr{D};a)$, note that $id * \gamma = \gamma = \gamma * id$ for $\gamma \in [\gamma]$. Hence, $\pi_f(\mathscr{D};a)$ has the identity element id.

Now, we need only show that $\pi_f(\mathscr{D};a)$ has inverses. Suppose that for $[\gamma] \in \pi_f(\mathscr{D};a)$ that $\{f, a\} = \{f_\gamma, a\}$. By definition, the inverse $(\gamma)^{-1}$ exists, and thus is a loop for which $\{f, a\} = \{f_{(\gamma)^{-1}}, a\}$. Hence $\{f_{(\gamma)^{-1}}, a\} \in \pi_f(\mathscr{D};a)$ has inverses and $\pi_f(\mathscr{D};a)$ is a group. □

Proposition 3.3.5. $\pi_f(\mathscr{D};a) \trianglelefteq \pi(\mathscr{D};a)$.

Proof. Define $\psi : \pi(\mathscr{D};a) \to \pi_f(\mathscr{D};a)$ so that $[\gamma^\dagger] \mapsto [\gamma] * [\gamma^\dagger] * [\gamma]^{-1}$. Under ψ, we are essentially taking any $\gamma^\dagger \in \mathscr{C}(\mathscr{D};a)$ and any $\gamma \in \mathscr{C}_f(\mathscr{D};a)$ to show $\gamma^\dagger \mapsto \gamma * \gamma^\dagger * \gamma^{-1}$. Lemma 3.11 says that the analytic continuation of f along $\gamma^\dagger$ is the same as the analytic continuation of f along $\gamma * \gamma^\dagger * \gamma^{-1}$ for all $\gamma \in \mathscr{C}_f(\mathscr{D};a)$ and all $\gamma^\dagger \in \mathscr{C}(\mathscr{D};a)$, and vice versa, also. This is true for all branches of F, hence ψ is a bijective homomorphism, and $\pi_f(\mathscr{D};a) \trianglelefteq \pi(\mathscr{D};a)$. □

This proposition allows us to define the Monodromy group as $G \cong \frac{\pi(\mathscr{D};a)}{\pi_f(\mathscr{D};a)}$. The theory provided in Section 3.1 and 3.2 proves that it does not matter where our loops are based, nor what branch of the global analytic function we take to determine $\mathscr{C}_f(\mathscr{D};a)$. Rather, it is sufficient enough to define the Monodromy group of a set using only a global analytic function.

In our next examples, we will relate G to geometric properties of their respective global analytic functions.

3.4 EXAMPLE 1. A POWER SERIES (PART 2)

Recall the function element $\{f_0, 1\}$, where the function f_0 is the equation found in Section 2.5. We found that the largest set f_0 can be analytically continued to is $\mathbb{C}\backslash\{0\}$, so let us consider all loops around the origin. Using Equation (12), we can transform any such loop based at $a \in \mathbb{C}\backslash\{0\}$ to the unit circle S^1 based at 1. Let loop $\gamma_0(t) = e^{-2\pi i t}$ where $t \in [0, 1]$ and $\gamma_0(0) = \gamma_0(1)$. The loop $\gamma_0(t)$ traverses S^1 once in the positive direction, or equivalently has the winding number of $n = 1$. Further, let $\gamma_1(t) = e^{-2n\pi i t}$ to traverse S^1 n-times in the positive direction, whenever $n < 0$. Then its winding number is $n \in \mathbb{Z}$. All curves homotopic to nS^1 give rise to their equivalence class $[\gamma_{nS^1}]$. Therefore, the fundamental group of the loops on $\mathbb{C}\backslash\{0\}$ is isomorphic to $\mathbb{Z}$.

From Section 2.5, we know that by analytically continuing f_0 along S^1 once, $\{f_{S^1}, 1\} \neq \{f_0, 1\}$ and $\{f_{2S^1}, 1\} \neq \{f_0, 1\}$. However, by analytically continuing f_0 along S^1 three times, we find $\{f_{3S^1}, 1\} = \{f_0, 1\}$. By the Monodromy Theorem, curves homotopic to $\gamma_n(t)$ so that $n \equiv 0(\mod 3)$ are equivalent and form an equivalence class. By Proposition 3.1.6, a loop whose base point is 1 is the same as a loop whose base point is $e^{2\pi i/3}$; hence, by the Monodromy Theorem, curves homotopic to $\gamma_n(t)$ so that $n \equiv 1(\mod 3)$, are equivalent and form an equivalence class. Then argument applies for $n \equiv 2(\mod 3)$ also. Therefore, the fundamental group on the set loops, relative to the analytic continuation of f_0, is isomorphic to $3\mathbb{Z}$.

Thus the Mondoromy group defined as $\pi(\mathbb{C}\backslash\{0\}; a)/\pi_{f_0}(\mathbb{C}\backslash\{0\}; a)$ for any $a \in \mathbb{C}\backslash\{0\}$ must be isomorphic to $\mathbb{Z}\backslash 3\mathbb{Z}$.

Consider the function $f : \mathscr{H} \to \mathscr{R}$ and its global analytic function $F : \mathbb{C}\backslash\{\pm 1, \pm 1/k\} \to \mathscr{D}$ as defined in section 2.7. Any loop in $\mathbb{C}\backslash\{\pm 1, \pm 1/k\}$ may go around one or more points of singularity and these loops can be equated to a motion (defined in Proposition 2.6.1) or a composition of motions of the plane. For $m, n \in \mathbb{N}$, define $(g_1)^{-m} : z \mapsto z - 4mK$ and $(g_2)^{-n} : z \mapsto z - 2inK'$.

Proposition 3.5.1. *Let $G^* = \{g_1^m g_2^n | m, n \in \mathbb{Z}\}$. Then G^* is an abelian group isomorphic to $\mathbb{Z}^2$.*

Proof. Take

$$\begin{aligned} \phi \quad & : \mathbb{Z}^2 \to G^* \\ m \times n & \mapsto g_1^m g_2^n \end{aligned}$$

Then for any $m_0, m_1, n_0, n_1 \in \mathbb{Z}$,

$$\begin{aligned} \phi((m_0 + m_1) \times (n_0 + n_1)) = g_1^{m_0+m_1} g_2^{n_0+n_1} &= g_1^{m_0} g_1^{m_1} g_2^{n_0} g_2^{n_2} \\ &= g_1^{m_0} g_2^{n_0} g_1^{m_1} g_2^{n_2} = \phi(m_0 \times n_0) * \phi(m_1 \times n_1). \end{aligned}$$

Hence ϕ is a homomorphism. It is injective since $\phi(m_0 \times n_0) = \phi(m_1 \times n_1)$ for some $m_0, m_1, n_0, n_1 \in \mathbb{Z}$, which implies $g_1^{m_0} g_2^{n_0} = g_1^{m_1} g_2^{n_1} \Rightarrow g_2^{n_0 - n_1} = g_1^{m_0 - m_1}$. Hence $m_0 = m_1$ and $n_0 = n_1$. Now take any $g_1^m, g_2^n \in G^*$. By definition $g_1^m g_2^n = \phi(m \times n)$ is uniquely defined by m, n. Hence the there exists $m, n \in \mathbb{Z}$ so that $\phi(m \times n) = g_1^m g_2^n$. Therefore ψ is also surjective. $\square$

Proposition 3.5.2. *Let $G = \{g_1^m g_2^n h^j | m, n \in \mathbb{Z}$ and $j = 0, 1\}$ be the set of motions of $\mathbb{C}$. Then G is an group.*

Proof. As a reminder, these motions are $g_1 : z \mapsto z + 4K$ and $g_2 : z \mapsto z + 2iK'$, and $h : z \mapsto -z + 2K$. Proposition 2.6.1 gives us closure and linear maps are associative. Therefore, we need only show G has an identity, and each map has an inverse. We claim that the map $id : z \mapsto z$ is the identity element of G, and trivially, this works. Lastly, assume without loss of generality that $g_1 a = id$, where a is the inverse element of g_1. Since g_1 is a linear map, a must be $a : z \mapsto z - 4K$. As defined above, this is exactly $(g_1)^{-1} \in G$. The composition of linear maps is commutative, thus g_1 has an inverse. The proof of an inverse map for g_2 is nearly the same. Without loss of generality, let $hb = id$, where b is the inverse element of h. Notice, $bh : z \to -(-z + 2K) + 2K = z$, meaning $b = h$. Since $g_1,\ g_2,\ h$ have linear inverses, any composition of the three is linear, and thus also has a linear inverse. Namely, if $g_1^m g_2^n h^j (abc) = id$, then $a = h^j$, $b = (g_2)^{-n}$, and $c = (g_1)^{-m}$. Thus G is a group. □

We now show $G \cong \mathbb{Z}^2 \rtimes C_2$.

Proposition 3.5.3. *Let G and G^* be as defined above. Then $G^* \trianglelefteq G$.*

Proof. Let $\phi : G^* \to G$ so that for all $a \in G$ and all $b \in G^*$ that $b \mapsto aba^{-1}$. If $a = g_1^m$ or $a = g_2^n$ for $m, n \in \mathbb{Z}$, then certainly for all a, b, $aba^{-1} \in G^*$. Without loss of generality, take $b = g_1^m$ and $a = h^j$, where $j = 0, 1$. Under $\phi(g_1^m) = h^j g_1^m (h^j)^{-1}$, the rectangle $\mathscr{R}$ is translated to a black rectangle under Proposition 2.6.1. If $j = 1$, then $(h^j)^{-1}$ will rotate all rectangles once by $-\pi$, and after the horizontal translation, h^j will rotate all rectangles by π. This is to say that $h^j g_1^m (h^j)^{-1} = g_1^{-m} \in G^*$. A similar proof works for g_2^n for $n \in \mathbb{Z}$. Since $aba^{-1} \in G^*$ for all $a \in G$ and all $b \in G^*$, $G^* \trianglelefteq G$. □

As a consequence of Proposition 3.5.2, $G \cong \pi(\mathbb{C} \backslash \{\pm 1, \pm 1/k\}; a)$ by the First Isomorphism Theorem. Notice, $\mathbb{Z}^2 \cap C_2 = \{id\}$. By applying Proposition 3.5.2 and Proposition 3.5.1, we see that $G \cong \mathbb{Z}^2 \rtimes C_2$ (Rotman, 1973). Finally, we consider the set $G^\dagger = \pi_f(\mathbb{C} \backslash \{\pm 1, \pm 1/k\}; a)$. The motions on the plane which send $\mathscr{R}$ to $\mathscr{R}$

are given by the identity map; hence $G^* = \{id\}$, and thus, the Monodromy group is exactly G.

3.6 EXAMPLE 3. THE ELLIPTIC MODULAR FUNCTION (PART 2)

Recall the analytic function $\chi : \mathscr{H} \to \mathscr{T}$ as defined in Section 2.7, and the (many-valued) global analytic function $X : \mathbb{C}\backslash\{0,1\} \to \mathscr{D}$ generated by analytically continuing χ using the iterated application of the Schwarz Reflection Principle.

Suppose that $\gamma : I \to \mathbb{C}\backslash\{0,1\}$ is a loop that begins and ends at the fixed point $z_0 \in \mathscr{H}$. As t varies from 0 to 1, the loop γ may cross any or all of the three real line segments $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$, possibly several times; as in Section 2.7, this defines a corresponding sequence of reflections over the sides of the triangles in the tessellation of $\mathscr{D}$, and hence the loop γ defines a unique black triangle to which this chain of reflections takes $\mathscr{T}$. As in Section 2.7, this defines a unique $g \in G$, since this final black triangle is uniquely given as $g\mathscr{T}$ for some $g \in G$. Analytic continuation along γ therefore takes an initial function element centered at z_0 with values in $\mathscr{T}$ to a final function element also centered at z_0, but with values in $g\mathscr{T}$. Note that if γ_0 and γ_1 are two homotopic loops at z_0 then the outcomes of continuing an initial functional element along either loop are the same, by the Mondoromy theorem of $\pi(\mathbb{C}\backslash\{0,1\})$.

Proposition 3.6.1. *Let $\pi(\mathbb{C}\backslash\{0,1\})$ be the fundamental group of loops and let G be the set of sequential reflections g which analytically continue $\mathscr{T}$ everywhere. Then $\psi : \pi(\mathbb{C}\backslash\{0,1\}) \to G$ is a group homomorphism.*

Proof. Take arbitrary loops $\gamma_1 \in [\gamma_1]$ and $\gamma_2 \in [\gamma_2]$ which map to $g_1, g_2 \in G$. Then $g_1\mathscr{T}$ is formed by the map $\chi^{(1)} : \mathscr{H} \to g_1\mathscr{T}$. By applying g_2 to $g_1\mathscr{T}$, we get the map $\chi^{(2)} \circ \chi^{(1)} : \mathscr{H} \to g_2(g_1\mathscr{T})$. The triangle $g_2(g_1\mathscr{T}) = (g_2g_1)\mathscr{T}$ is unique, since each sequence of reflections $g_1,\ g_2$ are also unique.

Define $\gamma_1 * \gamma_2$ to be the product of loops in $\pi(\mathbb{C}\backslash\{0,1\})$. Now define $g_3 = g_2g_1$ and let $\psi(\gamma_0 * \gamma_1) = g_3$. This means that by applying g_3 to $\mathscr{T}$, we get $\chi^{(3)} = \chi^{(2)(1)} = \chi^{(2)}\chi^{(1)}$. Therefore $g_3\mathscr{T} = (g_2g_1)\mathscr{T} = g_2(g_1\mathscr{T})$. Hence ψ is a homomorphism of groups. □

Given the group homomorphism ψ, the kernel of ψ will be precisely those (homotopy classes of) loops such that the corresponding g is the identity; in other words (see Section 2.7), those homotopy classes such that direct analytic continuation of a function element of χ at z_0 gives the same functional element when the loop returns to z_0. This defines a subgroup π_χ of π, and by the first isomorphism theorem

$$\frac{\pi(\mathbb{C}\backslash\{0,1\})}{\pi_X(\mathbb{C}\backslash\{0,1\})} \cong G.$$

Recalling the definitions from this shows that the Monodromy group of χ is G.

Finally we would like to recognize G as a subgroup of a well-known group. This is easier if one uses the map from $\mathscr{H}$ to the hyperbolic plane; in other words, we replace $\mathscr{D}$ by $\mathscr{H}$, and $\mathscr{T}$ by the triangle with vertices at $i\infty,\ 0,\ 1$ and sides given by the vertical half-line of real part 0 and 1, and the half-circle between 0 and 1. Reflecting about the imaginary axis gives the first “white" triangle, and the group G can be generated by the Möbius transformations

$$z \mapsto z+2,\ z \mapsto \frac{z}{-2z+1},\ z \mapsto \frac{z-2}{2z-3}.$$

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