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Sperner's Lemma, The Brouwer Fixed Point Theorem, the Kakutani Fixed Point Theorem, and Their Applications in Social Sciences

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SPERNER'S LEMMA, THE BROUWER FIXED POINT THEOREM, THE KAKUTANI FIXED POINT THEOREM, AND THEIR APPLICATIONS IN SOCIAL SCIENCES

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Can a cake be divided amongst people in such a manner that each individual is content with their share? In a game, is there a combination of strategies where no player is motivated to change their approach? Is there a price where the demand for goods is entirely met by the supply in the economy and there is no tendency for anything to change? In this paper, we will prove the existence of envy-free cake divisions, equilibrium game strategies and equilibrium prices in the economy, as well as discuss what brings them together under one heading.

This paper examines three important results in mathematics: Sperner's lemma, the Brouwer fixed point theorem and the Kakutani fixed point theorem, as well as the interconnection between these theorems. Fixed point theorems are central results of topology that discuss existence of points in the domain of a continuous function that are mapped under the function to itself or to a set containing the point. The Kakutani fixed point theorem can be thought of as a generalization of the Brouwer fixed point theorem. Sperner's lemma, on the other hand, is often

described as a combinatorial analog of the Brouwer fixed point theorem, if the assumptions of the lemma are developed as a function. In this thesis, we first introduce Sperner's lemma and it serves as a building block for the proof of the fixed point theorem which in turn is used to prove the Kakutani fixed point theorem that is at the top of the pyramid.

This paper highlights the interdependence of the results and how they all are applicable to prove the existence of equilibria in fair division problems, game theory and exchange economies. Equilibrium means a state of rest, a point where opposing forces balance. Sperner's lemma is applied to the cake cutting dilemma to find a division where no individual vies for another person's share, the Brouwer fixed point theorem is used to prove the existence of an equilibrium game strategy where no player is motivated to change their approach, and the Kakutani fixed point theorem proves that there exists a price where the demand for goods is completely met by the supply and there is no tendency for prices to change within the market.

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CHAPTER 1

INTRODUCTION AND BACKGROUND

In this paper, we introduce and prove three important results in mathematics, namely Sperner's lemma, the Brouwer fixed point theorem and the Kakutani fixed point theorem. In addition, we present applications of these results in the social sciences.

These three results are interdependent and the topics are developed in a manner that the combinatorial result, Sperner's lemma, is introduced first and it serves as the main tool in the proof of the Brouwer fixed point theorem, which in turn is the basis for the proof of the Kakutani fixed point theorem. This interconnection runs deeper than merely an aid in proving the theorems. Sperner's lemma can be thought of as a combinatorial analog to the Brouwer fixed point theorem, and the Kakutani fixed point theorem can be seen as an extension of the Brouwer fixed point theorem from point-valued functions to set-valued functions.

In a way, all three results allow us to prove the existence of a fixed point for continuous functions, that is, a point that maps to itself under that function. In the case of Sperner's lemma, we do not work with an explicit function. Instead, Sperner defined a labeling on simplices and Sperner's lemma accounts for the existence of at least one "point" in the set (a sub-simplex in the simplex) with the same labeling as the whole set (simplex). We discuss the concepts of simplex and sub-simplex in the next section.

Another reason we chose to present these three results is that they all can be applied to model human behavior and choices. In this paper, we look at their application to the fields of economics and game theory. Economics is the study of problems of choice, that is, how to allocate scarce goods and resources so as to maximize welfare, growth or other objectives. Game theory, on the other hand, though often considered a branch of economics, focuses more on strategies and outcomes. However, both economics and game theory study human behavior, and one of the main ideas studied in these fields is that of equilibrium, a state of rest where counteracting forces balance. We find that fixed point theorems have interesting applications in determining the equilibria and distributions in social sciences.

We use Sperner's lemma to prove the existence of an envy-free division in a distribution of goods. Fair-division problems have long been discussed in economics, and using this lemma, we can guarantee that a division exists where each person is content with her/his share and no one has a tendency to envy another person for their share. This idea can be equated to that of a Nash equilibrium in game theory, where each participant is content with the strategy they choose and has no motivation to change it, given the strategies of the other participants of the game. We use the Brouwer fixed point theorem to prove the existence of such an equilibrium in non-cooperative games and look at example games. Finally, we use the Kakutani fixed point theorem to prove the existence of an equilibrium in an economic system where m consumers are endowed with a fixed supply of n goods that they can exchange amongst themselves. It is not obvious that equilibrium exists in such an exchange economy; however, the theorem allows us to prove that values can be structured in a manner that each player's demand can be met, given the supply.

In the next chapter, we state and prove Sperner's lemma in one, two and n dimensions and look at its application in a fair-division problem, addressing the envy-free division of a cake. In the following chapter, we state and prove the Brouwer fixed point theorem in one, two and n dimensions using Sperner's lemma for the latter two cases. Furthermore, we apply the Brouwer fixed point theorem to prove the existence of a Nash equilibrium in game theory. In the last chapter, we state and prove the Kakutani fixed point theorem in one and n dimensions, followed by a discussion of its application to prove that an equilibrium exists in an exchange economy.

There are many, essentially equivalent for each, versions of Sperner's lemma, the Brouwer fixed point theorem, and the Kakutani fixed point theorem in the mathematics literature. The versions we present are adapted for the setting as we develop it here.

Before we delve into the theorems and their applications, we need to familiarize ourselves with some definitions and theorems that will form the background for the work presented in subsequent chapters. We assume that the reader is familiar with introductory topology, analysis and linear algebra. In this section, we introduce and state the basic concepts and results that we use in the main part of this paper.

1.1 Background

Much of the work we do in this paper involves simplices in Euclidean space. It will be helpful to look at some definitions that will be of use to us and that will make the concepts clearer.

Definition. *An* n ***- simplex*** S *is the convex hull in* $\mathbb{R}^m$, *with* $m \geq n+1$, *of* $n+1$ *geometrically independent points* $v_0, v_1, ..., v_n$. *We call* $v_0, v_1, ..., v_n$ *the* ***vertices*** *of the simplex* S.

Note that the points $v_0, v_1, ..., v_n \in \mathbb{R}^m$ are geometrically independent if the vectors $\overrightarrow{v_0v_1}$, $\overrightarrow{v_0v_2}$, ..., $\overrightarrow{v_0v_n}$ are linearly independent. So a 1 - simplex is a line segment in $\mathbb{R}^2$, a 2 - simplex is a triangle in $\mathbb{R}^3$, a 3 - simplex is a tetrahedron in $\mathbb{R}^4$, and so on.

Definition. *The* ***standard*** n ***- simplex*** σ_n *in* $\mathbb{R}^{n+1}$ *is the convex hull of the points* $(1,0,...,0)$, $(0,1,...,0)$, ..., $(0,0,...,1)$. *The points* $(1,0,...,0)$, $(0,1,...,0)$, ..., $(0,0,...,1)$ *are the vertices of* σ_n.

Note that if $x = (x_0, x_1, ..., x_n) \in \sigma_n$, then $0 \leq x_i \leq 1$ for all i and $\sum_{i=0}^{n} x_i = 1$. More generally, if S is an n - simplex with vertices $v_0, v_1, ..., v_n$ and $x \in S$, then we can express x uniquely as

$$x = \alpha_0 v_0 + \alpha_1 v_1 + ... + \alpha_n v_n$$

for $\alpha_i \in \mathbb{R}$ such that $\alpha_i \geq 0$ for all $i \in \{0,1,...,n\}$ and $\sum_{i=0}^{n} \alpha_i = 1$.

Definition. *Let* S *be an* n *- simplex with vertices* $v_0, v_1, ..., v_n$. *Let* f *be a function defined on* $\{v_0, v_1, ..., v_n\}$ *such that* $f(v_i) = w_i \in \mathbb{R}^m$. *Then the* ***linear extension of*** f *is the function* $f : S \to \mathbb{R}^m$ *defined so that if* $x = \sum_{i=0}^{n} \alpha_i v_i$, *then* $f(x) = \sum_{i=0}^{n} \alpha_i w_i$.

Throughout the paper, we will be working with subdivisions of simplices. Before we define what is meant by subdivision of an n - simplex, let us understand the terms *face* and *facet* of an n - simplex.

Definition. *An* m ***- face*** *of an* n *- simplex* S *is the* m *- simplex formed by* $m + 1$ *vertices out of the* $n + 1$ *vertices of* S.

Definition. *An* $(n-1)$ *- face of an* n *- simplex is called a* ***facet****. In other words, a facet is an* $(n-1)$ *- simplex formed by* n *vertices out of the* $n + 1$ *vertices of an* n *- simplex.*

By giving a specific name to the $(n-1)$ - faces of an n - simplex, we are distinguishing them from the other faces in our discussion. We can see that an n - simplex has $n + 1$ distinct facets. For instance, the two facets of a 1 - simplex are the two end points of the line segment. The three facets of a 2 - simplex are the three sides of the triangle. The four facets of a 3 - simplex are the four triangular faces of a tetrahedron and so on.

Definition. *A* ***subdivision*** *of an* n *- simplex* S *(Figure 1.1) is a collection of subsets of* S*, each an* n *- simplex, called a* ***sub-simplex*** *of* S*, such that:*

1. *The union of all the sub-simplices is* S.
2. *Any two sub-simplices either do not intersect or have an intersection that is a common face.*

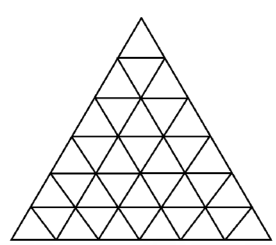

Figure 1.1. An example of a subdivision of a 2 - simplex S

A subdivision of a $2-$ simplex is called a **triangulation**. Note that in Figure 1.1, if we subdivide each small triangle in a manner similar to how S is subdivided, we obtain another subdivision of S with triangles whose diameter is $\frac{1}{6}$ of those in the initial subdivision. In this manner, we can construct sequences of subdivisions whose sub-simplex diameters go to zero in the limit. Such sequences will be of benefit to us in limiting arguments we make in the paper.

Another important notion we work with is continuity. In topological spaces, we say that a function f is continuous if pre-images of open sets under f are open. For functions mapping between Euclidean spaces, there are various equivalent definitions, including the traditional $\epsilon - \delta$ definition. We use the definition of continuity of functions that is as follows.

Definition. *If $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$, then f is* ***continuous*** *if for every sequence (x_j) in D converging to $x \in D$, the sequence $(f(x_j))$ converges to $f(x)$.*

So far, we have been talking about point-valued functions. However, both in the application of Sperner's lemma and for the Kakutani fixed point theorem, we work with set-valued functions. Let us first understand what is meant by a set-valued function and then define continuity for such functions.

Definition. *Given sets X and Y, a* ***set-valued function*** *is a function $f : X \to \mathbb{P}(Y)$ where $\mathbb{P}(Y)$ is the power set of Y. Thus, for each $x \in X$, its image $f(x)$ is a subset of Y.*

Continuity for set-valued functions is a natural extension of the definition of continuity for point-valued functions.

Definition. *Assume* $X \subset \mathbb{R}^m$ *and* $Y \subset \mathbb{R}^n$. *A function* $f : X \to \mathbb{P}(Y)$ *is* ***continuous*** *if for every pair of convergent sequences* $x_n \to x$ *in* X *and* $y_n \to y$ *in* Y *such that* $y_n \in f(x_n)$ *for all* n, *it follows that* $y \in f(x)$.

1.2 Important Results

The following results are straightforward convergence results that will be of use to us. We also state and prove a lemma that will be useful to us in the proof of the Brouwer Fixed Point Theorem.

Theorem 1.1. *If D* $\subset \mathbb{R}^n$ *is closed and bounded, and* (x_n) *is a sequence in D, then* (x_n) *has a convergent subsequence.*

Theorem 1.2. *Let* (x_n) *and* (y_n) *be sequences in* $\mathbb{R}^n$. *If* $(x_n) \to x$, *and* $\mid x_n - y_n \mid \to 0$, *then* $(y_n) \to x$.

Theorem 1.3. *If* $f : D \to \mathbb{R}$ *is continuous,* $(x_j) \to x$ *in* D, $(a_j) \to a$ *in* $\mathbb{R}$ *and* $f(x_j) \leq a_j$ *for all* j, *then* $f(x) \leq a$.

Lemma 1.1. *Let* σ_n *be the standard* n *- simplex in* $\mathbb{R}^{n+1}$ *and let* x, y *be points in* σ_n. *If* $x_i \leq y_i$ *for all* $i \in \{0, 1, ..., n\}$, *then* $x = y$.

Proof. Let σ_n be the standard n - simplex in $\mathbb{R}^{n+1}$. Let $x = (x_0, x_1, ..., x_n)$ and $y = (y_0, y_1, ..., y_n)$ be points in σ_n such that $x_i \leq y_i$ for all i. Suppose $x \neq y$. Then there exists at least one $j \in \{0, 1, ..., n\}$ such that $x_j \neq y_j$. This means $x_j < y_j$.

However since $x,\ y\ \in\ \sigma_n$,

$$\sum_{i=0}^{n} x_i \quad = \quad \sum_{i=0}^{n} y_i \quad = \quad 1.$$

This means that there exists $k\ \in\ \{0, 1, ..., n\}$ such that $x_k\ >\ y_k$. This is a contradiction since $x_i\ \leq\ y_i$ for all i. Thus, there is no $j\ \in\ \{0, 1, ..., n\}$ such that $x_j\ \neq\ y_j$. Therefore, $x\ =\ y$. $\square$

CHAPTER 2

SPERNER'S LEMMA AND FAIR DIVISION

Emanuel Sperner (1905 - 1980) was a German mathematician who received much recognition for his contribution to topology, analytic geometry, algebra and matrix theory. He is best known for two results, namely Sperner's theorem and Sperner's lemma, that he developed at an early age of 22 years. He was awarded a doctorate with distinction for his thesis titled *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes* in 1928. Sperner's lemma was written as a part of his thesis, and later he used it to give a simple proof of the Lebesgue covering theorem and the invariance of dimension and domain theorems. This lemma gained popularity among topologists, and *B. Knaster*, *C Kuratowski*, and *S Mazurkiewicz* used it to prove the Brouwer fixed point theorem [1]. Later applications of the Sperner's lemma came up in fair division problems, including a recent one by Francis Edward Su [2] who used it to calculate envy-free rent division.

2.1 Sperner's Lemma

In this chapter, we focus on Sperner's lemma and its applications to fair-division problems. In the next chapter, we use it to prove theorems that help us understand the topological connections to economics and game theory. The idea outlined by the lemma is very intuitive, especially in dimensions one and two. In higher dimensions, it is more difficult to visualize, however the idea is exactly the same, as we shall see. We discuss the one-dimensional and two-dimensional cases next and then separately provide proofs of the one-dimensional, two-dimensional and n - dimensional cases.

The one-dimensional case is as follows. If we write down 0's and 1's in a single line with at least one of each, it is obvious that at least once a 0 is written next to a 1. Moreover, if we start and end with the same number, clearly the number of times a 0 comes next to a 1 is even since we switch between 0 and 1 an even number of times to make sure we end with the number that we started with. Similarly, if we start with a 0 and end with a 1 (or vice-versa), the number of times a 0 comes next to a 1 is odd.

We can extend this idea to two dimensions using a main triangle T that has vertices distinctly labeled $0,$ $1,$ and $2.$ Suppose we have a triangulation of T and we were to label the sub-triangle vertices with the numbers $0,$ $1,$ or 2 following the single rule that sub-triangle vertices on a side of the main triangle T are not labeled with the number on the main triangle vertex opposite to that side. Then we find that no matter how we label the sub-triangles, we end up with at least one sub-triangle with all its vertices distinctly labeled $0,$ $1,$ and $2.$ In fact, we always end up with an odd number of such triangles. Two examples of this idea can be seen in the Figure 2.1, where sub-triangles labeled with $0,$ $1,$ and 2 are highlighted.

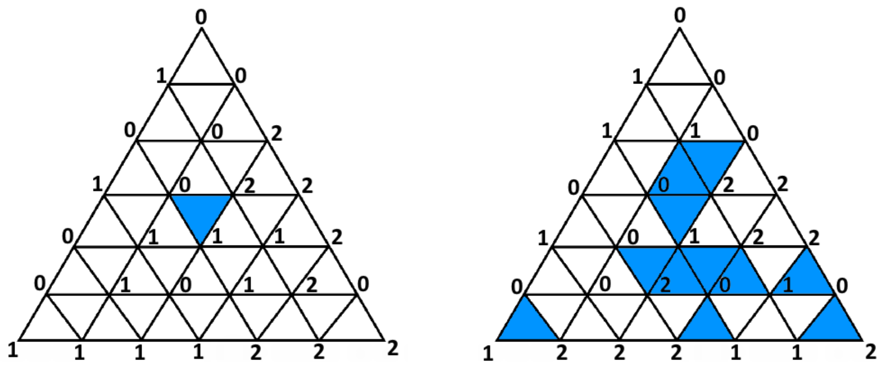

Figure 2.1. A look at the two-dimensional Sperner's Lemma

It might be interesting to take it up as an exercise to try to find some labeling that does not give an odd number of sub-triangles with vertices distinctly labeled $0,\ 1,$ and $2.$ You will find that it is not possible to do so. We prove this idea in this chapter as the two-dimensional Sperner's lemma.

Before we discuss the idea in higher dimensions, let us state and prove the one-dimensional and two-dimensional versions of Sperner's lemma.

2.1.1 One-Dimensional Sperner's Lemma

Here we consider the one-dimensional Sperner's lemma. The lemma asserts that in an n - tuple of 0s and 1s, the number of times a 0 comes next to a 1 is even if we start and end with the same number and odd if we start and end at different numbers. Formally, we have:

Lemma 2.1. ***(The One-Dimensional Sperner's Lemma)***

For $n \geq 2,$ in an n - tuple $(a_1, a_2, ..., a_n)$ of 0s and 1s, the size of the set $\{\ j \mid a_{j-1} \neq a_j\ \}$ is even if $a_1 = a_n$ and odd if $a_1 \neq a_n$.

Proof. We prove the lemma using induction. Let J_n denote the set $\{ j \mid a_{j-1} \neq a_j \}$ for an n - tuple $(a_1, a_2, ..., a_n)$ of 0s and 1s, and let $\mid J_n \mid$ represent the size of the set.

Base Case $(n = 2)$: When $a_1 = a_2$, the possible 2 - tuples are $(0, 0)$ and $(1, 1)$. In both these cases, $\mid J_2 \mid$ is 0 which is even. When $a_1 \neq a_2$, the possible 2 - tuples are $(0, 1)$ and $(1, 0)$. For both cases, $\mid J_2 \mid = 1$, which is an odd number. Hence, the lemma holds for $n = 2$.

Induction hypothesis: In a k - tuple $(a_1, a_2, ..., a_k)$ of 0s and 1s,

$$\mid J_k \mid = \begin{cases} even & a_1 = a_k, \\ odd & a_1 \neq a_k. \end{cases}$$

Now consider the $(k+1)$ - tuple $(a_1, ...a_k, a_{k+1})$.

Case I: $a_1 = a_{k+1}$

1. If $a_1 = a_k$, then $\mid J_k \mid$ is even (by the inductive hypothesis). Since $a_1 = a_k$ and $a_1 = a_{k+1}$, this means $a_k = a_{k+1}$. Thus, $k+1 \notin J_{k+1}$. Then $J_{k+1} = J_k$ and hence $\mid J_{k+1} \mid$ is even.
2. If $a_1 \neq a_k$, then J_k is odd (by the inductive hypothesis). Since $a_1 = a_{k+1}$ and $a_1 \neq a_k$, this means $a_k \neq a_{k+1}$. Thus, $k+1 \in J_{k+1}$. Then $J_{k+1} = J_k \cup \{k+1\}$ and hence $\mid J_{k+1} \mid = \mid J_{k+1} \mid +1$. Therefore, $\mid J_{k+1} \mid$ is even.

Case II: $a_1 \neq a_{k+1}$

1. If $a_1 = a_k$, then $\mid J_k \mid$ is even (by the inductive hypothesis). Since $a_1 \neq a_{k+1}$ and $a_1 = a_k$, we get $a_k \neq a_{k+1}$. Therefore, $k+1 \in J_{k+1}$. So $J_{k+1} = J_k \cup \{k+1\}$ and $\mid J_{k+1} \mid = \mid J_{k+1} \mid + 1$. Thus, $\mid J_{k+1} \mid$ is odd.

2. If $a_1 \neq a_k$, then J_k is odd (by the inductive hypothesis). Since $a_1 \neq a_{k+1}$ and $a_1 \neq a_k,$ it follows that $a_k = a_{k+1}.$ Thus $k+1 \notin J_{k+1}.$ This implies $J_{k+1} = J_k$ and hence $\mid J_{k+1} \mid$ is odd.

Hence we can conclude that

$$\mid J_{k+1} \mid = \begin{cases} even & a_1 = a_{k+1} \\ odd & a_1 \neq a_{k+1} \end{cases}$$

Thus, if the lemma holds for $n = k,$ then it holds true for $n = k + 1.$ By the process of induction, we can conclude for all n that in an n-tuple $(a_1, a_2, ...a_n)$ of 0s and 1s, the size of the set $\{ j \mid a_j \neq a_{j+1} \}$ is even if $a_1 = a_n$ and odd if $a_1 \neq a_n.$ With this, the proof of one-dimensional Sperner's lemma is complete. □

2.1.2 Two-Dimensional Sperner's Lemma

The one-dimensional Sperner's lemma is crucial to understanding and proving Sperner's lemma in two dimensions. However, first let us outline what we mean by some of the terms.

Assume that we have a triangulation of T and that the vertices of the sub-triangles are labeled with $0,$ $1,$ or $2.$ Then an edge of T or of a sub-triangle of T is called an (a_0, a_1) edge if its endpoints are labeled with a_0 and $a_1.$ Moreover, a triangle in T is called an (a_0, a_1, a_2) triangle if its vertices are labeled $a_0,$ a_1 and $a_2.$

For the purpose of the discussion, the **regions** associated with the triangulation of T are the interiors of the sub-triangles and the exterior of $T.$

Definition. *A triangulation of a triangle* T *with all of its sub-triangle vertices labeled with* $0,$ $1,$ *or* 2 *is said to have a* ***Sperner labeling*** *if the sub-triangle vertices are labeled according to the following rules (refer to Figure 2.2):*

1. T *is a* $(0,1,2)$ *triangle,*
2. *Sub-triangle vertices on a side of* T *are not labeled with the same number as the* T *vertex opposite to that side.*

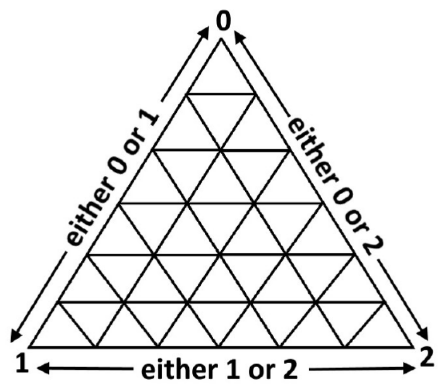

Figure 2.2. A Sperner Labeling

If a triangulation of T has a Sperner labeling, then by property (2) of the labeling, all the sub-triangle vertices on the side of T that is opposite to the T vertex labeled 2 are labeled with either 0 or 1. By the one-dimensional Sperner's lemma, that side has an odd number of $(0,1)$ sub-triangle edges. Also, by property (2), there are no $(0,1)$ sub-triangle edges on the other two sides of T. Thus, *there is an odd number of* $(0,1)$ *sub-triangle edges on the boundary of* T*.*

There is another concept that we need to define before we can state and prove Sperner's lemma, namely, p-paths.

Definition. *A **permissible path** or **p-path** is a path going from one region to another region such that (refer to Figure 2.3):*

1. *It begins in a* $(0,1,2)$ *sub-triangle or in the exterior of* T*,*
2. *It ends in a* $(0,1,2)$ *sub-triangle or in the exterior of* T*,*
3. *It passes from one region to an adjacent region only through a* $(0,1)$ *sub-triangle edge,*
4. *It does not pass through a* $(0,1)$ *sub-triangle edge more than once.*

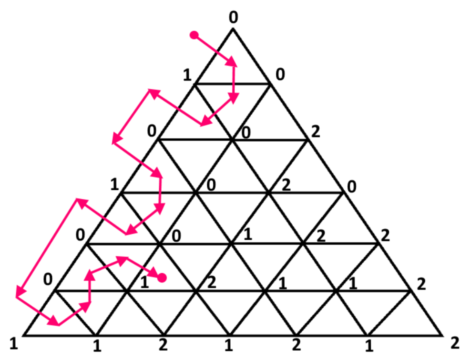

Figure 2.3. An example of a p-path

A collection of p-paths in T is called **complete** if for every $(0,1)$ sub-triangle edge, there is a unique path in the collection that crosses it. It is not difficult to show that p-paths and complete collections of them exist.

Assume we have a complete collection of p-paths. Since the collection of p-paths is complete, every $(0,1)$ edge is crossed by some p-path. And therefore, every $(0,1,2)$ sub-triangle is visited by a p-path. This enables us then to "count" all $(0,1,2)$ sub-triangles in T. We can now use the concepts we have recently outlined to state and prove Sperner's lemma in two dimensions.

Lemma 2.2. ***(The Two-Dimensional Sperner's Lemma)***

If a triangulation of a triangle T has a Sperner labeling, then there exists at least one $(0,1,2)$ sub-triangle in T. Moreover, there is an odd number of such sub-triangles.

Proof. Given a triangulation of a triangle T labeled with a Sperner labeling, we show the existence of an odd number of $(0,1,2)$ sub-triangles by working with p-paths through T. Assume we have a complete collection of p-paths.

For any sub-triangle in T, the number of $(0,1)$ edges can be 0, 1 or 2 as we can see in Figure 2.4 below. Here, $(0,1)$ edges of a sub-triangle are highlighted.

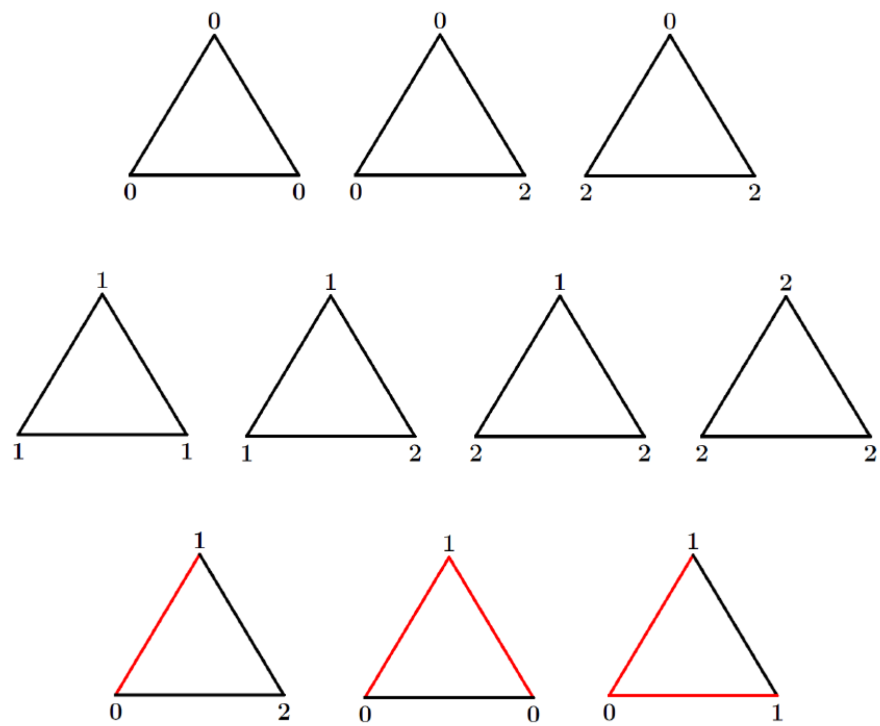

Figure 2.4. Number of $(0,1)$ edges in a sub-triangle is 0, 1 or 2

Note that if a sub-triangle has no $(0,1)$ edge, then no p-path meets it. If a sub-triangle has exactly one $(0,1)$ edge, then a p-path that meets it must begin or end in that sub-triangle, and that sub-triangle is a $(0,1,2)$ triangle.

Furthermore if a sub-triangle has two $(0,1)$ edges, then every p-path that enters it also leaves it.

There are three types of p-paths in the collection (Figure 2.5):

1. **Type I:** p-paths that begin and end inside T. This means the path begins and ends inside two distinct sub-triangles with exactly one $(0,1)$ edge in each. Thus, each p-path of this type accounts for two $(0,1,2)$ sub-triangles. Also, note that such a p-path crosses an even number (possibly none) of $(0,1)$ edges on the boundary of T.

2. **Type II:** p-paths that begin and end outside T. Each p-path of this type does not account for any $(0,1,2)$ sub-triangles and crosses an even number of $(0,1)$ edges on the boundary of T.

3. **Type III:** p-paths that begin inside T and end outside T or vice-versa. Each p-path of this type accounts for one $(0,1,2)$ sub-triangle and crosses an odd number of $(0,1)$ edges on the boundary of T.

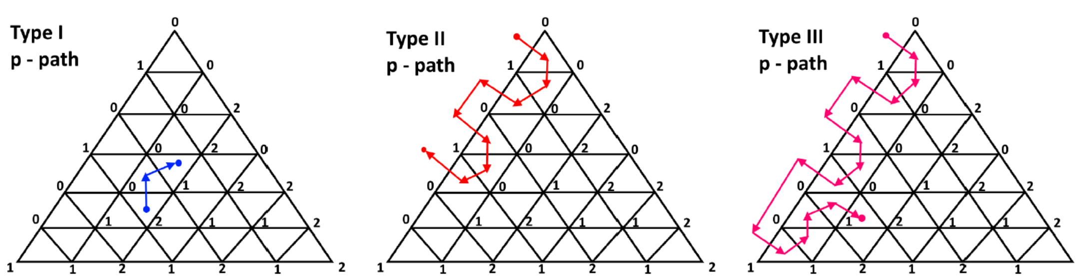

Figure 2.5. Types of p-paths

Let n_i denote the number of paths of Type i. Let the total number of $(0,1)$ sub-triangle edges on the boundary of T be denoted by n_E and the total number of $(0,1,2)$ sub-triangles in T be denoted by n_T. From the above discussion we know that Type I and Type II p-paths together account for an even number of $(0,1)$ sub-triangle edges on the boundary of T and each Type III p-path accounts for an odd number of $(0,1)$ sub-triangle edges on the boundary of T.

Then for some $\alpha,\ \beta,\ \gamma\ \in\ \mathbb{Z}_{\geq\,0}$:

$$n_E \ = \ 2\alpha n_1 \ + \ 2\beta n_2 \ + \ (2\gamma+1)n_3. \tag{2.1}$$

Moreover, each Type I p-path accounts for two $(0,1,2)$ sub-triangles and each Type III p-path accounts for one $(0,1,2)$ sub-triangle. Type II p-paths account for no $(0,1,2)$ sub-triangles. Then we get that:

$$n_T \ = \ 2n_1 \ + \ n_3. \tag{2.2}$$

As we already saw, there is an odd number of $(0,1)$ edges on the boundary of T; i.e. n_E is odd. Then by Equation 2.1, n_3 is odd. It follows from Equation 2.2 that n_T is odd. In other words, there is an odd number of $(0,1,2)$ sub-triangles in T.

The completeness of the collection of p-paths in T ensures that every $(0,1,2)$ sub-triangle is included in the count. Thus there is an odd number of $(0,1,2)$ sub-triangles in T. This guarantees there is at least one $(0,1,2)$ sub-triangle in T and the proof of the two-dimensional Sperner's lemma is complete. □

Let us now look at Sperner's lemma in dimension n.

2.1.3 n-Dimensional Sperner's Lemma

In the previous sections, we showed that Sperner's lemma holds for the one-dimensional and two-dimensional cases. The proof of Sperner's lemma in dimension n is very similar to the proof of the two-dimensional Sperner's lemma. However, before we state the lemma in n - dimensions, we need some vocabulary to visualize and understand it better. Building upon the definitions in the introductory chapter on simplices, faces, facets and subdivisions, here we introduce some more terminology.

Assume we have a subdivision of an n - simplex S and that the vertices of sub-simplices of S are labeled with numbers from $\{0,\ 1,\ ...,\ n\}$. A facet in S is called an $(a_0, a_1, ..., a_{n-1})$ **facet** if its vertices are labeled $a_0,\ a_1,\ ...,\ a_{n-1}$. Also an n - simplex in S is called an $(a_0, a_1, ..., a_n)$ **simplex** if its vertices are labeled $a_0,\ a_1,\ ...,\ a_n$.

As before, the **regions** associated with the subdivision of S are the interiors of the sub-simplices and the exterior of S. Note that a subdivision that subdivides a simplex also subdivides all the facets of the simplex. Then a **sub-facet** refers to a facet of a sub-simplex, and the collection of all the sub-facets that lie on a facet F of S gives a subdivision of F.

Definition. *Given a subdivision of an n - simplex S with sub-simplex vertices in S labeled with any number from the set $\{0, 1, ..., n\}$, the subdivision is said to have a* ***Sperner labeling*** *if the sub-simplices are labeled according to the following rules:*

1. *S is a $(0, 1, ..., n)$ simplex,*
2. *The vertices of sub-simplices on a facet of S do not have the same label as the vertex opposite the facet.*

Assume we have a subdivision of S with a Sperner labeling. Note that the $(0, 1, ..., n-1)$ facet of S is opposite the vertex of S labeled n and by (2), all the sub-facet vertices that lie on the $(0, 1, ..., n-1)$ facet are not labeled n. Also, by (2), we have that no other facet of S has $(0, 1, ..., n-1)$ sub-facets. Thus, on the boundary of S, $(0, 1, ..., n-1)$ sub-facets lie only on the $(0, 1, ..., n-1)$ facet of S.

As in the two-dimensional case, our proof of the n - dimensional Sperner's lemma uses p-paths, defined as follows:

Definition. *A **p-path** in an n - simplex is defined as a path that:*

1. *begins in a $(0, 1, ..., n)$ sub-simplex or outside S,*
2. *ends in a $(0, 1, ..., n)$ sub-simplex or outside S,*
3. *crosses from one region to an adjacent region only through a $(0, 1, ..., n-1)$ sub-facet,*
4. *crosses each $(0, 1, ..., n-1)$ sub-facet exactly once.*

Just like in the case of a two-dimensional simplex, a collection of p-paths is **complete** if every $(0, 1, ..., n-1)$ sub-facet is crossed by a unique path in the collection. It is not difficult to see that complete collections of p-path exist. As in the two-dimensional case, p-paths allow us to count the total number of $(0, 1, ..., n)$ sub-simplices in S.

We are now ready to state and prove the n - dimensional Sperner's lemma.

Lemma 2.3. ***(Sperner's Lemma)***

For each $n = 1, 2, ...,$ *there exists an odd number of* $(0, 1, ..., n)$ *sub-simplices in a subdivision of an* n *- simplex* S *that has a Sperner labeling.*

Proof. We will prove by induction on n that given a subdivision of an n - simplex S and a Sperner labeling of the subdivision, there exists an odd number of $(0, 1, ..., n)$ sub-simplices in S. The base case $n = 1$ follows from the one-dimensional Sperner's lemma that we proved before (see Lemma 2.1). The induction hypothesis is that the lemma holds for $n - 1,$ that is, for subdivided $(n - 1)$ - simplices with a Sperner labeling. With that assumption, we will prove the result for n - simplices. So assume we have a subdivision of an n - simplex S with a Sperner labeling.

By induction, the $(0, 1, ..., n - 1)$ facet of S has an odd number of $(0, 1, ..., n - 1)$ sub-facets. Since on the whole boundary of $S,$ $(0, 1, ..., n - 1)$ sub-facets lie only on the $(0, 1, ..., n - 1)$ facet of $S,$ the total number of $(0, 1, ..., n - 1)$ sub-facets on the boundary of S is odd.

Assume now that we have a complete collection of p-paths for the subdivision of $S.$ A sub-simplex inside S may have no $(0, 1, ..., n - 1)$ sub-facets, in which case no p-path meets it. If a sub-simplex has at least one $(0, 1, ..., n - 1)$ facet, then there are two possibilities for the label on the remaining vertex:

1. If the remaining sub-simplex vertex opposite the $(0, 1, ..., n - 1)$ sub-facet is labeled $n,$ the sub-simplex is a $(0, 1, ..., n)$ simplex and it has exactly one $(0, 1, ..., n - 1)$ sub-facet. Then, a p-path either begins or ends in this sub-simplex.

2. If the remaining sub-simplex vertex opposite the $(0, 1, ..., n-1)$ sub-facet is labeled with any number other than $n,$ that is any number from $\{0,\ 1,\ ...,\ n-1\},$ then it has exactly two $(0, 1, ..., n-1)$ sub-facets and it is not a $(0, 1, ..., n)$ sub-simplex. Furthermore, a p-path enters and leaves the sub-simplex.

Like the two-dimensional case, there are only three types of p-paths possible:

1. **Type I:** p-paths that begin and end in two distinct $(0, 1, ..., n)$ sub-simplices. Thus, a Type I p-path guarantees the existence of two $(0, 1, ..., n)$ sub-simplices and crosses an even number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S (each time it exits $S,$ it does so through a $(0, 1, ..., n-1)$ sub-facet and it must return to S through another).
2. **Type II:** p-paths that begin and end outside $S.$ This type of path gives no $(0, 1, ..., n)$ sub-simplices and crosses an even number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S (each time it enters $S,$ it does so through a $(0, 1, ..., n-1)$ sub-facet and it must exit S through another).
3. **Type III:** p-paths that begin inside a $(0, 1, ..., n)$ sub-simplex and end outside S or vice-versa. Each Type III p-path gives exactly one $(0, 1, ..., n)$ sub-simplex and crosses an odd number of $(0, 1, ..., n-1)$ sub-facets on the boundary of $S.$

Let n_i denote the number of p-paths of Type $i.$ Let the total number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S be denoted by n_F and the total number of $(0, 1, ..., n)$ simplices in S be denoted by $n_S.$

Then from the above discussion, we know that each Type I and Type II p-path accounts for an even number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S

and each Type III p-path accounts for an odd number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S. So for some $\alpha,\ \beta,\ \gamma\ \in\ \mathbb{Z}_{\geq\,0}$:

$$n_F \ = \ 2\alpha n_1 \ + \ 2\beta n_2 \ + \ (2\gamma + 1)n_3. \tag{2.3}$$

Moreover, each Type I p-path accounts for two $(0, 1, ..., n)$ simplices in S and each Type III p-path accounts for one $(0, 1, ..., n)$ simplex in S. Type II p-paths give no $(0, 1, ..., n)$ simplices in S. Then:

$$n_S \ = \ 2n_1 \ + \ n_3. \tag{2.4}$$

As we already saw, there is an odd number of $(0, 1, ..., n-1)$ sub-facets on the boundary of S; i.e., n_F is odd. Then by Equation 2.3, n_3 is odd. It follows from Equation 2.4 that n_S is odd. In other words, there is an odd number of $(0, 1, ..., n)$ simplices in S. The completeness of the collection of p-paths in S ensures that every $(0, 1, ..., n)$ simplex is included in the count. Thus there is an odd number of $(0, 1, ..., n)$ simplices in S.

Hence, Sperner's lemma holds for the n - dimensional case, assuming the $(n\ -\ 1)$ dimensional case, and by induction on n, we get that for all n, the Sperner labeling of a subdivision of an n - dimensional simplex S gives an odd number of $(0, 1, ..., n)$ sub-simplices in S. With that, the proof of the n -dimensional Sperner's lemma is complete. □

Now that we have established Sperner's lemma for all dimensions, let us examine its application to fair-division problems, a branch of economics that has gained prominence in the last half-century. In the next chapter, we will be using this lemma to prove the Brouwer fixed point theorem, which in turn has various applications in social sciences.

2.2 Envy-Free Division in Economics

Economics is a behavioral science and in essence, it studies the *problem of choice*, namely how to allocate resources, goods (or desirables) and bads (or undesirables) in an economy. Distribution is basically a *fair-division problem*, where fairness depends upon the outcome desired. For instance, the goal of distribution may be to maximize social welfare or minimize negative externalities (like pollution). Economists have been grappling with these problems for centuries. Lately, there has been a growing focus on achieving *envy-free distribution* in which each individual feels satisfied with their share and does not vie for another individual's share. Envy-free distribution sounds ideal since it minimizes discontent and so maximizes happiness, however the challenge lies in measuring a quality like envy in order to be able to make considerations about it.

Many approaches have been proposed by mathematicians and economists who have constructed models that lead to approximately envy-free distribution. However, it was only after the development of Sperner's lemma that the existence of an envy-free division was established, keeping in mind certain assumptions. When we talk about fair division of goods, it is difficult to guarantee envy-free distribution as each individual vies for the best piece, and here it has to be noted that the term "best" is subjective, depending on individual preferences. Similarly, fair distribution of bads is difficult as every individual wants the least share in it. A division which satisfies everyone simultaneously seems hard to attain, yet as we shall see, Sperner's lemma guarantees the existence of such a division. To prove the existence of a fair distribution of goods, we are going to use the approach used by Francis Edward Su in [2] who attributed it to Forest Simmons.

2.2.1 Cake-Cutting Dilemna

When a cake is divided between two people, fair division is possible using a single cut if one person gets to cut the cake into two pieces while the other person gets to choose the piece. This process motivates the first person to cut the cake in a manner that she/he is indifferent between the two pieces and the second person's choice does not really affect them. Also, the second person gets to choose the piece first so they will not envy the other person's piece. In this manner, an envy-free distribution can be achieved.

Is it possible to have an envy-free division of the cake between three people? It does not seem obvious, yet given certain assumptions, there exists such a division as we shall see in the approach given by Simmons. We have a cake which is to be cut into three pieces using two cuts as shown in Figure 2.6.

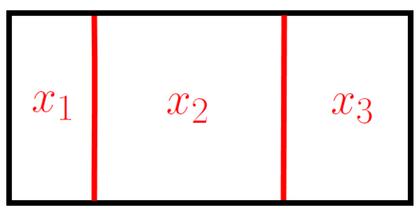

Figure 2.6. Using two cuts to cut a cake into three pieces of size $x_1,\ x_2,$ and x_3

There are many ways in which these two cuts can be made. A set of cuts is completely defined by the size of the three pieces it generates. We assume that the total size of the cake is 1. Then a **cake-cut** is defined as a point (x_1, x_2, x_3) where x_i is the size of the i^{th} piece, such that

$$0 \le x_i \le 1 \qquad \forall \ i \ \in \{1,2,3\}, \tag{2.5}$$

$$\sum_{i=1}^{3} x_i = 1. \tag{2.6}$$

Let the three individuals be denoted by A, B and C. Plotting all possible cake-cuts, we find that the space T of cake-cuts is the standard 2-simplex in $\mathbb{R}^3$. The coordinates of the vertices of T are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

We shall now make some general assumptions that form the base of the model we are constructing.

Assumptions:

1. People are *rational* and their choice for a piece or pieces for a cake-cut depends only on which piece(s) they prefer the most and not on other people's choices. Note, this means that a person always chooses at least one piece for each cake-cut.

2. People are *hungry* and so they always choose some piece over no piece, i.e. a piece of any size is chosen over a piece of size 0.

3. The set of choices is *closed*, i.e., if an individual chooses a piece for a convergent sequence of cake-cuts, the individual chooses that piece at the limiting cake-cut.

Note that a piece is not necessarily chosen based on its size. A person may prefer a particular piece of cake because it has more candy flowers on it or more chocolate or some such reason. The only assumption that is made about piece size is that nobody chooses pieces of size 0.

Given the above assumptions, it is possible to find at least one cake-cut where each individual prefers a different piece, and this gives us the envy-free partition we desire. We establish this formally in what follows.

Definition. *For* $\alpha = A$, B, *and* C, *and* $x = (x_1, x_2, x_3) \in T$, *let* $C_\alpha(x)$ *be a subset of* $\{1, 2, 3\}$. *We call the set-valued function* C_α *a* ***choice function for person*** α *if:*

1. $C_\alpha(x) \neq \emptyset$ *for all* $x \in T$.
2. *For all* $x = (x_1, x_2, x_3)$, *if* $x_i = 0$, *then* $i \notin C_\alpha(x)$.
3. C_α *is continuous.*

Note that the three properties of a choice function in the above definition reflect the corresponding assumptions listed previously. Also, if $i \in C_\alpha(x)$ for a choice function C_α, we say that **person** α **chooses piece** i **at cake-cut** x.

Lemma 2.4. *There exist arbitrarily fine triangulations of* T *with vertices labeled* A, B, *and* C *such that each sub-triangle has vertices distinctly labeled* A, B, *and,* C.

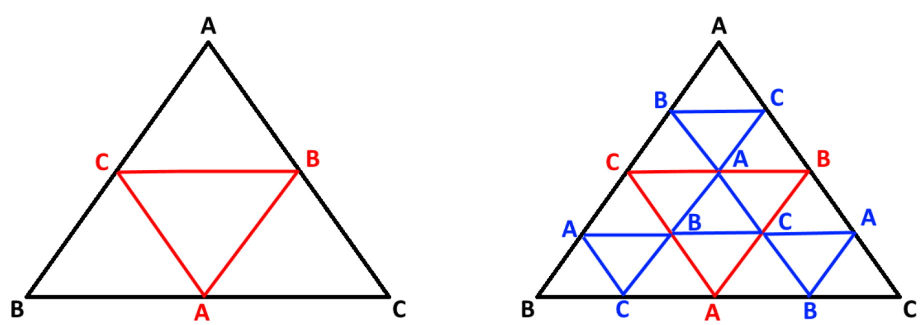

Figure 2.7. An example of arbitrarily fine triangulations of T

Proof. Given T with the vertex $(1, 0, 0)$ labeled A, vertex $(0, 1, 0)$ labeled B, and vertex $(0, 0, 1)$ labeled C, join the mid-points of AB, BC, and CA to form a triangle and label these midpoints as C, A and B respectively (refer to Figure 2.7). As we observe, each sub-triangle has vertices distinctly labeled

A, B and C. We can continue to further divide each sub-triangle with vertices distinctly labeled A, B and C into finer and finer sub-triangles and label them as outlined above.

In this manner, for each $n \in \mathbb{N}$, we have a triangulation $\mathscr{T}_n$ of T into equal-sized equilateral sub-triangles such that the side lengths of the sub-triangles go to 0 as n becomes infinite. Furthermore, each sub-triangle has vertices labeled A, B, and C. □

Then, the existence of the desired envy-free partition is established via the following:

Theorem 2.1. *Let C_A, C_B, C_C be choice functions on T. There exists $x \in T$ and distinct α, β, γ among A, B, C such that $1 \in C_\alpha(x)$, $2 \in C_\beta(x)$, and $3 \in C_\gamma(x)$. Thus, at $x \in T$, each person can have a piece of cake that they choose.*

Proof. Assume we have a sequence of labeled triangulations of T as described in Lemma 2.4. Consider a triangulation $\mathscr{T}_n$ of T. Since each point in T represents a cake-cut, we can create a new secondary labeling of the triangulation using 1's, 2's, and 3's such that if the vertex v is labeled with person α, then we give it a secondary label i such that $i \in C_\alpha(v)$.

We claim that the labeling is a Sperner labeling (Figure 2.2). This is because no one chooses a piece of size 0. So we can see that:

1. At the vertices, the only piece chosen (by any of the three - A, B or C) is 1 at $(1, 0, 0)$, 2 at $(0, 1, 0)$, and 3 at $(0, 0, 1)$. Hence, the vertices are labeled accordingly, and T is a $(1, 2, 3)$ triangle.

2. Now consider the side of T connecting the vertices $(1,0,0)$ and $(0,1,0)$. All sub-triangle vertices on that side have coordinates $(x_1, x_2, 0)$. At such a cake-cut, none of A, B or C chooses piece 3. Therefore, no sub-triangle vertex on that side is labeled with 3, that is, with the label on the vertex $(0,0,1)$ opposite to that side. Similarly, we can see that no vertex on the side of T connecting the vertices $(1,0,0)$ and $(0,0,1)$ is labeled 2 and no vertex on the side of T connecting the vertices $(0,1,0)$ and $(0,0,1)$ has the label 1. Thus, vertices of sub-triangles on a side of T will not have the same label as the vertex of T opposite to the side.

Thus, the labeling is a Sperner labeling. By Sperner's lemma (Lemma 2.2), there is at least one $(1,2,3)$ sub-triangle in the triangulation $\mathscr{T}_n$ of T.

Each triangulation of T yields at least one $(1,2,3)$ sub-triangle. For the triangulation $\mathscr{T}_n$, we denote that sub-triangle by T_n. The vertices of T_n are labeled A, B, C and have the secondary labeling 1, 2, 3. For now, assume A is labeled 1, B is labeled 2, and C is labeled 3. We denote the overall labeling on such T_n by $A1B2C3$.

This means that at one vertex of T_n, A chooses piece 1 in that cake-cut, at another vertex B chooses piece 2 for that cake-cut, and for the last vertex C chooses piece 3 for that cake-cut. Note that this is not yet our desired envy-free division because these are three different cake-cuts and not a single one where A, B, and C choose distinct cake pieces. However, since we get smaller and smaller $(1,2,3)$ triangles for finer and finer triangulations, we can reach this desired cake-cut through a limiting argument we use below.

Note that there are six choice distributions possible for each T_n:

$$A1B2C3,\ A1B3C2,\ A2B1C3,\ A2B3C1,\ A3B1C2,\ A3B2C1$$

Since there are finitely many choice distributions and infinitely many $T_n,$ at least one choice distribution must repeat infinitely often. Without loss of generality, assume it is $A1B2C3;$ i.e., A chooses piece $1,$ B chooses piece 2 and C chooses piece 3 at different vertices of $T_n.$ Then for each of these sub-triangles, take the vertex labeled $A.$ They result in a sequence: $v_1,\ v_2,\ v_3,\ \ldots$. Since this is a sequence in a closed and bounded space $T,$ it must have a sub-sequence that converges to a point, say $v.$ Since the size of sub-triangles of T becomes smaller and smaller as n becomes infinite, the distance between the three vertices of the triangles tends to $0.$ Thus, the corresponding sub-sequences of B and C vertices also converge to v (by Theorem 1.2). At this limit point $v,$ since the choice functions are continuous, A chooses piece $1,$ B chooses piece 2 and C chooses piece $3.$ Thus, v represents a cake-cut of T such that $A,$ $B,$ and C prefer distinct pieces of the cake.

Hence, we are able to find at least one point in $T,$ i.e. at lease one cake-cut, that gives an envy-free division of the cake. With that, the proof of the theorem is complete. □

Note that a similar method can be adopted to find an envy-free division for $n + 1$ individuals, using the n - dimensional Sperner's lemma in the same way that the two-dimensional lemma was used here.

CHAPTER 3

THE BROUWER FIXED POINT THEOREM

In this chapter, we state and prove the Brouwer fixed point theorem, a theorem of topology that was given in the 1912 publication *Über Abbildung Von Mannigfaltigkeiten* [3] by the Dutch mathematician L. E. J. Brouwer. This theorem was inspired by the work done by French mathematician Henri Poincaré [4] in the field of differential equations [5]. The Browuer fixed point theorem proves the existence of fixed points for certain continuous functions. First let us define what is meant by a fixed point of a (point-valued) function.

Definition. *For any set* X, *a* ***fixed point*** *of a function* $f : X \to X$ *is a point* x^* *such that*

$$f(x^*) = x^*.$$

The knowledge of existence of fixed points for a function has various applications in social sciences (as well as other fields, though we are not discussing them in this paper). One such application is to prove the existence of equilibrium strategies for games, an important topic of discussion in game theory. In this chapter, we will use the Brouwer fixed point theorem to prove the existence of Nash equilibria for non-cooperative games. In the next chapter, we will use the Kakutani fixed point theorem to prove the existence of general equilibria in economics.

3.1 The Brouwer Fixed Point Theorem

Let us begin by formally stating and proving the Brouwer fixed point theorem in one, two, and n dimensions. Just as in the case of Sperner's lemma, the Brouwer fixed point theorem in one-dimension is fairly intuitive. It can be easily proved using the Intermediate Value theorem. For dimensions two and higher, we use the corresponding Sperner's lemma to prove the theorem. In particular, the proof of the two-dimensional Brouwer fixed point theorem provides greater understanding of the proof in higher dimensions.

3.1.1 The One-Dimensional Brouwer Fixed Point Theorem

The standard 1 - simplex is a line-segment in $\mathbb{R}^2$ given by

$$y = 1 - x \; ; \; x \geq 0 \; , \; y \geq 0.$$

This is homeomorphic to the line-segment

$$y = 0 \; ; \; 0 \leq x \leq 1.$$

Thus, it is enough to prove the Brouwer fixed point theorem for functions mapping $[0,1]$ to $[0,1]$. Let $f : [0,1] \to [0,1]$ be continuous. Intuitively it is clear that the graph of f must intersect the line $y = x$ at some point (see Figure 3.1), and this point is a fixed point of f since $f(x) = x$ at such a point. We will formally prove the one-dimensional Brouwer fixed point theorem using the Intermediate Value theorem from calculus.

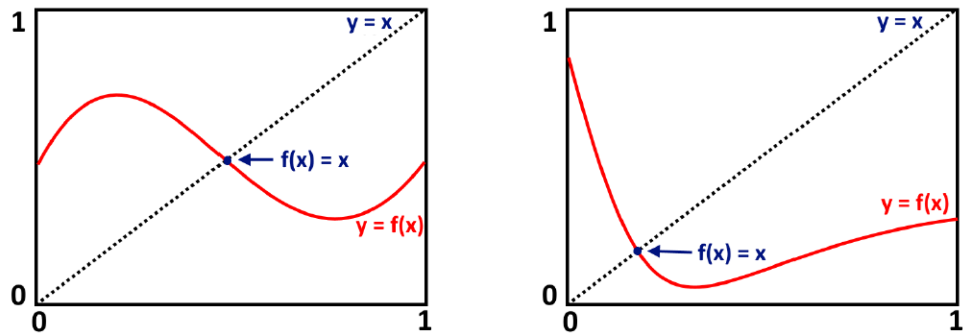

Figure 3.1. The idea behind the One-Dimensional Brouwer Fixed Point Theorem

Theorem 3.1. *Every continuous function* $f : [0, 1] \to [0, 1]$ *has a fixed point.*

Proof. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Note that $f(0) \geq 0$ and $f(1) \leq 1$. Define $g(x) = f(x) - x$ for all $x \in [0, 1]$. Since f is continuous, g is also continuous. Moreover, $g(0) \geq 0$ and $g(1) \leq 0$. Then by the Intermediate Value theorem, there exists at least one point $c \in [0, 1]$ such that $g(c) = 0$. This implies $f(c) - c = 0$. Therefore $f(c) = c$, and c is a fixed point of f.

Hence, there exists at least one fixed point of $f : [0, 1] \to [0, 1]$. Thus, the Brouwer fixed point theorem holds for dimension one. □

3.1.2 The Two-Dimensional Brouwer Fixed Point Theorem

Just like in the case of Sperner's lemma, the proof of the n-dimensional Brouwer fixed point theorem is very similar to the proof of the theorem in dimension two. Since it is easy to visualize a 2 - simplex in $\mathbb{R}^3$, we will first look at the two-dimensional Brouwer fixed point theorem and then proceed to prove the theorem in dimension n. Note that the standard 2-simplex, T, is the equilateral triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The Brouwer fixed point theorem is applicable to any space homeomorphic to T, for instance, any 2-disk in the plane.

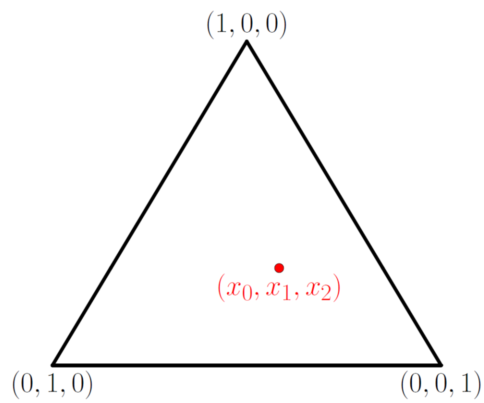

Figure 3.2. A point $x \in T$

Let $x = (x_0, x_1, x_2)$ be a point in T. For the purpose of the proof, define the regions associated with x as below:

1. $R_0(x) := \{y \in T \mid y_0 < x_0\}$,
2. $R_1(x) := \{y \in T \mid y_0 \geq x_0 ;\ y_1 < x_1\}$,
3. $R_2(x) := \{y \in T \mid y_0 \geq x_0 ;\ y_1 \geq x_1 ;\ y_2 < x_2\}$.

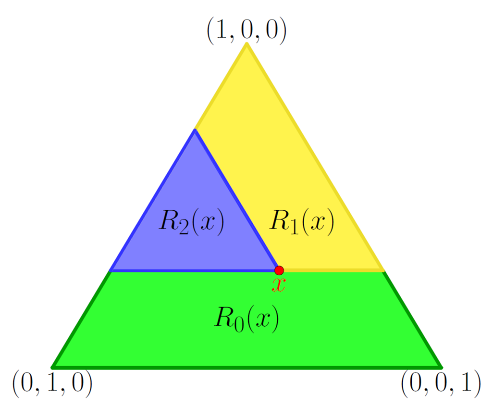

Figure 3.3. Regions associated with a point $x \in T$

Figure 3.3 depicts the three regions for an arbitrary point $x \in T$. Before we state and prove the Brouwer fixed point theorem in two dimensions, here are some easily seen properties of these regions.

Lemma 3.1. *For any point $x \in S$,*

1. $R_0(x),\ R_1(x),\ R_2(x)$ *are mutually disjoint.*

2. $R_0(x) \cup R_1(x) \cup R_2(x) = T - \{x\}$.

3. $Cl\ (R_0(x)) \cap Cl\ (R_1(x)) \cap Cl\ (R_2(x)) = \{x\}$.

4. $R_0((1,0,0)) = S - \{(1,0,0)\}$,
   $R_1((0,1,0)) = S - \{(0,1,0)\}$,
   $R_2((0,0,1)) = S - \{(0,0,1)\}$.

5. $R_2((x_0,x_1,0)) = R_1((x_0,0,x_2)) = R_0((0,x_1,x_2)) = \varnothing$.

We do not prove Lemma 3.1. Instead we can see these relationships in the figures as follows: Parts (1) and (2) are illustrated in Figure 3.3 ; Part (3) is demonstrated in Figure 3.4; Part (4) is represented in Figure 3.5 ; and Part (5) is illustrated in Figure 3.6.

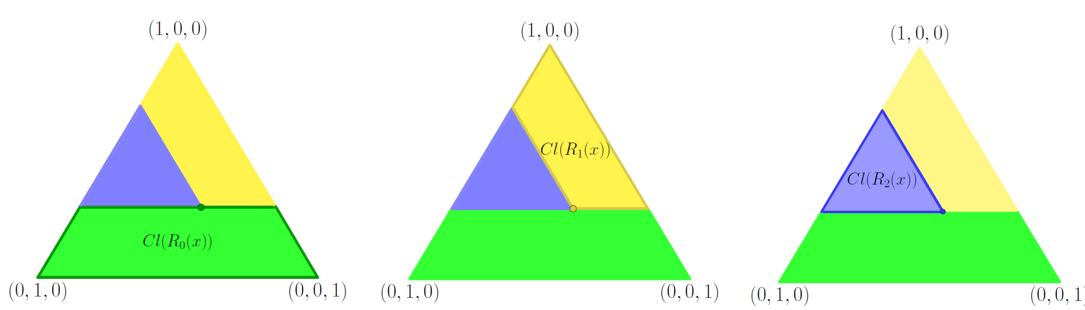

Figure 3.4. $\{x\} = Cl\ (R_0(x)) \cap Cl\ (R_1(x)) \cap Cl\ (R_2(x))$

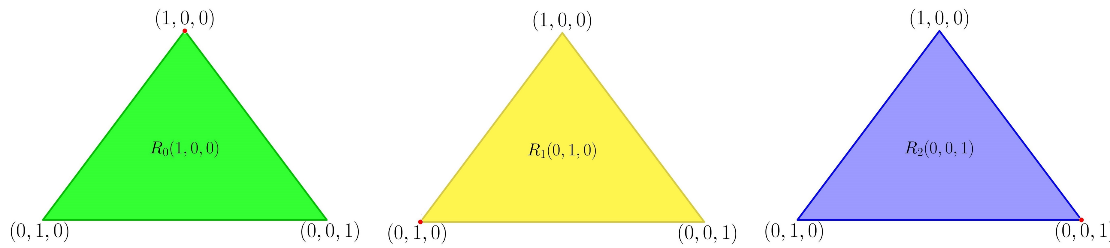

Figure 3.5. Regions associated with the vertices of S

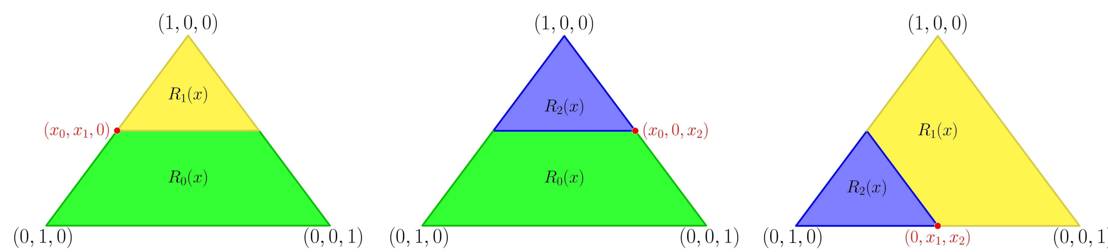

Figure 3.6. Regions associated with a point on a side of S

Next we introduce an important limit relationship for the regions.

Lemma 3.2. *For* $j = 0, 1, 2$; *if* $(x_n) \to x$, $(y_n) \to y$, *and* $x_n \in R_j(y_n)$ *for all* n, *then* $x \in Cl\ (R_j(y))$

The proof for this lemma is straightforward and it follows directly from Theorem 1.3.

Let us now formally state and prove the two-dimensional Brouwer fixed point theorem using the above lemmas.

Theorem 3.2. *Let T be the standard 2-simplex in $\mathbb{R}^3$. Every continuous function $f : T \to T$ has a fixed point.*

Proof. Given the standard simplex T, let $f : T \to T$ be an arbitrary continuous function. Let $\mathscr{T}_1, \mathscr{T}_2, ...$ be triangulations of T where each $\mathscr{T}_{i+1}$ is generated by further triangulating sub-triangles formed under $\mathscr{T}_i$ for all i. Also assume that as i increases and tends to ∞, the diameter of subdivision simplices tends to 0.

If some sub-triangle vertex of T (over all triangulations) is a fixed point of f, then we are done. Now assume that no sub-triangle vertex v for any of the triangulations of T is a fixed point of f. In other words, $f(v) \neq v$ over all vertices v in all triangulations of T. Since $f(v) \neq v,$ parts (1) and (2) of Lemma 3.1 imply that $f(v)$ lies in exactly one of $R_0(v),\ R_1(v),\ R_2(v)$.

Label the vertex v with j if $f(v) \in R_j(v)$. Then all sub-triangle vertices in T are labeled with the numbers $0,\ 1,$ or 2. Now by Lemma 3.1, $R_0((1,0,0)) = S - \{(1,0,0)\}$. This means $f((1,0,0)) \in R_0((1,0,0))$. Thus, $(1,0,0)$ is labeled 0. Similarly, $(0,1,0)$ is labeled 1 and $(0,0,1)$ is labeled 2. (See Figure 3.5.) Hence T is a $(0,1,2)$ triangle.

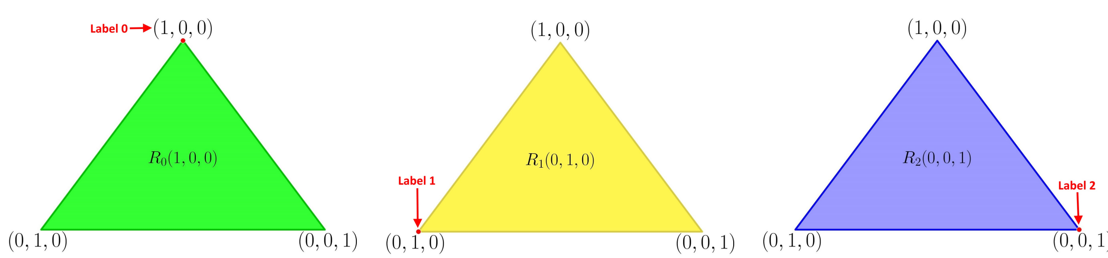

Figure 3.7. T is a $(0,1,2)$ triangle

Furthermore, sub-triangle vertices on a side of T do not have the same label as the vertex opposite the side. For instance, vertices on the side of T connecting the vertices $(1, 0, 0)$ and $(0, 1, 0)$ are not labeled 2. This is because a sub-triangle vertex v on that side has the coordinates $(x_0, x_1, 0)$. Then, $R_2(v) = \varnothing$ by Lemma 3.1, so $f(v) \notin R_2(v)$, and hence v is not labeled 2.

Therefore, each triangulation $\mathscr{T}_i$ has a Sperner labeling (Figure 2.2). By Sperner's lemma, there exists at least one $(0, 1, 2)$ sub-triangle in each triangulation of T.

For each triangulation $\mathscr{T}_i$, choose a $(0, 1, 2)$ sub-triangle T_i. Let v_k^i denote the vertex of the sub-triangle T_i that is labeled k. By our labeling,

$$f(v_k^i) \in R_k(v_k^i). \tag{3.1}$$

Consider the sequence (v_0^i) over all i. This is a sequence in a closed and bounded space T, and hence, it must have a convergent subsequence. For simplicity, assume the sequence itself converges. Let $(v_0^i) \to V$ as $i \to \infty$ where $V \in T$. By continuity of f, we get that $f(v_0^i) \to f(V)$. Furthermore, since $f(v_0^i) \to f(V)$ and $f(v_0^i) \in R_0(v_0^i)$, it follows by Lemma 3.2 that

$$f(V) \in Cl\,(R_0(V)). \tag{3.2}$$

Also, the sequences of vertices of T_i labeled 1 and 2 converge to V since the distance between the sub-simplex vertices tends to 0 as $i \to \infty$. Hence, $(v_1^i) \to V$ and $(v_2^i) \to V$ (by Theorem 1.2). As before, by continuity of f and Lemma 3.2, we get:

$$f(V) \in Cl\,(R_1(V)) \quad \text{and} \quad f(V) \in Cl\,(R_2(V)). \tag{3.3}$$

Combining 3.2 and 3.3, we get that:

$$f(V) \in Cl\,(R_0(V)) \cap Cl\,(R_1(V)) \cap Cl\,(R_2(V)). \tag{3.4}$$

By Part (3) of Lemma 3.1, this means $f(V) = V$. Thus, V is a fixed point of f. So if no subdivision vertex is a fixed point of f, then some other point in T is. Hence, the proof of the Brouwer fixed point theorem in two dimensions is complete. □

3.1.3 The n - Dimensional Brouwer Fixed Point Theorem

Let us prove the Brouwer fixed point theorem for the standard n - simplex S in $\mathbb{R}^{n+1}$. Note that if the theorem applies to the n - simplex, then it extends to all spaces that are homeomorphic to the simplex, for instance, the closed unit ball B_n in $\mathbb{R}^n$ where $B_n = \{\, x \in \mathbb{R}^n : |\, x \,| \leq 1 \,\}$.

Theorem 3.3. *Let S be the standard n - simplex in $\mathbb{R}^{n+1}$. Then every continuous function $f : S \to S$ has a fixed point.*

Proof. Given the standard n -simplex S in $\mathbb{R}^{n+1}$, we want to show that every continuous function $f : S \to S$ has a fixed point. Let $\mathscr{S}_1, \mathscr{S}_2, ...$ be subdivisions of S where each $\mathscr{S}_{i+1}$ is generated by further subdividing sub-simplices formed under $\mathscr{S}_i$ for all i. Assume that the diameter of sub-simplices tends to 0 as $i \to \infty$.

Let $f : S \to S$ be an arbitrary continuous function. If some sub-simplex vertex (over all subdivisions $\mathscr{S}_i$) is a fixed point of f, then we are done. Now assume none of the sub-simplex vertices for any subdivision is a fixed point of f. In other words, $f(v) \neq v$ over all vertices v in all subdivisions $\mathscr{S}_i$ of S.

For any $x = (x_0, x_1, ..., x_n) \in S$, let

$$f(x) = f(x_0, x_1, ..., x_n) = (f_0(x), f_1(x), ..., f_n(x)).$$

Assume x is not a fixed point of f. So $(x_0, x_1, ..., x_n) \neq (f_0(x), f_1(x), ..., f_n(x))$. Then by Lemma 1.1, there exists some $k \in \{0, 1, ..., n\}$ such that $f_k(x) < x_k$. Label the point x with the number p such that

$$p = \min\{\, k = 0, 1, ..., n \mid f_k(x) < x_k\}.$$

Note that the $n+1$ vertices of S have the coordinates $(1, 0, ..., 0)$, $(0, 1, ..., 0)$, ..., $(0, 0, ..., 1)$. Since these vertices of S are not fixed points (as assumed before), then according to the labeling, they must be labeled $0, 1, ..., n$ respectively. For example, with $f(0, 0, 1, ..., 0) = (a_0, a_1, ..., a_n)$, we have $0 \leq a_0$, $0 \leq a_1$, and $a_2 < 1$ (the latter holds because otherwise we would have a fixed point). So $(0, 0, 1, 0, ..., 0)$ is labeled with 2. Thus, S is a $(0, 1, ..., n)$ simplex.

Now fix a subdivision $\mathscr{S}_i$ of S. Consider the vertex of S with 1 in the j^{th} entry. Then by our labeling, this vertex is labeled j. Let x be a point on the facet opposite this vertex. Clearly, $x_j = 0$. Then since $f_j(x) \geq 0$, it follows that x is not labeled j. Therefore, over all subdivisions $\mathscr{S}_i$, no sub-simplex vertex is labeled j on the facet opposite the S vertex labeled j . Hence, each subdivision $\mathscr{S}_i$ has a Sperner labeling.

Therefore by Sperner's lemma, there exists at least one $(0, 1, ..., n)$ sub-simplex in S . Choose a $(0, 1, ..., n)$ sub-simplex S_i from each sub-division $\mathscr{S}_i$.

Let v_k^i be the vertex of S_i labeled k . Consider the sequence (v_0^i) over all i. Then this sequence, in a closed and bounded space S, must have a convergent subsequence. For simplicity, assume the sequence itself converges. Let $v_0^i \to v^*$ as $i \to \infty$ where v^* is a point in S. Then the sequences formed by the other vertices (v_1^i), (v_2^i), ..., (v_n^i) also converge to v^* since the distance between the sub-simplex vertices tends to 0 as $i \to \infty$ (by Theorem 1.2). We claim that the limiting point $v^* \in S$ is actually a fixed point of f.

To prove the claim, let the point $v^* = (v_0^*, v_1^*, ..., v_n^*)$. Let $v_{k,m}^i$ denote the m^{th} coordinate of v_k^i. Note that for all k and i,

$$(v_{k,0}^i) \to v_0^*, \quad (v_{k,1}^i) \to v_1^*, \quad ..., \quad (v_{k,n}^i) \to v_n^*. \tag{3.5}$$

Now v_k^i has the label k. Then, by the labeling, we must have

$$f_k(v_k^i) < v_{k,k}^i. \tag{3.6}$$

Thus, from Equations 3.5 and 3.6 and Theorem 1.3,

$$f_k(v^*) \leq v_k^* \quad \text{for all } k.$$

By Lemma 1.1, we have that

$$f_k(v^*) = v_k^* \quad \text{for all } k.$$

Hence,

$$f(v^*) = v^*.$$

In other words, v^* is a fixed point of f. So if no subdivision vertex is a fixed point of f, then some other point in S is. Therefore, the Brouwer fixed point theorem holds for all n-dimensions $(n \in \mathbb{N})$.

□

3.2 Using the Brouwer Fixed Point Theorem to Prove the Existence of Nash Equilibria for Non-Cooperative Games

Equilibrium is often defined as a state of rest, where all counteracting forces balance. In game theory, equilibrium means a state where all players are content with the strategy they employ and have no reason to change it, given the strategies of the other opponents. One of the main questions in game theory is to determine whether equilibrium is attainable, and if so, what strategy must one employ in a game in order to achieve it. John Nash proved the existence of equilibria in a finite non-cooperative game using the generalized Kakutani fixed point theorem, and later he presented a more straightforward proof of the same using the Brouwer fixed point theorem [6]. In this section, we will first introduce game theory and what is meant by an equilibrium in a finite non-cooperative game. Then we will use the Brouwer fixed point theorem to prove the existence of equilibria. Further, we will see its application in example games that we introduce.

3.2.1 Introduction to Game Theory

The formal establishment of game theory as a field of study is credited to the 1944 publication *Theory of Games and Economic Behavior* by mathematician John von Neumann and economist Oscar Morgenstern [7]. However, the roots of this science can be traced as far back as two thousand years. The Babylonian Talamud, the basis of all codes of Jewish laws and ethics, has the oldest known reference to what is now referred to as the theory of cooperative games [8]. Even the Spanish conquest of the Aztecs reflects the subtle use of game theory in military tactics, when the Spanish conqueror Cortes scuttled his ships so that his military did not mutiny and flee back and also, to induce fear into the minds of the Aztec people. As it happened, his strategy was successful and he won the conquest [9]. It is an age-old question whether or not a situation necessarily has an "outcome" that maximizes the welfare of everyone, given the choices made by each person involved. And so the formal theory of games emerged to find answers to these conundrums.

To begin with, a **game** is an interaction between participants of the game, defined by a set of rules. Participants of a game are called **players** and these players are responsible for making decisions, or **choices**, that determine the outcome of the game. Note that the final outcome of the game is determined by the combination of choices made by all players. A **move** is an action taken by a player in a game. A **non-cooperative game** is a game without any coalition or communication between the players that can influence their decision. Thus, in a non-cooperative game, each player acts independently and tries to maximize his or her own objective.

In this paper, we will focus our attention on single-move non-cooperative games with finite players and finite choices for each player. Let us look at an example two-player game. Suppose the two players are player X and player Y. The game is played by having each player make exactly one move. In this game, each player is asked to throw either a nickel or a quarter into a hat. The players make the move simultaneously and cannot communicate their choice with the other player beforehand. The final outcome of the game depends upon the rules of the game and the choices made by both the players.

For instance, suppose the game rule was that if the coins match, player X gets the coin player Y played (in addition to getting back the coin she/he played) and if the coins do not match, player Y gets both player X's coins (in addition to getting back the coin she/he played). Then depending upon the coin each player chooses to put into the hat, either player X earns a profit or player Y earns a profit. Note that the players do not know what the other player chooses. However, it is safe to assume that, knowing the rules of the game, they both are fully aware of all possible outcomes of the game and the question here is - What are the best choices for each player when playing the game?

We are interested in examining the situation where a game is played multiple times. Given the possible outcomes, the question is, what are the best overall strategies the players can employ so that they can maximize their gains or minimize their losses?

Let the i^{th} player have n_i available choices in the game. In our example, let N and Q represent the choice of playing a nickel and of playing a quarter, respectively. Then there are two choices available to each player, $\{N, Q\}$.

When a player makes the same choice for a move in every play of the game, the player is said to have a **pure strategy**. A pure strategy of a player is independent of the pure strategies of other players. In our example, if player X plays N all the time, this strategy can be written as $(1, 0)$ to indicate that player X plays only N and never plays Q.

A player's **strategy set** is the set of all the pure strategies of the player. Let the strategy set of the i^{th} player be $\{ p_{i,\tau} \mid \tau = 1, ..., n_i\}$. Note that each pure strategy $p_{i,\tau}$ is written as an n_i - dimensional vector, that is, $p_{i1} = (1, 0, ..., 0)$, $p_{i,2} = (0, 1, ..., 0)$, ..., $p_{i,n_i} = (0, 0, ..., 1)$. In our example, as we saw, the strategy set of player X is $\{p_{1,1}, p_{1,2}\}$ where $p_{1,1} = (1,0)$ and $p_{1,2} = (0,1)$. Similarly, the strategy set of player Y is $\{p_{2,1}, p_{2,2}\}$ where $p_{2,1} = (1,0)$ and $p_{2,2} = (0,1)$.

In general, playing a pure strategy is not an ideal tactic for a player because making the same choice in every play of the game allows other players to guess the player's strategy and use that knowledge to maximize their own payoff at the expense of this player. Instead, each player generally uses a **mixed strategy**. This means that the player assigns probabilities to the pure strategies available to her/ him in order to incorporate an ambiguity in their decision making and keep the other players guessing. Mixed strategies model real life situations since they allow players to deviate from a fixed path, accounting for the human decision-making element.

For player i, let s_i denote a mixed strategy where she/he plays the available n_i choices with differing probabilities over a large number of plays of the game. Let the probability associated with the τ^{th} choice be α_τ. Clearly, each $\alpha_\tau \geq 0$ and

$$\sum_{\tau=1}^{n_i} \alpha_\tau = 1. \tag{3.7}$$

Then we can represent this mixed strategy s_i as the n - tuple $(\alpha_1, ..., \alpha_\tau, ..., \alpha_{n_i})$. Each mixed strategy s_i can be thought of as a linear combination of the pure strategies of player i, written as

$$s_i = \sum_{\tau=1}^{n_i} \alpha_\tau p_{i,\tau} \tag{3.8}$$

where n_i = number of choices available to the i^{th} player,

α_τ = probability associated with the τ^{th} choice, and

$p_{i,\tau}$ = pure strategy of the i^{th} player to play the τ^{th} choice.

It is obvious that there are infinitely many mixed strategies available to each player, even when the strategy set is finite. By Equations 3.7 and 3.8, we can see that the set of all possible mixed strategies for a player forms an $(n_i - 1)$ - simplex S_i whose vertices represent the pure strategies for that player. Thus, a mixed strategy of a player is a point in the vector space formed by the span of all the pure strategies of that player.

Again, referring back to our example of the two-player, two-choice game, we see that the mixed strategies for player X is the set of points on the line segment in $\mathbb{R}^2$ connecting $p_{1,1}$ to $p_{1,2}$. Similarly, the mixed strategies for player Y is the set of points on the line segment in $\mathbb{R}^2$ connecting $p_{2,1}$ to $p_{2,2}$.

Let $\mathbb{s} = (s_1, ..., s_n)$ denote the n-tuple of mixed strategies of the n players. Since each s_i is a point in a vector space spanned by the pure strategies of the i^{th} player, $\mathbb{s}$ can be geometrically understood as a point in the vector space that is the product space of all vector spaces formed by the span of pure strategies of each player. In our example, $\mathbb{s} = (s_1, s_2) \in \mathbb{R}^4$, where $s_1 \in \mathbb{R}^2$ is a mixed strategy of player X and $s_2 \in \mathbb{R}^2$ is a mixed strategy of player Y.

Each n - tuple of mixed strategies $(s_1, ..., s_n)$ has an associated **payoff**, $\pi = (\pi_1, ..., \pi_n)$ that represents the outcome of the game if the players were to choose that mixed strategy combination. Here, π_i denotes the payoff for the i^{th} player, and it is determined, not only by the move made by the i^{th} player, but also by the moves of the other players. Thus, $\pi_i,$ the payoff to the i^{th} player, is a function of mixed strategies of all players,

$$\pi_i(\mathbb{s}) = \pi_i(s_1, ..., s_i, ..., s_n).$$

Since a mixed strategy for a player is a linear combination of the player's pure strategies and each player's payoff depends not only on the choice made by them but also on the choices made by other players, it is natural to assume that each payoff function $\pi_i(\mathbb{s})$ is a linear combination of the payoffs associated with the pure strategies of each of the players. That is, each payoff function is a linear function of the payoffs corresponding to the pure strategies of all players, $p_{i,\tau}$ (where $i = 1, ..., n$ and τ ranges from $1, ..., n_i$ for each i).

Definition. *A* ***Nash equilibrium*** *is a strategy combination* $\mathbb{s}$ *such that the strategy* s_i *chosen by the* i^{th} *player maximizes the player's payoff when the strategies of the other players' are assumed fixed. Thus,* $(s_1, ..., s_n)$ *is a Nash equilibrium if for all* i*,*

$$\pi_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) = \max_{s \in S_i} \{\pi_i(s_1, ..., s_{i-1}, s, s_{i+1}, ..., s_n)\}. \tag{3.9}$$

This equilibrium is known alternately as the best response equilibrium since "no player can improve his expectation by changing his choice, the others being held fixed" [10].

In 1950, John Nash proved the existence of a Nash equilibrium for an n - player finite-choice non-cooperative game using the generalized Kakutani fixed point theorem [11]. However, in 1951, he published an alternate proof in the paper *Non-Cooperative Games* that in his words was a "considerable improvement over the earlier version and is based directly on the Brouwer theorem" [6]. Our objective is to give a detailed perspective of how the Brouwer fixed point theorem helps establish the existence of a Nash equilibrium for a 3 - player, 3 - choice non-cooperative game. The method of proof carries over to n players where each has n_i choices in a game. Later, we will work out the algorithm to find a Nash equilibrium for our two-person game outlined earlier in this section.

3.2.2 Nash Equilibrium for 3 - Player, 3 - Choice Non-Cooperative Game

Assume we have a game with three players $\{1, 2, 3\}$, and assume that each player has exactly three choices a_i, b_i and c_i they can make for their move. Let us represent the pure strategy vectors of the i^{th} player by $\{A_i, B_i, C_i\}$. Here, $A_i = (1, 0, 0)$ represents the pure strategy of player i to make the choice a_i one hundred percent of the time. Similarly, $B_i = (0, 1, 0)$ and $C_i = (0, 0, 1)$. Thus each player has a 2 - simplex of mixed strategies, given by:

$$T_i = \{\alpha A_i + \beta B_i + \gamma C_i \mid \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1\}. \tag{3.10}$$

Here, α, β, γ represent the various frequencies (probabilities) at which each choice a_i, b_i, c_i is played by the i^{th} player.

Each player has a payoff function $\pi_i : T_1 \times T_2 \times T_3 \to \mathbb{R}$. As discussed previously, we assume that the payoff function of the i^{th} player is linear on each T_i. For instance,

$$\pi_1(\alpha A_1 + \beta B_1 + \gamma C_1, s_2, s_3) = \alpha\pi_1(A_1, s_2, s_3) + \beta\pi_1(B_1, s_2, s_3) + \gamma\pi_1(C_1, s_2, s_3). \tag{3.11}$$

As a consequence, for fixed s_2 and s_3, one of the following must hold:

1. The maximum of π_1 occurs at a single vertex of T_1,
2. The maximum of π_1 occurs all along an edge of T_1,
3. The maximum of π_1 occurs over all of T_1, i.e., π_1 is constant on T_1.

Similar statements can be made for the maximum of π_2 (for fixed s_1 and s_3) and for the maximum of π_3 (for fixed s_1 and s_2).

Note that (s_1, s_2, s_3) is a Nash equilibrium if

$$\begin{aligned}
\pi_1\,(s_1, s_2, s_3) \;&=\; \max_{s\,\in\,T_1}\,\{\pi_1\,(s, s_2, s_3)\},\\
\pi_2\,(s_1, s_2, s_3) \;&=\; \max_{s\,\in\,T_2}\,\{\pi_2\,(s_1, s, s_3)\},\\
\pi_3\,(s_1, s_2, s_3) \;&=\; \max_{s\,\in\,T_3}\,\{\pi_3\,(s_1, s_2, s)\}.
\end{aligned} \tag{3.12}$$

The ideas of "perturbation" and "improvement" that we introduce next will assist us in our proof of the existence of Nash equilibria. Now, for $\alpha',\ \beta',\ \gamma'\ \geq\ 0,$ if we adjust a mixed strategy $s_i\ \in\ T_i$ by relative amounts $\alpha',\ \beta',\ \gamma'$ towards the pure strategies $A_i,\ \ B_i,\ C_i,$ the resulting mixed strategy $s_i',$ defined as

$$s_i' \;=\; \frac{s_i\ +\ \alpha' A_i\ +\ \beta' B_i\ +\ \gamma' C_i}{1\ +\ \alpha'\ +\ \beta'\ +\ \gamma'}\ \ \in\ \ T_i \tag{3.13}$$

is called the **perturbation** of s_i by $[\alpha',\ \beta',\ \gamma']$.

Define

$$I_{A_1}\,(s_1, s_2, s_3) \;=\; \max\,\{0,\ \pi_1(A_1, s_2, s_3)\ -\ \pi_1(s_1, s_2, s_3)\}. \tag{3.14}$$

We call I_{A_1} the A_1-**improvement** of π_1. It measures by how much the payoff of Player 1 increases, if any, by switching from mixed strategy s_1 to pure strategy $A_1,$ given that Players 2 and 3 continue to use mixed strategies s_2 and $s_3,$ respectively. We similarly define $I_{B_1},\ \ I_{C_1},\ \ I_{A_2},\ \ I_{B_2},\ \ I_{C_2},\ \ I_{A_3},\ \ I_{B_3},$ and $I_{C_3}.$

Theorem 3.4. (s_1, s_2, s_3) *is a Nash equilibrium if and only if* $I_{A_i}\ =\ I_{B_i}\ =\ I_{C_i}\ =\ 0$ *for* $i\ =\ 1, 2, 3.$

Proof. Clearly, if $I_{A_i}\ =\ I_{B_i}\ =\ I_{C_i}\ =\ 0$ for all $i,$ this means the payoff each player receives from the mixed strategy s_i is maximum, given that the other players' strategies are fixed. Hence, (s_1, s_2, s_3) is a Nash equilibrium.

Now, let us assume that (s_1, s_2, s_3) is a Nash equilibrium. We want to show that $I_{A_i} = I_{B_i} = I_{C_i} = 0$ for all i. Without loss of generality, let us assume the strategies of players 2 and 3 are fixed as s_2 and s_3 respectively. (s_1, s_2, s_3) is a Nash equilibrium, so by Equation 3.12, we know that

$$\pi_1(s_1, s_2, s_3) \geq \pi_1(A_1, s_2, s_3),$$
$$\pi_1(s_1, s_2, s_3) \geq \pi_1(B_1, s_2, s_3),$$
$$\pi_1(s_1, s_2, s_3) \geq \pi_1(C_1, s_2, s_3).$$

Hence, by equation 3.14, $I_{A_1}(s_1, s_2, s_3) = I_{B_1}(s_1, s_2, s_3) = I_{C_1}(s_1, s_2, s_3) = 0.$

We can make similar arguments for players 2 and 3. Thus (s_1, s_2, s_3) being a Nash equilibrium implies that for all i,

$$I_{A_i}(s_1, s_2, s_3) = I_{B_i}(s_1, s_2, s_3) = I_{C_i}(s_1, s_2, s_3) = 0.$$

Therefore, we conclude that (s_1, s_2, s_3) is a Nash equilibrium if and only if $I_{A_i} = I_{B_i} = I_{C_i} = 0$ for $i = 1, 2, 3$. □

Theorem 3.5. *A Nash equilibrium exists for a* 3*-player,* 3*-choice, non-cooperative game.*

Proof. For $(s_1, s_2, s_3) \in T_1 \times T_2 \times T_3$, let s_i' denote the perturbation of s_i by $[I_{A_i,}\ I_{B_i,}\ I_{C_i,}]$ in T_i for $i = 1, 2, 3$. Then,

$$f(s_1, s_2, s_3) = (s_1', s_2', s_3') \qquad (3.15)$$

defines a continuous function $f : T_1 \times T_2 \times T_3 \to T_1 \times T_2 \times T_3$. Since $T_1 \times T_2 \times T_3$ is homeomorphic to the 6-dimensional closed ball, it follows by the Brouwer fixed point theorem that f has a fixed point (s_1^*, s_2^*, s_3^*).

We claim that (s_1^*, s_2^*, s_3^*) is a Nash equilibrium and we prove this by showing that at (s_1^*, s_2^*, s_3^*), we have $I_{A_i} = I_{B_i} = I_{C_i} = 0$ for $i = 1, 2, 3$.

Note that since (s_1^*, s_2^*, s_3^*) is a fixed point of f,

$$f(s_1^*, s_2^*, s_3^*) = (s_1^*, s_2^*, s_3^*). \tag{3.16}$$

Then, by Equation 3.13, for each i,

$$s_i^* = \frac{s_i^* + I_{A_i} A_i + I_{B_i} B_i + I_{C_i} C_i}{1 + I_{A_i} + I_{B_i} + I_{C_i}}. \tag{3.17}$$

Let us consider the payoffs for player 1, keeping the strategy choices of players 2 and 3 fixed as s_2^* and s_3^*. In T_1, without loss of generality, we may assume that

$$\pi_1 (A_1, s_2^*, s_3^*) \le \pi_1 (B_1, s_2^*, s_3^*) \le \pi_1 (C_1, s_2^*, s_3^*). \tag{3.18}$$

We divide the analysis into three distinct cases:

1. $\pi_1 (A_1, s_2^*, s_3^*) = \pi_1 (B_1, s_2^*, s_3^*) = \pi_1 (C_1, s_2^*, s_3^*)$,
2. $\pi_1 (A_1, s_2^*, s_3^*) = \pi_1 (B_1, s_2^*, s_3^*) < \pi_1 (C_1, s_2^*, s_3^*)$,
3. $\pi_1 (A_1, s_2^*, s_3^*) < \pi_1 (B_1, s_2^*, s_3^*) \le \pi_1 (C_1, s_2^*, s_3^*)$.

We consider each case separately.

Case I : $\pi_1 (A_1, s_2^*, s_3^*) = \pi_1 (B_1, s_2^*, s_3^*) = \pi_1 (C_1, s_2^*, s_3^*)$**.**

By linearity of the payoff function over T_1, the payoff of player 1 is constant over T_1. Thus, $I_{A_1} = I_{B_1} = I_{C_1} = 0$.

Case II : $\pi_1(A_1, s_2^*, s_3^*) = \pi_1(B_1, s_2^*, s_3^*) < \pi_1(C_1, s_2^*, s_3^*)$**.**

Since the payoff function is linear over T_1, the payoff of player 1 is constant and minimum along the edge of T_1 that connects the two pure strategy vertices A_1 and B_1. Hence, $I_{A_1} = I_{B_1} = 0$. Then, by Equation 3.17,

$$s_1^* = \frac{s_1^* + I_{C_1} C_1}{1 + I_{C_1}}. \tag{3.19}$$

Now, s_1^* is a mixed strategy and can be represented as

$$s_1^* = \alpha^* A_1 + \beta^* B_1 + \gamma^* C_1 \ , \quad \text{where} \quad \alpha^* + \beta^* + \gamma^* = 1. \tag{3.20}$$

Thus, in Equation 3.19,

$$\alpha^* A_1 + \beta^* B_1 + \gamma^* C_1 = \frac{\alpha^* A_1}{1 + I_{C_1}} + \frac{\beta^* B_1}{1 + I_{C_1}} + \frac{(\gamma^* + I_{C_1})\, C_1}{1 + I_{C_1}}.$$

Since A_1, B_1, and C_1 are independent vectors, we have

$$\alpha^* = \frac{\alpha^*}{1 + I_{C_1}} \ , \ \beta^* = \frac{\beta^*}{1 + I_{C_1}} \ , \ \gamma^* = \frac{\gamma^* + I_{C_1}}{1 + I_{C_1}}. \tag{3.21}$$

Here there are two sub-cases:

1. If either α^* or β^* is non-zero, then $1 + I_{C_1} = 1$, so $I_{C_1} = 0$.
2. If both $\alpha^* = \beta^* = 0$, then $\gamma^* = 1$ and $s_1^* = C_1 = (0, 0, 1)$. Then, by definition (as seen in Equation 3.14), $I_{C_1} = 0$.

Thus, in Case II, we get $I_{A_1} = I_{B_1} = I_{C_1} = 0$.

Case III : $\pi_1\left(A_1, s_2^*, s_3^*\right) < \pi_1\left(B_1, s_2^*, s_3^*\right) \leq \pi_1\left(C_1, s_2^*, s_3^*\right)$**.**

Linearity of the payoff function over T_1 implies that the payoff of player 1 is minimum for the pure strategy A_1. Hence, $I_{A_1} = 0$. By Equation 3.17,

$$s_1^* = \frac{s_1^* + I_{B_1}B_1 + I_{C_1}C_1}{1 + I_{B_1} + I_{C_1}}. \tag{3.22}$$

As before, s_1^* is a mixed strategy and can be represented as in Equation 3.20. Thus,

$$\alpha^* A_1 + \beta^* B_1 + \gamma^* C_1 = \frac{\alpha^* A_1}{1 + I_{B_1} + I_{C_1}} + \frac{(\beta^* + I_{B_1})B_1}{1 + I_{B_1} + I_{C_1}} + \frac{(\gamma^* + I_{C_1})C_1}{1 + I_{B_1} + I_{C_1}}.$$

Since $A_1,\ B_1,\ C_1$ are independent vectors, we have

$$\alpha^* = \frac{\alpha^*}{1 + I_{B_1} + I_{C_1}}\ ,\ \beta^* = \frac{(\beta^* + I_{B_1})}{1 + I_{B_1} + I_{C_1}}\ ,\ \gamma^* = \frac{(\gamma^* + I_{C_1})}{1 + I_{B_1} + I_{C_1}}. \tag{3.23}$$

Again, there are two sub-cases here:

1. If $\alpha^* \neq 0$, then $1 + I_{B_1} + I_{C_1} = 1$. Hence, $I_{B_1} + I_{C_1} = 0$. Since I_{B_1} and I_{C_1} are non-negative, this forces $I_{B_1} = I_{C_1} = 0$.

2. If $\alpha^* = 0$, then by Equation 3.20, $\beta^* + \gamma^* = 1$. This means that s_1^* is on the segment connecting B_1 and C_1. Since $\pi_1\left(B_1, s_2^*, s_3^*\right) \leq \pi_1\left(C_1, s_2^*, s_3^*\right)$, it follows by the linearity of the payoff function over T_1 that on the segment, the payoff of player 1 is minimum for the pure strategy B_1. Hence, $I_{B_1} = 0$. Then by equation 3.23, we have

$$\beta^* = \frac{\beta^*}{1 + I_{C_1}}\ ,\ \gamma^* = \frac{(\gamma^* + I_{C_1})}{1 + I_{C_1}}. \tag{3.24}$$

   Now β^* can be zero or non-zero.

(a) If $\beta^* \neq 0$, then $1 + I_{C_1} = 1$ and hence, $I_{C_1} = 0$.

(b) If $\beta^* = 0$, then $\gamma^* = 1$ in Equation 3.20. In other words, $s_1^* = C_1 = (0, 0, 1)$. As before, $I_{C_1} = 0$ by definition.

Hence, in Case III, we have $I_{A_1} = I_{B_1} = I_{C_1} = 0$.

Summarizing over all three possible cases, we conclude that

$$I_{A_1}(s_1^*, s_2^*, s_3^*) = I_{B_1}(s_1^*, s_2^*, s_3^*) = I_{C_1}(s_1^*, s_2^*, s_3^*) = 0. \quad (3.25)$$

The same argument holds for players 2 and 3, and therefore,

$$s_i^* = \frac{s_i^* + I_{A_i}A_i + I_{B_i}B_i + I_{C_i}C_i}{1 + I_{A_i} + I_{B_i} + I_{C_i}}$$

implies that $I_{A_i}(s_1^*, s_2^*, s_3^*) = I_{B_i}(s_1^*, s_2^*, s_3^*) = I_{C_i}(s_1^*, s_2^*, s_3^*) = 0$ for $i = 1, 2, 3$. Thus, by Theorem 3.4, (s_1^*, s_2^*, s_3^*) is a Nash equilibrium. □

Note that this proof can be extended to establish existence of a Nash equilirium for n players in a non-cooperative game where the i^{th} player has τ_i pure strategies.

3.2.3 Example of a 2 - Player, 2 - Choice Game

Working with our example of a 2 - player game that we have been referring to, let us look at an algorithm that will enable us to determine Nash equilibria for the game.

Recall that the two players X and Y had two choices each $\{N, Q\}$. Also, the pure strategies for players X and Y are $\{p_{11}, p_{12}\}$ and $\{p_{21}, p_{22}\}$ respectively.

Suppose X chooses to play N with probability $p.$ Since there are only two choices available to player $X,$ it means that player X chooses Q with the probability $1 - p.$ Similarly, player Y chooses N with probability q and Q with probability $1 - q.$ So the mixed strategies of players X and Y are $(p,\ 1 - p)$ and $(q,\ 1 - q),$ respectively.

For each combination of strategies of the two players, there is an attached payoff. Tabulating all this data gives us the **payoff matrix** for the game. Any entry in the matrix, written as $(\pi_1, \pi_2),$ shows the payoffs to the two players given the choice combination. Note that $\pi_1,$ and π_2 depend upon the choices made by both players X and $Y.$ In this paper, we will limit ourselves to examining two example games and the Nash equilibria in each case.

3.2.3.1 Game I

As we saw before, suppose the game rule is that if the coins match, player X gets player Y's played coin and if the coins do not match, player Y gets both of player X's coins. Then the payoffs, or profits here, will be given by:

	Available Choices	**Player Y**	
		N	Q
Player X	N	$(5, -5)$	$(-30, 30)$
	Q	$(-30, 30)$	$(25, -25)$

Table 3.1. Payoff matrix I

Here, the entry $(5, -5)$ means that if player X plays N and player Y also plays $N,$ then the payoff to player X is 5 cents and to player Y is -5 cents (that is, player Y loses 5 cents).

To find a Nash equilibrium, we begin by examining the payoffs to each player when they play pure strategies and the other player can play any mixed strategy. In our example, the payoff for player X from choosing to play the pure strategy $p_{11} = (1, 0)$ (that is, play a nickel every time) depends upon the probability q that player Y chooses to play N and the probability $1 - q$ that player Y chooses strategy Q. Thus the payoff for player X from choosing the pure strategy p_{11} is given by:

$$\pi_1 = (5)q + (-30)(1 - q) = 35q - 30 . \tag{3.26}$$

By the same argument, the payoff for player X from choosing the pure strategy p_{12} is given by:

$$\pi_1 = (-30)q + (25)(1 - q) = 25 - 55q . \tag{3.27}$$

If we graph both these equations (Figure 3.8), they intersect at $q^* = \frac{11}{18}$. We claim that this is the Nash equilibrium strategy for player Y, that is, to play a nickel 11 out of 18 times.

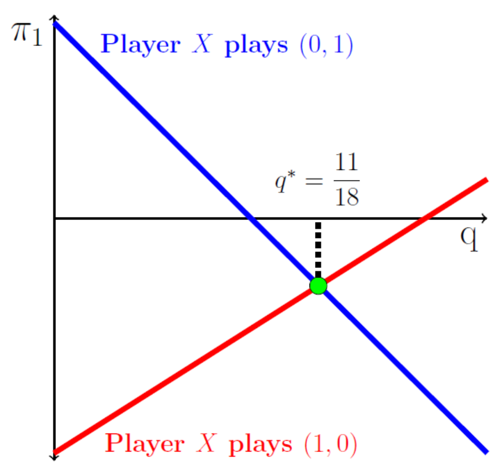

Figure 3.8. $q^* = \frac{11}{18}$ is a Nash equilibrium strategy

Let $q_0 < \frac{11}{18}$ be fixed. Given this mixed strategy for player $Y,$ all possible payoffs for player X lie on the part of the line $q = q_0$ between the two pure strategy payoff lines of player $X.$ Given that player Y is playing mixed strategy $(q_0,\ 1 - q_0),$ player X's best response then is to play the pure strategy $p_{12} = (0, 1).$ This does not yield a Nash equilibrium strategy, however, because if player X plays pure strategy $p_{12},$ player Y will maximize her/his payoff by playing the pure strategy $p_{21} = (1, 0),$ and not playing $(q_0,\ 1 - q_0).$

So a Nash equilibrium does not occur when player Y plays mixed strategy $(q_0,\ 1 - q_0)$ with $q_0 < \frac{11}{18}.$ A similar argument shows there is no Nash equilibrium possible when player Y plays mixed strategy $(q_0,\ 1 - q_0)$ with $q_0 > \frac{11}{18}$ either. Note that when player Y plays the mixed strategy $(q^*,\ 1 - q^*)$ with $q^* = \frac{11}{18},$ all mixed strategies for player X provide the same payoff, so in this case, the payoff to player X is maximized with every choice of mixed strategy. Thus, π_1 is maximized at mixed strategy $\left(\frac{11}{18}, \frac{7}{18}\right)$ for player $Y,$ that is, when player Y chooses to play the nickel 11 out of every 18 games.

A similar analysis yields that when player X plays the mixed strategy $\left(\frac{11}{18}, \frac{7}{18}\right),$ the payoff for player Y is maximized for every choice of mixed strategy for player $Y.$

It follows that the Nash equilibrium for the game is at the mixed strategy combination

$$s = \left(\left(\frac{11}{18}, \frac{7}{18}\right), \left(\frac{11}{18}, \frac{7}{18}\right) \right).$$

3.2.3.2 Game II

Suppose the game rule is that if the coins played by the two players match, each player gets twice the amount they put in. If they do not match, the coins are exchanged, that is, each player gets the coin played by the other. Then the payoffs will be given by:

	Available Choices	**Player** Y	
		N	Q
Player X	N	$(5, 5)$	$(20, -20)$
	Q	$(-20, 20)$	$(25, 25)$

Table 3.2. Payoff matrix II

When we analyze the best response of each player to the strategies chosen by the other player, we find that there are two pure-strategy Nash equilibria for this game. This is because if player Y plays N all of the time, then player X receives maximum payoff by playing N all the time as well, and vice versa. Thus, the pure strategy combination $((1,0),(1,0))$ is a Nash equilibrium.

By a similar argument, there is also a pure strategy Nash equilibrium at $((0,1),(0,1))$, having both players play Q all of the time. Thus, the two pure-strategy Nash equilibria for this game are at (p_{11}, p_{21}) and (p_{12}, p_{22}). Note that in Game I, there are no pure strategy Nash equilibria since there is no pure strategy combination that is a mutual best response to the other players' strategy.

By an analysis similar to the one we did for the first example, we find that the game has a mixed strategy Nash equilibrium at

$$s = \left(\left(\frac{1}{6}, \frac{5}{6}\right), \left(\frac{1}{6}, \frac{5}{6}\right)\right).$$

There is a notion of stability for Nash equilibria. We don't address it formally here but in a 2 - player game, a Nash equilibrium is **stable** if when either player makes a small change in strategy, the other player has no incentive to change her/his strategy and the player who made change is then compelled to return to playing the strategy at Nash equilibrium that they deviated from. On the other hand, a Nash equilibrium is **unstable** if a small change in strategy by one player induces the other player to make a major change in her/his strategy.

In our example game, Game II, we can see that the pure strategy Nash equilibria are stable. For instance, if the two players are playing with Nash equilibrium strategy combination (p_{11}, p_{21}) and player Y chooses to play Q instead of N, player X will continue to play N as she/he is actually getting a higher payoff with the change. Player Y on the other hand will lose 20 cents instead of earning 5 cents and will be forced to revert to the original strategy of playing N all the time. Also, by a similar argument, it can be seen that the mixed strategy Nash equilibrium for this game is unstable since even a small change in strategy by one player will create responses that will cause both players to move away from that Nash equilibrium strategy to one of the pure strategy Nash equilibria.

Clearly, depending on the game rules and the resultant payoffs, equilibrium strategies for games will vary. It is an interesting exercise to determine the Nash equilibria for different games and examine their stability.

CHAPTER 4

THE KAKUTANI FIXED POINT THEOREM

Another theorem with important economic applications is the Kakutani fixed point theorem given by Shizuo Kakutani in 1941 as a generalization of the Brouwer fixed point theorem [12]. Though it is not as intuitive as the Brouwer fixed point theorem since it deals with set-valued functions and hence is more difficult to visualize, the theorem found popularity in mathematical economics, especially after John Nash used it to prove the existence of equilibrium points in n player games [11]. In this paper, we use the Brouwer fixed point theorem to prove the Kakutani fixed point theorem and then use the latter to prove the existence of equilibria in a pure exchange economy.

Before that, let us first understand what is meant by fixed points of set-valued functions.

Definition. *For a set-valued function, $f : X \to \mathbb{P}(X)$, a **fixed point** is a point that is mapped to a set containing itself. In other words, x is a fixed point of f if $x \in f(x)$.*

4.1 The Kakutani Fixed Point Theorem

Recall that the Brouwer fixed point theorem proves the existence of fixed points for point-valued functions. We will see in the subsequent sections that the Kakutani fixed point theorem asserts the existence of fixed points for set-valued functions. As we proceed, we will find that the former theorem is helpful in understanding and proving the latter. Let us begin by stating and proving the Kakutani fixed point theorem in one dimension.

4.1.1 The One-Dimensional Kakutani Fixed Point Theorem

Theorem 4.1. *Let* $f : [0,1] \to \mathbb{P}([0,1])$ *be a continuous function such that for every* $x \in [0,1]$, *the image* $f(x)$ *is an interval in* $[0,1]$. *Then there exists* $x \in [0,1]$ *such that* $x \in f(x)$ *.*

[Note: we assume that for each x in $[0,1]$, its image is an interval $[a,b]$, $(a,b]$, $[a,b)$, or (a,b) in $[0,1]$. The importance of this assumption will become clear in the proof.]

Proof. Let $n \in \mathbb{N}$ be arbitrary. Subdivide $[0,1]$ into n sub-intervals of equal width $\left[\frac{j}{n}, \frac{j+1}{n}\right]$ where $0 \leq j \leq n-1$. For each $\frac{j}{n}$, where $j = 0, ..., n$, pick a point $q_{j,n} \in f\left(\frac{j}{n}\right)$.

Define a point-valued function $g_n : [0,1] \to [0,1]$ such that

$$g_n\left(\frac{j}{n}\right) = q_{j,n}$$

and g_n maps linearly on each sub-interval $\left[\frac{j}{n}, \frac{j+1}{n}\right]$ (See Figure 4.1).

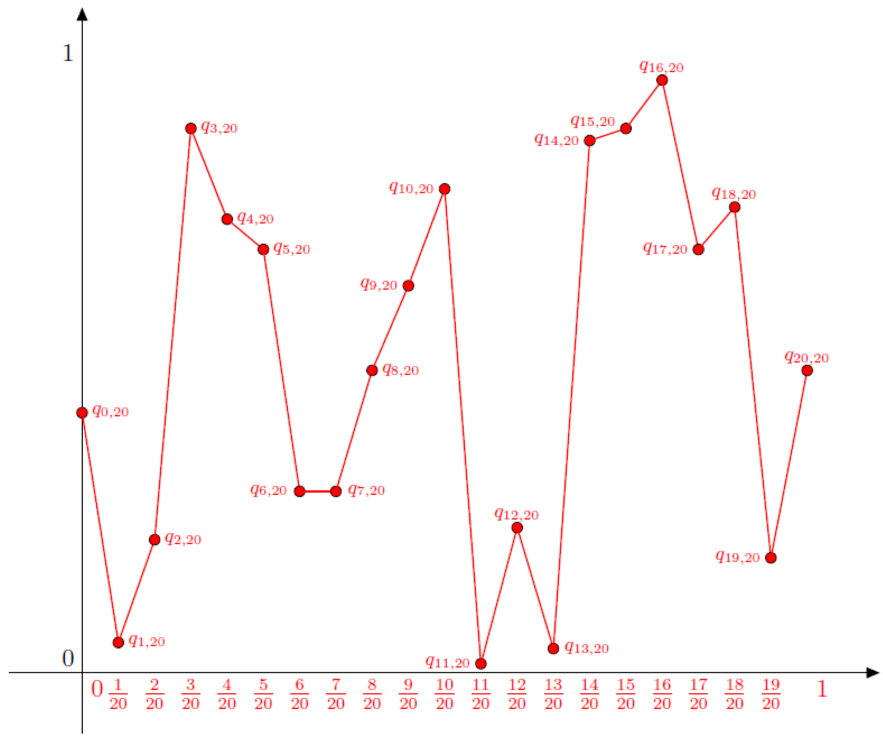

Figure 4.1. An example of constructing g_n for $n = 20$

Note that g_n is a point-valued continuous function mapping $[0,1] \to [0,1]$. Then by the one-dimensional Brower fixed point theorem, g_n has a fixed point x_n.

Suppose $x_n \in \left[\dfrac{i_n}{n}, \dfrac{i_n+1}{n}\right]$. Then

$$x_n = \alpha_n \left(\frac{i_n}{n}\right) + (1 - \alpha_n)\left(\frac{i_n+1}{n}\right) \qquad \text{for some } \alpha_n \in [0,1]. \tag{4.1}$$

Since g_n is a linear function on sub-intervals, we have

$$g_n(x_n) = \alpha_n \left[g_n\left(\frac{i_n}{n}\right)\right] + (1 - \alpha_n)\left[g_n\left(\frac{i_n+1}{n}\right)\right]. \tag{4.2}$$

Recall that

$$g_n\left(\frac{i_n}{n}\right) \;=\; q_{i_n,n} \quad \text{and} \quad g_n\left(\frac{i_n+1}{n}\right) \;=\; q_{i_n+1,n}$$

are chosen such that

$$q_{i_n,n} \;\in\; f\left(\frac{i_n}{n}\right) \quad \text{and} \quad q_{i_n+1,n} \;\in\; f\left(\frac{i_n+1}{n}\right).$$

Thus, by Equation 4.2, we get

$$g_n(x_n) \;=\; \alpha_n\, q_{i_n,n} \;+\; (1 \;-\; \alpha_n)\, q_{i_n+1,n}\;. \tag{4.3}$$

Now, x_n is a fixed point of g_n and hence, by Equation 4.3,

$$x_n \;=\; g_n(x_n) \;=\; \alpha_n\, q_{i_n,n} \;+\; (1 \;-\; \alpha_n)\, q_{i_n+1,n}\;. \tag{4.4}$$

Combine the sequences (x_n), (α_n), $(q_{i_n,n})$, and $(q_{i_n+1,n})$ into a sequence $(x_n\,,\;\alpha_n,\;q_{i_n,n}\,,\;q_{i_n+1,n})$ in $[0,1] \times [0,1] \times [0,1] \times [0,1]$. This is a sequence in a closed and bounded space, so it has a convergent subsequence. Just like we did before, for simplicity, assume that the original sequence converges. So assume (x_n) converges to $x \in [0,1]$, $(q_{i_n,n})$ converges to $q_0 \in [0,1]$, $(q_{i_n+1,n})$ converges to $q_1 \in [0,1]$ and (α_n) converges to $\alpha \in [0,1]$.

We claim that x is a fixed point of f, that is, $x \in f(x)$. To prove the claim, we begin by taking the limit in Equation 4.4 in order to obtain

$$x \;=\; \alpha\, q_0 \;+\; (1 \;-\; \alpha)\, q_1. \tag{4.5}$$

Note that, by Theorem 1.2,

$$\left(\frac{i_n}{n}\right) \;\to\; x \quad \text{and} \quad \left(\frac{i_n+1}{n}\right) \;\to\; x$$

since $x_n \in \left[\frac{i_n}{n}, \frac{i_n + 1}{n}\right]$ and the length of the sub-intervals $\left[\frac{i_n}{n}, \frac{i_n + 1}{n}\right]$ goes to 0. Also, since $q_{i_n,n} \to q_0$ and $q_{i_n,n} \in f\left(\frac{i_n}{n}\right)$, it follows by the continuity of f that

$$q_0 \in f(x).$$

Similarly, $q_1 \in f(x)$. Since $f(x)$ is an interval in $\mathbb{R}$, it follows that

$$\alpha q_0 + (1 - \alpha) q_1 \in f(x). \tag{4.6}$$

In other words, $x \in f(x)$ as claimed. Hence, x is a fixed point of f, and the one-dimensional Kakutani fixed point theorem is proved. □

4.1.2 Kakutani Fixed Point Theorem in n Dimensions

Generalizing the one-dimensional result, we have the following n - dimensional Kakutani fixed point theorem.

Theorem 4.2. *Given the standard n - simplex S and a continuous set-valued function $f : S \to \mathbb{P}(S)$ such that $f(x) \subset S$ is convex for all $x \in S$, then f has a fixed point in S.*

Proof. Let $f : S \to \mathbb{P}(S)$ be a continuous function such that $f(x)$ is a convex subset of S for all $x \in S$.

As we did before for an n - dimensional simplex, take finer and finer subdivisions $\mathscr{S}_1, \mathscr{S}_2, ..., \mathscr{S}_i, ...$ such that the diameter of sub-simplices goes to 0 as $i \to \infty$. Let $\mathbb{V} = \{v_{i,1}, ..., v_{i,n_i}\}$ be the set of all of the vertices of the sub-simplices of S in the i^{th} subdivision.

Define $f_i : S \rightarrow S$ such that $f_i(v_{i,j}) \in f(v_{i,j})$ for all j and such that f_i extends linearly over sub-simplices. Note that f_i is a continuous point-valued function mapping S to itself. Thus, by the n-dimensional Brouwer fixed point theorem, f_i has a fixed point x_i in S.

Since x_i lies in some sub-simplex in the i^{th} subdivision of S, it can be expressed as:

$$x_i = \alpha_{i,0}w_{i,0} + ... + \alpha_{i,n}w_{i,n} \tag{4.7}$$

where $w_{i,0}, ..., w_{i,n} \in \mathbb{V}$ are the vertices of the sub-simplex containing x_i, each $\alpha_{i,k} \in [0,1]$, and

$$\sum_{k=0}^{n} \alpha_{i,k} = 1. \tag{4.8}$$

Let $f_i(w_{i,k}) = u_{i,k}$. Since $w_{i,k} \in \mathbb{V}$, by definition of f_i, it follows that $u_{i,k} \in f(w_{i,k})$.

From Equation 4.6, we get that

$$f_i(x_i) = \alpha_{i,0}f_i(w_{i,0}) + ... + \alpha_{i,n}f_i(w_{i,n}) = \alpha_{i,0}u_{i,0} + ... + \alpha_{i,n}u_{i,n} \tag{4.9}$$

since f_i is a linear function on subsimplices in $\mathscr{S}_i$. Moreover, we know that x_i is a fixed point of f_i and so, $f_i(x_i) = x_i$. Thus,

$$x_i = \alpha_{i,0}u_{i,0} + ... + \alpha_{i,n}u_{i,n}. \tag{4.10}$$

We have sequences indexed by i : (x_i) in S, $(\alpha_{i,k})$ in $[0,1]$ for each k, $(u_{i,k})$ in S for each k. By our usual convergence arguments, we can find sub-sequences of each of these sequences converging simultaneously to limits for each. For simplicity, we assume that the above sequences themselves converge.

So assume $x_i \to x \in S$, $\alpha_{i,k} \to \alpha_k \in [0,1]$ for each k, and $u_{i,k} \to u_k \in S$ for each k. We claim that x is a fixed point of f. That is, $x \in f(x)$.

First note that x_i is in the subdivision simplex with vertices $w_{i,0}, ..., w_{i,n}$ and the subdivision simplices have diameters that go to 0 as i becomes infinite. It follows, by Theorem 1.2, that each sequence $(w_{i,k})$ also converges to x as $i \to \infty$.

So for each k, $u_{i,k} \in f(w_{i,k})$, $u_{i,k} \to u_k$, and $w_{i,k} \to x$. By the continuity of f, it follows that $u_k \in f(x)$ for all k.

Furthermore, taking the limit as $i \to \infty$ in Equations 4.8 and 4.10, we obtain

$$\sum_{k=1}^{n} \alpha_k = 1 \tag{4.11}$$

$$x = \alpha_0 u_0 + ... + \alpha_n u_n \tag{4.12}$$

where $\alpha_k \in [0,1]$ for all k . Thus, since each $u_k \in f(x)$, and $f(x)$ is convex, it follows that $x \in f(x)$.

Hence, x is a fixed point of f and we have proved the Kakutani fixed point theorem for dimension n. □

4.2 An Application of the Kakutani Fixed Point Theorem to Equilibrium in Economic Models

Recall that an equilibrium is a state of rest where opposing forces balance. In economic models, equilibrium in a goods market refers to a point where the supply of goods meets the demand for them. At this stage, there is no tendency within the market for the price to change. Thus, the balance of demand and supply remains undisturbed at equilibrium, unless affected by external forces. In this section, we use the Kakutani fixed point theorem to show the existence of equilibria in a pure exchange economy, that is, an economy where there is no production.

Let us assume the economy is a pure exchange economy. All economic agents are consumers and have an initial endowment of goods. Thus, the total amount of each good in the economy is fixed and consumption depends upon initial endowments as well as exchange (trade) between consumers.

Suppose there are m consumers and n goods in the economy. Each consumer is endowed with a bundle of goods. Let the **bundle of goods** owned by the i^{th} person be

$$b^i = (b^i_1,\ b^i_2,\ ...,\ b^i_n)$$

where b^i_j represents the quantity of good j that the i^{th} consumer has.

Since there is no production, the supply of goods in the economy is fixed and can be found by summing up the bundles of goods owned by each consumer. So the fixed **supply vector** is

$$s = (s_1, s_2, ..., s_n) \quad \text{where} \quad s_j = \sum_{i=1}^{m} b^i_j \quad \text{for all} \quad j = 1, ..., n. \tag{4.13}$$

Each good has a relative price or value, p_j, associated to it. We assume these prices are non-negative $(p_j \geq 0)$ and are normalized to sum to 1 $\left(\text{that is, } \sum_{j=1}^{n} p_j = 1\right)$. In the latter sense, they are relative prices.

We call the vector $p = (p_1, p_2, ..., p_n)$ a **price vector**. The set of all price vectors forms an $(n-1)-$ simplex, $S \in \mathbb{R}^n$ with vertices $(1,0,...,0), (0,1,...,0), ...,$ $(0,0,...,1)$.

In an exchange economy, the **wealth** of the i^{th} consumer, w^i, is completely determined by the value of the goods owned by her/him. It is defined by

$$w^i = b^i \cdot p. \tag{4.14}$$

A **demand vector** for the i^{th} individual is of the form

$$d^{\,i}(p) = (d_1^{\,i}(p),\ d_2^{\,i}(p),\ ...,\ d_n^{\,i}(p))$$

where $d_j{}^i$ is the demand for good j by the i^{th} consumer and it is a function of the prices of all the goods in the economy. In our analysis, we are making the reasonable assumption that the demand for a good by a consumer is not a fixed amount but can be within a range. This means that the consumer has a minimum and maximum amount of the good that she/he wants. However, since the wealth of every consumer is fixed (it depends upon the initial endowment), when a consumer demands more of one good, they automatically have to demand less of another good. Therefore, the budget constraint coupled with the range of preferences of the i^{th} consumer gives rise to a set of demand vectors, $D^i(p)$, from which she/he can choose, corresponding to a given price vector, p. That is,

$$D^i(p) = \{\ d^{\,i}(p)\ \mid\ d^{\,i}(p) = (d_1^{\,i}(p), d_2^{\,i}(p), ..., d_n^{\,i}(p))\ \}. \tag{4.15}$$

We call $D^i(p)$ the **demand set (for consumer i at price p)**. Given that each consumer cannot demand more than they can afford, the value of each demand vector equals the wealth of the consumer. Thus, for each $d^{\,i} \in D^i(p)$,

$$d^{\,i} \cdot p = b^i \cdot p\,. \tag{4.16}$$

Each $D^i(p)$ is set valued. Given the price p, $D^i(p)$ represents the different bundles of goods the i^{th} consumer can afford and would be equally happy possessing; in other words, she/he would be indifferent between these bundles.

We assume that the mapping from p to the set $D^i(p)$ is continuous, reflecting that a small change in p should result in only a small change in the demand.

Given any two demand vectors v_1 and v_2 in the set $D^i(p)$, we claim that the consumer can afford any linear combination of the two demand vectors, that is any v such that

$$v = t\,v_1 + (1-t)\,v_2\,; \quad t \in [0,1].$$

To prove this claim, we get from Equation 4.16 that

$$v_1 \cdot p = b^i \cdot p = v_2 \cdot p$$

and therefore, using Equation 4.14,

$$v \cdot p = t\,v_1 \cdot p + (1-t)\,v_1 \cdot p = t\,b^i \cdot p + (1-t)\,b^i \cdot p = b^i \cdot p = w^i.$$

Thus, the consumer can afford any linear combination of two demand vectors. We assume that if the consumer prefers two bundles of goods, then the consumer will be equally happy with a linear combination of the two bundles. By this assumption, any linear combination of two demand vectors is also a demand vector at price p. In other words, we assume that each $D^i(p)$ is convex.

The assumption that $D^i(p)$ is a convex set will come in useful later in establishing the premise for the Kakutani fixed point theorem, as we will see.

For a given price, p, the aggregate demand in the economy can be found by summing up all possible individual demands over all consumers. Thus, we define the **overall demand** by

$$D(p) = \left\{ \sum_{i=1}^{m} d^{\,i} \;\mid\; d^{\,i} \in D^i(p) \quad \text{for all} \quad i \right\}. \tag{4.17}$$

Since each $D^i(p)$ is a convex set, we can see that $D(p)$ is convex in $\mathbb{R}^n$.

The **excess demand** for a good is the difference between its demand at a given price and the supply. Thus, excess demand is a function of price. For a price p, the excess demand is the set

$$E(p) = \{ e \mid e = d - s \,;\, d \in D(p) \}. \tag{4.18}$$

Note that $E(p) \subset \mathbb{R}^n$ is a convex set, since the supply vector s is fixed and $D(p)$ is a convex set for each p.

In Equation 4.16, summing over all m consumers, we get that for each $d \in D(p)$,

$$p \cdot d = p \cdot s\,. \tag{4.19}$$

Thus, for each $e \in E(p)$,

$$p \cdot e = 0. \tag{4.20}$$

Geometrically speaking, each $e \in E(p)$ is orthogonal to p in $\mathbb{R}^n$. This is known as the Walrasian law [13] in general equilibrium theory. This means the value of excess demand in the economy is always zero, whether or not there is equilibrium in the economy. So, the sum of values of excess demand across all goods must be zero; that is, if positive excess demand exists for some goods, then negative excess demand must exist for other goods to balance it out due to budget constraints.

Equilibrium in the economy exists at a price p for which there exists $d \in D(p)$ such that $d_j \leq s_j$ for all $j = 1, 2, ..., n.$ At this price, the overall demand for each good is not more than the supply of that good in the economy and so the demand can be met. Thus, there will be no tendency for the price to change within the market and the economy would be at rest. This price p is called an **equilibrium price vector for the economy**. We use the Kakutani fixed point theorem to show the existence of a price equilibrium vector for an economy such as the one described above.

Theorem 4.3. *Assume we have a pure exchange economy with n goods and m consumers. Further assume we have $p,$ $D^i(p)$ and $s = (s_1, ..., s_n)$ as defined above. In particular, assume $D^i(p)$ is set-valued and continuous and such that $D^i(p)$ is convex. Then there exists an equilibrium price vector for the economy.*

We prove Theorem 4.3 through a series of results below. To begin, for each $e \in E(p),$ let $f(e, p) \in \mathbb{R}^n$ define a **price tendency vector** whose j^{th} component is

$$f_j(e, p) = \frac{\max\{p_j + e_j, 0\}}{\sum_{j=1}^{n} \max\{p_j + e_j, 0\}}. \tag{4.21}$$

Thus, $f_j(e, p)$ represents the tendency of the price of good j to change if there is excess demand for the good in the economy at price $p.$ It reflects the idea that if there is excess demand for a good, then the relative price of it increases. Note that $f_j(e, p)$ is only a tool we use in the proof of Theorem 4.3 and is not necessarily of direct importance in the economy.

We need to check that $f(e, p)$ is defined. In other words, we need to show that

$$\sum_{j=1}^{n} \max \{p_j + e_j, 0\} \neq 0.$$

Lemma 4.1. *For each* $e \in E(p)$, *there exists* j *such that* $p_j + e_j > 0$.

Proof. By Equation 4.20,

$$(p + e) \cdot p = p \cdot p > 0.$$

Since the dot product $(p + e) \cdot p$ is positive, and $p_j \geq 0$ for all j, there exists j such that $p_j + e_j > 0$. □

By Lemma 4.1, for a given p, $f(e, p)$ is defined for all $e \in E(p)$ since the denominator is non-zero. By definition of $f(e, p)$, it follows that $f(e, p) \in S$. Then, for a price p, the set of all the price adjustment vectors, $F(p)$, is given by

$$F(p) = \{ f(e, p) \mid e \in E(p) \} \subset S. \tag{4.22}$$

Therefore $F : S \to \mathbb{P}(S)$ defines a set valued function. Since $E(p)$ is convex in $\mathbb{R}^n$, it is not difficult to see that $F(p)$ is a convex set in S for each p. Also, it is not difficult to show that F is continuous since each mapping p to $D^i(p)$ is continuous.

Thus, the assumptions of the Kakutani fixed point theorem are satisfied, and it follows that there exists a fixed point $p^* \in S$. By definition of fixed point,

$$p^* \in F(p^*).$$

In other words,

$$p^* \; = \; f(e^*, p^*) \quad \text{for some} \quad e^* \; \in \; E(p^*). \tag{4.23}$$

Then by Equation 4.21, for all $j \; = \; 1, 2, ..., n,$

$$p_j^* \; = \; \frac{\max \, \{p_j^* \; + \; e_j^*, \; 0\}}{\sum_{j=1}^{n} \max \, \{p_j^* \; + \; e_j^*, \; 0\}}. \tag{4.24}$$

We want to show that p^* is an equilibrium price vector. In order to prove that, we will begin by stating and proving a simple lemma that will help us.

Lemma 4.2. $\sum_{j=1}^{n} \max \, \{p_j^* \; + \; e_j^*, \; 0\} \; = \; 1.$

Proof. Let $k \; = \; \sum_{j=1}^{n} \max \, \{p_j^* \; + \; e_j, \; 0\}.$ We claim that for all $j,$

$$p_j^* \, e_j^* \; = \; (k \; - \; 1) \, p_j^* \, p_j^*.$$

Clearly, this equation is true if $p_j^* \; = \; 0.$ Consider the case $p_j^* \; > \; 0.$ Then by Equation 4.24,

$$\frac{\max \, \{p_j^* \; + \; e_j^*, \; 0\}}{k} \; = \; p_j^* \; > \; 0.$$

It follows that $p_j^* \; + \; e_j^* \; > \; 0$ and $k \, p_j^* \; = \; p_j^* \; + \; e_j^*.$

Hence $e_j^* \; = \; (k \; - \; 1) \, p_j^*,$ and therefore $p_j^* \, e_j^* \; = \; (k \; - \; 1) \, p_j^* \, p_j^*.$ Since this holds for all $j,$ we conclude that

$$p^* \; \cdot \; e^* \; = \; (k \; - \; 1) \, p^* \; \cdot \; p^*.$$

However, $p^* \; \cdot \; p^* \; > \; 0$ and by Equation 4.20, $p^* \; \cdot \; e^* \; = \; 0.$ Therefore, $k \; - \; 1 \; = \; 0$ and hence, $k \; = \; 1.$ □

Lemma 4.2 will help us in our objective of showing that p^* is an equilibrium price vector.

Theorem 4.4. *p^* is an equilibrium price vector.*

Proof. Since p^* is a fixed point of F, $p^* = f(e^*, p^*)$ for $e^* = d^* - s$ in $E(p^*)$ where $d^* \in D(p^*)$. Hence,

$$d^* = e^* + s. \tag{4.25}$$

We want to show that $d_j^* \leq s_j$ for all $j = 1, 2, ..., m,$ and therefore p^* is an equilibrium price vector.

By Lemma 4.1, $\sum_{j=1}^{n} \max\{p_j^* + e_j^*, 0\} = 1.$ Then by Equation 4.24, for all $j = 1, 2, ..., n,$

$$p_j^* = \max\{p_j^* + e_j^*, 0\}. \tag{4.26}$$

For each $j,$ we have two possibilities:

1. $p_j^* = 0$:

   Then, in Equation 4.26, $\max\{p_j^* + e_j^*, 0\} = 0.$ Therefore, $e_j^* \leq 0.$

   By Equation 4.25, $d_j^* = e_j^* + s_j,$ and hence $d_j^* \leq s_j$

2. $p_j^* > 0$:

   Then, in Equation 4.26, $p_j^* = p_j^* + e_j^*.$ Therefore, $e_j^* = 0.$

   By Equation 4.25, $d_j^* = e_j^* + s_j,$ and hence $d_j^* = s_j.$

Thus, $d_j^* \leq s_j$ for all $j = 1, 2, ..., n.$ Hence, the fixed point p^* of the function F is an equilibrium price vector. □

Via the results above, we have now proven Theorem 4.3.

Thus, the Kakutani fixed point theorem allows us to establish the existence of a price equilibrium in a pure exchange economy. Since the price equilibrium vector is a fixed point of the function $F(p)$ as defined above, it is possible to have an equilibrium price vector where the economy is at rest and there is a demand vector at that price that is met by the supply.

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Ayesha Maliwal was born in Delhi, India on July 25, 1988. She was raised in Noida, India and graduated from Amity International School in 2005. She attended The University of Delhi and graduated in 2009 with a Bachelor's degree in Economics. She became interested in the field of alternate education and trained to be a Montessori educator in Mumbai, India under Association Montessori International certification. She returned to Delhi and worked as a Montessori facilitator for a year before deciding to join graduate school and further her education. Ayesha entered the Economics graduate program at Amity University, India in the fall of 2012. While she was pursuing her graduation in economics, she applied and was accepted into the Mathematics graduate program at The University of Maine. After completing her graduation in economics, she travelled to Maine and enrolled in the Mathematics graduate program in the fall of 2014. After receiving her second graduate degree in Mathematics, Ayesha will be teaching as an adjunct instructor in the Department of Mathematics at The University of Maine, to begin her career as a math educator. Ayesha is a candidate for the Master of Arts degree in Mathematics from the University of Maine in December 2016. Ayesha Maliwal is a candidate for the Master of Arts degree in Mathematics from The University of Maine in December 2016.