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Investigating Student Mental Models at the Intersection of Mathematics and Physical Reasoning in Physics

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INVESTIGATING STUDENT MENTAL MODELS AT THE INTERSECTION OF MATHEMATICS AND PHYSICAL REASONING IN PHYSICS

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An Abstract of the Thesis Presented

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A significant challenge in learning science and mathematics is coordinating different types of mental models, such as mathematical and physical mental models, that represent different aspects of a given phenomenon. This challenge is illustrated in the present study, in which we observed a small number of college students reasoning about forces as both physical and mathematical quantities as they reasoned about a physical system. Using video analysis of the students' gestures when they thought qualitatively and mathematically about the system, we documented the construction and coordination of participants' physical and mathematical mental models. It was found that the participants readily constructed mathematical mental models as imagined vector arrows or lines, but they less readily constructed physical force mental models as imagined pulls. Moreover, students rarely exhibited coordinated vector (mathematical) and force (physical) mental models needed to represent the force vector component, which was key to understanding the overall system. Taken together with the assumption that coordinated mathematical and physical mental models support robust understanding, these findings suggest that instruction in physical-mathematical quantities, such as force vectors, would benefit from

greater emphasis on building mental models of physical aspects of such quantities and coordinating these with mental models of mathematical aspects.

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INTRODUCTION

In education, we are constantly looking for ways to enhance our practices and better ourselves as educators. In education research, we strive to find information that can show us how best to teach science for understanding (National Resource Council's Framework for K-12 Science Education, 2012). One of the common threads in such research is the study of what kinds of cognitive skills are needed to understand science concepts (Reif, 1987b). Science has many difficult disciplines with which students struggle, and the more we learn, the better students will successfully learn such concepts (Glynn and Muth, 2006) One such area of difficulty is on the concept of force vectors in physics. This topic constantly eludes students (Halloun, 1996). Breaking this topic into its roots (vectors and forces) finds research that shows each components presents similar difficulties to students, as will be discussed in length in the literature review to follow. Such difficulties lie most prominently at the intersection of mathematical and conceptual reasoning.

One way to conceptualize student thinking is through mental models. Mental models are a reinstatement of a perceptual experience and are used to reason about systems (Hegarty and Waller, 2005). The National Resource Council's Framework also states, "Better mental models…lead to a deeper understanding of science and enhanced scientific reasoning." (National Resource Council's Framework for K012 Science Education, 2012) This illustrates the importance of mental models in the learning of science, and necessitates further understanding therein.

To better teach the topics of force vectors in physics, we must first learn more about the role of mental models in students' understanding of force vector concepts. This study aimed to shed light on several different types of mental models that students generate when encountering force vectors by observing the mental models generated by eight introductory physics students at a public university. A video recording of the students performing a laboratory activity was transcribed and analyzed to discover the relationship between mental model generation and the difficulties students encounter at the intersection of mathematics and force concepts in introductory college physics. A discussion of the variety, frequency, and effectiveness of these models, as well as several case studies of individual experiences, provide insight into the types of mental models exhibited by students learning force vectors in an introductory college physics course.

LITERATURE REVIEW

Within this section existing research of mathematics and physics education and mental models will be presented and discussed. An emphasis is made on the juxtaposition of algebraic and conceptual understandings in each of the aforementioned topics. However, the literature will show that this juxtaposition hinders student understanding in a variety of topics. The discussion of mental model research will suggest that such models be used as a theoretical framework to help the researcher understand student thinking, as they are useful in describing student spatial understanding. Using mental models to depict student comprehension could provide insight into why students struggle with the connection between algebraic and conceptual rationales, which would allow teachers to better address such difficulties.

In mathematics research, there are data to suggest children have great difficulty structuring space when talking about math problems, as well as within specific spatial mathematics topics such as length, area, and statistics (Battista 1999; Cohn 1997; Clements 1999; Curry 2005; Hollenbrand 2004; Sharma 1993; Tarte 1990; Yelland 1997). In Harris' 2011 study on student understanding of area, the authors implemented experience-based activities instead of teaching the formula directly. They showed that students were more confident in calculating area when using a visual tool. This tool provided a spatial reference for the students when doing their calculations. Similarly, Lehrer et al. (2002) studied various instructional strategies when teaching students about similarity of rectangles. The author showed first how elementary students were able to use algebraic tools to determine "rules" for rectangle similarity. Students were then able to separately identify similar rectangles when they were able to plot them on a coordinate plane. The similarity line of the rectangles showed students that the line could "go on forever" which supported the students' ability to consider similarity of rectangles. The students used the graphical model to improve their thinking beyond a simple mathematical sentence, yet never combined the ideas of algebraic notation and the graphical representation of similar rectangles. Rather, the graphical analysis was a tool used by the instructor to further children's understanding of geometric concepts when they had not yet been taught the formal algebraic equivalent. Frank (1991) further discussed these tools through the example of cardinal directions (*e.g.*, "geometric intuition" similar to that of a coordinate plane), which the author states are an application of cognition outside of numbers. Beyond a coordinate plane with numerical foundations, students are aware of cardinal directions, showing that graphing (also directional) doesn't necessitate numeric values. This adds evidence to the separation of algebraic concepts from Cartesian coordinates discussed in the Lehrer (2002) study. The usage of both algebraic and spatial skills (that are simple suggestions in mathematics) is necessitated in subjects like physics where you are attempting to interpret or describe a physical quantity. Vectors exemplify this situation, as students must evaluate the magnitude and direction simultaneously.

Though literature on vector understanding in math is limited, there are a few studies that demonstrate the difficulty vectors present. Neerings and Vergari (2008) discuss how procedural (algebraic) processes in vectors do not necessarily result in conceptual understanding. Vector concepts are inherently spatial, and their relationship with algebraic processes is disjointed. The Forster (2000) case study showed that even when vector problems were given in a real life spatial context (*e.g.,* distance and direction from home), algebraic testing revealed a lack of understanding. The Maracci (2005) study pointed out that vectors present a duality between a spatial object and an algebraic process; students had trouble associating these two. This study illustrates that within math, even at the collegiate level, a cognitive separation occurs as students attempt to apply vector objectivity with the use of vector processes. These studies repeatedly show a disconnect between spatial reasoning and algebra, as was noted earlier in more general mathematical topics. However, there have been studies that showed that students who initially struggled with the algebraic processes improved when taught using spacial (embodied) strategies (Watson et al. 2003). Reconciling this disconnect between algebraic and spatial aspects of mathematics is important to ensuring the success of students. This reconciliation is crucial in a two-dimensional force context in physics.

It has been shown that students have historically had difficulty with static force problems in one and two dimensions (Minstrell 1982 and Clement 1982). Some authors seem to imply that the key to understanding vector quantities in physics is to achieve a high degree of fluency with vectors and vector operations in mathematics. For instance Knight et al. (1995) explains the importance of student vector knowledge prior to force vector instruction: "Regardless of the method or methods employed, beginning physics students need explicit instruction in and proactive use of vectors. The majority do not bring a working knowledge of vectors with them to the course." So perhaps prior instruction is the reason students have such issues with force vectors. It should also be noted that students without high scores in math tend to struggle with introductory physics concepts (Meltzer 2002). Based on previous discussion about difficulties students have with vectors, it may be inferred that these students would also struggle with applications of vectors in physics. Indeed, in a study by Nguyen (2003), a significant amount of students who began a calculus-based college physics course with a rudimentary knowledge of vector quantities and manipulations saw no change in their level of understanding in vectors even after a full semester of instruction in physics. The course was one "in which students are assumed from the very first day to have considerable expertise with vector methods" (Nguyen

2003). The study shows that a physical context for vector knowledge is not sufficient for gains in this subject. Further, Shaffer and McDermott (2005) suggest one of the issues students have is "not associating the direction of the acceleration with that of the net force." They also note "the difficulties that the introductory students had with kinematics extended beyond vector formalism."

There is currently, to the author's knowledge, no study explicitly linking the dissociation between spatial and algebraic components seen within the mathematics literature, with the struggles experienced by students in force vector problems in physics. However, based on the analysis of the literature presented above, it may be possible that these two individually observed phenomena are related. Indeed, if this is the case, then the inability to associate algebraic and spatial quantities within vector contexts, may compound with the inability to understand vector applications in physics, further causing issues within the comprehension of two dimensional physics problems. In Flores's study (2004) physics students were given a picture of a gymnast holding herself up by 2 ropes with her arms at an angle. The gymnast weighed 500N and students were asked if the force exerted by the left is less than, greater than or equal to 250N. Only 20% answered correctly, and 70% said it would be equal to 250N (results were similar in both algebra and calculus based physics courses). The results show students can find it difficult to reason about static force problems in two dimensions. Further, students in physics have increased difficulty transferring their vector knowledge into a physical context in two dimensions, again pointing to the historical lack of association between spatial and algebraic reasoning. The teaching implication here is apparent: spatial reasoning is imperative in student comprehension of physics. Pallrand et al. (1984) studied the relationship between spatial reasoning and success in physics courses. Indeed, they found that students possessing spatial reasoning skills did in fact perform better in physics courses, as expected. However, implementing teaching methods to emphasize student spatial reasoning skills will only be successful if it is understood how students envision these spatial concepts in physics. To do so, one can look at cognitive trends across students with regards to how they attempt to understand the subject matter being taught.

Theoretical Framework

Student cognition in the subject of physics has been studied in the areas of gesture, representations, embodied cognition, and spatial reasoning (Hegarty 1992; Schwartz 1996; Shapiro 2010; Nersessian 2008; Shepard and Metzler 1971). In Segal's (2011) study continuous vs. discrete actions are discussed. Each is important for different tasks such as number line vs. counting. Halloun (1998) describes a similar dichotomy; wherein schemata and constructs are non-visual and visual models, respectively, formed by students when attempting to explain phenomena. These models are composed of various stages and forms, and involve mathematical equations, organizations, etc. In a study by Shemwell (2012), the author defines the term cognitive representation as "thoughts by which a person's conception of some aspect of a scientific entity or process takes on meaning." He discusses two types of representations: depictive, using pictorial and symbolic operators; and propositional, using rule based methods such as equations. A consistent theme in the studies reviewed here is a divide within means of comprehension between algebraic and spatial understanding. As discussed previously, force vector quantities are a prime example of this, and must clearly use the non-visual models described above. Yet, these trends do not answer whether students can use discrete and continuous thinking together to solve problems at the intersection of these fields.

Symbolic (algebraic) operators in mathematical vector knowledge are only half of what students need in order to understand vector quantities in physics. The other half is not being addressed and leads to an incomplete picture for students in many areas of science. As will be discussed below, mental models are a neglected aspect of spatial reasoning in instruction.

The National Resource Council's Framework for K-12 Science Education states, "mental models are internal, personal, idiosyncratic, incomplete, unstable, and essentially functional" tools with which students generate ideas about concepts (National Resource Council's Framework for K-12 Science Education, 2012). These tools are, predictably, the missing piece for students to have a clear understanding of the vector knowledge mentioned above. According to theory, mental models are mostly depictive, or a "reinstatement of a perceptual experience (Schwartz and Black, 1996)." They are used to reason about systems (Hegarty and Waller, 2005) and are generated in perceptual systems, rather than language-based rules (Nersessian, 2008), and are therefore not based on rules or equations. If mental models are tools that allow students to reason spatially, then there should be specific instructional materials that aid teachers in supporting student expression of mental models.

As seen in studies about vectors and physics such as those noted above, many curricula, college level and before, include little to no strategies to aid students in the development of mental models. If these strategies are present, it is most often the formulation of algebraic constructs. Science classes will often give students objects, structures, and processes to aid them in their understanding of certain concepts (Doerr, 1996). If these models could be observed and with algebraic and spatial coordination in mind, instructors could know what to look for and could therein apply teaching strategies based on their occurrence.

Research Questions

As elaborated upon previously, the mental models that students generate while learning deeply influence their ability to learn. Indeed, the National Resource Council's Framework states, "better mental models…lead to a deeper understanding of science and enhanced scientific reasoning" (National Resource Council's Framework for K012 Science Education, 2012). The literature regarding mathematics and physics education and cognition, as has been discussed at length above, details that these mental models tend to align with either algebraic or spatial notions. To further understand the types of mental models observed during physics instruction, the goal was to reveal students' mental models when solving a force vector problem within a model-rich instructional context, to observe the interactions between the mathematical (algebraic) and physical (spatial) reasoning skills of students. Therefore, the following research question was asked:

- How do introductory physics students use mental models to produce explanations about force vector interactions?

In an attempt to answer this question, the current study will look closer at students' production of mental models and the categories that lie within, in the hopes of better understanding how students think about force vectors in introductory physics courses at the college level. By placing students into a model-rich instructional context, there should be opportunities to observe a variety of mental models. Once addressed, a potential outcome is a productive inclusion of mental models into science curricula, which could be a partial solution to the problem of incomplete science understanding among students of all academic levels. Identifying students' knowledge through their production of mental models would allow for individualized instruction. Additionally, instructors could incorporate a variety of teaching methods to allow for the production of specific commonly expressed mental models to better facilitate student association of mathematical vector principles and their associated physical meaning.

METHODS

Students were selected to participate in a laboratory activity for the purpose of determining the role of mental models in force vector understanding. Participant selection methods and laboratory activity instructions are detailed below. Guidelines for analysis of the results are also specified.

Participants

The study involved eight participants recruited from a calculus-based introductory physics course at a state university. Participants were a few weeks into their course, and had covered a varied amount of topics in forces and vectors. A class of 191 students answered a screening question, during lecture that tested their knowledge of two-dimensional force interactions. The screening item was adopted from the gymnast question by Flores and Kanim (2004), as seen below in Figure 1. The question was also used in this study to gage student ability. Screening revealed students struggled to represent forces in two-dimensions, as 73% answered incorrectly, consistent with Flores and colleagues' findings of about 70%. Participants were recruited from this majority pool. Emails were sent to the 150 students who answered the screening question incorrectly asking for their participation in this study. Twelve responses of interest were received, and eight participants were selected due to scheduling. In return for their one hour of participation in the learning activity, one hour of physics tutoring was offered. However, only one of these eight participated requested tutoring as compensation.

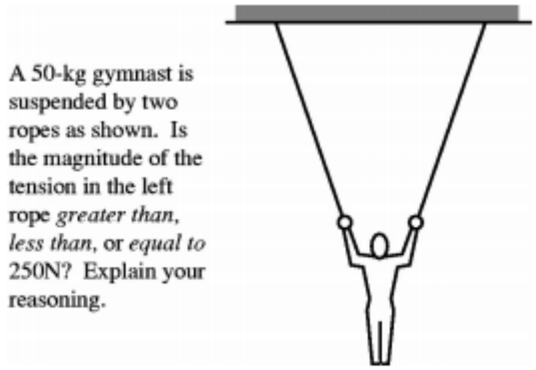

Figure 1 The stationary hanging gymnast question from Flores (2004), used as a screening question to gauge student prerequisite understanding of force vectors in physics

The rationale for taking participants who answered incorrectly was that engaging them in the learning activity would provide an opportunity to observe their construction of mental models. Because this topic was difficult and relatively new to the learners, students would likely be constructing newer mental models, which, because of their newness, would be more visible than more expert models (Dixon, 2011).

Overview and Rationale for the learning activity

When participants arrived they completed a pretest, three activities, a posttest, and an exit interview, each of which will be described in detail in later sections. The entire experience took approximately 45 minutes. The first of the three activities will be henceforth referred to as the learning activity, as it is the focus of this study. The learning activity used an apparatus constructed for the study in which each participant arranged a set of strings and force meters to measure the effects of a changing angle on a static force in two dimensions (Figure 2). In situations like Figure 2, the tension in the angled string increases with a larger angle to the vertical of T_2, because the horizontal component of the string's tension increases. The design of the learning activity assumed that a robust explanation for why the tension increased would depend upon the construction of a mental model that accessed both the vector formalism of components and the organic physical idea of tension as a force (*e.g.,* push or pull).

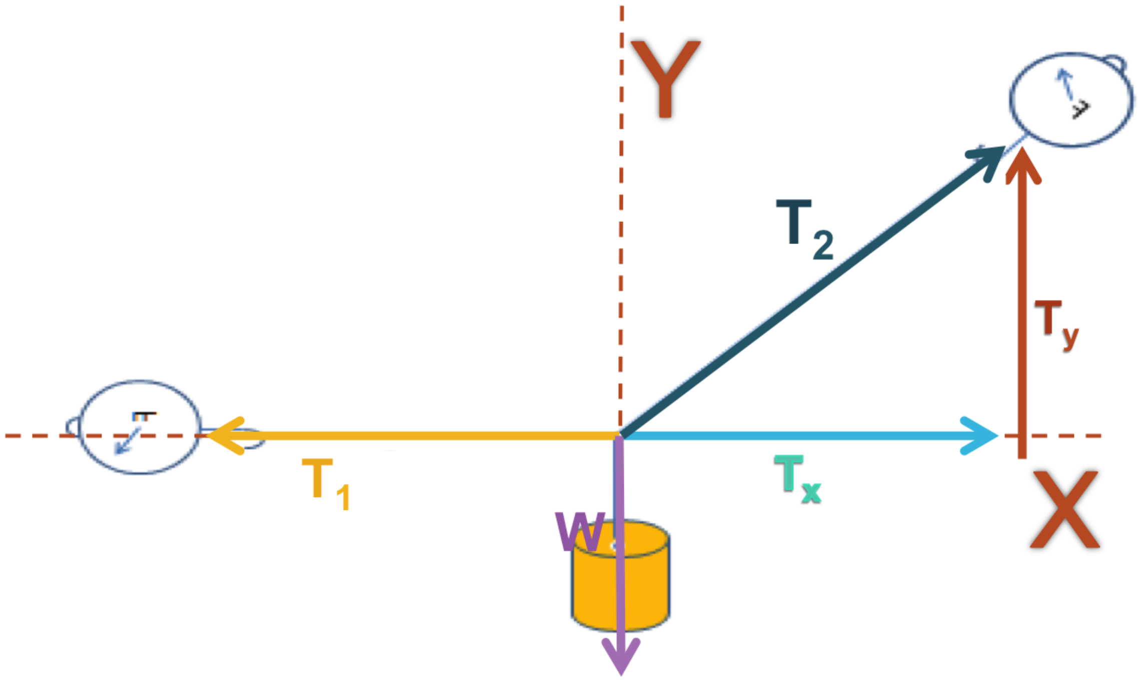

Figure 2 Vector analyses of the physical apparatus set up in the learning activity. T_2 is broken into components along a superimposed axis.

It was assumed that successful mental models would somehow embody this combination of the mathematical and the physical. Such mental models may be displayed by gestures involving imagined lines of a horizontal and/or vertical component and increasing exerted force (*e.g.,* pushes and pulls) balancing other forces in the system. This expectation was based on observations from preliminary work for this study using a similar apparatus. However, the mental models looked for were those that in any way pertained to participants reasoning about forces or vectors.

Through the calculation of *x* and *y* components in the angled string for both situations (completed by the participant), participants had in front of them data that showed the x component of the tension was increasing with the increasing angle, while the y component stayed more or less the same. This was intended to decrease the cognitive demand of participants. By seeing a declarative statement of the phenomenon in numerical form, there was no need to theorize about the x component of the angled string growing. The participants were being shown via data that the tension in the angled string increased as the x component increased. Their only task was to explain why the x component growing increased the tension in the angled string. It was expected that the physical arrangement of the apparatus together with the measurements, values for the x and y components of the tension in the angled string, and the demand for verbal explanations (described below), would facilitate the construction and coordination of necessary mental models and that those models would be visible via participants' gestures and speech.

Instrumentation

Pretest

When participants arrived, they first completed a second pretest. The pretest had one question similar to the screening test (Figure 1), but also asked students to decompose a vector given a magnitude and angle. The purpose of this pretest was to determine each participant's knowledge of vectors and vectors as forces at the time of the activity.

Posttest

The posttest asked students about a setup similar to the learning activity with some extensions. There were two strings with one weight hanging between them, but unlike the learning activity, the two strings were of different lengths. They were then asked, "Consider the above situation. Angle A was less than angle B. Predict which string has more tension?" String A is much shorter than string B (see Figure 3). Once they answered this question, they were then given a second paper asking a similar question about the same setup. This second question asked them what is happening to the x and y components separately (*e,g.,* Which is larger? String A x-component or string B x-component).

The purpose of this test was to see if students could take what they learned from the learning activity and apply it to another situation with a different set of criteria. The reason this question is different is due to the differing angles off axis. As seen in the figure below, these strings are not centered, causing unequal lengths and unequal angles. This question was eventually removed from the analysis due to its complexity. Students were generally not able to apply their knowledge to such a situation, resulting in low scores for all participants. Instead, an analysis of participants' level of understanding was conducted as will be described later.

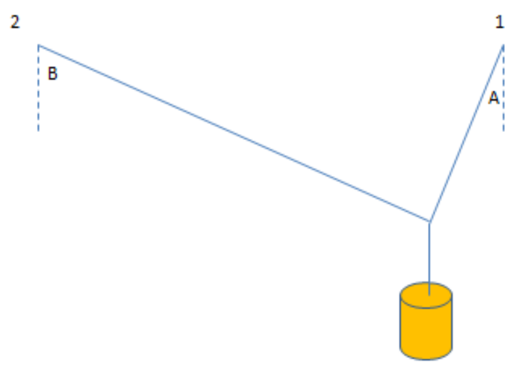

Figure 3 Picture presented to participants on posttest

Procedure

The learning activity involved strings hanging at two distinct angles. For both the small and large angle setup (A and B in Figure 4), participants recorded the force meter readings for both strings. They also measured the angle to the vertical for the angled string in both situations. They were then asked to calculate the x and y components of the angled string's tension for both situations. These measurements and calculations showed that, with the larger angle, the x component of the angled string's tension increased. All of the participants were able to make and record the necessary vector component calculations without intervention from the researcher.

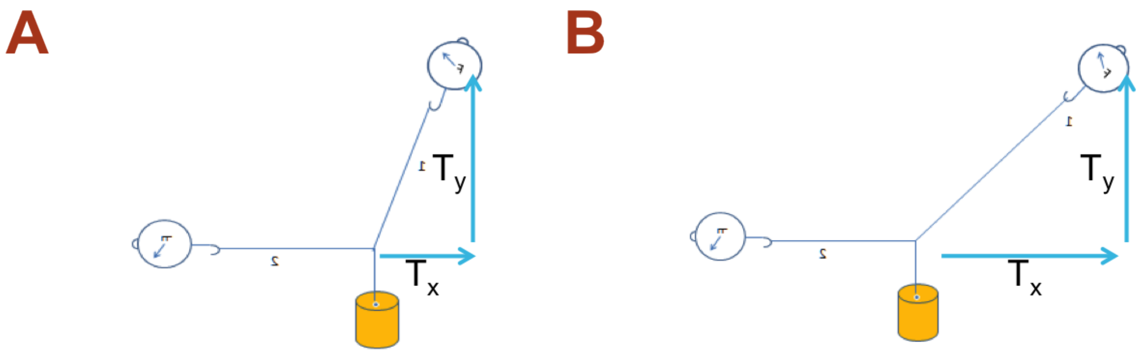

Figure 4 The two apparatus setups for the learning activity, showing an increasing angle from setup A to set up B, as well as an increasing x component. The y component remains more or less the same

After this process was complete for both setup A and B, students were asked, “Comparing 1A and 1B, explain what is happening to the tension in the angled string, and why? Please discuss with your partner.” The partner in this case was the researcher present.

Participants then verbalized their thinking about the situation. The relationship between the x–components in each scenario sought by the question, was demonstrated by the numerical results previously calculated. However, the question is really asking them why this phenomenon of increasing tension occurred. This put a constraint on their thinking, as they didn’t require full range of thought about what could be happening physically. They were only answering why based on the answer drawn directly from their data (that the x-component was growing while the y-component was not).

The researcher’s role in the learning activity was threefold. First, the apparatus setup was not intended to cause the participant stress or be at all difficult, so the researcher aided in setup as needed. Second, clarifying and elaborative questions during the explanation part of the learning activity were asked when the researcher felt it necessary. These included phrases such as: “What do you mean by that?” “Say more about that,” and “can you restate that?” Finally, the researcher was also present to tell participants when to move on. This was determined when the researcher felt the participant had answered the question to the best of his or her ability, and when elaborative questions had been exhausted.

In the exit interview, participants were asked to discuss what they were learning about in their current physics class, and if any of that information (from lab or lecture) had informed their knowledge during the activity. The information from this exit interview was not used in the present study. The researcher also debriefed participants about the research project and answered any questions.

The learning activity took anywhere from 10 to 15 minutes to complete. Participants were asked to complete the activity to the best of their ability, and were assured that correct answers were not important. The researcher present stated, “Remember, we aren’t looking for correct answers, but rather that you to describe your thinking in detail.”

Each participant was videotaped while completing the learning activity. Two cameras recorded from two different angles, giving coders the ability to observe gesture from two perspectives when necessary to eliminate ambiguity.

Methods of Analysis

<u>*Rationale for Analysis - Mental Models*</u>

The primary analysis of this study focused on observing mental models expressed when students were immersed in an intentionally model-rich learning activity focused on force vectors. This analysis was adapted from research tradition for think aloud activities, in which a subject is asked to answer a question and detail his or her thinking out loud. A researcher was present to support participants in expressing their ideas through speech and gesture. This strategy was consistent with Cohen's finding that students tend to gesture more when discussing a topic face to face (Cohen, 1977). Analysis of student behavior from the learning activity yielded similar results to Kita (2002), who inferred non-linguistic representations from gestures during a think aloud situation. Additionally, Alibali (2001) described a coding scheme similar to that used in this study. Alibali distinguished between representational and beat gestures, where representational gestures depict semantic content, and beat gestures are non-explanatory gestures such as pointing and touching. Specifically, Alabali's study coded for iconic gestures, a sub-category of representational, which are described as gestures that held both spatial and functional meaning. The analysis of the present study will focus on these iconic gestures, referred to as expressed mental models, pertaining to force and vector content.

Analysis Procedure - Observational Evidence.

Each participant's interview video was transcribed and segmented according to each gesture the participant made, along with the speech used when that gesture was expressed. Gestures that were continuous, or periods of time over which the gesture is maintained, were grouped together using speech as an aid, defining the segment. This included all gestures and speech, even those that seemed to hold no meaning (as discussed below). However, regardless of their quickness, consecutive gestures regarding new ideas or speech were segmented separately. At times, this required the researcher to slow down the video to better observe certain series of gestures, or to better understand what was being said. The segmented gestures and speech will be referred to as gesture reasoning units (GRUs). Additionally, time spent on calculations, drawing diagrams, pauses, and speech without gesture were grouped with the previous GRU. Grouping these non-gesturing times with the previous gestures allowed the researcher to focus on the gesture. The reasoning units ranged in length from less than a second up to 10 seconds. This procedure yielded 124 total GRUs distributed across the eight participants.

As analysis of these GRUs progressed, evidence was gathered based on gestures, hand placement, and speech, which led the emergence of variation between the GRUs. Two researchers segmented the 124 GRUs (see distribution between participants in

Table 1), where it was determined that 39 of the initial GRUs were non-explanatory gestures. Non-explanatory gestures are analogous to the beat gestures observed by Alibali, described previously, in that they depicted no content, such as a flailing arms or crossing of the arms, or were merely indexical, such as pointing to a particular piece of the apparatus, touching the apparatus, drawing on paper, or flipping between pages. From the 86 remaining GRUs, 30 were discarded because they were merely comparisons or clarifications, such as big (arms stretched out wide) vs. small (arms in near body with hands close together), and were not pertinent to this study. The remaining 56 GRUs were then determined to be either compelling or ambiguous. Ambiguous gestures were those in which even one coder was uncertain that the gesture could be assigned to one of the mental models that were ultimately defined. There were 22 such gestures, all of which were discarded.

Table 1 The distribution of GRU variations between the participants in this study

Participant Pseudonym	Non-explanatory	Comparison/Clarification	Ambiguous	Compelling	All Gestures
Ashley	7	6	2	3	18
Grace	3	1	5	2	11
Jake	15	10	1	9	35
Jennifer	2	0	3	6	11
Karen	2	2	2	1	7
Katherine	5	0	2	3	10
Meredith	5	9	3	5	22
Stanley	0	2	4	4	10
Total	39	30	22	33	124

Only 33 compelling GRUs were deemed applicable to this study. This meant that two researchers agreed that the reasoning unit fit a mental model and that the unit's inclusion into another mental model type outside of those ultimately defined was implausible.

Analysis Procedure - Coding for Frequency

Once the 33 compelling GRUs were agreed upon, the units were coded into distinct categories of expressed mental models by each researcher. A third researcher independently coded a 20% random sample (eight units) to ensure appropriate categorization. Agreement with the first two researchers was 87.5%. With the categorization of the researchers validated, it was then possible to determine a frequency relation between participants, and to then define each category by representative characteristics found within, as will be described in the next chapter.

Rationale for Analysis - Assessing Conceptual Understanding

As a final method of analysis, participants' level of conceptual understanding was determined. Level of conceptual understanding is a designation of how successfully participants reasoned about the system. This was obtained by the end of the learning activity based on speech alone, ignoring gesture. Gesture was neglected during this analysis to ensure the distinction between methods of analysis for the level of understanding and the observational evidence. To do so, line-by-line coding was used (Charmaz' 1995). Line-by-line coding allowed the researcher to "defamiliarize the familiar" by forcing the researcher to actively interpret distinctive words within a transcript (as demonstrated in a later section). This process forced a re-examination of the transcripts and ultimately determined further results. Once transcripts were analyzed, participants were assigned to tiered levels of understanding based upon how well they were able to verbalize and describe the results from the learning activity.

Together, each of these methods of analysis contributed greatly to the discerning of the valuable results obtained in the current study. The next chapter presents an in depth description of the results of the analysis performed in this study, followed by a discussion of the overall impact and implications of these results.

RESULTS

The method of analysis discussed in the previous chapter revealed that the participants in this study exhibited distinct mental models as they reasoned about the presented force vector scenario. Some key differences were observed regarding what these mental models represented. Some participants represented the lines or arrows showing the geometry of the situation before them, which will be referred to as mathematical models. Other participants represented the physical phenomenon of force as a pull or tug, which will be referred to as physical models. The specific gestures that occurred within each general subset of observed models are described below.

Mathematical Mental Models

A representative depiction of a mathematical mental model is shown in Figure 5. Within Figure 5, the red circles were created using the program "Tracker" to track the motion of an object at a given frequency. It should be noted that this program was not used in the determination of mental models and was simply used to establish a visual aid for the reader. Participant Katherine is describing the similarity between force vectors in physics and the vectors without context that one would decompose in trigonometry class, as she attempts to calculate the x and y components of the tension in the angled string. Her hand moves along these imagined x and y components of the angled string. She is using her hands to describe where (in space) these components are located (see Figure 4, in previous section, of apparatus). However, she is not discussing how these components are pushing or pulling, or how they relate to the tension force.

Figure 5 Participant Katherine exhibiting a mathematical mental model. The participant hand motion is steady and not rapidly accelerating, as seen easily by the consistent spacing of the red dots

Because Katherine is moving her hand slowly and with even speed (as is visually apparent by the red circles in Figure 5) in the horizontal (not pictured) and vertical plane of the apparatus (as is visually apparent by the pink axes in Figure 5) while saying “x and y component,” she appears to be perceiving the projected lines of a triangle one would conceptualize while decomposing a vector. As fits a definition of a mental model, it may be inferred that she is exhibiting one via her perception of the lines of the horizontal and vertical vector components. Such a mental model does not include a physical aspect, as are described in the section to follow, because there is no rapid acceleration embodying some push or pull, as her hands are moving slowly and steadily, and are within the defined vector component space of the apparatus. Further, in conjunction with the gesture, the participant vocalized a geometric monologue confirming the designation. Katherine’s mental model is defined as a line or arrow with some relative magnitude or number attached to it (i.e., a scalar quantity) and without any expression of physical exertion.

Additionally, it was noted that Katherine's hand is flat as if she is tracing or drawing in the air. Indeed, it was observed that some other participants who generated mathematical mental models had a flat hand, pointed finger, or used their writing utensil to trace out the geometric space, as seen in Figure 6.

Figure 6 Examples of hands of participants exhibiting mathematical mental models. Flat hand, pointed finger, and utensil pointing are seen

Katherine's mathematical model occurred when calculating the components of the tension, and examining the collected data in regards to the tension's components and attempting to explain the phenomenon. The evidence above suggests that Katherine is generating a perceptual experience regarding imagined vector components in geometric space. Her hand motion, hand position on the apparatus, hand shape, and the question they are addressing support this suggestion. However, an important possibility is that the participant may also perceive a physical situation that is not observable through gesture and speech; this important possibility will be examined in a later section.

Physical Mental Models

In contrast to the example above, physical mental models include rapid hand movement, as depicted by participant Ashley in Figure 7. It is observed in Figure 7 that Ashley moves her hands rapidly accelerating downward motion (seen visually by the inconsistent spacing of the red circles) outside the defined geometric vector component space. During this expression, she uses the phrase “the force of gravity.” Because of Ashley’s rapid acceleration of her hand, the location of her hands in regards to the apparatus, and her specific verbiage, it may be inferred that she is reinstating the perceptual experience of the exertion of force due to gravity on an object.

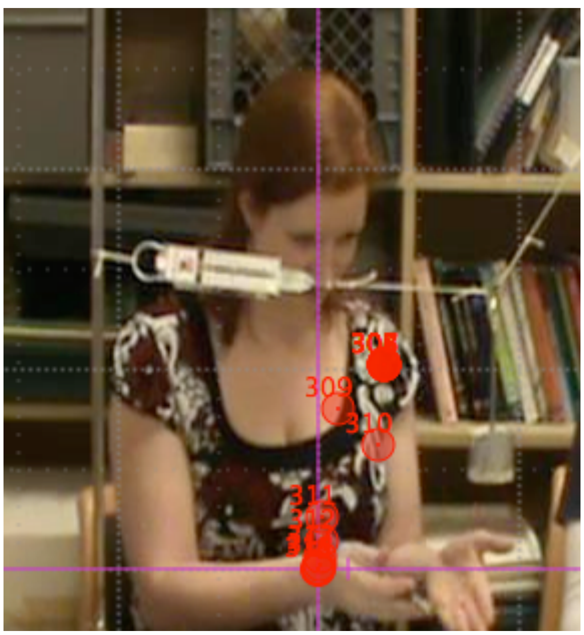

Figure 7 Participant Ashley exhibiting a physical mental model. The rapid acceleration of the hands is represented by the varied spacing of the red circles

In addition to rapid hand acceleration, Ashley had a distinct hand shape, in which her hands were cupped. Other participants even showed grasping or clenched, as depicted in Figure 8. Such hand shapes were interpreted as attempts to physically move or manipulate some imaginary or perceived object. The way, in which the hand was formed when these physical mental models are expressed differed among participants, ranging from curled fingers, to touching fingers to thumb, or making a fist. Comparison of the hand shapes in Figure 6 and Figure 8 reveal stark differences in how participants were expressing their mental models, further affirming the differences between the observed mathematical and physical models.

Figure 8 Examples of hand shapes of participants exhibiting physical mental models. Cupped, pinching, and clenched hands are seen

It may then be inferred that Ashley is exhibiting a mental model capturing aspects of the physical situation in front of her. Though it is possible Ashley may be thinking about a vector (pointing downward with the motion of her hands), the motion does not exhibit the vector formalism previously described. Regardless, this instance seems to inform her knowledge about a force acting on the weight. Alternatively, one could argue participants perceive a geometric vector component that is not observable. This important possibility will be examined in a later section. However, there are instances in which simultaneous expression of mathematical and physical mental models was observed and will be described below.

Simultaneous Expression of Mental Models

In the two cases just presented, the models were not necessarily used to represent physical and mathematical ideas together. For the purposes of this work, the use of the phrase "simultaneous expression" with regards to mental models refers to a GRU wherein the participant coordinated components of both the mathematical and physical mental models described previously, when reasoning about the system. This does not mean that two distinct mental models were expressed simultaneously, but rather the expression of mathematical and physical elements was observed. When Katherine represented the vector component, there was no indication that she was also representing force. When Ashley represented the force of gravity, she may or may not have also been thinking of a vector component. In contrast to these cases, components of the two models were sometimes expressed simultaneously. Meredith provides an example.

Meredith moves her hands in a quick accelerating motion (seen visually by the inconsistent spacing of the red circles in Figure 9) in an appropriate geometric space for the vector component (seen visually by the pink axes). During this expression, she clarified her statement about the horizontal component by saying, "because it's pulling it further." The indicators of this combination of mental models included aspects of each of the two unique mental models when expressed separately. She was clearly moving her hand in the x direction while motioning a push or pull, classifying this mental model as both physical and mathematical.

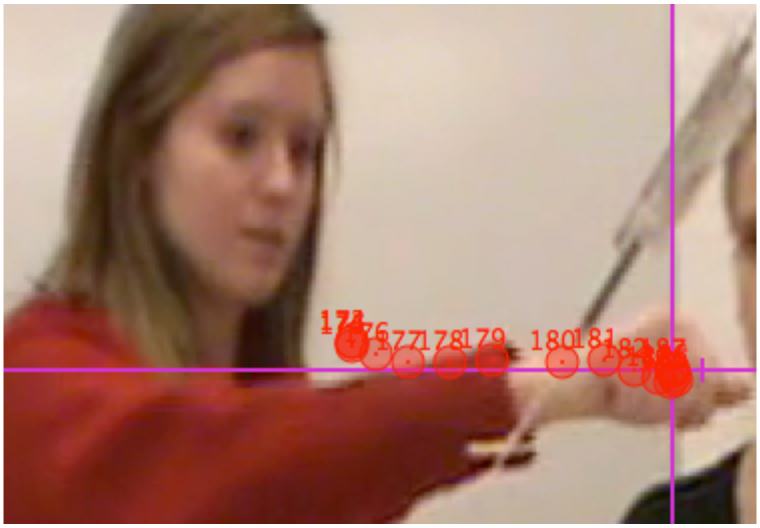

Figure 9 Participant Meredith exhibiting the simultaneous expression of a mathematical and physical mental model. The rapid acceleration of her hands is represented by the inconsistent spacing of the red circles on the defined axis. The pink axes show her exhibiting this model within the geometric space of the perceived x axis

Meredith's hand looks as if it were attempting to manipulate or move an imaginary object. Like the physical model, participants often had a distinct hand position, in which their hands were cupped, grasping, or clenched, which is interpreted as an attempt to physically move some imaginary or perceived object. Also, similarly to the mathematical mental models, these hand motions were in a distinct x or y direction in regards to the apparatus. From the previous discussions of the mental model depictions, it is important to emphasize that these mathematical or physical mental models may be expressed individually or simultaneously. Further, it should be noted that these perceptual experiences - whether they be mathematical, physical, or both - are not isolated incidents, as they manifest throughout the population of students interviewed, as discussed below.

Frequency of occurrence

As previously mentioned, two distinct categories of mental models arose, one relating to vectors (mathematical) and the other to forces (physical), and that these models could be expressed either together or separately. With regards to mathematical models, observable commonalities became apparent within a subset of gestures, specifically, participants who moved their hand(s) slowly and steadily in a straight line in space (without rapid acceleration). These lines were traced out via flat hand, pointed finger, or pointed object, within the constraint of the apparatus, and were often interpreted as the imagined components of the angled string's tension within this space on the apparatus, as described in the following section. These fluid mathematical gestures and speech were in stark contrast to those that did not represent a vector component, as physical gestures in regards to force were not fluid.

Gestures representing physical models were frequently observed and involved the rapid acceleration of the hand(s). Participants exhibiting these gestures would move their hands in a way as if to represent a push or a pull. One can imagine "shoving" or "tugging" on an object requires some acceleration of the hands for a force to be exerted (i.e., $F=ma$, where the mass is a constant). These forceful motions, in conjunction with specific verbal cues, helped to reinforce the differences between the two categories of mental models.

Given the specific observations associated with each of these types of models, a coding process was utilized to enumerate the specific occurrences of each of these types of mental models (Alibali, 2001). The defining characteristics of each are presented in Table 2. Each of the 33 GRUs were examined and placed into the corresponding category.

Table 2 Specific attributes of mental model categories to function as operational coding tools

Mental Model	**Hand Shape**	**Hand Speed**	**Verbal Cues**	**Position on apparatus**
Mathematical	Flat, pointed, or utilizing utensil	Slow/ Constant	Component language	Within confines of imagined components of tension
Physical	Cupped, clenched, pushing, pulling, or grasping	Rapid/accelerating	Force language	Anywhere outside the imagined components of tension
Simultaneous	Cupped, clenched, pushing, pulling, or grasping	Rapid/accelerating	Force and Component language	Within confines of imagined components of tension

Analysis of the participants performing the learning activity did indeed result in the display of behaviors corresponding to the suggested categories of mental models throughout the GRUs. The frequency with which the different mental models occurred is presented in Table 3. It can be seen from the table that physical mental models were expressed far less often than were mathematical mental models in this context. Only half (4/8) of the participants exhibited a mental model involving force, while most (7/8) participants exhibited a mental model involving vector components (mental model occurrences for individual participants are discussed later). This could suggest that mathematical mental models are easier, more prevalent, or more readily constructed then are physical mental models among the population studied here, as discussed later. As stated above, it is also of note that four of the mental models simultaneously incorporated mathematical and physical components, and are therefore represented in both frequency categories.

Table 3 Frequency of exhibited mental models, four of which were simultaneously expressed and therefore represented in both columns

Mental Model	Mathematical	Physical
Total Occurrences	23	14
Number of Participants	7	4

Table 3 also provides evidence that the two categories of mental models were repeatedly observed and fairly well distributed among the participants. The expressed mental models were not singular behaviors, but rather consistently recurring models. Noting that the same categories of mental models, as identified previously, could be identified over a range of participants within this context, supports the validity of the observational evidence.

Alternatively, one could conclude that, though these models have distinction and repetition, the categories are indeterminate of the participants' understanding about vectors, forces, and their relationship. However, as previously discussed, learners who are unable to relate the mathematical and physical aspects of force vectors tend to have difficulties achieving high levels of understanding. Therefore, it is important to explore the relationship between mental model expressions and the effectiveness with which the participant reasons about the system in order to ascertain whether or not the perceived experience was mathematical, physical, or a combination of the two, as is addressed in the section to follow.

Level of Understanding

In addition to coding through observation and frequency, the level of understanding of the participants was investigated and analyzed. The purpose of this analysis was to attempt to determine if students were thinking about mathematical or physical situations while expressing observable models about them. Line by line coding (as noted in previous chapter) consisted of paraphrasing the portion of the transcript in which participants explained why the tension was increasing, for each sentence or phrase using video data to aid in the interpretation of pronouns (*e.g.,* this, that), as depicted in Table 4.

Table 4 Representation of line-by-line coding for the purpose of ascertaining a participant's level of understanding

Speech	**Paraphrase**	**Narrative**	**Level**
Ashley The tension need to be greater because there's a	Tension is more with a larger angle because of sine	Something like a rule.	Mid-level
greater difference in the angle so, whenever you take the sine of a bigger angle you get a smaller number.			
So the resultant tension	The tension compensates for the force of gravity. I'm getting confused	Tries to say why; gets confused	
in order to compensate for the normal force of gravity that would be it just hanging alone. [Pause here] I think I'm doing this wrong. I'm getting confused now. Instructor: Ok, just talk it out. (Both giggle) Ashley: my, my, I was sick on Wednesday so my migraine is still somewhat befuddled. Instructor: That's ok, we can just talk through it			
Ashley: Cuz like, this is, probably 500 grams? Instructor: Yep it is 5 Newtons, so yep, approximately. Ashley: Mmhm, so it just being down like that it is 5 newtons. And so some something else is taking part of the weight off which is this one and so you have the component that is 5 Newtons. Which it is each time because that is gravity on the block mumbles.	Something is taking part of the weight off. This one (horizontal string).	Load sharing the weight from the two strings	

From these data, descriptive narratives of the participants' thinking were developed for the purpose of ranking the participants' levels of understanding. Analysis of the descriptive narratives revealed 3 distinct tiers of level of understanding. Two of the participants were deemed to possess a high-level understanding; four with a mid-level understanding, and the remaining two had a low-level understanding. Table 5 below shows the level of understanding of the individual participants alongside the types and frequency of mental model they exhibited. More detailed descriptions of these categories with examples are detailed to follow.

Table 5 Participants' mental models by category, and their level of understanding reached during the first task of the learning activity

Participant Pseudonym	Physical	Mathematical	Mathematical and Physical	Level of Understanding
Katherine	**0**	**3**	**0**	Low
Jennifer	**0**	**6**	**0**	Low
Ashley	**3**	**0**	**0**	Mid
Grace	**0**	**2**	**0**	Mid
Stanley	**0**	**4**	**0**	Mid
Karen	**0**	**0**	**1**	Mid
Meredith	**1**	**2**	**2**	High
Jake	**6**	**2**	**1**	High
Total	**10**	**19**	**4**	**-------------------**

Low-level understanding

One of the participants, Katherine, was assigned a low-level understanding because of her inability to discuss what was happening in regards to force in her explanation. She eventually described the vector component situation correctly (visible in data collection), but was unable to use correct terminology and reach a complete analysis. She never came to a conclusion about what occurred:

"If we're doing [this calculation] like in like in the pretest where you use like a triangle to find like the x and y components, then maybe like this [string] would be just like the hypotenuse part and then, like the x and the y component [would be here]…But I'm not sure if you can do that with like forces if like the same as with measurements… I guess that the great, like the greater angle, the tension was greater. But ya it seems like when we increase the angle the tension was greater for both strings…I guess, I guess it was ah, when you increase the angle the distribution of like of the mass is different… But I, I would think that it wouldn't, the tension wouldn't increase like that much. I would think that like the tension would be the same, I guess. I don't know…Cuz the mass is the same…I think the force would be the same. But, for some reason it's not so I don't even know… Or maybe it's just like an angle that makes a difference. Not really sure."

Katherine was constantly unsure of the statements she made as seen by her comments such as "I'm not sure" and "I don't know" which appear often in her transcript. She even expressed that she thought the forces should be the same as the angle increases, showing that she did not have a firm understanding of the situation. She was not able to express her mathematical knowledge in a physical situation. Cases such as this were deemed low-level understanding.

High-level understanding

Participant Jake is an example of high-level understanding, because he explained the situation using correct terminology, accurate analyses, and used both the concept of force and vector components correctly.

"It looks like the larger the angle the more weight or the more tension is in the rope… I know if you get to 120 degrees in the center [of the ropes], both of

> them would equal 5 [Newtons], approximately. Like if this [string] snapped right now … it would have a lot of force due to gravity which would put even more force on that [other] anchor, which would also possibly break due to that [extra force]… If [the strings] were really close to each other they would basically just be holding up the single load, but as they're being pulled apart they're each pulling it on the x axis compared to just the y… And its still holding up the same amount of weight um in the y direction just adding more force in the x direction."

Jake used his experience rock climbing to infer about the angle between the two strings, and then theorized the learning activity situation, coming to the correct conclusion that the x-component of the tension increasing. He used terminology like force, components, pulling, and direction to analyze the situation. Unlike Katherine, Jake discussed the correct conclusion and was able to verbalize it completely. Such cases were labeled high-level understanding.

Mid-level understanding

Participant Ashley reached a mid-level understanding due to her inability to use components in her answer.

> "The tension needs to be greater because there's a greater difference in the angle so, whenever you take the sine of a bigger angle you get a smaller number… In order to compensate for the normal force of gravity that would be it just hanging alone. I think I'm doing this wrong. I'm getting confused now… with the bigger angle has the bigger force… I'm thinking with potential energy and kinetic energy… So being higher up it needs more energy to stay up. I think… I know how to work things out very well, but I don't know the theoretical stuff behind it."

Here Ashley seemed to focus on the force concepts and the physical understand that the weight is being held up. She did not come to a conclusion using this approach since she did not bring components into her answer, and then began to search for other plausible physics ideas (energy). However, she did relate the angle to the increasing force when stating that the sine of a bigger angle will be a smaller number. It is evident that she had a higher level of understanding than Katherine, but she did not master the situation like Jake. These cases were categorized as mid-level understanding.

The data presented in Table 5 suggests that those who generated only mathematical mental models did not reason effectively about the system. This reaffirms that mathematical models can exist independently from physical mental models, even in a situation discussing forces. The above data further support the claim that mathematical and physical mental models are unique categories. Subsequently, these data also imply that students can generate mathematical or physical mental models without reasoning effectively about the situation. It also appears as if, in the admittedly small sample size, students who exhibited simultaneous expression of components of mathematical and physical mental models reasoned more effectively about the system. The limitations of this evidence are not insignificant and will be discussed at length in the chapter that follows. Annotated transcripts of select participants depicting the evolution and sequence various mental model exhibition are available in the Appendix.

Summary

Analysis of student behavior when completing a laboratory activity demonstrates that students exhibited two distinct mental models (mathematical and physical) when thinking about force vectors in introductory physics contexts at the college level. The data above presented observational evidence of the existence of these mental models and their frequency of occurrence, along with an analysis of the level of understanding of participants. The observational evidence supported the differences between the two categories based on hand shape, hand motion, position in regards to the apparatus, and the question being addressed, while also presenting evidence that these two categories can be expressed independently or simultaneously (as previously described). The frequency of these occurrences as seen in Table 3 provided evidence that the mental models were fairly well distributed across participants and that these models arose not as isolated occurrences, but rather as repeated observations, solidifying the distinctions of the categories of mental models exhibited. Finally, Table 5 shows that the effectiveness with which participants reasoned about the system and the type of mental models generated have some relation. The implications of these findings are discussed at length in the following chapter.

DISCUSSION

The previous chapter highlighted the important results from the current study. These results include observational evidence, frequency data, and an analysis of participant level of understanding. Indeed, there are several claims that can be made from this data, along with abundant implications for instruction. The current chapter will discuss these in depth these while considering the potential for future work.

It is most evident from the results discussed in the previous chapter that two distinct mental models (mathematical or physical) and ways of interpreting the learning activity were utilized by various participants in this study. As mentioned, there were various traits unique to each of these models, including specific language used, hand movement, hand shape, and position of the hands on the apparatus. Therefore, the following is a reasonable assertion:

- Claim 1: Mathematical and physical mental models are two possible methods students may utilize in order to reason about a force vector problem presented to them.

The observational evidence seen in the detailed examples from the previous chapter defends this claim. Specifically, participant Katherine expressed a mathematical mental model when she used a slowly and steadily moved a flat hand across the imagined component of the tension while discussing vector components. In contrast, Ashley generated a physical model when she used cupped hands in a rapidly accelerating motion outside the defined vector component space while talking about the force of gravity. Indeed, these cases show that the participants reasoned about the system in distinct ways, either mathematical or physical, the mathematical being a tracing out of components of a vector, and the physical a rapid accelerating motion to indicate a force. It should be noted, as seen in participant Meredith, that components of these distinct models may also be exhibited together, leading to the following claim:

- Claim 2: The two observed mental models (mathematical and physical) can be utilized both separately and simultaneously.

The attributes, similarities, and differences of the mathematical, physical, and simultaneous models are well characterized in the previous chapter. Specifically, participant Meredith used a clenched hand that rapidly accelerated (indicative of a physical model) within the defined vector component space on the apparatus (indicative of a mathematical model) while using language that applied to both physical and mathematical situations. An important implication of these claims involves the variety of ways participants reasoned about the system. In one case, a physical phenomenon was reasoned about solely using mathematical model, despite the presence of an apparatus that would tend to suggest students consider a physical reasoning; vector components representing the geometry of the given situation were modeled, but not necessarily with a corresponding physical mental model. In another case, when reasoning about the system, no particular attention was given to the nature of the geometric space (representing the mathematical aspect of the situation) the apparatus possessed as considerations were solely of a physical nature. It is important to note, again, that the students were placed into an intentionally model-rich instructional context. Even within this context, there were students who only expressed mental models referring to mathematical contexts.

The methodology generating the data supporting these two claims does have some limitations. The participants' precise thinking is never observable and must be understood as such; therefore, it is important to note that though this is an overall limitation of the methodology for data collection and interpretation, such techniques are widely accepted and applicable within the field of education research. With that in mind, if participants only exhibit mathematical mental models, one might assume they are not able to reason effectively about the physical aspects involved in the system at the end of the learning activity.

This reflects what was discussed in the literature, as described by Maracci (2005), Segal (2011), Halhoun (1998), and Shemwell (2012), wherein a separation exists between the mathematical and physical understanding. In the current work, it was demonstrated that Katherine expressed a uniquely mathematical model, while Ashley expressed a uniquely physical model. However, as the case of Meredith shows, a participant was able to simultaneously employ components of both mental models to analyze the situation that was occurring. These individualized scenarios speak to the three distinct ways in which learners may use mental models. Teachers can expect these models when approaching the topic of physics situations through an interactive scenario designed to support the understanding of both mathematical and physical aspects of, specifically those of force vectors in regards to the current work. Each participant generated a unique set of mental models and developed his or her learning differently. This certainly shows the existence of the possible ways that students can respond to physics problem solving situations meant to support physical and mathematical thinking in combination.

One question that arises in the three specific cases of mental model usage is whether they may have been idiosyncratic to those particular participants. Indeed, the frequency data generated supports the idea that these mental models were not isolated occurrences and that the models were present across the population of students interviewed. Extrapolating this idea further leads to another claim:

- Claim 3: These types of mental models could possibly be expressed among a similar student body performing a similar learning activity.

The frequency data derived from the coding methodology employed in the present work show that 7 out of 8 participants exhibited uniquely mathematical mental models, 4 out of 8 expressed uniquely physical mental models, while 3 of the participants displayed a coordination of said models. Specifically, as noted in Table 2, there were differing hand shapes, hand movements, hand placement, and vocabulary used within the contexts of the mathematical and physical mental models employed by the participants. The mathematical models involved pointed fingers, slow and steady movements along the apparatus in the space of the geometrically imagined tension component space, while participants discussed the components. In contrast, the physical models involved cupped or clenched hands that rapidly accelerated outside the defined component space on the apparatus while the participants discussed force. This plurality suggests these models are not isolated occurrences, and could possibly present in a similar setting in which students are trying to make sense of a hands-on situation designed to help them reason about the physical aspects of force vectors. All of this evidence does indeed suggest that these mental models could be exhibited across an array of students when faced with similarly designed, intentionally model-rich, hands-on force vector situations.

Further, there may be more mental models that we did not discover. For example, more experienced learners may exhibit completely different mental models or even none at all, as suggested by research into so-called "expert learners" (Dixon, 2011). Regardless, it is evident that these individualities exist and therefore instructors should make an effort to ensure that their curriculum supports these mental models.

If physics learners are able to generate and reason about physical aspects of physics quantities such as force, then the support of mathematical and physical mental models may be necessary for the success and understanding of some students. Instructors should take note of the mental models identified in this study because they show a glimpse of the individuality of the student. Further and more importantly, these findings show instructors that they cannot assume that students generate physical models even in situations optimized to support that type of model generation. Since these are models that could reasonably appear in any physics classroom in this context, there should be specific instructional materials that can be used with in a physics curriculum to support their development and usage. Standard textbook problems involving force vectors provided on a page are not designed to foster the usage of physical models. If students may not generate physical models in a context designed to elicit them, then instructors should not assume that students would generate said models when given a situation that does not support these models. When instructors construct lab exercises similar to the one in this study, designed to support the generation of models, those instructors will want to find a way to formally assess the generation of these models, as that model generation in this context cannot be assumed. Finally, as suggested in the National Research Council's K-12 Framework for Science Education (2012), instructors should make pedagogical changes to further support the generation of mental models throughout this context. These could involve improvements to assignments that evoke physical movement, the inclusion of real world problems to reinstate experiences, and the formative assessment of students' mental model generation throughout their learning process.

Further analysis of the frequency of occurrence data reveals that in only four of the individual expressions of mental models did the participants utilize the simultaneous expression of the mathematical and physical models. These were also across only three of the eight participants. This low frequency may imply there is difficulty involved in the coordination of these mental models, and suggests the following:

- Claim 4: The ability to coordinate mathematical and physical mental models may be rarer than previously assumed in similar populations to those employed in the current study.

If coordination of mental models within the force vector situation presented in the context of the current work is difficult, and mental models aid in student understanding, it is worthwhile for instructors to develop curricula that aid students in the coordination of models at the intersection of these two topics (vectors and forces). One such way to do so, specifically with regards to force vector situations, is to modify the apparatus utilized in the present work. Perhaps if the participants could physically see the connective medium (strings in the current work) stretch, it would support them in constructing physical mental models. This invokes the use of springs in lieu of strings on the apparatus. Seeing the stretch of the spring could invite participants to imagine the tension better than the strings. Imagining stretching a spring is easier than imagining stretching a string because the spring's stretch is within participants' experience. Further, in the focal part of the learning activity, there was no point at which the students were asked to reflect or reflect upon what they had done. Such a reflection could also aid the students in coming to a more complete conclusion about the force vectors. Metacognition, thinking about one's thinking, can help students by forcing them to look within their reasoning about a situation. Without this pause to reflect, it is plausible that students may not discover errors within their reasoning. This reflection could lead to greater success, and/or new and notable outcomes.

Indeed, the presented data may suggest that coordination of mental models of different topics in general is difficult for students within the population studied here, without this period of reflection. In this case, there is more to explore in other topics within and outside of physics and mathematics. However, this idea does lead to an important observation from the data obtained in the present work, in that the students who were indeed able to coordinate the mathematical and physical models were more likely to have a higher level of understanding of the force vector situation presented to them. It may therefore be posited that:

- Claim 5: *Students who can utilize simultaneous expression of mathematical and physical mental models may be more successful in solving force vector situations.*

The level of understanding data displayed in Table 5 of the results section also presents points of discussion involving how the participants reasoned about the situation. Though the claims brought about by this evidence are not central to the current study, they do provide potential insights into questions that deserve further exploration, and are worthy of initial articulation and discussion.

The level of understanding data are evidence of several findings. If there are distinct mental models as the current study suggests (via the observational and frequency data) then the type of mental model employed by the student should relate to their ability to reason about a particular situation; in this instance, force vectors. To reason effectively about force vectors in physics, one would assume students must utilize both mathematical and physical understanding. If students who only displayed one of the mental models did reason effectively about this topic, one might wonder whether the participants' mathematical mental models involved forces that were unobservable to the researcher. However, participants who solely utilized mathematical mental models were unable to effectively reason about the phenomenon. Additionally, it was also observed, as is true in the case of participant Ashley, that a solely expressed physical mental model is also not enough to come to an accurate conclusion about this force vector situation. Therefore, the assumption that participants who only displayed characteristics of one mental model did not coordinate them is reasonable.

It should be mentioned that a student's mathematical/procedural vector ability did not define his or her success in this situation. Every participant in this study was able to decompose a vector into components. However, this was not enough to inform their thinking about forces in that context. This could be an opportunity to refine the ideas presented in Knight et al. (1995), which proposed that vector knowledge was sufficient for success with force vectors. Perhaps this skill is necessary for the learning of force vectors, but it may not be sufficient.

The idea that vector knowledge isn't sufficient background for success in force vector problems does not mean that mathematical knowledge did not aid these participants. The ability to decompose these vectors supplied each participant with data that showed them the correct answer for the scenario; they knew that that x component of the tension was increasing, and merely needed to explain why this was so. This mathematical ability narrowed the scope of possibility in their answers, which guided their explanation in part because they knew how the explanation would culminate.

The limitations of claim 5 are not insignificant. The coding of the level of understanding is not necessarily reliable. Though it has been used in other research in other fields, the coding itself relies heavily on the researcher's interpretations. Additionally, there is an interdependence of the mental models observed and the level of understanding of each participant. These mental models were determined within the same learning activity as was the level of understanding. This could lead to unreliable and circular observations. Indeed, this may be observed in instances wherein participants gesture concurrently with their vocalization, or there may be a slight pause between gesture and speech. Such instances may suggest dependence or interdependence respectively, between the mental model assignment and level of understanding in the analysis methods. Finally, the sample size of this case is of concern. Eight participants may not be enough to make a larger claim about the understandings of students. However, this claim is only being stated preliminarily as further research is required. Additional studies could address claim 5 within larger similarly derived samples, as well as samples of learners who did not initially struggle, as it would be of interest to examine the model expression of non-novice learners in this subject area.

The implications of the presented claims abound. If the coordination of mathematical and physical mental models does lead to greater understanding of force vector situations, instructors could be trained to identify situations in which these individual models were being utilized and adjust practices in real time to aid in student understanding.

In further studies, participants could be asked to reflect about what they know about vectors as forces at the beginning, what they would need to know to solve the problem in front of them, and then what types of conceptions they utilized to arrive at their final conclusions. Doing so may allow the participants to reach a deeper level of thinking and, hopefully, lead to notable results.

Summary

The most significant claims from the data presented in this study are the presence of two types of mental models (mathematical and physical) and the ability of some students to coordinate them. Observation of the participants performing the learning activity demonstrated that these mental models were not isolated occurrences and most likely would manifest themselves throughout a class of students, although the coordination of these models may be rare among novice learners (such as those who participated in this study). The implications of these claims present evidence that informing instructors of the individualization of learning that occurs amongst populations, and the recognition of particular classes of models that may occur in similar circumstances could be beneficial to student learning. Finally, the level of understanding data suggest further research on coordination of these mental models and successful reasoning therein.

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APPENDIX - CASE STUDIES: SEQUENCE AND EVOLUTION

The following section details the individual experiences of three participants. What follows are narratives as described in the methods sections. These case studies are of interest in that they tell a complete story of how mental models are sequenced and how learning evolves throughout the activity. I have chosen three participants who illustrate the variety and uniqueness of each experience. In each case, I will also be examining how the participant uses physical and mathematical reasoning, physical reasoning in regards to force, mathematical in regards to vector components (non-force). Jake uses all three types of mental models frequently throughout the learning activity and is constantly evolving his learning from the mathematical to the physical. Katherine expresses all of her mental models while calculating the components of the tension and the evolution of her learning is then halted by a misconception leaving her only able to regurgitate the mathematical descriptions detailed in her data. Ashley used only three force exertion mental models in no particular sequence, and is thwarted by her confusion at regular intervals never expanding her thinking beyond the physical. Each of these is described in detail below.

Jake. This case is an interesting one because the participant expressed mental models in each of the three variations of models. Additionally, Jake uses his understanding of the physical and mathematical separately before combining these ideas in his final explanation. As his understanding evolves so does his use of mental models and the abundance with which he expresses them. His mental models are underlined in the narrative below.

Jake begins his explanation by pointing out that the tension is increasing as the angle increases. In this explanation, he discusses the internal angle's importance in the tension on each of the strings. When prompted for why he knows this, Jake then uses his experiences rock climbing, and how the internal angle will affect force to explain the situation, and also exhibits his first mental model:

> *"You wanna have the narrowest central angle so incase...so first off when you're attached to the rope both anchors have equal amount of weight compared to each other ...and also if one of the anchors snaps you don't ...like if this [horizontal string] snapped right now then this [angled string] would come down and it would have a lot of force due to gravity*[1] *which would put even more force on that anchor which would also possibly break due to that and that wouldn't be good."*

Jake expresses many gestures as he talks, but only some of them were classified as mental models in analysis. As he discusses his rock climbing experience he exhibits one of his many force exertion mental models[1]. Jake is then prompted for why this phenomenon occurs and then addresses the situation at the apparatus:

> *"It isn't just holding up*[2] *the, um, if [the strings] were really close to each other they would basically just be holding up the single load*[3]*, but as they're being pulled apart they're also pulling them*[4]*, they're each pulling it on*[5] *like the x-axis, compared to just the y*[6]*. So you have two different um forces. Or not forces, just two different directions that they're pulling."*

During this discussion, he exhibits four more force exertion without vector component mental models[2,3,4,5] as well as one of the two vector component without force exertion mental models[6].

Finally, Jake is prompted for his final explanation and sums up his explanation. He exhibits one more force exertion without vector component mental model[7], his final vector component without force exertion mental model[8], and his one vector component with force exertion mental model[9]. He states:

> *"The string would be pulling it more towards the left, and it's still holding up the same amount of weight[7] in the y direction[8] just adding more force[9] in the x direction."*

He exhibits one of each type of mental model in this last section within a span of about ten seconds.

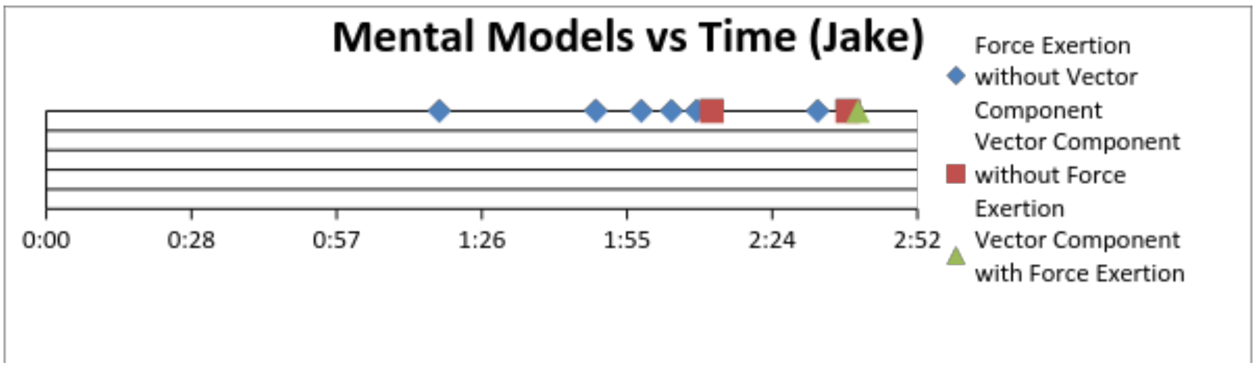

Figure 10 Jake's mental models over time during the focal part of the learning activity. These correspond to the underlined phrases in the text above

Jake's mental models are spread throughout his explanation and occur often. He begins his explanation with a real life experience of which he is familiar (lines 4-10). He then applies this knowledge to the given situation making firm and explicit analogies (lines 14-17). Finally he summarizes all of his thinking in one final and complete explanation (lines 26-27). Jake's narrative illustrates how abundant force mental models can be coordinated with prior knowledge and vector component mental models to ultimately produce a force vector component mental model.

Jake is able to use both physical and mathematical reasoning to come to a conclusion about the situation. In his final conclusion, he exhibits all three types of mental models, using both a physical understanding of the tension and mathematical thinking about the tension in the rope. The vector component with force exertion mental model is the last mental model exhibited, after he was able to easily express the force exertion and vector component mental models alone. It is as if the force vector component concept was difficult or less familiar, leading to the initial expression of the easier other two mental model types. Once these more familiar ideas had been approached, he was able to use that knowledge to construct the final, more difficult, force vector component explanation.

Katherine. Katherine's case is of interest because she only constructs vector component mental models. Her somewhat solid understanding of vector components is not enough to lead her to an explanation of what is happening to the tension in the strings. Her mathematical knowledge at points leads her to question if certain physical situations can occur, and ultimately end in confusion.

During the focal part of the learning activity Katherine first had trouble calculating the x and y components of tension in the angled string. She is initially confused about the similarities between basic trigonometry and calculating x and y components:

"If we're doing it like in like in the pretest where you use like a triangle to find like the x and y components then maybe like this would be just like the hypotenuse part and then like the x and the y component and if we know like this is 30 degrees then and I guess we know that we could say like this is 5.4 for the tension. But I'm not sure if you can do that with like forces if like the same as with measurements. Um, cuz I know you can, you can, ah, with measurements and like a triangle you can have, you could use like the, the sides to find other sides for values but, I'm not sure if you can do that with forces or not. "

All three of Katherine's mental models occur as she is calculating these components. Katherine stumbles on the idea that a tension can be not only described as a vector but also decomposed as such. After a while, she decides to decompose the force as if it were a vector and *see what happens.*

Once Katherine comes to the focal part of the activity and is asked why the tension increases she starts discussing the angle changing. She then stops to discuss what her prediction would have been and how that compares to what she is seeing:

"But I I would think that it wouldn't the tension wouldn't increase like that much. I would think that like the tension would be the same I guess. I don't know.... Cuz the mass is the same. And the mass is the same and like the downward like the gravitational force is the same. So like the mass and the acceleration are the same so I would like the gravitational force is the same. So like the mass and the acceleration are the same so I would I think the force would be the same. But for some reason it's not (giggle) so I don't even know."

Here Katherine states that she would think the tension would remain constant regardless of the angle since the mass remains constant. After this Katherine then returns to her theory that the angle must have some effect on the tension but is unable to construct a coherent response by the end of the focal part of the learning activity. Additionally, she expresses no more mental models in this part of the activity.

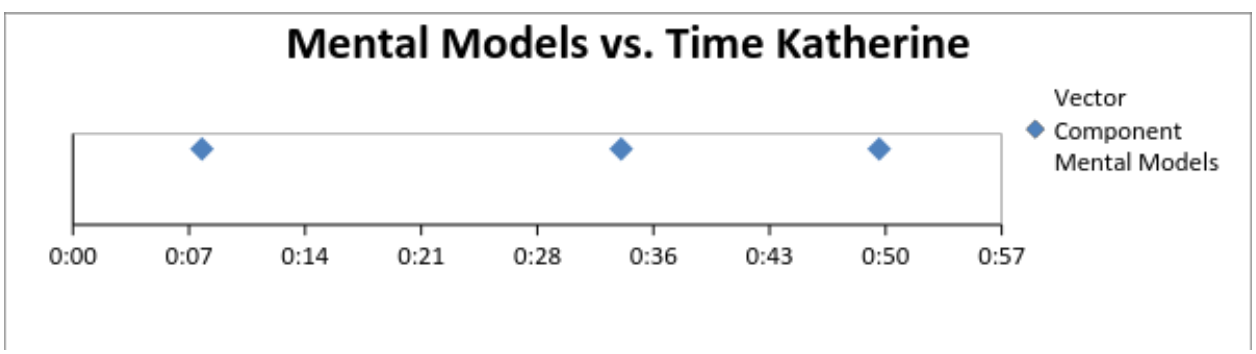

Figure 11 Katherine's mental models over time during the focal part of the learning activity

Katherine's narrative illustrates the sole use of vector component without force mental models. Additionally, all of her mental models occur during the calculation of components of the tension. Katherine's thinking never evolves. She has no resources to make any progress with her understanding, so her evolution is stopped by her misconception and she is never able to reconcile. She begins confused (lines 3-7) goes ahead with a guess, and then is contradicted by the data. However, this contradiction is not enough to inform her knowledge. Katherine can only access her mathematical knowledge, which she is uncertain can be applied to physical situations. She is never able to use physical reasoning to make progress in her thinking.

Ashley. This case is of interest because during the activity, Ashley expresses only force mental models representing the force of gravity on the hanging mass. She seems to come to an accurate physical understanding, but is unable to apply mathematical reasoning to the situation and therefore unable to accurately explain the situation.

Ashley started off with some trouble using the spring scale, but after some aid was able to accurately collect data. She had slight difficulty calculating the vector components but after checking her answer was able to correct her mistake and accurately calculate the components without aid. In the explanation section of the learning activity, Ashley begins by discussing the idea that the sine of a larger angle will always give you a larger number, followed by the discussion that the component of the tension will increase with a larger angle. When asked why she thinks this is, Ashley states:

> *"the resultant tension in order compensate for the normal force of gravity*."

This is when she generates two of her force mental models. She then stumbles and says she is getting confused. She then draws an accurate free body diagram, without the x and y components of force, for both situations on her paper. She then states:

> *"[The force] greater with the angle because you still want that fragment to come down."*

This is when she expresses her final force mental model. When asked once again why this is she begins talking about energy (kinetic vs. potential). Ashley then states:

> *"I know how to work things out very well, but I don't know like the theoretical stuff behind it."*

She never states why the tension is increasing.

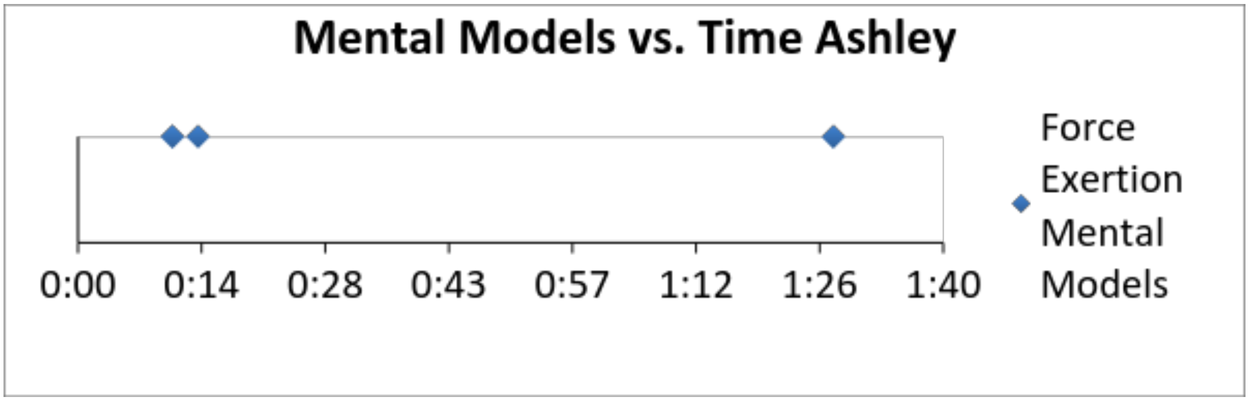

Figure 12 Ashley's mental models over time during the focal part of the learning activity

Ashley's sequencing is interesting because she uses few mental models, which are all force exertion without vector component. She exhibits two quickly in a row at the very beginning of her explanation and then one more a bit later on. These three mental models are in conjunction with her discussion of gravity (lines 13 and 18). Additionally, her evolution of thinking is halted, in contrast to Katherine, by her inability to connect her physical thinking with a mathematical explanation. She begins with the correct understanding that the weight is compensating for gravity, but then becomes confused the explanation for these phenomena beyond the simple idea that it must be compensating even more at larger angles (line 13). She then attempts to find a new idea, but can only come back to her compensation idea. Ashley's thinking never evolves beyond this point. As she mentions (lines 22-23), there is some aspect of her thinking that is missing. This "theoretical stuff" she mentions she doesn't know how to do is actually a mathematical understanding of the physical situation.

Summary. Each of these cases tells a different story about the experience this learning activity presented. The sequencing and evolution was unique for each participant. Jake used a plethora of mental models throughout his explanation, while Katherine and Ashley used only 3 each packed toward the beginning of their explanation. This could have contributed to Katherine and Ashley's lacking explanation about the situation, Katherine with a lack of physical reasoning and Ashley with a lack of mathematical thinking. Jake's explanation evolved nicely and used the overlapping of physical and mathematical reasoning.

BIOGRAPHY OF THE AUTHOR

Savannah Elizabeth Lodge-Scharff was born in Portsmouth, NH on April 25, 1989. She was raised in southern Maine and graduated from Marshwood High School in South Berwick, ME in 2007. After high school she went on to Colby College where she majored in Physics with minors in Mathematics and Administrative Science. While at Colby Savannah also participated in the Broadway Musical Revue, and was a member of Colby's oldest all-female acapella group, the Colbyettes. Upon graduation in 2011, Savannah entered the MST program at the University of Maine. In 2013 she accepted an offer to work in education outreach for the Museum of Science in Boston, MA. After a wonderful year there, she moved on to teach high school physics for three years in Massachusetts. She currently lives in Chelsea, MA with her partner Alexander Demers (a UMaine Alumnus) and their dog Pythagoruff (Pi for short). In the fall of this year, Savannah will start a new position teaching physics at Madison Park High School in Roxbury, MA (part of the Boston Public Schools). She is a candidate for the Master of Science in Teaching degree in physics from The University of Maine in August 2017.