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# Calculus Students' Reasoning About Slope and Derivative as Rates of Change

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# CALCULUS STUDENTS' REASONING ABOUT SLOPE AND DERIVATIVE AS RATES OF CHANGE


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**THESIS ACCEPTANCE STATEMENT**

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# CALCULUS STUDENTS' REASONING ABOUT SLOPE AND DERIVATIVE AS RATES OF CHANGE

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An Abstract of the Thesis Presented
in Partial Fulfillment of the Requirements for the
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Students' low success rates in college calculus courses are a factor that leads to high attrition rates from science, technology, engineering and mathematics (STEM) degree programs. To help reach our nation's goal of one million additional STEM majors in the next decade, we must address the conceptual difficulties of our students. Studies have shown that students have difficulty with the concepts of slope and derivative, especially in cases when students are asked to utilize these concepts in real-life contexts.

For this study, written surveys were collected from 69 differential (first semester) calculus students. Follow-up clinical interviews were performed on 13 integral (second semester) calculus students. Through the surveys and interviews, students' understanding of slope and derivative using real-life contexts was explored. On the surveys, students answered questions about linear and nonlinear relationships and interpretations of slope and derivative. They also critiqued the reasoning and accuracy of a hypothetical person's predictions based on values of slope and derivative. In interviews, students explained their thought process and reasoning for the problems, and answered follow-up questions.

Results indicate that students struggle with knowing what the slope and derivative represent and how to use them appropriately to make predictions. The dominant incorrect reasoning by students (one-third of surveyed students and two-thirds of interviewed students) was to think of slope as the ratio-of-totals $\left(\frac{y}{x}\right)$ instead of the ratio-of-differences $\left(\frac{\Delta y}{\Delta x}\right)$. Thinking of slope as a ratio-of-totals implies that all linear relationships are directly proportional (of the form $f(x) = mx$, with a y-intercept of zero); students went on to interpret the slope as something that can be used to calculate the value of the dependent variable (by multiplying it by the value of the independent variable).

This incorrect thinking about slope influences students' understanding of the derivative. As a result, they often interpreted the derivative as something that could be used to find the value of the dependent variable (by multiplying the derivative by the value of the independent variable). This led to the incorrect relationship, $f\ (x) = f'(x) * x$. Furthermore, when students were asked to critique the reasoning of a hypothetical person's predictions, they showed little knowledge of how the derivative can be used to make valid predictions. Instead of demonstrating understanding that the derivative can be used to estimate change only near the input value, 54% of interviewed students said once again that they could use the derivative to calculate the total value ($f\ (x) = f'(x) * x$). Students' impoverished views of slope are adversely impacting their ability to understand the more advanced related topic of derivative.

Knowing more about students' understanding of slope and derivative as rates of change can help educators improve our instruction, with the overall goal of retaining our STEM majors. Instructional implications of this study, as well as limitations and future avenues for research, are discussed.

**DEDICATION**

To my husband Andy, for his love and support

To my children Patrick, Eleanor, and Maureen, for lighting up my life

And to my dad Ron, who was the first professor in my life, and my biggest cheerleader

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# 1. INTRODUCTION

## 1.1 The national context and STEM education

America's international competitiveness in the areas of science and mathematics is undermined by the declining mathematics and science literacy of Americans (Seymour & Hewitt, 1997). The President's Council of Advisors on Science and Technology (PCAST) forecasted that an additional one million science, technology, engineering and mathematics (STEM) graduates are needed in the next decade to meet the United States' demand (Holdren & Lander, 2012). Although throughout the $20^{th}$ century the United States led the way in research and development, facilitated by post World War II increases in the number of STEM graduates, the number of STEM graduates has been falling during the past decade. A 34% increase over our current annual STEM degree production rates is needed in order to keep pace with our economy's growing need for STEM graduates.

Combinations of task forces, conferences, commissions, and workgroups, all sponsored by a variety of different organizations, have focused on the causes and consequences of high attrition rates from mathematics and science. One such focus is on the pedagogical context of undergraduate learning and the unmet needs of students (Seymour & Hewitt, 1997). Faculty pedagogy, curriculum design, and student assessment practices are the dominant sources of problems for students who eventually switch from STEM to other majors. However, an underlying reason is the conceptual difficulties that students have that are not adequately addressed over the course of their mathematics career. Seymour and Hewitt (1997) point out:

> The experience of conceptual difficulty at particular points in particular classes, which might not constitute an insuperable barrier to progress if addressed in a timely way, commonly sets in motion a downward spiral of falling confidence, reduced class attendance, falling grades, and despair – leading to exit from the major. (Seymour & Hewitt, 1997a, p.35).

Our introductory undergraduate courses must be improved in order to address the conceptual difficulties students bring to our classrooms. Many of our students have not had adequate opportunities to learn key concepts in secondary school, which influences their learning of new, more advanced concepts in college. In addition, the ideas of college-level mathematics are difficult for many students, and we need to improve our capacity to provide high-quality learning opportunities for our students. In order to reach the goal of one million additional STEM majors, it is not enough to attract new STEM majors; we need to focus on improving our instruction in order to retain the students who start out in STEM majors (Holdren & Lander, 2012).

### 1.2 The importance of calculus in STEM education

Calculus is used to model, analyze, and understand changing quantities in our world. In the STEM fields, it is important for students to be able to apply their knowledge of change to analyze and make sense of various phenomena, from the marine biologist who needs to estimate the population growth of a species, to the medical researcher who is approximating the volume of air being inhaled by an asthmatic patient.

These change concepts are foundational for STEM students. Focusing on engineering students, but applicable to all STEM majors, Moore (2005) notes calculus is the gateway course for engineers, and therefore academic success in engineering depends on the successful completion of the calculus sequence (typically, three semester-long courses).

Since calculus is a gateway course for STEM students, it is not surprising that it is also often the roadblock to students' overall success in their chosen STEM field. While lack of preparation for postsecondary mathematics is often cited as a reason for low success rates in college mathematics (Thomasian, 2011), responsibility for addressing and improving the deficiencies lies with the entire mathematics education community. We must do better as a community in providing our students with the solid mathematics education needed for STEM degrees, and for their careers following graduation.

### 1.3 Students' difficulties with learning calculus

While the importance of calculus is apparent, first semester calculus courses have high failure rates and are a huge barrier to success in a STEM field (Bressoud, Mesa, & Rasmussen, 2015; Ferrini-Mundy & Graham, 1991; Habre & Abboud, 2006; Steen, 1987). In their overview of research in calculus learning, Ferrini-Mundy and Graham (1991) categorize student understanding of calculus into four key areas: function, limits and continuity, the derivative, and the integral. Research into student thinking and difficulties in these four areas is extensive. In this study, we look at one of these four categories, the derivative, and two concepts that are foundational to its understanding: rates of change and slope.

## 1.4 Motivation to study students' understanding of slope and derivative

My motivation to focus on these specific areas stems from both education research findings and my own experiences in the classroom, in both a general education algebra course where a focus is on slope, and a differential (first semester) calculus course where a focus is on the derivative.

In the algebra course, developed to provide students with an alternative to a traditional college algebra class, we ask students to do more than just use linear relationships. For example, given an equation such as $C = 9.8g + 750$, where $C$ is the cost in dollars to produce $g$ gallons of a chemical, students are asked for the slope, the units on the slope, and the interpretation of what the slope means in the context of the problem. In this case, the slope is 9.8 dollars per gallon, and as we increase the number of gallons produced by one gallon, the cost increases by $9.80 (Franzosa & Tyne, 2010). Students must understand slope as a rate of change to answer this question. My students often struggle with these questions. Common mistakes include stating that the units on the slope are "costs in dollars per gallons of a chemical," thus not recognizing what is the variable and what is the unit. For the interpretation question, students will often say that the slope means, "It costs $9.80 for each gallon produced," thus implying a directly proportional relationships of the form $C = 9.8g$ (a linear relationship that goes through the origin).

Similarly, in differential calculus courses, we often ask calculus students to interpret the derivative in comparable ways. For example, given that $C = f(g)$ is the cost in dollars of producing $g$ gallons of the chemical, what are the units on $f'(g)$? And, what does $f'(200) = 6$ represent? In this case, the units are dollars per gallon, and when the number of gallons produced is 200, the cost is increasing at a rate of $6 per gallon (Hughes-Hallet, 2013). In order to fully understand how to answer these questions, students must know that the derivative is an instantaneous rate of change that is only applicable at 200 gallons, and that the derivative can be used to approximate the cost of producing the 201st gallon.

This study was designed to examine student thinking on these types of questions. Slope and derivative are both concepts that rely on understanding rates of change. Slope is a precollege idea (first encountered in middle school) that appears throughout one's mathematical career, including in a differential calculus course. In addition, the derivative, a concept students encounter in calculus, relies on a strong foundational understanding of slope. As discussed below, examining these ideas separately, researchers have already found slope and derivative to be difficult for students. However, these concepts are highly interconnected and examining whether students' incomplete knowledge about one shapes their developing knowledge of the other is the focus of this study.

**1.5 The importance of rates of change, slope, and derivative in the study of calculus**

Rates of change and slope are initially encountered in middle school mathematics. Rates are mentioned first in the Common Core State Standards for Mathematics (CCSSM) in sixth grade (National Governors Association Center for Best Practices, 2010), where students are called to "use ratio and rate reasoning to solve real-world and mathematical problems" (Grade 6, Ratios and Proportional Relationships, CCSS.Math.Content.6.RP.A.3). The rate of change concept is used to describe relationships between changing quantities in all sorts of fields, such as biology (e.g., changes in population with respect to time), physics (e.g., the relationships between position, velocity, and acceleration), and economics (e.g., changes in production costs with respect to the number of units produced).

Slope, a specific rate of change, is covered in eighth grade, where students are called to "interpret the rate of change and initial value of a linear function in terms of the situation it models" (Grade 8, Functions, CCSS.Math.Content.8.F.B.4). Interpreting the rate of change in context (for both slope and derivative) is a key part of this study, and is directly tied to this 8$^{th}$ grade content standard.

Slope is a fundamental mathematical concept that can be represented in many different ways, for example as a geometric ratio (i.e., "rise over run") or as a physical property (using the words "steepness," "incline," etc.). It is the functional representation of slope (understanding slope as a rate of change) that is the focus of this study. To understand slope functionally, strong covariational reasoning is necessary. Covariational reasoning is defined as "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). For example, for the function $f(x)$, when input value $x_1$approaches $x_2$, does the function value $f(x)$ increase or decrease? Covariational reasoning is again necessary for understanding derivative as a rate of change (Nagle, Moore-Russo, Viglietti, and Martin, 2013).

Derivative is a calculus concept that is deeply connected to slope, but at a much more advanced level. The derivative is formally defined as a limit of a difference quotient, in other words a limit of a slope of a line (specifically the limit of the slope of the secant line as it approaches the slope of the tangent line). The derivative extends our knowledge of an average rate of change to an instantaneous rate of change, one in which the rate of change is varying.

It is essential for university-level mathematics faculty to understand students' understanding of slope coming into calculus, and to expand on that knowledge in teaching the derivative. "If students do not understand average rate of change, it is hard to imagine they have anything but a superficial understanding of instantaneous rate of change" (Hackworth, 1994, p. 154). Not only must students understand the derivative as an instantaneous rate of change, they must also have an understanding of continuously changing rates (the derivative is different at each value of the independent variable), as well as have strong covariational reasoning skills to interpret dynamic situations modeled with calculus (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

It is not enough for students to be able to calculate slope and derivative; we also want them to understand what they mean in a specific context and understand the rate of change they represent. Understanding how students think about slope and derivative can lead to better instruction, and hence stronger conceptual knowledge for our students. Therefore, examining calculus students' understanding of slope and derivative as rates of change is important. We know the concept of rate of change in general is not well understood by students (Bezuidenhout, 1998). Since understanding slope as a rate of change is the foundation that calculus students need to bring to the learning of derivatives as instantaneous rates of change (Hackworth, 1994), we seek to examine what the relationships are between students' knowledge of slope and their knowledge of derivative.

Also important is examining students' understanding of these rates of change in the context of real life situations. The importance of modeling real life situations is reflective in CCSSM. One of the Common Core's Standards for Mathematical Practice is to "model with mathematics," which focuses on the importance of solving problems that arise in everyday life. Likewise, research points to the use of context problems, defined as "problems of which the problem situation is experientially real to the student" (Gravemeijer & Doorman, 1999, p. 111) as key to students' understanding of formal mathematics and their improved mathematical reasoning.

The concepts of rate of change, slope, and derivative are tightly interconnected. Students' knowledge of derivatives is built on their foundational knowledge of slope. One cannot effectively study students' understandings of derivative as an instantaneous rate of change without also exploring their notions of slope as a constant rate of change.

### 1.6 The call for the ability to critique the reasoning of others

This study also focuses on students' abilities to critique the reasoning of others. As called for by the CCSSM, critiquing the reasoning of others is the ability to "distinguish correct logic or reasoning from that which is flawed and – if there is a flaw in an argument – explain what that is" (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP3, National Governors Association Center for Best Practices, 2010). While the National Council of Teachers of Mathematics (NCTM) Principles and Standards (NCTM, 2000) recognized the importance of communicating about mathematics, specifically calling for explanations that “include mathematical arguments and rationales, not just procedural descriptions or summaries,” (Process Standards, Communication) the call for critiquing the reasoning of others goes further, adding another level of reasoning.

Since this additional level of reasoning is a relatively new standard for mathematical practice, it has not been studied widely in the mathematics education community, nor have any studies to date focused on critiquing the reasoning of others in calculus contexts. This study is also one way to see whether the building blocks ideas that students, ideally, learn in K-12 are robust in their undergraduate mathematics learning.

### 1.7 Research questions

This study builds on previous research on students’ difficulties with slope and derivative, focusing on the connections between students’ verbal interpretation of slope and their verbal interpretation of the derivative. It also probes into students’ abilities to critique the reasoning of others’ predictions, through questions focusing on the appropriate use of a rate of change to make a prediction. Both are new research areas at the college level.

My two principal research questions are:

1. Is there a relationship between calculus students’ understanding of slope and their understanding of derivative? Specifically, do students’ abilities to correctly interpret the slope as a constant rate of change make them more likely to be able to interpret the derivative as an instantaneous rate of change?

2. Given predictions based on slope and derivative, can students appropriately critique the reasoning?

By “interpret” I mean to provide a description of the meaning in the context of the problem. By “real life situations” I mean application problems that model realistic circumstances. Such applications require students to be able to translate from the context to the abstract level of calculus and then back to the context, skills that require conceptual knowledge (White & Mitchelmore, 1996). “Not only do real-world situations provide meaningful opportunities for students to develop their understanding of mathematics, they also provide opportunities for students to communicate their understanding of mathematics” (Stump, 2001, p.88). By “use of slope and derivative” I mean do students understand the difference between a constant rate of change (which can be used to interpret change at any x-value) and an instantaneous rate of change (which is only valid for a specific x-value, and can only be used to approximate around that x-value). Lastly, by “appropriately critique” I mean distinguish correct reasoning from flawed reasoning, and be able to explain what the flaw, if any, is in an argument.

My research questions are directly related the larger goal of student thinking of slope and derivative. The focus emphasizes linear and non-linear, one-variable relationships, concepts that are necessary for first-year calculus students. The goal is that by answering my two research questions, I will contribute to this collection of research, adding understanding about the connection between students’ understanding of slope and how it impacts their ability to understand the derivative, and also students’ understanding of how rates of change can be appropriately used to make predictions. This knowledge will in turn help to inform instruction, both in the calculus classroom (where students are expected to have understanding of slope and rates of change) and in the middle grades (where students are first exposed to slope and rates of change).

We know that far too many students start in STEM majors, only to drop out due to experiences in early courses (Holdren & Lander, 2012). We must better understand students' knowledge coming into calculus, and how that knowledge can affect their success in calculus. Otherwise, if left unaddressed, students' views of important concepts (such as slope) may adversely impact their learning of more advanced concepts (such as derivative). By understanding the knowledge students bring to our classes, we will better be able to meet their individual needs, thus providing them a more successful calculus experience. Hopefully this will lead to more persistence in STEM majors, an ultimate goal for us as mathematics educators.

## 2. RESEARCH ON STUDENT THINKING ABOUT RATES OF CHANGE, SLOPE, DERIVATIVE, AND LINEAR APPROXIMATION

Research shows that students have difficulties with rates of change, both rates of change of linear functions (i.e., slope) and rates of change of non-linear functions (as represented by the derivative) (Barr, 1980, 1981; Carlson, 1998; Crawford & Scott, 2000; Ferrini-Mundy & Graham, 1994, 2004; Habre & Abboud, 2006; Hackworth, 1994; Lobato & Thanheiser, 2002; Orton, 1984; Park, 2013; Stump, 2001; Thompson, 1994; White & Mitchelmore, 1996; Zandieh, 2000). In many textbooks and mathematics classrooms, the overarching concept of rate is often ambiguous and confusing to students (Confrey & Smith, 1994). At the same time, rates of change are fundamental to understanding the relationships between various quantities, and many concepts in higher-level mathematics. Understanding rates of change is necessary not just to be able to succeed in mathematics courses; it is essential to be able to understand the relationships encountered in just about any major that uses quantitative analysis (biology, economics, business, etc.).

Rate of change is the overarching topic of this research study. Within rate of change are the related concepts of slope and derivative. Calculus instructors may assume that students are entering their classrooms with a strong foundation in slope as a constant rate of change. From this foundation, they work to build knowledge of derivative as an instantaneous rate of change. The slope of a secant line (average rate of change) is the basis for understanding slope of a tangent line (instantaneous rate of change). Slope is a necessary building block for understanding derivative, but unfortunately slope is a concept that is very difficult for students (Barr, 1980, 1981; Crawford & Scott, 2000; Lobato & Thanheiser, 2002; Stump, 2001).

Also important to this research is the idea that a constant rate of change can be used to predict function values over any interval, but a varying rate of change (as is found with a derivative for non-linear functions) can only be used to estimate function values around the point where the derivative was calculated. Linear approximation, the concept of using the tangent line to approximate the value of the function at nearby points, is typically covered in the differential calculus curriculum. While there have been calls to bring linear approximation and approximation in general to the forefront of calculus teaching (Bivens, 1986; Sofronas et al., 2011), there has not been much research into student thinking and difficulties with this topic.

The following example is demonstrative of how incorrect reasoning around slope can lead to potential problems in understanding the derivative. Let $C = 2.5n + 25$ be the cost (in dollars) to rent a bowling alley lane for a birthday party, where *n* is the number of people attending. As described later in this section, research shows that many students take a ratio-of-totals $\left(\frac{y}{x}\right)$ approach to slope (as opposed to a ratio-of-differences $\left(\frac{\Delta y}{\Delta x}\right)$ approach). A student with a ratio-of-totals approach to slope would conclude that the slope of 2.50 dollars per child means that the "total cost is $2.50 for each person who attends." This is different than the correct reasoning of "for each *additional* person who attends, the cost would increase by $2.50." The incorrect ratio-of-totals approach assumes that the relationship is directly proportional (in other words a linear relationship with a y-intercept of zero), so that $C = 2.5n$.

In calculus, the students might go on and apply the incorrect ratio-of-totals reasoning to a nonlinear relationship, $C = f(n)$. When we explore instantaneous rates of change, we want students to understand that $C'(10) = 2.50$ means that when 10 people are attending the party, the total cost is increasing at a rate of $2.50 dollars per person. In other words, we can use the derivative of $2.50 per person to estimate the cost increase for the next person (the 11th person). The student with the incorrect ratio-of-totals approach from before might incorrectly conclude that $C'(10) = 2.50$ means that when the number of people attending is 10, the total cost is $2.50 for each person.

With the important concepts of rates of change, slope, derivative, and linear approximation in mind, and the example of how incorrect reasoning in one can impact understanding of another, we examine the research into students' understanding of these ideas.

## 2.1 Student thinking about rates of change

With their basis in everyday experiences, rates of change are fundamental for understanding the relationships between various quantities (Confrey & Smith, 1994). Many researchers claim that students' success in higher level mathematics depends on a deep understanding of rate (Carlson et al., 2002; Zandieh, 2000). As rates of change play a significant part in describing and understanding changing quantities in biology, physics, chemistry, economics, and other areas, rates of change are a critical mathematical topic.

Research findings about students' difficulties with rates of change fall into three categories: (1) students' underdeveloped concepts of rates, (2) students' difficulties interpreting rates of change, and (3) students' incorrect view of rates of change as the ratio-of-totals.

### 2.1.1 Students' underdeveloped concepts of rates

Since rates of change play such an important role in calculus, an underdeveloped understanding of rates will impact students' abilities to understand slope and derivative. The concept of rate is based in proportionality. One of the first rates explored in middle school is the unit rate, also called the constant of proportionality; it is the slope of the linear relationship of the form $y = mx$. The research documents the difficulties students have with ratio and proportion (e.g., Heller, Post, Behr, & Lesh, 1990; Thompson & Saldanha, 2003; Tourniaire & Pulos, 1985). Proportional reasoning is often considered the cornerstone of middle school mathematics, and therefore more advanced mathematics (Lesh, Post, & Behr, 1988). It is not surprising that if students do not understand ratio and proportion, their ability to understand rates will be compromised. "Fundamentally, rate of change is a manifestation of proportionality" (Orton, 1984, p. 184).

Rates of change and proportionality are linked directly. Ben-chaim, Fey, Fitzgerald, and Benedetto (1998) formalize three broad categories of proportional reasoning, the second being what we think of as rates: "comparing magnitudes of different quantities with an interesting connection" (p. 249), for example miles per gallon, kilometers per minute, or people per square mile.

Thompson (1994) and Hackworth (1994) both speak to students' underdeveloped concepts of rates. In studying student understanding of the Fundamental Theorem of Calculus, difficulties in understanding were often tied to underdeveloped understanding of rates of change (Thompson, 1994). Specifically, Thompson found that many students did not have the understanding of average rate of change that is needed to move onto instantaneous rate of change. He defines this average rate of change knowledge to be "that if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of the change in the dependent quantity as actually occurred" (p. 50). Hackworth (1994) found that students who did poorly in calculus seemed to come in with an underdeveloped understanding of rate of change; for these students, instruction about derivatives failed to substantially change their reasoning about rate situations. He found that since rate of change requires envisioning two variables co-varying systematically, a strong understanding of function is necessary. However, many students did not appear to understand that "for a given amount of time there is a unique distance value and for any change in time there is a corresponding change in distance" (p.157).

Orton (1983) found that many of the calculus students he interviewed did not think about rate of change in derivative problems, hence losing the connection to the original graphical representation of a derivative as a limit of an average rate of change, and instead had moved on to the techniques for calculating derivatives. He examined this lack of connection through interviews, but did not provide insight into the student thinking behind it. He expressed concerns with being able to make the concept of derivative accessible to our students without revisiting the very basic ideas of ratio and proportion, well-documented problematic concepts for students.

Many students are entering calculus with underdeveloped understandings of rate, often developed in middle school and connected to misunderstandings surrounding ratio and proportionality. While we know that ratio and proportionality are difficult concepts for students and the building blocks needed for understanding rate of change, we do not yet know fully how these misunderstandings affect the way students make sense about constant versus changing rates of change.

### 2.1.2 Students' difficulties with interpreting rates of change

Understanding what a rate of change (for example a slope or derivative) tells us about the relationship that is being modeled is an important outcome for students. In addition to understanding them in purely mathematical contexts, students should able to recognize and understand rates of change that they encounter in other classes, in the news, and in their future employment. Many studies, however, point to students' difficulties with being able to interpret rates of change (Carlson, 1998; Teuscher & Reys, 2007; Wilhelm & Confrey, 2003).

Teuscher and Reys (2007) studied Advanced Placement calculus students and concluded that students lack an understanding of the interpretation of rate of change. They found that students successfully calculated the rate of change of linear functions, but when the function was not linear they had difficulty, both in representing the relationship on a graph, and interpreting the rate of change in relation to the real-world context. These struggles were evident when students were asked to graph a function of the amount of water in a tank at a given time after reading a verbal description of the flow rates into the tank, and less than 50% were able to correctly graph the function. A typical mistake was that students drew a straight line when the water flow decreased gradually, signifying in their graph that the water flow was still constant. The researchers concluded that the vocabulary used by teachers and textbooks might have contributed to student misunderstandings, where often the terms slope, rate of change, and steepness were used interchangeably. They also pointed out that if teachers are not aware of the ways of thinking about rate of change that their students are bringing to the classroom, effectively teaching topics such as limits and derivatives would be difficult.

Wilhelm and Confrey (2003) reported that most research on the rate of change concept involved motion and speed, the context most dealt with in calculus textbooks and courses. There are, however, many different contexts in which students encounter rate of change in other courses or in everyday life. By structuring their study with multiple contexts, Wilhelm and Confrey were able to examine students' abilities to apply their knowledge of rate of change and accumulation (i.e., the amount of the quantity that changes over time for a particular rate of change) from one context to another context. Specifically they used a motion context and a money context, and examined students' abilities to see the similar aspects in a contextually unlike situation. They found that students did not have to master the relationship between the rate of change and accumulation graphs within a single context before applying concepts of rate of change and accumulation separately in another context. Also, they found that students who can easily go back and forth between the rate of change and accumulation graphs might not be able to apply these concepts from one contextual situation to another. In other words, one did not have to fully understand the relationship between rate of change and accumulation in order to apply to different contexts, and if one did have what seemed like a full understanding of the relationship, it did not mean they could apply it elsewhere. They advise that instead of always focusing on motion, rate of change instruction should be approached in multiple contexts (for example, the rate of change of money going into a bank account or water going into a swimming pool), allowing the "learner the opportunity to see the 'like' in the contextually unlike situation, so that the learner might later be able to project these rate of change and accumulation concepts into novel situations" (p. 904).

Carlson (1998) studied students' development of the function concept as they progressed through undergraduate mathematics. She found that even the most talented second-semester calculus students had trouble interpreting rate of change information from a dynamic situation, as well as demonstrating an awareness of the impact of a change in one variable has on another variable. She found that current calculus curricula gave very little opportunity for students to interpret the covarying aspects of functions. Instead, the rapid pace at which new material is presented (especially with evolving concepts like function and rate) does not leave time for reflection, even for the strongest students, and instead allows students to come away with a very superficial understanding.

We want our students to be able to interpret and understand what a rate of change means in the context of the situation, but many students struggle with this interpretation. Students are not given many opportunities to explore rate of change, especially in contexts other than motion. While others' studies touched upon this interpretation as one small part, the present study directs attention specifically at students' abilities to interpret slope and derivative in the context of the problem.

### 2.1.3 Students' incorrect view of rates of change as the ratio-of-totals

Rates of change of linear functions can be viewed as a ratio of differences ($\frac{\Delta y}{\Delta x}$ or $\frac{y_2 - y_1}{x_2 - x_1}$) but some students view it incorrectly as the ratio-of-totals $\left(\frac{y}{x}\right)$, possibly because the expressions $\frac{\Delta y}{\Delta x}$ and $\frac{y}{x}$ are so similar (Hauger, 1995). Understanding slope as a ratio-of-totals is appropriate for directly proportional relationships (relationships of the form $y = mx$), but not for linear relationships where $b$ is not equal to zero (such as in equations of the form $y = mx + b$). In Hauger's study of high school and college students, 11% of the students estimated a rate of change of a graph using just one point and calculating $\frac{y}{x}$. He notes that in many real-life situations the initial values are zero, and therefore $\frac{y}{x}$ is appropriate, but "unless these subtle distinctions are made in the minds of students, it is a small wonder that they use $\frac{y}{x}$ when they should be using $\frac{\Delta y}{\Delta x}$" (p. 27).

Whether or not a line passes through the origin also results in difficulties for students regarding slopes and rate of change. Beichner (1994) found that students were much less successful in calculating slope when the line did not pass through the origin. Students would regularly divide a single ordinate value by a single abscissa value, "forcing" the graph through the origin, in other words calculating slope as $\left(\frac{y}{x}\right)$ instead of $\left(\frac{\Delta y}{\Delta x}\right)$. Like mentioned previously, for many real life relationships, the initial value is zero, thus $\frac{y}{x}$ is an appropriate interpretation of slope in many (but not all) situations. This has the potential to impact students' understanding of derivative, the slope of a curve at a point.

#### 2.1.4 Summary of students' thinking about rates

While rates of change are fundamental for understanding the relationships between various quantities, and higher-level mathematics depends on a deep understanding of rate, research shows that rate is a difficult concept for students. Students' underdeveloped concept of rate is often rooted in misunderstandings surrounding ratio and proportion, the foundation for rates of change. Students also fail to see the connection between rates of change and concepts such as derivative, and instruction in concepts such as the derivative typically fails to reinforce a correct interpretation of the more basic concept of rate.

Research also shows that students have trouble interpreting what a rate of change represents in context, a key skill that students need in order to understand changing quantities in other areas (such as economics or biology). Oftentimes, however, students are presented with rate of change questions focused on just motion and kinematics, instead of a diverse mix of rate of change contexts that they might see in other areas of study. While research shows that students had more success with linear functions than non-linear ones, they still struggled when interpreting constant rates in real-life contexts.

Lastly, and most importantly for this study, research shows that students often misinterpret slope as a ratio-of-totals $\left(\frac{y}{x}\right)$ instead of a ratio-of-differences $\left(\frac{\Delta y}{\Delta x}\right)$. While many relationships are directly proportional and therefore have a slope that can be interpreted as $\frac{y}{x}$, research shows that students misinterpret and miscalculate slope of linear relationships with non-zero initial values using $\frac{y}{x}$.

This current study attempts to fill in some of the gaps in the research on student understanding of rate of change. Specifically, it focuses on interpretation of two different rates of change (slope of linear functions and derivative for non-linear functions) in the context of real-life situations that do not involve motion. Through students' interpretations and follow-up questions in the interviews, the common incorrect ratio-of-totals approach to interpreting a rate of change is explored to see what kind of implications such incorrect understanding has on more complex concepts such as the derivative.

### 2.2 Student thinking about slope

The overarching concept of rate of change is difficult for students, as summarized in the previous section. Slope, the constant rate of change associated with linear functions, is one rate of change that forms the foundation needed to understand instantaneous rates of change in calculus. Researchers have documented difficulties students have with slope (e.g., Barr, 1980, 1981; Crawford & Scott, 2000; Lobato & Thanheiser, 2002; Stump, 2001). Research findings about student difficulties fall into two categories: (1) students' inabilities to interpret the slope as a rate of change, (2) students' underdeveloped conceptions of slope.

#### 2.2.1 Students' inabilities to interpret slope as a rate of change

Lobato and Thanheiser (2002) found that students can correctly calculate slope using the "rise over run" formula, but only view slope as a number, not as a measure of rate. For example, a high-performing algebra student calculated the slope correctly as ½, but when asked "a half of what?" she responded, "It isn't a half of anything, I think. It just determines the measurements on how high it is rising" (p. 162). They proposed ratio-as-measure tasks that can "help students develop an understanding of slope that is more general and applicable" (Lobato & Thanheiser, 2002, p.174).

Stump (2001) also found that high school students "had trouble interpreting slope as a measure of rate of change" (p. 81). In interviews, when she asked students "what does the slope of the line represent?" many students gave general observations such as "the steeper the slope, the more resistance," and only one student gave a specific answer with numbers and units. Within the interview transcripts, many students described the slope as an “angle,” some said, “rise over run,” and some also quoted the formula. Stump suggests instruction that focuses on providing opportunities for students to communicate their understanding of slope.

Stump (1999) categorized pre-service and in-service teachers’ responses to the answer “What is slope?” into seven different categories: geometric ratio, algebraic ratio, physical property, parametric coefficient, trigonometric conception, calculus conception, and functional property. While the functional representation is key to understanding slope as a rate of change, less than 20% of the teachers in Stump’s study thought of slope as a functional concept. “In other words, the majority of teachers did not think of slope as a rate of change between two variables” (Stump, 1999, p. 140). She notes that the teachers might have been capable of making the connection, but that they did not incorporate the connections into their definitions. Functional knowledge of slope (understanding slope as a rate of change) is necessary to understand the derivative as an instantaneous rate of change; calculus instructors likely expect students to enter with this functional knowledge of slope. The most common definitions by teachers were that of slope as a geometric ratio (i.e., “rise over run”), or as a physical property (using the words “steepness,” “incline,” etc.).

Also addressing the known difficulties students have understanding the slope as a rate of change, Crawford and Scott (2000) call for having students communicate and reason about slope, as well as using real-world examples to introduce the concept of rate of change prior to introducing slope. They suggest using multiple representations (words, tables, graphs, and equations) to explore the patterns of change, starting with directly proportional relationships, then introducing linear relationships where the vertical intercept is non-zero, and lastly discussing rates of change that are not constant.

### 2.2.2 Students' underdeveloped conceptions of slope

Nagle, Moore-Russo, Viglietti, and Martin (2013) studied slope conceptualizations in both college students (from one university) and college instructors (from a number of colleges and universities in the same region). They classified student and instructor responses among 11 conceptualizations of slope. They found that instructors' responses often contained evidence of understanding slope functionally (as a rate of change), and often used more formal "textbook" descriptions of slope. On the other hand, students often recalled various ideas about slope from their past without a precise definition. In terms of conceptualizations, students were most likely to talk about slope as a behavior of the graph ("a number that tells you if the line is increasing, decreasing, or staying flat"), though no instructors took the behavior approach. Likewise, instructors' most common conceptualizing was functional ("how fast one thing changes as something else changes"), though less than one-fifth of all students used a functional approach. They conclude that it is imperative that instructors understand the conceptualizations commonly held by their students in order to build advanced ideas.

### 2.2.3 Summary of students' thinking about slope

Slope, a key rate of change concept necessary for understanding the derivative, is difficult for students. There are many conceptions of slope, and often students and instructors bring different conceptions to the classroom. Whereas the functional concept of slope as a rate of change is necessary for understanding the derivative, it is not the conception that many students bring to calculus. It is well documented that students struggle with understanding slope as a rate of change, a necessary part of understanding the derivative as an instantaneous rate of change.

While it is known that students must understand slope functionally (as a rate of change) in order to understand derivative as an instantaneous rate of change, it is not yet known how students' interpretations of slope affect their interpretation of the derivative. This current study fills in some of the gaps in the research on student understanding of slope. Specifically, it focuses on students interpreting the slope in real world contexts and using the slope appropriately to make predictions. Through students' written interpretations and follow-up questions in the interviews, students' understandings of slope as a rate of change (as well as derivative as a rate of change) were explored to examine how the slope and derivative understandings are related.

### 2.3 Student thinking about derivatives

Students' difficulties with the derivative are well documented in the literature (e.g., Asiala et al., 1997; Bingolbali et al., 2007; Ferrini-Mundy & Graham, 1994, 2004; Habre & Abboud, 2006; Park, 2013; White & Mitchelmore, 1996; Zandieh, 2000). Students are often able to compute derivatives using algorithms, but may have very little conceptual knowledge about the derivative (Ferrini-Mundy & Graham, 1994; White & Mitchelmore, 1996). Research about difficulties students have understanding the derivative falls into three categories: (1) student weaknesses with underlying concepts, (2) students' difficulties with covariational reasoning, and (3) the difficulties which stem from the multi-faceted nature of the derivative.

### 2.3.1 Students' weaknesses with underlying concepts

There are many underlying concepts necessary for full understanding of the derivative (for example slope, variable, function, limit, etc.). The two concepts of variable and function are precalculus concepts of which calculus instructors often expect students to come in with a strong foundation. These, however, are difficult concepts for students. White and Mitchelmore (1996) cite the need for a mature view of variable as a prerequisite to a successful study of calculus. They found that many of the students treated the variables in the problems as symbols to be manipulated and instead ignore the meaning behind the symbols. For example, in one of their tasks, students were asked to maximize the area, thus first finding $A(x)$ in terms of one variable, and then taking the derivative $\frac{dA}{dx}$. Instead, many students focused on the constraint, $y = 12 - x^2$, and instead found $\frac{dy}{dx}$. The authors conclude that students were remembering the procedure for maximizing a function in terms of the symbols first used (i.e., $\frac{dy}{dx}$), instead of on a process based on the rate of change of a specific function (in this case, area).

Many students also come to calculus with a very primitive understanding of functions (Carlson, 1998; Ferrini-Mundy & Graham, 1994; Monk, 1994). Monk (1994) looked at students' understanding of functions from two approaches – point-wise and across-time. Point-wise understanding is what students first attain in their learning about functions, thinking of particular values of the independent variable corresponding to particular values of the dependent variable. However, in calculus, students must have across-time understanding of functions, where changes in one variable lead to changes in another variable. Monk found, however, that many calculus students are much more successful in answering point-wise questions about functions than across-time ones. As an example from his research, he found that when given a graph of a non-linear function and a secant line between two points, 87% of the students were able to find the slope of the secant line, but only 57% were able to answer whether the slope of the secant line increases or decreases as one independent variable value changes and one point moves toward another point.

### 2.3.2 Students' difficulties with covariational reasoning

Rate of change knowledge is strongly linked to the notion of covariational reasoning, described by Carlson et al. (2002) as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Such reasoning requires someone to hold an image of two quantities' values simultaneously (Saldanha & Thompson, 1998).

Research has shown that students lack the understanding necessary to deal with these co-varying quantities efficiently. These difficulties fall into two categories: (1) students often using a point-wise approach instead of an across-time approach (Carlson, 1998; Park, 2013), and (2) students not having the understanding of average rates of change necessary for full covariational reasoning.

Students need to understand the covarying nature of the derivative function, not just the point-wise interpretation, in order to fully make sense of this key calculus concept. Research shows, however, that many students only have a point-wise understanding. For example, in Carlson (1998), students were asked to sketch the graph of height of water in a bottle as a function of the volume of water, as water is poured into the bottle (a picture of the bottle was provided). To complete this task successfully, students must understand the covariant aspects of the variables (how change in the height affects change in the volume). Students who did not display knowledge of the covariant aspects were not able to visualize the change in the height, and responded with, for example, a sketch of a straight line. She concluded that even the most talented second-semester calculus students exhibited difficulties with “demonstrating an awareness of the impact change in one variable has on the other” (p. 142).

Park (2013) also found that most differential calculus students used a point-specific understanding of the derivative. She designed her interview study to explore students’ explanations about the derivative as a function. In task-based interviews, she asked calculus students to explain what the derivative is, and then answer whether their response was more similar to the definition of the derivative of a function or the derivative of a function at a point. She concluded that to define the derivative function, students use a point-specific concept of the derivative over an interval, and that the transition from the derivative of a function at a point to the derivative of a function is nontrivial for students.

In looking more closely at covariational reasoning and the mental actions needed for success, Carlson et al. (2002) created five classifications for covariational reasoning with five mental actions (MA) as follows: MA1 (coordination of the variables), MA2 (coordination of the direction of change), MA3 (coordination of the amounts of changes of the variables), MA4 (coordination of average rate of change across uniform increments of the domain, and MA5 (coordination of instantaneous rates of change). They found that calculus students were able to perform at the MA1, MA2, and MA3 levels, but consistently had difficulty at MA4 (average rates of change). Carlson et al. tie students’ struggles with covariational reasoning to their incomplete understanding of average rates of change.

### 2.3.3 Students' difficulties with multi-faceted nature of the derivative

The derivative is a multi-faceted idea, with students needing to connect many underlying concepts in order to think fully about the derivative (Ferrini-Mundy & Graham, 1994; Zandieh, 2000). The concept of the derivative can be represented graphically as the slope of a tangent line, verbally as the instantaneous rate of change, physically as velocity, and symbolically as the limit of the difference quotient (Zandieh, 2000). Researchers have found that students often do not connect a function's derivative with its rate of change, which leads to the inability to understand differentiation as an operator that measures a rate of change (Weber, Tallman, Byerley, & Thompson, 2012). Even when they can connect the derivative with rate of change, students often confuse derivative at a point with the derivative function (Ubuz, 2007).

### 2.3.4 Summary of students' thinking about derivatives

The derivative is a complex concept, and research points to many difficulties in students' understanding of the derivative. There are multiple underlying concepts that are necessary for understanding the derivative, and students often come into calculus with under-developed conceptions and misunderstandings about these concepts. Also, in order to understand the derivative as a function, students must have strong covariational reasoning, a well-documented struggle among calculus students. Like slope, the derivative is multi-faceted and can be represented in different ways, adding to the complexity of the concept and leading to students' challenges.

While it is known that students struggle with the derivative, this current study fills in some of the gaps in the research on student understanding of derivative. Specifically, it focuses on interpreting the derivative in real world contexts and using the derivative appropriately to make predictions. Through students' written interpretations and questions in the interviews, students' understandings of derivative as a rate of change, and their ability to interpret and use appropriately, were explored. Also, through comparison of students' responses to questions about slope and derivative, it attempts to reveal how misunderstandings about slope can impact students' misunderstandings about derivative.

**2.4 Student thinking about linear approximation**

While there have been calls to bring linear approximation and approximation in general to the forefront of calculus teaching (Bivens, 1986; Sofronas et al., 2011), there has been very little research as to student thinking and difficulties with this topic (Asiala et al., 1997).

Asiala et al. (1997) did look at student understanding of the tangent line while studying students' graphical understanding of the derivative. They found that many students calculated and used the equation of the tangent line but did not show any understanding of it as an approximation of the function near the point.

Sofronas et al. (2011) reported that 33.3% of the 24 calculus experts in their study identified student understanding of approximation as central to a deep understanding of first-year calculus. While the idea of linear approximation is mentioned in just about all calculus textbooks, it is “always as an afterthought and the intuitiveness of the tangent-as-limit-of-a-secant remains unquestioned” (Schremmer & Schremmer, n.d.).

Bivens (1986) calls to bring linear approximation to the forefront of calculus teaching, where “the interpretation of the tangent line as the ‘best linear approximation’ can be used with profit in the beginning college calculus course” (p. 142).

While linear approximation is not specifically stated in the current study’s survey, the questions pertaining to the appropriateness of using the derivative to make a prediction are designed to probe students’ understanding of linear approximation.

### 2.5 What is not yet known about student understanding

This study focuses on slope and derivative in real-life contexts. Specifically, it centers on students’ abilities to interpret the slope and derivative in context, students’ understanding of appropriate uses of slope and derivative to make predictions, and students’ abilities to critique the reasoning of others. Researchers have documented that students often have incomplete conceptions of rates of change, slope, and derivative, all key concepts in understanding the tasks in the current study.

Recall that this study is designed to answer the following two research questions:

1. Is there a relationship between calculus students’ understanding of slope and their understanding of derivative? Specifically, do students’ abilities to correctly interpret the slope as a constant rate of change make them more likely to be able to interpret the derivative as an instantaneous rate of change?
2. Given predictions based on slope and derivative, can students appropriately critique the reasoning?

Focusing on the first question, while some research has already examined students' abilities to interpret slope in real-life contexts, it has mostly involved high school students. Research shows that college students also struggle with slope, and this study extends these findings by examining whether college students also struggle with these interpretations. Also, very little research has been done on students' verbal interpretation of the derivative as a rate of change; most studies focus on slope. Lastly, by focusing on both slope and derivative interpretation, this study aims to examine how (mis)understandings of slope interpretation can impact students' interpretation of the derivative and to examine how students interpret both slope and derivative in real-life contexts.

A second focus of the study raised in research question two, and one that has not been examined previously, is student understanding of appropriate uses of slope and derivative. In particular, when can the slope and derivative be used appropriately to estimate a change in the dependent variable? Though focused on functions in general, and not derivatives, Carlson (1998) found that second-semester calculus students "had difficulty interpreting and representing covariant aspects of a function situation" (p. 115). The current study also focuses on covariant aspects of functions, in particular students' understanding of how changes in one variable affect changes in another.

The appropriate use of the derivative to make predictions also indirectly relates to students' understanding of linear approximation, an important topic in calculus for which very little research around student understanding exists. Prior research points to students being able to find and use the tangent line, but does not address appropriate use of the tangent line approximation (e.g., how close to the point of tangent is an appropriate approximation?).

The third focus of this study is the ability of students to critique the reasoning of others, also a gap in the current research on slope, derivative, and rates of change. The CCSSM states that it is important for students to be able to critique the reasoning of others, and “distinguish correct logic or reasoning from that which is flawed and – if there is a flaw in an argument – explain what that is” (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP3, National Governors Association Center for Best Practices, 2010). This study has students critique the reasoning of others in the context of the questions about appropriate use of slope and derivative so students’ ways of reasoning on these kinds of tasks can be investigated.

In summary, this study is designed to focus on some unanswered questions in the literature surrounding students’ interpretations of slope and derivative in real-life contexts, students’ understandings of appropriate uses of slope and derivative to make predictions, and students’ abilities to critique the reasoning of others. My research questions begin to fill in two gaps in the current research: (1) existing literature highlights that interpreting slope and derivative are hard for students, but how specifically do students’ misunderstandings about slope impact their misunderstandings about derivative?; (2) do students understand when it is appropriate (and not appropriate) to use a rate of change to make predictions?; and can they critique the appropriateness of others’ predictions? By working to fill these gaps, the mathematics education community will understand better the students who enter our classroom, which will in turn lead to the improvement of the teaching and learning of calculus.

# 3. RESEARCH DESIGN

## 3.1 Theoretical framework

The present study was conducted within a cognitivist framework (Byrnes, 2001; Siegler, 2003), which posits that students make sense of the mathematics they are doing based on their experiences and that their answers are rational and subject to explanation (Ferrini-Mundy & Graham, 1994). Ferrini-Mundy and Graham describe the intention of such studies as “to provide rich and defensible descriptions of student understandings that can serve as springboards for acknowledging the great complexities to be understood in learning about student knowledge” (p. 32). Because a cognitive lens focuses on individual thinking, it is useful for investigating how students think about slope and derivative and use them to make predictions. The focus here is on detailed analyses of student understanding of a few key concepts, gained from direct student responses. Hence this study used a written survey instrument and follow-up interviews as data sources. Since my goal is to understand how individual students are thinking about the ideas of slope and derivative, I utilized this theoretical perspective, using the current findings in the field to guide the questions in this study.

## 3.2 Written survey data collection

The data for this study were collected from 69 students enrolled in differential (e.g., first semester) calculus at a public university in the northeast. Over 50% of the students had taken calculus in high school, and all needed to either pass a placement exam or complete precalculus at the university with a C or better to gain enrollment into differential calculus. Since there is only one flavor of calculus at the university, it contains a mixture of majors (engineering, science, mathematics), as well as those taking the course as a general education requirement. Students completed the surveys during class time, approximately 80% through the course.

The survey instrument consisted of questions about slope and derivatives, including questions about linear and nonlinear relationships between the yield of a crop of corn (bushels) as a function of the amount of nitrogen put on the field (lbs.). There were two versions of the survey; one version asked students to sketch the graphs of the relationships in question (Figure 3.1 and Appendix A) and one version provided graphs of the relationships (Figure 3.2 and Appendix B). The two different versions were created to examine whether students performed differently when the graphs were provided. Thirty-seven of the surveys included the graphs, and thirty-two of the surveys asked students to sketch the graphs.

The survey questions are not mechanical in nature and therefore do not assess computational skills; instead, they are questions about students' interpretations of slope and derivative and their ability to critique others' reasoning, and therefore try to uncover their understanding about these topics.

The questions were informed by the typical presentation of slope and derivative in textbooks, the Common Core Standards for Mathematical Practice, and the call for assessing students' across-time view of functions (Monk, 1994). In textbooks and in instruction, when focus is given to students' understanding of slope and derivative, usually the questions asked are similar to A1, A2, B1, and B2 (Figure 3.1). These questions address units (Bezuidenhout, 1998) and students' point-wise understanding of rates of change (Monk, 1994). Based on the Common Core's call for critiquing the

**Let B(n) be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen.**

A. Assume that B(n) is a linear function with a slope equal to 2 (m = 2)

0. On the graph to the right, give a rough sketch of what the function B(n) looks like. Label the axes, but no need to scale them.
1. What are the units on the slope, m = 2?

2. Explain what this slope (m = 2) means in the context of the problem.
3. Using the slope (m = 2), Farmer Jim predicts that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn. How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident     Somewhat Confident     Not Confident

   Explanation:
4. Farmer Jim purchases 20 pounds of nitrogen for his tract this season. At the last minute, he decides to invest more in nitrogen and increases the application to 30 pounds. Based on his model, he predicts that will get him 20 additional bushels (2 bushels for each additional pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident     Somewhat Confident     Not Confident

   Explanation:

B. Now, assume that B(n) is a non-linear function.

0. On the graph to the right, give a rough sketch of what the function B(n) looks like, assuming that the nitrogen is helpful to the crop up until a certain point and then too much is harmful. Label your axes, but no need to scale them.
1. What are the units on $\frac{dB}{dn}$? (also known as $B'(n)$)
2. Explain the meaning of the statement $B'(20) = 2$ in the context of the problem.
3. Using the fact that $B'(20) = 2$, Farmer Jim predicts that his corn yield will increase by 2 bushels when his nitrogen application increases from 20 pounds to 21 pounds. How much confidence do you have in his reasoning? (circle one and provide explanation)

   Very Confident     Somewhat Confident     Not Confident

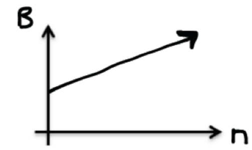
   Explanation:
4. Farmer Jim purchases 20 pounds of nitrogen for his tract this year. Last minute, he decides to invest more in nitrogen and raises it to 30 pounds. Since $B'(20) = 2$, he predicts that the additional nitrogen will yield him 20 additional bushels (2 bushels for each pound of nitrogen) . How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident     Somewhat Confident     Not Confident

   Explanation:

Figure 3.1. Survey instrument, graphs not provided

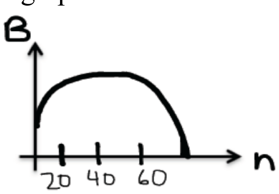
**Let B(n) be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen.**




A. Assume that B(n) is a linear function with a slope equal to 2 (m = 2), shown in the graph.

1. What are the units on the slope, m = 2?
2. Explain what this slope (m = 2) means in the context of the problem.
3. Using the slope (m = 2), Farmer Jim predicts that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn. How much confidence do you have in Jim's reasoning? (circle one and provide explanation)
   Very Confident Somewhat Confident Not Confident
   Explanation:
4. Farmer Jim purchases 20 pounds of nitrogen for his tract this season. At the last minute, he decides to invest more in nitrogen and increases the application to 30 pounds. Based on his model, he predicts that will get him 20 additional bushels (2 bushels for each additional pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)
   Very Confident Somewhat Confident Not Confident
   Explanation:

B. Now, assume that B(n) is a non-linear function, such that nitrogen is helpful to the crop up until a certain point and then too much is harmful, as show in the graph.




1. What are the units on $\frac{dB}{dn}$? (also known as $B'(n)$)
2. Explain the meaning of the statement $B'(20) = 2$ in the context of the problem.
3. Using the fact that $B'(20) = 2$, Farmer Jim predicts that his corn yield will increase by 2 bushels when his nitrogen application increases from 20 pounds to 21 pounds. How much confidence do you have in his reasoning? (circle one and provide explanation)
   Very Confident Somewhat Confident Not Confident
   Explanation:
4. Farmer Jim purchases 20 pounds of nitrogen for his tract this year. Last minute, he decides to invest more in nitrogen and raises it to 30 pounds. Since $B'(20) = 2$, he predicts that the additional nitrogen will yield him 20 additional bushels (2 bushels for each pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)
   Very Confident Somewhat Confident Not Confident
   Explanation:

Figure 3.2. Survey instrument, graphs provided

reasoning of others, as well as students' across-time understanding of rate of change (Monk, 1994) the survey included questions A3, A4, B3, and B4. The linear questions (A3 and A4) were included to gain an understanding of students' knowledge of predictions based on linear change and are similar to typical textbook/instruction presentation of slope.

### 3.3 Responses of an ideal knower

In order to frame the analysis, I have included in this section what an "ideal knower" would answer, what students would be thinking about while solving each task, and what the question is designed to generate data on. These descriptions were used to inform the data analysis.

*A: Assume that B(n) is a linear function with a slope equal to 2 (m = 2).*

*1. What are the units on the slope, m = 2?*

The ideal knower would respond that the units are bushels (of corn) per pound (of nitrogen). This question gets at students' understanding of the units on the slope, which are the units on the dependent variable over the units on the independent variable.

*2. Explain what this slope (m = 2) means in the context of the problem.*

The ideal knower would respond that the slope of 2 means that for each additional pound of nitrogen, the number of bushels of corn will increase by 2. This question gets at students' understanding of slope as a constant rate of change, where the ratio of *changes in variables* is constant. This is different than a directly proportional relationship where the ratio of *amounts* is constant, which implies a vertical intercept of zero. They also have to understand the covarying nature of the variables, in other words how the dependent variable changes with changes in the independent variable.

*3. Using the slope (m = 2), Farmer Jim predicts that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn. How much confidence do you have in Jim's reasoning?*

The ideal knower would respond, "very confident," by understanding that a slope of 2 represents the increase in bushels per pound of nitrogen, and that it is a constant rate of change. As the pounds increase by 1, the yield increases by 2 bushels. This question begins to get at students' across-time understanding of functions, as they have to understanding how the dependent variable changes as the independent variable increases by one.

*4. Farmer Jim purchases 20 pounds of nitrogen for his tract this season. At the last minute, he decides to invest more in nitrogen and increases the application to 30 pounds. Based on his model, he predicts that will get him 20 additional bushels (2 bushels for each additional pound of nitrogen). How much confidence do you have in Jim's reasoning?*

The ideal knower would respond, "very confident" and explain that the increase of 2 bushels per pound of nitrogen is constant and would be applied to the ten-pound increase. This question is designed to get at students' knowledge of the slope as a constant rate of change, and how it can therefore be applied to any change in the independent variable.

*B: Now, assume D(w) is a non-linear function, such that nitrogen is helpful to the crop up to a certain point and then too much is harmful.*

*1. What are the units on* $\frac{dB}{dn}$? (also known as $B'(n)$)

The ideal knower would respond that the units are bushels per pound. This question gets at students' understanding of the units on the derivative, which are the units on the dependent variable over the units on the independent variable.

2. *Explain the meaning of the statement* $B'(20) = 2$ *in the context of the problem.*

The ideal knower would respond that when 20 pounds of nitrogen is applied, the corn yield is increasing at a rate of 2 bushels per pound of nitrogen. This question gets at students' understanding of the derivative in the context of the problem, and their ability to demonstrate a point-wise understanding of the derivative at a point.

*3. Using the fact that* $B'(20) = 2$*, Farmer Jim predicts that his corn yield will increase by 2 bushels when his nitrogen application increases from 20 pounds to 21 pounds. How much confidence do you have in Jim's reasoning?*

The ideal knower would respond "somewhat confident," with some explanation of the instantaneous rate of change as an appropriate approximation for the marginal change, or for input values very close to the input value of the derivative. Students might also discuss linear approximation and how the tangent line is a good approximation for the function near the point of tangent. This problem is designed to get information about students' understanding of the use of instantaneous rate of change in predicting marginal change. The important thing for students to show an understanding about is that the non-linear nature of the function means the derivative gives an estimate of the change (and because information is not given about the type of non-linear function, we cannot be sure how much error is involved).

*4. Farmer Jim purchases 20 pounds of nitrogen for his tract this year. Last minute, he decides to invest more in nitrogen and raises it to 30 pounds. Since* $B'(20) = 2$*, he* *predicts that the additional nitrogen will yield him 20 additional bushels (2 bushels for each pound of nitrogen). How much confidence do you have in Jim's reasoning?*

The ideal knower would respond, "Not confident because 2 bushels per pound is the instantaneous rate of change for a 20 pound application. Because the function is non-linear, one can not use the instantaneous rate of change to make a prediction so far away from 20-pounds." This ideal knower would understand that the instantaneous rate of change is not a constant rate of change, and cannot be used as an estimate of the rate of change except at or around the specific input value. This question is designed to get at students' across-time understanding of instantaneous rates of change.

### 3.4 Coding of survey data

I took a modified Grounded Theory approach (Strauss & Corbin, 1990) to analyzing the written surveys. Grounded Theory is a qualitative data analysis method widely used in similar mathematics education studies (e.g., Byerley & Thompson, 2014; Ferrini-Mundy & Graham, 1994; Oehrtman, 2009; Orton, 1983). In pure Grounded Theory, the researcher does not look at literature until after the data analysis. After an earlier literature review, I had an idea of possible categories that would emerge but used techniques of Grounded Theory to identify and refine my analysis categories.

I examined data from the written surveys by categorizing answers from the unit questions (A1 and B1), the slope and derivative interpretation questions (A2 and B2), and the linear and non-linear critiquing questions (A3, A4, B3, and B4). These categorizations helped in identifying themes to be addressed in interviews.

### 3.5 Analysis of survey data

Before I categorized the written explanations, I examined the correctness of student responses, summarizing these as Monk (1994) did by creating 2x2 contingency tables to display combinations of right or wrong answers. A sample contingency table is shown in Table 3.1.

| | | Question B | | |
|---|---|---|---|---|
| Question A | | Correct | Incorrect | Total |
| | Correct | 30% | 62% | 92% |
| | Incorrect | 2% | 6% | 8% |
| | Total | 32% | 68% | N=69 |

Table 3.1. Sample 2x2 contingency table

The focus in the contingency tables is on the shaded diagonal cells, namely those students who got exactly one of the questions correct. In the case of the sample contingency table in Table 3.1, it is evident that there is an asymmetry in the success levels on questions A and B, since a large percentage of students got question A correct but went on to get question B incorrect (62%) but only a small percentage of students got question B correct and question A incorrect (2%). In other words, the conditional probabilities (the probability of an event occurring given that another event has already occurred) are not the same.

Three question comparisons were done: (1) the unit questions for both linear (question A1) and non-linear (question B1), (2) the interpretation questions for slope (question A2) and derivative (questions B2), and (3) the critiquing Farmer Jim's reasoning questions for linear context (questions A3 and A4) and non-linear context (questions B3 and B4). These comparisons were done to get insight as to whether there were significant differences between the success of students on the slope (linear) and derivative (non-linear) questions, a key part of my first research question about the relationship between students' understanding of slope and derivative.

For the unit questions for both slope and derivative (questions A1 and B1), an answer was marked as correct if the student stated that the units were "bushels per pound" or "bushels of corn per pound of nitrogen." They must have included the actual units for both variables in order for it to be considered a correct answer. For example, "corn per nitrogen" or "bushels per nitrogen" were marked as incorrect. Because contingency tables compare just "correct" and "incorrect" answers, I did not have any "partially correct" codes for this aspect of the analysis.

For the slope and derivative interpretation questions (questions A2 and B2), the linear slope question was marked as correct if the student stated something like "for each additional pound of nitrogen, two more bushels of corn are produced." The key language here is that students recognize that the slope represents a constant ratio in the *changes* in variables, thus the "additional" language. For the non-linear question, an answer was marked correct if the student stated something like "when the nitrogen is equal to 20 pounds, the corn yield is increasing at a rate of 2 bushels per pound of nitrogen." The key language here is that students recognize that the derivative represents a rate of change at a point. In order for an answer to be marked correct, students had to include units and include the context of the problem. For example, answers like "when the nitrogen is equal to 20, the corn is increasing at 2" or "it means the slope of the tangent line is equal to 2" were marked incorrect. Again, because contingency tables compare just "correct" and "incorrect" answers, I did not have any "partially correct" codes for this aspect of the analysis.

For the critiquing questions in the linear context (A3 and A4) I was looking for students with correct reasoning to circle "very confident" in both cases, since the relationship is linear and the slope of 2 bushels per pound is a constant. For the critiquing questions in the non-linear context (B3 and B4), I was looking for students to have more confidence in the 1-pound increase than the 10-pound increase. Therefore, I was looking for "somewhat confident" for the 1-pound increase and "not confident" in the 10-pound increase as evidence of strong understanding of the ideas. I also accepted the combination of "very confident" for the 1-pound and "somewhat confident" for the 10-pound, or "very confident" for the 1-pound and "not confident" for the 10-pound. The most important aspect of their answer was that they had more confidence in the one-pound increase than in the 10-pound increase. For the 2x2 contingency tables, I was not looking at student reasoning for these critiquing questions. I just looked to see whether they circled the correct level of confidence.

### 3.5.1 Comparing surveys with graphs provided versus those without graphs

I first compared the data from the surveys that provided students with graphs (called *graph*) and the surveys for which students sketched graphs (called *no graph*) to see whether there were differences in student responses. For each of the question comparisons, I created two 2x2 contingency tables, one of the *graph* data and one for the *no graph* data. For each table, I created a 95% confidence interval for the proportion of students who had the non-linear question correct given that they had the linear question wrong (in other words, a confidence interval around the lower left cell in the contingency tables). To do this, I calculated the adjusted Ward interval (Agresti & Coull, 1998) by adding two successes and two failures to my data. The adjusted Ward interval is suggested because the use of the normal curve to build a confidence interval is only approximate when the data are counts, and the adjusted Ward interval is a better approximation, especially when the cell count is very small.

If the confidence interval for the two proportions (*graph* and *no graph*) overlapped, then I could conclude that there is no significant difference between the results on the two surveys. If they did not overlap, I could conclude there are differences between the *graph* and *no graph* results.

Details of these analyses are in Chapter 4, but an example is presented here for clarification. Shown are the contingency tables (Tables 3.2 and 3.3) for the slope and derivative unit questions (questions A1 and B1). The 95% adjusted Ward confidence interval for the proportion of students who had the derivative question correct given that they had the slope question incorrect is (0.005, 0.495) for the surveys with the graphs and (0.119, 0.769) for the surveys without graphs. Since the confidence intervals overlap, there are no significant differences between the responses and therefore I combined the data from these surveys.

| | | Derivative | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| Slope | Correct | 32% | 19% | 51% |
| | Incorrect | 3% | 46% | 49% |
| | Total | 35% | 65% | N=37 |

Table 3.2. Results from unit questions (surveys with graphs)

| Derivative | | | | | |
|---|---|---|---|---|---|
| Slope | | Correct | Incorrect | Total |
| | Correct | 38% | 6% | 44% |
| | Incorrect | 9% | 47% | 56% |
| | Total | 47% | 53% | N=32 |

Table 3.3. Results from unit questions (surveys without graphs)

### 3.5.2 Combining the surveys

In the cases where there were no significant differences between the results from the surveys with graphs and without graphs, I combined the data and looked at one contingency table for each of the question comparisons. I performed McNemar's test ($\propto = 0.05$), which tests whether there is an asymmetry in the success levels of students on the two questions. A significant result shows that the conditional probabilities are not the same (Agresti, 2007). In addition to performing the test, I calculated the conditional probabilities to highlight their values.

To continue the unit example above, the contingency table for the combined data is presented in Table 3.4. The McNemar test was not significant ($p = 0.27$), which one would expect since the values in the shaded shells are quite similar. The conditional probabilities were also similar, with the probability of a student getting the slope correct given that the derivative is correct is 0.86, and the probability of a student getting the derivative correct given that the slope is correct is 0.77. This indicates that students were only slightly more successful at knowing the slope's units than the derivative's units, and not significantly so.

| Derivative | | | | |
|---|---|---|---|---|
| Slope | | Correct | Incorrect | Total |
| | Correct | 35% | 13% | 48% |
| | Incorrect | 6% | 46% | 52% |
| | Total | 41% | 59% | N=69 |

Table 3.4. Results from unit questions (all surveys)

After performing McNemar's test and calculating the conditional probabilities, I categorized the common incorrect responses, using a modified Grounded Theory approach. I did not focus on grammar or spelling. Categories that emerged were similar for the different types of problems; for example, the slope interpretation and derivative interpretation questions had similar categories. The specific categories that emerged, and examples of the responses that fit each category, are presented in conjunction with the results in Chapter 4. The categories that emerged from the data analysis formed the basis of my interview questioning.

### 3.6 Interviews

The interview instrument was very similar to the written survey, except the context was the amount of drug given to a patient as a function of the patient's weight (Figure 3.3 and Appendix C).

Follow-up clinical interviews (Hunting, 1997) were done during the first half of the semester following the survey data collection. Thirteen students participated, eight of whom had completed the written survey the previous semester. All the interviewees were enrolled in Calculus 2 when the interviews were conducted, and all had either taken

Calculus 1 at the university or received Advanced Placement credit for Calculus 1 in high school.

**For certain drugs, the amount of dose given to a patient, D (in milligrams), depends on the weight of the patient, w (in pounds).**

A. Assume that D(w) is a linear function with a slope equal to 2 (m = 2).
   0. On the graph below, give a rough sketch of what the function D(w) looks like. Label the axes, but no need to scale them.
   1. What are the units on the slope, m = 2?
   2. Explain what this slope (m = 2) means in the context of the problem.
   3. Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident    Somewhat Confident    Not Confident
      Explanation:
   4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident    Somewhat Confident    Not Confident
      Explanation:

B. Now, assume D(w) is a non-linear function.
   0. On the graph below, give a rough sketch of what the function D(w) might look like.
   1. What are the units on $\frac{dD}{dw}$? (also known as $D'(w)$)
   2. Explain the meaning of the statement $D'(140) = 2$ in the context of the problem.
   3. Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
      Very Confident    Somewhat Confident    Not Confident
      Explanation:
   4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation).
      Very Confident    Somewhat Confident    Not Confident
      Explanation:

Figure 3.3. Interview instrument

Students were asked to "think out loud" while they worked through the problems. I asked follow-up questions throughout the interview. For example, if a student sketched their graph but did not follow the direction that stated "label your axes," I made sure to remind them that the question asked to label their axes. As another example, if a student drew their linear function through the origin, then after answering the slope interpretation question I asked whether their answer would change if their graph did not go through the origin.

Interviews allowed me to probe student thinking more deeply, especially focusing on themes that emerged in the survey data analysis. I kept the survey data's findings in the forefront as I conducted the interviews. For example, one of the categories from the written survey data analysis was the need for another derivative to predict the change in dosage. If an interviewee engaged in this sort of reasoning, I asked, "Why do you need a different derivative to answer the question? What could you do with that information if you had it?"

The interviews lasted 30-45 minutes and written work and audio were recorded with a Livescribe™ pen. Students were given a $25 gift card to the university's bookstore for participation in the interview.

### 3.7 Analysis of interview data

Immediately following each interview, I wrote a reflective field note of the interview. I summarized the interview question by question, transcribing student answers that were pertinent to the themes and unanswered questions from the survey data analysis. I copied the graphs from the Livescribe™ recording into the field notes, writing comments about what was being said during the sketching of the graphs.

I also made notes about whether the answers were "ideal knower-like," fell into one of the categories from the survey data, and/or whether they were similar to previous interviewee responses. I examined whether the responses were similar to the larger sample of written survey data, and documented in the field notes any new themes that were emerging from the interview data. This was an interactive process that went on after each individual interview; I did not wait until all the interviews were completed to start this analysis. The process of creating the field notes and analyzing the interview helped inform future interview questions, as themes emerged that were not explicit in the survey data.

# 4. SURVEY DATA RESULTS

The 69 surveys were analyzed using the methods outlined in Chapter 3. While many of the students answered parts of the survey like an "ideal knower," none of the students answered all questions correctly using "ideal knower" reasoning.

Results are presented below for the three types of questions: units (questions A1 and B1), slope and derivative interpretation (questions A2 and B2), and linear and non-linear critiquing of Farmer Jim's predictions (questions A3-A4 and B3-B4). One overarching theme throughout the analyses is that students' underdeveloped understanding of slope (seen by many as a ratio-of-totals) seems to impact their abilities to both interpret the derivative and understand how it can be used to make predictions.

## 4.1 Students' understanding of units on slope and derivative

Questions A1 and B1 addressed the units on the slope and derivative, respectively. In question A1, students were asked, "*what are the units on the slope, m = 2?*" and on question B1, students were asked, "*what are the units on* $\frac{dB}{dn}$ *(also known as* $B'(n)$*)*?" The correct answer for both the slope and derivative questions was "bushels per pound."

### 4.1.1 Students perform similarly on unit questions with and without graphs provided

As discussed earlier, there were two versions of the survey, one that provided students with the graphs of the linear and non-linear relationships, and one where students had to sketch the graphs. Whereas one would perhaps expect that students would perform better on the surveys where the graphs were provided for students, this was shown to not be the case for the unit questions (see section 3.5.1 for a discussion of the methods used to make this determination). The 2x2 contingency tables for the percentage of students who got the questions correct are in Table 4.1 (for the surveys where the graphs were provided to students) and Table 4.2 (for the surveys where students had to sketch the graphs).

| | | Derivative | | |
|---|---|---|---|---|
| Slope | | Correct | Incorrect | Total |
| | Correct | 32% | 19% | 51% |
| | Incorrect | 3% | 46% | 49% |
| | Total | 35% | 65% | N=37 |

Table 4.1. Results from unit questions (surveys with graphs)

| | | Derivative | | |
|---|---|---|---|---|
| Slope | | Correct | Incorrect | Total |
| | Correct | 38% | 6% | 44% |
| | Incorrect | 9% | 47% | 56% |
| | Total | 47% | 53% | N=32 |

Table 4.2. Results from unit questions (surveys without graphs)

For the graph data, the 95% adjusted Ward confidence interval for the proportion of students who got the derivative question correct given that they got the slope question incorrect (in other words, a confidence interval around the lower left cell in the contingency tables) is (0.005, 0.495). For the *no graph* data, the confidence interval is (0.119, 0.769). The fact that these confidence intervals overlap signal that there are no significant differences between the responses for the *graph* and *no graph* surveys. In the analysis that follows, I combine the data from these questions.

### 4.1.2 No significant differences between students' success on slope and derivative units

Whereas one might expect students to perform better on questions involving slope than questions involving derivative, this was not the case in the questions that required students to identify the units. The 2x2 contingency table for the combined data, comparing the slope unit question (A1) with the derivative unit question (B1) is shown in Table 4.3.

| | | Derivative | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| Slope | Correct | 35% | 13% | 48% |
| | Incorrect | 6% | 46% | 52% |
| | Total | 41% | 59% | N=69 |

Table 4.3. Results from unit questions (all surveys)

Performing the McNemar test and examining conditional probabilities gives us information about whether the students were more successful at one of the types of problems. The McNemar test was not significant ($p = 0.27$), which one would expect from looking at the table since the values in the shaded cells are quite similar. The conditional probabilities were also similar, with the probability of a student getting the slope correct given that the derivative is correct being 0.86, and the probability of a student getting the derivative correct given that the slope is correct being 0.77. This indicates that students were only slightly more successful at knowing the slope's units than the derivative's units, and not significantly so.

### 4.1.3 Categories of student responses to slope unit question

After looking at answers in terms of just "correct" ("bushels per pound") and "incorrect," the incorrect answers were examined in more detail. The answers were examined and common patterns emerged. While almost 50% of the students answered the question correctly, it became evident from the incorrect responses that many students struggled with identifying the units of the slope of a linear function, a concept first seen in middle school. In order for students to recognize the units on the slope, they must understand that the slope is a ratio of the dependent variable to the independent variable, and therefore recognize that the units on the slope are the units on the dependent variable over the units on the independent variable.

The struggles that students displayed were categorized into five large types: (1) students who knew that the unit of the slope is a ratio, but did not use units in their ratio or switched the order of the ratio, (2) students who knew that the units involved either "bushels" and/or "pounds" but combined them incorrectly or used just one, (3) students who introduced "acres" into their answer, (4) students who left the question blank or answered "unitless," and (5) students who gave an answer that showed no knowledge of units.

Some students used variable names instead of units, for example, "B/n" or "corn/nitrogen," showing some knowledge of slope as change in the dependent variable over the change in the independent variable, but not recognizing the difference between a variable name and the unit of measurement. Some students used the units, but put them in a different order ("pounds per bushel") or used one variable name and one unit ("bushels per nitrogen"), once again recognizing that slope is a ratio and has to do with division.

Some students just used one of the units (“bushels” or “pounds”), and three students used something other than division to combine the units, either multiplying (bushels x pound), or squaring a unit (bushels per pound-squared). These students used a unit name (as opposed to a variable name), but did not show knowledge of slope as a ratio of the dependent variable to the independent variable.

Quite a few students introduced acres into the units (for example, bushels per acre, acre per square foot, or pounds per acre). It seems likely that this is because in the description of the problem it said that the tract of land that the corn was grown on was 10 acres. While the size of the tract was not a relevant piece of information, it appears that because the size of the tract was given (including its units), 10% of the students used it in their answer.

Some students left the question blank or answered that the slope is unitless. Another group of students gave answers that did not demonstrate knowledge of units (for example, “up 2 over 1,” “(2, 0) and (3, 2),” or “(B, n)”).

Close to half the students produced the correct answer; another 11.6% recognized it as a ratio. The distribution of response types is summarized in Table 4.4. There does not seem to be any one dominant incorrect answer.

| Correct | Gave answer as ratio | Used correct parts but combined incorrectly | Introduced acres into answer | Left blank, or stated “unitless” | No demonstrated knowledge of units |
|---|---|---|---|---|---|
| 33<br>47.8% | 8<br>11.6% | 12<br>17.4% | 7<br>10.1% | 4<br>5.8% | 5<br>7.2% |

Table 4.4. Student responses for the slope unit question

### 4.1.4 Categories of student responses to unit derivative question

For the derivative unit question, the same categories emerged. While the percentage of students answering correctly was lower (40.6% for derivative versus 47.8% for slope), more students who answered incorrectly for the derivative question gave their answer as a ratio (21.7% of derivative responses versus 11.6% of slope responses). Samples of answers that fell into the "no demonstrated knowledge of units" category included things like "B = 1, n = 20" and "20, 40, 60." The distribution of responses is presented in Table 4.5.

| Correct | Gave answer as ratio | Used correct parts but combined incorrectly | Introduced Acres into answer | Left blank, or stated "unitless" | No demonstrated knowledge of units |
|---|---|---|---|---|---|
| 28<br>40.6% | 15<br>21.7% | 11<br>15.9% | 3<br>4.3% | 6<br>8.7% | 6<br>8.7% |

Table 4.5. Student responses for the derivative unit question

Of the 36 students who did not answer the slope unit question correctly, only four answered the derivative question correctly. For those four students, on the slope unit question two of them switched the order of the variables (stating pounds per bushel), one said the slope was unitless, and one introduced acres into the answer (corn per acre). There does not seem to be a pattern among these students' incorrect slope answers.

### 4.1.5 Summary of students' successes and struggles with the unit questions

There were no significant differences between students' success on the slope unit problem and the derivative unit problem. In both cases, less than half of the students answered the question correctly. Without knowledge of the units on these rates of change, it seems unlikely that a student would be able to interpret the meaning of the slope and/or derivative in the context of the problem, which was asked of them next in the survey.

On both the slope and derivative questions, some students who answered correctly were able to give their answer as a ratio, demonstrating some knowledge of slope and derivative as a ratio. However, many more students gave answers that were not ratios, did not include the correct variables, and demonstrated very little knowledge of units. While my research questions do not directly deal with units, students need knowledge of units in order to interpret the slope and derivative in context, the next question in the survey.

### 4.2 Students' abilities to interpret the slope and derivative in context

Questions A2 and B2 addressed interpretation of the slope and derivative, respectively. In question A2, students were asked to "*explain what this slope (m = 2) means in the context of the problem*" and on question B2, students were asked to "*explain the meaning of the statement* $B'(20) = 2$ *in the context of the problem.*" On the slope question, for the answer to be correct, students had to demonstrate that they understood that a slope of 2 means that for each additional pound of nitrogen, the number of bushels will increase by 2. For correct answers to the derivative question, students had to state that when 20 pounds of nitrogen is applied, the corn yield is increasing at a rate of 2 bushels of corn per pound of nitrogen.

#### 4.2.1 Students perform similarly on interpretation questions with and without graphs provided

As was the case with the unit questions, it was shown that students performed similarly on the interpretation questions with and without the graphs provided. The 2x2 contingency tables for the slope and derivative interpretation results are in Table 4.6 (for the surveys giving the graphs) and Table 4.7 (for the surveys where students had to sketch the graphs).

| | | Derivative | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| Slope | Correct | 8% | 19% | 27% |
| | Incorrect | 3% | 70% | 73% |
| | Total | 11% | 89% | N=37 |

Table 4.6. Results from interpretation questions (surveys with graphs)

| | | Derivative | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| Slope | Correct | 0% | 6% | 6% |
| | Incorrect | 16% | 77% | 93% |
| | Total | 16% | 83% | N=32 |

Table 4.7. Results from interpretation questions (surveys without graphs)

For the *graph* data, the 95% adjusted confidence interval for the proportion of students who got the non-linear question correct given that they got the linear question incorrect (in other words, a confidence interval around the lower left cell in the contingency tables) is (0.005, 0.495). For the *no graph* data, the confidence interval is (0.491, 0.781). As before, overlapping confidence intervals demonstrate no significant differences between the *graph* and *no graph* data, and so from this point on, I combine the data from these questions.

### 4.2.2 No significant differences between students' success on slope and derivative interpretations

One might expect that students would be more successful interpreting the slope than derivative, since students have been exposed to slope since middle school. However, based on the survey data, this was not the case. The 2x2 contingency table for the combined data, comparing the slope interpretation question (A2) with the derivative interpretation question (B2) is shown in Table 4.8.

| | | Derivative | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| Slope | Correct | 5% | 13% | 18% |
| | Incorrect | 9% | 74% | 83% |
| | Total | 14% | 87% | N=69 |

Table 4.8. Results from interpretation questions (all surveys)

Performing the McNemar test and examining conditional probabilities can give us information about whether the students were more successful at one of the types of problems. The McNemar test was not significant ($p = 0.61$). The conditional probabilities were also similar, with the probability of a student getting the slope interpretation correct given that the derivative interpretation is correct being 0.33, and the probability of a student getting the derivative interpretation correct given that the slope interpretation is correct being 0.25. Students were not more successful in interpreting the slope than in interpreting the derivative and students' success at interpreting slope does not seem to make them more likely to interpret derivative correctly, as seen by the results of the McNemar test and examination of the conditional probabilities.

### 4.2.3 Categories of student responses to the slope interpretation question

In examining student responses to the question of what a slope of 2 means in the context of the linear problem, 17.4% of students responded correctly, stating something like "for each additional pound of nitrogen, two more bushels of corn are produced." The key language here is that students recognize that the slope represents a constant ratio in the *changes* in variables, thus the "additional" language.

The incorrect answers were categorized into two groups. Many students (39.1% of the total number of students) responded using language that implied they were assuming a directly proportional relationship that goes through the origin, in other words that the slope represents a constant ratio in the variable values $\left(\frac{y}{x}\right)$. The most common response was that "2 represents the number of bushels produced per pound of nitrogen" or "for every pound of nitrogen, 2 bushels of corn are produced," implying a directly proportional relationship of $B(n) = 2n$. This was coded as a "ratio-of-totals" interpretation, and denoted as $B(n) = 2n$. It is important to note that the difference between the correct answer and a "ratio-of-totals" answer is very subtle. The difference could be as subtle as the word "additional" being absent from the statement "for each *additional* pound of nitrogen, 2 bushels of corn are produced."

The remaining responses (43.4% of the total) were incorrect in some way other than the ratio-of-totals approach. For example, some gave answers out of context such as "the rate of increase of the function," or gave vague answers such as "it tells us that the corn is increasing," "it relates to how adding more N changes the bushel production," or "it tells us how the function grows over time." No strong patterns or groupings emerged in this large group of respondents.

Table 4.9 summarizes the results. While there is only a subtle difference between the correct answer ("for every additional pound of nitrogen, 2 more bushels of corn are produced") and the ratio-of-totals interpretation ("for every pound of nitrogen, 2 bushels of corn are produced"), this becomes an important distinction when analyzing the derivative survey data and the interview data.

| Correct | Ratio-of-Totals Interpretation $B(n) = 2n$ | Other Incorrect |
|---|---|---|
| 12<br>17.4% | 27<br>39.1% | 30<br>43.4% |

Table 4.9. Student responses for the slope interpretation question, N = 69

#### 4.2.4 Categories of student responses to the derivative interpretation question

In examining students' responses to the question of what $B'(20) = 2$ means in the context of the non-linear problem, 13.0% responded correctly, saying something like "when the nitrogen is equal to 20 pounds, the corn yield is increasing at a rate of 2 bushels per pound of nitrogen." The key here is that students recognized that the derivative represents a rate of change (2 bushels per pound) at a specific point (in this case, 20 pounds).

For the incorrect responses, five categories emerged. 10.1% of students responded similarly to a correct response, except with no units or incorrect units, answering, for example, "when the nitrogen is equal to 20 pounds, the corn is increasing at a rate of 2." These students demonstrated some knowledge of the derivative as an instantaneous rate of change, but their demonstration was not complete because it lacked units.

Even more students (15.9%) responded using language that implied they were assuming that the rate of change could be used to calculate the total yield, stating that the derivative means “that at 20 pounds, there are 2 bushels produced for each pound of nitrogen.” Some went on to conclude that this gave a total yield of 40 bushes, implying that $B(n) = B'(n) * n$. This language is similar to the “ratio-of-totals” response for the slope, where students responded that for each pound of nitrogen, there are 2 bushels of corn produced.

Other students (10.1%) interpreted the derivative as the function value, concluding that $B'(20) = 2$ means that when 20 pounds of nitrogen are applied, the total number of bushels is equal to 2. The final two categories were for responses from students who gave a correct answer but not in the context of the problem (for example, “it is the slope of the tangent line when n = 20”) and those who gave incorrect answers that did not fit into the other categories (18.8% and 31.9% respectively). Table 4.10 summarizes these results.

| Correct | Correct but no/wrong units | $B(n) = B'(n) * n$ | $B'(n) = B(n)$ | No context | Other Incorrect |
|---|---|---|---|---|---|
| 9<br>13.0% | 7<br>10.1% | 11<br>15.9% | 7<br>10.1% | 13<br>18.8% | 22<br>31.9% |

Table 4.10. Student responses for the derivative interpretation question, N = 69

### 4.2.5 Summary of students’ successes and struggles with the interpretation questions

Slightly more students were successful on the slope interpretation question (17.4%) verses the derivative interpretation question (13.0%). But students’ success at interpreting slope does not seem to make them more likely to interpret derivative correctly. This is a surprising finding, as it seems reasonable to assume that because slope and derivative are such similar concepts that a strong understanding of slope would translate into a higher likelihood of getting the derivative question correct.

Most relevant for this research is that of the 11 students (16%) who answered the derivative question using the "$B(20) = B'(20) * 20$" interpretation, 8 of them answered the linear question using the incorrect ratio-of-totals interpretation ($B = 2n$). In other words, 73% of the students who thought that the derivative could be used to calculate a total yield (by multiplying the rate of change by the number of pounds to get total bushels) also interpreted the slope using an analogous ratio-of-totals approach (total bushels equals slope times number of pounds). It seems like students' misunderstanding of slope as a ratio-of-totals is impacting their abilities to interpret the derivative as an instantaneous rate of change.

**4.3 Students' abilities to critique the reasoning of others and demonstrate understanding of using rates of change to make predictions**

Questions A3 and A4 addressed the ability to critique the reasoning of Farmer Jim in a linear context, using the slope to make predictions. In question A3, Farmer Jim used the slope of m = 2 to "predict that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn." Students were then asked whether they were "very confident," "somewhat confident," or "not confident" in his prediction and to explain why. Question A4 was similar, except Farmer Jim predicts that by adding an additional 10 pounds of nitrogen, he will get an additional 20 bushels of corn.

Questions B3 and B4 also had Farmer Jim making similar predictions, but this time in a non-linear context using the derivative. In B3, he uses the fact that $B'(20) = 2$ to predict that his yield will increase by 2 bushels when his nitrogen application increases by 1 pound. In B4, he uses $B'(20) = 2$ to predict that his yield will increase by 20 bushels when his nitrogen application increases by 10 pounds.

### 4.3.1 Students perform similarly on critiquing questions with and without graphs provided

As with previous questions, here too it was shown that students performed similarly on questions with and without the graphs provided. For the linear context (questions A3 and A4), answers were considered correct for the contingency table analysis if the students responded "very confident" in both cases. For the non-linear context (questions B3 and B4), students who responded with more confidence in B3 than in B4 were considered correct. Many students circled "somewhat confident" for the first part (since the derivative can be used to estimate a change in the dependent variable close to the given input value) and "not confident" for the second part (because the non-linear function's derivative can only be used around the given input value). Also considered correct were responses stating, for example, "very confident" for the first part and "not confident" for the second part. The important feature for a response to be considered correct was that there was more confidence in the one-pound increase than in the 10-pound increase. The student explanations were not considered for the contingency table analysis.

The 2x2 contingency tables for the results of critiquing Farmer Jim's predictions are in Table 4.11 (for the surveys where the graphs were given) and Table 4.12 (for the surveys where students had to sketch the graphs).

| Linear Context | | Non-Linear Context | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| | Correct | 30% | 35% | 65% |
| | Incorrect | 5% | 30% | 35% |
| | Total | 35% | 65% | N=37 |

Table 4.11. Results from critiquing questions (surveys with graphs)

| Linear Context | | Non-Linear Context | | |
|---|---|---|---|---|
| | | Correct | Incorrect | Total |
| | Correct | 31% | 44% | 75% |
| | Incorrect | 0% | 25% | 25% |
| | Total | 31% | 69% | N=32 |

Table 4.12. Results from critiquing questions (surveys without graphs)

For the graph data, the 95% adjusted Ward confidence interval for the proportion of students who had the derivative question correct given that they had the slope question incorrect (in other words, a confidence interval around the lower left cell in the contingency tables) is (0.028, 0.394). For the no graph data, the confidence interval is (0, 0.256). As once again the confidence intervals overlap showing no significant differences between the graph and no graph data, I combine the data from these questions.

#### 4.3.2 Students more successful on slope than on derivative critiquing questions

Like before, one might expect that students would be more successful with the critiquing questions involving linear context (using slope) than with a non-linear context (using derivative), since slope is a concept students have seen for many years. In the critiquing questions this was the case; students were more successful with the linear slope questions.

The 2x2 contingency table for the combined data, comparing the linear critiquing questions question (A3 and A4) with the non-linear critiquing questions (B3 and B4) is shown in Table 4.13.

| | Non-Linear Context | | | |
|---|---|---|---|---|
| Linear Context | | Correct | Incorrect | Total |
| | Correct | 30% | 39% | 69% |
| | Incorrect | 3% | 28% | 31% |
| | Total | 33% | 67% | N=69 |

Table 4.13. Results from critiquing questions (all surveys)

The McNemar test was significant ($p < 0.001$), which is evident from the table since the values in the shaded cells are very different. The conditional probabilities were also very different. The probability of a student getting the linear critiquing question correct given that the non-linear critiquing question is correct was 0.91, and the probability of a student getting the non-linear question correct given that the linear question is correct was 0.44. Students performed significantly better on the linear slope critiquing questions as compared to the non-linear derivative critiquing questions. This demonstrates an understanding of the slope as a constant rate of change that can be applied at any input-value, but shows misunderstandings in how to use the derivative to make predictions.

### 4.3.3 Categories of reasoning responses to linear critiquing questions

A large majority of students (69.9%) answered the two linear critiquing questions correctly, stating that they were confident about both the one-pound and ten-pound increase and providing reasoning like “the slope is a constant rate of change, where for each additional pound of nitrogen you get two bushels of corn.”

An analysis of the incorrect answers resulted in four categories. A few students (2.9%) stated that they did not have enough information to answer the question. Others (10.1%) incorrectly interpreted the slope stating, for example, that they were not confident in Jim's answers because a slope of 2 means that a 10-pound increase would yield an additional 10 bushels. Other students (10.1%) brought in additional ideas based on the context of the problem, saying that they were not confident in the 10-pound increase because they did not know when the yield would plateau, or at some point too much nitrogen would be bad for the crop. Lastly, there was a small percentage of students (7.2%, categorized as "other incorrect") who did not give a reason for their circled answers, were confused by the context of the problem, or interpreted "increased to 30 pounds" as "increased by 30 pounds," which led them to conclude that the drug should be increased by 60 mg.

Table 4.14 summarizes the results. Most interesting is the large percentage of students who answered these questions correctly, much higher than the other slope questions (units and interpretation). Students seem to have a good understanding of slope being a constant rate of change, and therefore being able to use the slope to make predictions over any interval.

| Correct | Not enough info to answer | Questions about context | Incorrect interpretation of slope | Other Incorrect |
|---|---|---|---|---|
| 48<br>69.6% | 2<br>2.9% | 7<br>10.1% | 7<br>10.1% | 5<br>7.2% |

Table 4.14. Student responses for critiquing with the linear context, N = 69

### 4.3.4 Categories of reasoning responses to derivative critiquing questions

Whereas 69.9% of students answered the two linear critiquing questions correctly, only 15.9% answered the non-linear critiquing questions correctly. These students stated, for example, that they were “somewhat confident” or “very confident” about the interpretation in the one-pound increase scenario. However, they were “not confident” in the ten-pound increase scenario, and gave explanations about the derivative being a good approximation close to 20 pounds. Only two students used linear approximation language in their reasoning, which is the language introduced in a differential calculus course.

An analysis of the incorrect answers resulted in four categories. Some students (14.4%) stated that they needed another derivative to answer the question, for example, they needed the derivative at 21 or 30. Still others (4.3%) said they needed to know where the derivative was zero (or where the critical point of the function was located).

Another category of answers was for responses from those students who stated they were “not confident” in both predictions. Thirteen students (18.8%) said this was because the relationship is non-linear, and therefore the derivative is different at each weight. Ideally, students should acknowledge more confidence in the one-pound increase (a marginal increase) than the ten-pound increase, but these students were equally skeptical about both.

A small percentage of students (4.3%) answered “confident” on both, and gave reasoning such as “for each pound of nitrogen, 2 bushels are produced,” which would only be appropriate for a linear function.

Lastly, a large number of students’ responses (42%) fell into the incomplete or incorrect category, and did not fall into the other categories. For example, one student mentioned, “too much nitrogen can be harmful” and then circled “somewhat confident” in both cases. Another student circled “not confident” in both cases and gave reasoning that stated that the farmer “is treating the rate of change like it’s the bushels/pound of nitrogen value B(n).”

Table 4.15 summarizes the results. The percentage correct in Table 4.15 does not match the 33% who answered correctly in the 2x2 contingency table because many of those who circled correct responses did not provide valid reasoning for their answers.

| Correct | Need another derivative to make prediction | Need to know maximum/critical value to make prediction | Not confident on both predictions because relationship is non-linear | Confident on both because for each pound of nitrogen, 2 bushels are produced | Other Incorrect |
|---|---|---|---|---|---|
| 11<br>15.9% | 10<br>14.4% | 3<br>4.3% | 13<br>18.8% | 3<br>4.3% | 29<br>42.0% |

Table 4.15. Student responses for critiquing with the non-linear context, N = 69

Given the struggles many students had with the units and interpretation questions, namely thinking of slope as the ratio-of-totals and using the derivative analogously to calculate the total yield ($B(n) = B'(n) * n$), it seemed likely that a higher percentage of students’ responses would be in the “confident on both because for each pound of nitrogen, 2 bushels are produced” category. But, perhaps many of the “need another derivative to make the prediction” students would want that other derivative in order to calculate the total yield. This is something that was explored in the interviews.

#### 4.3.5 Summary of students’ successes and struggles with the critiquing questions

Many more students were successful on the linear critiquing questions (69%) as compared to the non-linear critiquing questions (15.9%). Students showed an understanding of the slope as a constant rate of change that can be used to make predictions over any interval. Students, however, had large gaps in their understanding of when it was appropriate to use a derivative of a non-linear function to make predictions. Almost 20% of the students said that they were not confident in Farmer Jim's predictions at both the one-pound increase and the ten-pound increase, indicating they do not understand the distinction between approximations for a value near the point of interest and a value far away. Another 14% said they needed to know the derivative at a different point in order to estimate the change in bushels.

On the interpretation questions, while many students took a ratio-of-totals approach for the slope (39.1%) and an analogous approach to the derivative questions by using the derivative to calculate the total (15.9%), only 4.3% of students reasoned similarly on the non-linear critiquing questions. There were, however, 14.4% of students who stated they needed another derivative, and it is not clear from their explanations why they felt a need for this other derivative.

### 4.4 Summary of survey results

Some key themes emerge from the survey data. First, there were no significant differences between the findings based on data from the *graph* and *no graph* surveys for any of the questions. Originally, I had thought that by providing the graphs, students might be more successful in answering the questions, or might be less apt to make assumptions, such as the origin-assumption for the linear graph (which might lead to an incorrect ratio-of-totals approach to the slope). However, this was not the case, and because there were no significant differences, I was able to combine the data for subsequent analyses.

Second, whereas students performed similarly on the slope and derivative questions for the units and interpretation scenarios, students performed much better on the linear (slope) critiquing problems as compared to the non-linear (derivative) critiquing scenarios. This is apparent in Figure 4.1, which provides an overview of how students did on the slope and derivative questions for the three scenarios.

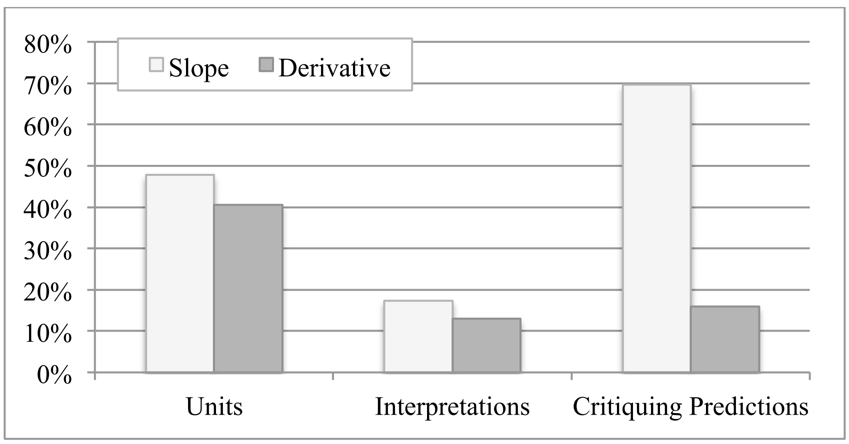



Figure 4.1. Students' success rates on the slope and derivative questions for the three types of scenarios (units, interpretation, and critiquing)

Third, looking at individual results for each scenario, students had difficulty with the unit questions, with only 35% of the students answering both the slope and derivative questions with correct units. Even more surprising was that students did not perform significantly better on the slope unit question than the derivative one. Units appear to be difficult for students, regardless of whether they are units for a linear relationship or units for a non-linear relationship. Units are a key part of understanding and interpreting rates of change, and by not recognizing the units on a slope or derivative, it seems unlikely that students would have a complete understanding of these rates of change.

Fourth, for the interpretation questions, students performed very poorly, with only 5% of students answering both the linear and non-linear questions correctly. The McNemar's test shows no significant differences in the marginal probabilities. For the linear interpretation, 17% of students answered correctly, and more than twice that many (39%) gave a ratio-of-totals answer, implying $B(n) = 2n$. This subtle difference (recognizing the slope as the "change in y over change in x," instead of "y over x") might be leading to misunderstandings with more complex concepts such as the derivative. Given that understanding slope in a functional context as a rate of change is necessary in understanding derivative, this is a very concerning result. The ability of students to interpret the slope and derivative in the context of the real-life situations is core to understanding these rates of change, but it is clear that most surveyed students are unable to correctly answer these questions.

For the non-linear interpretation question, only 13% of students answered correctly, with 16% taking the approach that you can use the derivative to find the total value ($B(n) = B'(n) * n$). Eight of the eleven students who answered using this approach also answered the slope interpretation question using the ratio-of-totals ($B(n) = 2n$) interpretation. While an ideal knower would understand that the slope is the ratio-of-changes, students who think of it as a ratio-of-totals seem to be bringing this misunderstanding to their interpretation of the derivative. Their incorrect view of slope as the ratio-of-totals seems to be clouding their ability to correctly interpret the derivative as an instantaneous rate of change.

Lastly, for the critiquing questions, in examining whether students correctly circled the level of confidence, as presented in the 2x2 contingency table, 30% correctly circled valid confidence levels for both the linear and non-linear. But, when examining their reasoning, many did not provide valid reasons. When I looked at the reasons for their responses, I found that 70% gave valid correct answers for the linear critiquing, but only 16% gave valid correct answers for the non-linear critiquing, with only two students mentioning linear approximation and the use of the derivative to make approximations near the known value.

Many students had the same level of confidence in the one-pound increase as in the 10-pound increase (23.1%), showing no understanding of losing accuracy in the prediction when moving away from the known value. Another group of students (14.4%) said that they could not answer the non-linear prediction questions because they needed another derivative, namely the derivative at 21 pounds or 30 pounds. Both of these fail to show understanding of the derivative as an instantaneous rate of change that can only be used at (or near) the known input value.

In summary, the survey results showed that students struggle with key ideas surrounding slope and derivative as rates of change. While student did significantly better on the linear critiquing question (as compared to the non-linear critiquing question), they did not perform significantly better on the slope unit and slope interpretation questions. Their struggles seem to be rooted in their knowledge of slope, as many of their misunderstandings of slope manifested themselves again in their answers to the derivative questions.

# 5. INTERVIEW DATA RESULTS

The written surveys gave much insight into my two research questions, though there were several findings that called for further investigation. In planning follow up interviews with survey respondents, I chose to delve more into the following questions that arose from the survey data:

- Did interviewees have similar difficulties to the surveyed students on the unit questions for slope and derivative, where only 35% of the surveyed students identified correct units on both the slope and derivative? If they had more success, could it be attributed to the different context (drug dose instead of corn yield)?
- Did the interviewees answer the slope interpretation question using the incorrect ratio-of-totals interpretation ($B(n) = 2n$), thus implying a directly proportional relationship, and the derivative interpretation question using the analogous explanation ($B(n) = B'(n) * n$)? If so, what was their reasoning?
- Did the interviewees give similar answers to the prediction questions as the survey responders? In particular, twenty-eight percent (22.1%) of the surveyed students did not distinguish between the 1-pound and the 10-pound increases for the non-linear context, giving similar confidence levels and reasoning. And, 14% stated that they did not agree with Farmer Jim because they needed a different derivative to answer the question. Did similar responses emerge from the interviewees, and if so, would follow-up questioning get at the reasoning behind these responses?

These questions were at the forefront of my mind as I interviewed the thirteen students. Of the thirteen interviewees, only one (Jarrod[1]) gave what I consider ideal answers for all questions. Excerpts of Jarrod's interview are presented throughout the chapter to illustrate ideal answers and to identify important points to keep in mind when examining other interviewee's responses and comparing them to the desired, ideal answers.

## 5.1 Students' understanding of units on slope and derivative

Students were asked to give the units on the slope and derivative using the context of the problem. Recall the context for the interview questions: "For certain drugs, the amount of dose given to a patient, $D$ (in milligrams), depends on the weight of the patient, $w$ (in pounds)." In the linear question, students were to assume that $D(w)$ is a linear function with a slope = 2 and were asked for the units on the slope. In the non-linear question, students were to assume that $D(w)$ is a non-linear function and were asked for the units on $\frac{dD}{dw}$ (also known as $D'(w)$). While not explicitly described in the research questions, understanding of units on rates of change is necessary in order to interpret the meaning of a rate of change in the context of a problem.

### 5.1.1 Summary of unit question results from the surveys

The survey results showed no significant differences between students' success on the slope unit question and the derivative unit question. On the individual slope and derivative unit questions, success rates were less than 50%, and only 35% of the students

[1] All names used are pseudonyms; gender was preserved in the name choices.

answered both unit questions correctly. With the low success rates on the unit questions, it was not surprising that students struggled with the interpretation questions that followed in the interviews. Approximately 10% of the students incorrectly introduced "acres" into their unit responses; I changed the context in the interview questions to one that did not contain extraneous units to see whether this might have influenced the success rates.

### 5.1.2 Ideal knower responses on the unit questions

Jarrod was the one interviewee who answered all questions correctly, and thus will be used as an example of an "ideal knower." At the start of the linear questions, when asked to sketch the function, Jarrod sketched a linear function with a positive slope and a vertical intercept of zero. He was then asked what the units were on a slope of 2:

<u>Jarrod:</u> *Ummm.. It'd be milligrams per pound I suppose.*

<u>Interviewer:</u> *Why did you get milligrams per pound?*

<u>Jarrod:</u> *The slope of a line is the change in the y over the change in the x. And the y is milligrams and the x is pounds.*

Key Point: Jarrod clearly stated the correct definition of slope, as the change in y over the change in x, not a common (incorrect) response of "y over x."

For the non-linear questions, he first sketched a concave down increasing function starting at the origin. He also drew a concave up increasing function and said you could argue it either way. For the units on the derivative, he struggled at first but talked himself through it:

<u>Jarrod</u>: *Umm…milligrams per pound per pound. Because….oh, hmm….I think it's milligrams per pound per pound because you are showing how the rate of change.*

*No, I'm wrong. I didn't just sketch the derivative. This [graph ] still shows the dosage, so the derivative of this would be….milligrams per pound. And the units on the derivative of the derivative would be milligrams per pound per pound.*

<u>Interviewer</u>: *So if I'm asking for the units on the derivative of this function* [I pointed to the graph of $D(w)$]*…that would be…*

<u>Jarrod</u>: *Milligrams per pound*

### 5.1.3 Interview responses on the unit questions

The interviewee success rates on the unit questions were higher than the survey participants' results (Table 5.1). Like the surveyed students, the interviewees had similar success rates on the slope and derivative unit questions. For both the surveys and the interviews, students performed slightly better on the slope units question. One reason for the higher success rates for the interviewees might be that the context on the written surveys could be interpreted as slightly more complicated, since it was the yield of corn (in bushels) that results from nitrogen (in pounds) added to a 10-acre tract of land, and some students introduced acres into their answers about units. Also, during the interviews, at the start of the linear and non-linear questions, students were asked to sketch the relationship. Before moving onto the unit questions I made sure students produced graphs that had well-labeled axes (including units). These well-produced graphs might have aided them in answering the unit questions.

| | Surveys (N=69) | Interviews (N=13) |
|---|---|---|
| Slope | 33<br>47.8% | 9<br>69.2% |
| Derivative | 28<br>40.6% | 8<br>61.5% |

Table 5.1. Survey and interview participants' success rates for unit questions

Table 5.2 summarizes the answers for the unit questions for all 13 interviewees. All the students except one either got both questions correct (62%) or both incorrect (31%). Only one student had difficulty with the derivative (answering milligrams per pound-squared) after getting the slope unit question correct.

| | | Derivative | | |
| --- | --- | --- | --- | --- |
| Slope | | Right | Wrong | Total |
| | Right | 8<br>61.5% | 1<br>7.7% | 9<br>69.2% |
| | Wrong | 0<br>0% | 4<br>30.8% | 4<br>30.8% |
| | Total | 8<br>61.5% | 5<br>38.5% | N=13 |

Table 5.2. Summary of interview unit questions

Of the thirteen interviewees, eight (62%) gave correct answers for the units on the slope (mg/pound) and derivative (mg/pound). When asked how they came up with mg/pound, six students responded that the slope was "change in y over change in x," "change in dosage over change in weight" or "rise over run," so they put the milligrams (the units on the y-variable) on top and the pounds (the units on the x-variable) on the bottom. For example, Kelly responded:

*Kelly: milligrams per pound*

*Interviewer: OK, milligrams per pound, and how did you come up with that?*

*Kelly: Ummm….the same thing. It's the rise over the run. So this is in milligrams* [points to dependent axis*], so it's milligrams over the independent variable, which is pounds.*

Two students gave the correct slope of milligrams per pound, but went on to explain their reasoning by saying that slope was "y over x," in other words a ratio-of-totals, instead of the correct answer of "change in y over change in x" or "rise over run," or a ratio-of-differences. For example, Chip claimed he chose milligrams per pound because "the slope is y over x." Missy also stated milligrams per pound because it's "y over x." While their reasoning led them to the correct units of milligrams per pound, their incorrect ratio-of-totals reasoning might lead to misunderstandings later on when they interpret the slope and derivative, as well as use the slope and derivative to make predictions. It is also possible, however, that it was just a short cut way to describe the slope and does not signal a true ratio-of-totals misunderstanding.

Two students switched the units on slope to be pounds per milligram (instead of milligrams per pound). Pam originally stated, "pounds" but quickly changed her answer to "pounds per milligram" with no prompting.

> *Pam: Oh the slope? The slope would be pounds per milligram.*
>
> *Interviewee: Pounds per milligram. How did you come up with that?*
>
> *Pam: The weight is pounds and the dosage is milligrams.*
>
> *Interviewee: OK, and why the pounds on the top and the milligrams on the bottom? Why not vice versa?*
>
> *Pam: Pounds is the x, so I was just dividing x by y.*
>
> *Interviewee: Dividing the x by y because……that's what you usually do?*
>
> *Pam: Yes.*

Later on, Pam interpreted the slope interpretation question correctly, and then recognized her mistake on the units and went back and corrected her answer to milligrams per pound.

Two students gave incorrect responses for the slope units (mg for one, and dosage per weight for the second), but after some prompting, they answered them correctly as milligrams per pound, and went on to answer the derivative units correctly too. Maddie's reasoning for switching her incorrect answer of "milligrams" to the correct answer of "milligrams per pound" was very interesting. Instead of taking the most common reasoning that slope is "the change in y over the change in x," so the units would be the "units on the y-axis over the units on the x-axis," she instead looked at the equation $y = mx$ and reasoned that the units on the slope times the units on the independent variable had to equal the units on the dependent variable.

*<u>Maddie</u>: So you're multiplying the pounds by the slope to get milligrams.*

*<u>Interviewer</u>: OK, I see what you are saying.*

*<u>Maddie</u>: So you have your slope, you multiply by pounds and you should get milligrams. So your slope has to be milligrams per pound.* (Figure 5.1).

*<u>Interviewer</u>: Because those pounds are going to cancel out?*

*<u>Maddie</u>: Yes.*

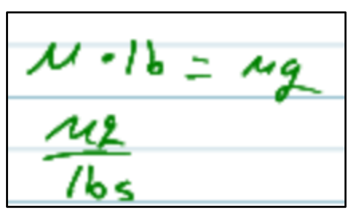



Figure 5.1. Maddie's reasoning for why the unit on slope is milligrams per pound

It was unclear from Maddie's answer whether she had a clear understanding of slope as the ratio-of-differences or whether she would take a ratio-of-totals approach to answer the interpretation questions. The graph that she drew had a y-intercept of zero (directly proportional) and although her calculations led her to the correct units, I was unsure as to whether she had a level of understanding as sophisticated as our ideal knower.

Harry answered the slope units correctly but went on to say that the units on the derivative were mg/lb$^2$. When probed he said, “I’m thinking about physics where, like, velocity is meters per second, and the derivative of velocity is acceleration which is meters per second squared.” This shows a lack of understanding of the units on the derivative as the units on the dependent axis over the units on the independent axis; instead Harry is reaching for something he recalls about kinematics. While the ideal knower understands that the units on the slope and derivative are the same, Harry does not.

### 5.1.4 Summary of interviewees’ successes and struggles with the unit questions

In summary, the success rates of the interviewees for the unit questions were better than the written survey results, perhaps signaling that the context in the interview questions was less confusing than the survey questions (where the extraneous unit of acres was used to describe the context). Still, 38% of the interviewees struggled with units, a key part of understanding slope and derivative as a rate of change. Even for the six students who answered correctly (milligrams per pound), two of them described the slope as “y over x,” implying a ratio-of-totals approach to slope, as opposed to the correct ratio-of-differences reasoning. Examining the possible impact of this on their thinking about other ideas (such as the interpretation of the slope and derivative) is part of what was done in the remainder of the interview.

## 5.2 Students' abilities to interpret the slope and derivative in context

After the questions about units, students were asked to give interpretations of the slope and derivative in the context of the problem. On the linear question, students were to explain the meaning of a slope of 2. On the non-linear question, students were asked to explain the meaning of the statement $D'(140) = 2$. These questions were directly tied to the first research question surrounding students' abilities to interpret the slope and derivative.

### 5.2.1 Summary of interpretation question results from the surveys

The survey results showed that students were slightly more successful on the slope interpretation question than on the derivative interpretation question, but not significantly so. Students' success at interpreting slope did not make them more likely to interpret the derivative correctly. The interpretation questions had success rates on the individual slope and derivative questions of less than 20%, and only 5% of the students answered both interpretation questions correct. One key finding was that 73% of the students who thought the derivative could be used to calculate the total yield (taking the $B(n) = B'(n) * n$ approach) also interpreted the slope using an analogous ratio-of-totals approach ($B(n) = m * n$). By asking follow-up questions to similar interviewee responses, I found out the thinking behind students' incorrect interpretations.

### 5.2.2 Ideal knower responses on interpretation questions

When asked to interpret the slope in the context of the problem, Jarrod (our ideal knower) responded:

<u>Jarrod:</u> *It means for every pound increase in the patient, you can expect an increase dose of 2 milligrams.*

Key Point: Once again, he refers to the change in the weight (the increase in weight) and the change in the dosage (the increase in dosage), not a common incorrect response of "the dosage is twice the weight."

Recall that Jarrod had initially drawn a linear function that went through the origin. After asking him to interpret the slope, I drew a second graph with a slope of 2 that did not go through the origin. I asked if his interpretation of slope would be any different if the graph looked like that.

> Jarrod: *Umm….if you are just looking from one point on the line to another, then how much it increases by, I'd say it would be the same……But you can't define it as twice as many milligrams as pounds. Over here* [points to graph that goes through the origin]*, you can say that any point on the line, numerically the milligrams dose is exactly twice as much as the weight in pounds. But over here* [points to graph that goes that has a positive y-intercept] *you can't because you are not starting at zero.*

Key Point: Jarrod was able to distinguish between a directly proportional graph (that goes through the origin and for which slope is equivalent to $\left(\frac{y}{x}\right)$) and a linear function that does not go through the origin (with a slope that is $\left(\frac{\Delta y}{\Delta x}\right)$).

When asked to interpret the derivative ($D'(140) = 2$) in the context of the problem, Jarrod responded:

> Jarrod: *So, 140 is the weight of the patient, and so that means at 140 (on that graph) that the rate at which the dosage is changing is 2 milligrams per pound. So, if you drew a tangent line at that point. You could say that, ummm if you are trying to estimate, like at 141 or 139, about how much the dosage is going to change from one point to the other, you can't exactly do because it's not linear and it's following some*

*sort of curve, but you would be able to add 2 or subtract 2 to approximate the change.*

Key Point: Jarrod was not only able to state the correct interpretation for the derivative concisely; he continued on to explain what the derivative can be used for (to predict changes near by).

### 5.2.3 Interview responses to the slope interpretation question

As with the unit questions, the interviewees did better than the surveyed students on the interpretation questions. Table 5.3 summarizes the success rates. This improved success could be due to the different context, the higher success on the unit questions (correct units could aid in answering the interpretation questions correctly), or the follow-up questions that were asked by the interviewer.

| Surveys (N=69) | Interviews (N=13) |
|---|---|
| 12<br>17.4% | 4<br>30.8% |

Table 5.3. Survey and interview participants' success rates for slope interpretation questions

For the interpretation of a slope of 2 in the context of the problem, 4 of the 13 interviewees gave correct responses. Excerpts from these four interviews are given below:

<u>Jarrod:</u> *It means for every pound increase in the patient, you can expect an increase dose of 2 milligrams.*

<u>John:</u> *For every one additional pound, add two milligrams.*

*Pam: For each more pound, you get two more milligrams.*

*Dawn: It means the change in the dosage in comparison the weight.*

*Interviewer: And where does the 2 come into play?*

*Dawn: The 2 is like the change in it. So the dosage will change twice as much per change in weight.*

*Interviewer: OK, so the dosage changes twice as much as the weight changes?*

*Dawn: Yes.*

*Dawn: For every unit of weight you increase, you will increase twice that amount in dosage.*

Eight other students gave ratio-of-total responses, using language such as “for every pound, they would need 2 mg of drug” or “dosage is twice the number of pounds.” As an example, here is an excerpt of Jackie’s interview:

*Jackie: It means that for, I guess, it means that for every pound, 2 milligrams of the dose. Depending on how much you weigh. How much it increases. So if you weigh 2 pounds, you would get 4 dosages.*

*Interviewer: OK, so 4 milligrams?*

*Jackie: Yes*

*Interviewer: OK, so if you weight a certain amount, you will get twice as much milligrams?*

*Jackie: Yes.*

Chip and Missy, who in their unit question gave the reasoning that “slope is y over x” both went on to interpret the slope incorrectly.

*Chip: Basically it means for someone's weight, you give them…their numerical value for the weight. If they weighed 100 pounds, their dosage would be 200…twice that.*

*Interviewer: So the dosage is twice the weight?*

*Chip: Yes, but not the same units. Dosage in milligrams, weight in pounds.*

*Missy: For every one pound, you go 2……you double it.*

*Interviewer: OK for every one pound you double what?*

*Missy: You multiply it by 2 to get the dosage.*

*Interviewer: OK, so what if I weighed 150 pounds?*

*Missy: 300 milligrams.*

These students thought of the slope as the ratio-of-totals (instead of the ratio-of-differences), that led them to an implied $D(n) = 2n$ relationship (where the y-intercept is zero). All students except one (Emily) graphed the linear equation going through the origin. Emily graphed a positive y-intercept but still interpreted the slope of 2 as a ratio-of-totals.

For seven students who graphed the linear equation going through the origin, I had a follow up question asking them whether their interpretation of the slope would be different if the equation did not go through the origin. I drew a graph of $D(w)$ with a positive y-intercept. Two of the seven students had answered the slope interpretation correctly, and five had answered it using the incorrect ratio-of-totals reasoning. Excerpts are below from a subset of these two groups.

For the two students who answered the interpretation question correctly with their original graph that went through the origin, they used correct language again to describe the slope for the newly drawn graph that did not go through the origin. Excerpts of one of the interviews follow:

> <u>Jarrod</u>*: Umm….if you are just looking from one point on the line to another, then how much it increases by, I'd say it would be the same……But you can't define it as twice as many milligrams as pounds. Over here* [points to graph that goes through the origin]*, you can say that any point on the line, numerically the milligrams dose is exactly twice as much as the weight in pounds. But over here* [points to graph that goes that has a positive y-intercept] *you can't because you are not starting at zero.*

These students had a solid understanding of the slope in both situations; Jarrod made the clear distinction between a linear relationship that has a non-zero y-intercept (where slope is a ratio-of-differences) and a directly proportional linear relationship (that goes through the origin, where slope is a ratio-of-totals).

For the five students who interpreted the slope using a ratio-of-total reasoning on their original graph that they drew through the origin, when I drew another graph with a non-zero y-intercept, four of the five said that the only change to their explanation would be that, in order to find a patient's dosage, you would have to add the y-intercept "at the end." They still used incorrect ratio-of-totals reasoning to describe the slope, and justified the different y-intercept as something that just needs to be added if you wanted to calculate the final dosage. To them, the y-intercept in and of itself had nothing to do with the slope interpretation. This is important to note because these students were not looking at their graph (drawn through the origin) and recognizing that for a directly proportional relationship, $\left(\frac{y}{x}\right)$ holds everywhere (like Jarrod did above). Instead, they were misinterpreting the meaning of the slope as $\left(\frac{y}{x}\right)$, instead of $\left(\frac{\Delta y}{\Delta x}\right)$, regardless of the y-intercept.

For example, I asked the students whether their interpretation of slope would be the same if the graph had a positive y-intercept. A few excerpts follow:

*Maddie: Yes.*

*Interviewee: So does it still mean "for every pound the patient weighs, the milligrams increase by 2?"*

*Maddie: Yeah, then you'd have to add the y-intercept.*

*Interviewee: How could you interpret the slope given that? Would you use the same interpretation?*

*Maddie: Yes, we're describing the slope, not the function.*

*Interviewee: So it still holds true?*

*Maddie: Yes.*

*Chip: The slope would be the same. You would have different number.... values associated with each tick mark because you have the graph running through a different point on the y-axes now. If the slope is the same, the change will be the same. It just has a different starting point.*

*Interviewer: So you could still say the dosage in milligrams would be twice the weight in pounds?* (Chip's response initially)

*Chip: Yes.*

Craig was the only student who changed his reasoning from a ratio-of-totals interpretation for the directly proportional relationship that he drew, to a ratio-of-differences interpretation for the graph that I drew (with a y-intercept of 50 mg and a slope of 2 mg per pound):

*Interviewer: Can you explain the slope in the context of this problem?*

*Craig: Of course. Initially, if someone weighs zero pounds, they need…theoretically if they weigh zero pounds, they need 50 milligrams of the drug. And then for every additional pound, they need 2, they need 2 milligrams more.*

In addition to the incorrect ratio-of-totals approach to the slope interpretation, there was one student, Brandon, who could not verbalize the interpretation of the slope, and instead kept talking about the steepness of the line. It is interesting to note that Brandon did go on to interpret the derivative correctly, but was unable to interpret the slope and got boggled down with the steepness of the line. This focus on "steepness" and trouble distinguishing between slope and steepness is something Teuscher and Reys (2007) noted in their study of AP calculus students.

In summary, only five of the 13 interviewees (38.5%) correctly interpreted the slope. Seven gave ratio-of-totals interpretations (53.8%), and one only student (7.7%) gave a different incorrect response. For those who initially drew a graph through the origin and then were later provided a graph with a non-zero y-intercept, those who answered correctly with their original graph continued to use the correct interpretation with the new graph, with Jarrod going on to point out the differences between the directly proportional relationship (where slope is equivalent to $\left(\frac{y}{x}\right)$) and a linear function that does not go through the origin (with a slope that is $\left(\frac{\Delta y}{\Delta x}\right)$). For those who answered incorrectly with their initial graph through the origin, four out of five continued to use an incorrect ratio-of-totals interpretation when given the graph with the non-zero y-intercept. The ratio-of-totals interpretation is by far the dominant incorrect interpretation of slope.

### 5.2.4 Interviewee responses to the derivative interpretation question

Similar to the slope interpretation questions, the interviewees performed better on the derivative interpretation question than the surveyed students (see Table 5.4).

| Surveys (N=69) | Interviews (N=13) |
|---|---|
| 9<br>13.0% | 5<br>38.5% |

Table 5.4. Survey and interview participants' success rates on derivative interpretation questions

For the derivative interpretation, five of the thirteen interviewees (38.5%) gave correct responses (at 140 pounds, the drug dosage is increasing at a rate of 2 mg per pound). Four of the five (Jarrod, John, Pam, and Craig) had also interpreted the slope correctly. First are two quotes from students who had interpreted the slope correctly:

*John: It is the slope of the tangent line at 140, and it's equal to 2.....the dosage is increasing at a rate of 2 milligrams per pound. When the weight is 140 pounds.*

Pam goes more in-depth to describe what the derivative can be used for, namely that an instantaneous rate of change at a point that can be used to figure out what will happen to the dependent variable with small changes in the independent variable.

*Interviewer: What does this derivative mean then for the 140-pound patient?*

*Pam: Well, if a patient were to weigh 141 pounds, if you go up by 1 pound, or something really close, like a half a pound, you would be going up by 2.*

*<u>Interviewer</u>: So, if you are 140-pounds, and you go up a little bit….your dosage would go up by how much?*

*<u>Pam:</u> Well, if it's 141, so we don't have to use decimals, then 141 would mean an additional 2 pounds.*

Craig uses similar reasoning to Pam, but makes one key mistake within his explanation when he refers back to how it would be different if it were a linear function:

*<u>Interviewer</u>: So what does it mean in the context of the problem?*

*<u>Craig</u>: It means, that, at 140 pounds, it will take 2 milligrams per pound to… that the dosage will increase by 2 milligrams per pound at 140 pounds. It's the instantaneous rate of change.*

*<u>Interviewer</u>: OK, so when I'm 140 pounds, then what's happening with the 2 and my rate?*

*<u>Craig</u>: The instantaneous rate of change of the function is 2 milligrams per pound. So if you are 140 pounds, the slope, in this case it's not linear. So the derivative won't give you. It can't just multiply the weight by the derivative to get the answer…..At that instant, if you could, gain an infinitely small amount of weight, that infinitely small amount of weight times 2 would give you the milligrams you'd need.*

Craig describes the derivative with what seems like ideal knower understanding, but then when he compares the instantaneous rate of change to the linear function, he says that we can no longer multiply the weight by the derivative to get the answer, a process that would hold true for only linear relationships that are directly proportional. As a reminder, Craig at first interpreted the slope as a ratio-of-totals, but then when I drew a linear function that did not go through the origin, he changed his answer to a correct ratio-of-differences interpretation. Here, however, he is implying that for all linear relationships, one can calculate the total by multiplying the slope by the input (in other words $y = mx$). The instability of Craig's answers suggests that even though he got the answers correct at the end, that his understanding might not be solid.

Eight students did not answer the derivative interpretation question correctly. Seven of these students gave responses stating that the derivative allows you to calculate how much dosage to give per each pound. In other words, they used a $D(n) = D'(n) * n$ approach. A few excerpts highlight this reasoning:

*Dawn: That, ummmm, it terms of slope it would mean that at 140, the line is equal to 2. In terms of function, the value of D-prime, at x = 140, would be equal to 2.*

*Interviewer: So, if I'm 140 pounds, what does the 2 represent?*

*Dawn: The 2 would represent….hmmm…..I'm not 100% sure.*

*Interviewer: OK, What if you think about the units from before? What if I say, I'm 140 pounds.*

*Dawn: Oh, it would be able to tell you how much drug to get. So, if it's 2 mg/pound, with an 140-pound patient, your dosage would be 2 * 140, or 280 mg.*

*Interviewer: So that 2 mg/pound tells you how much drug to give?*

*Dawn: Yes.*

*Interviewer: OK, umm….so how do we umm….what exactly does the 2 mean? When the patient is 140 pounds, what's going on in terms of the 2?*

*Harry: Meaning that at 140 pounds, it needs twice as many milligrams per pound. So it would need 280.*

*<u>Interviewer</u>: OK, so when a patient is 140 pounds, that means they need….*

*<u>Harry</u>: 280*

*<u>Interviewer</u>: 280 mg. So for each pound they need 2 milligrams? That's what it is saying?*

*<u>Harry</u>: Yeah*

Table 5.5 summarizes the slope and derivative interpretations. Four of the five students who interpreted slope correctly went on to interpret the derivative correctly. Six of the seven who interpreted slope as the ratio-of-totals (thus implying a directly proportional relationship) went on to interpret the derivative similarly as a rate of change that can be used to calculate the total.

| | | Derivative Interpretation | | |
|---|---|---|---|---|
| | | Correct | $D(n) = D'(n) * n$ | Other |
| Slope Interpretation | Correct | 4 | 1 | |
| | ratio-of-totals ($D = 2n$) | | 6 | 1 |
| | Other | 1 | | |

Table 5.5. Interview responses for the slope and derivative interpretations, N = 13

**5.2.5 Summary of interviewees' successes and struggles with the interpretation questions**

In summary, only five of the 13 (38.5%) correctly interpreted the slope. Seven gave ratio-of-totals interpretations (53.8%), and one only student (7.7%) gave a different incorrect response. For the written surveys, 17.4% answered correctly and 39.1% gave the ratio-of-totals reasoning. For the remaining 43.3% that were left blank or answered incorrectly using another reasoning in the written surveys, it is unclear what they would have concluded in an interview setting with some follow-up questions.

The two students who provided the correct units for slope but described their reasoning as slope being “y over x” instead of “change in y over change in x” went on to interpret the slope as a ratio-of-totals. This signals that students are not just leaving out the “change in” language, but instead think that for all linear relationships, the slope can be used to calculate the total (in other words are directly proportional relationships).

Also important to note is that four of the five students who interpreted the linear relationship as a ratio-of-totals after drawing a graph through the origin, went on to take the ratio-of-totals approach when I drew a linear relationship with a non-zero y-intercept. This again adds evidence to the claim that students’ incorrect understanding of slope as a ratio-of-totals is impacting their ability to make sense of linear relationships that have non-zero y-intercepts.

Building on the results of the survey data, the interviews provide additional evidence that students’ inabilities to interpret the slope as ratio-of-differences (and instead taking a ratio-of-totals approach) lead to struggles with interpreting the derivative. If a student takes the ratio-of-totals approach for slope (where $y = m * x$), he/she is more likely to hold an incorrect interpretation of the derivative as something that can be used to directly calculate the total ($D(n) = D'(n) * n$). Six of the seven interviewed students (85.7%) who took the incorrect slope approach (ratio-of-totals) went on to take the analogous incorrect derivative approach ($D(n) = D'(n) * n$).

### 5.3 Students' abilities to critique the reasoning of others and demonstrate understanding of using rates of change to make predictions

Students were asked in the interviews to critique the reasoning of Nurse Jodi's predictions; there were a total of four predictions, two based on the linear model and two based on the non-linear model. In all cases, students had to circle whether they were "very confident," "somewhat confident," or "not confident" in Nurse Jodi's predictions, and explain their reasoning.

For the linear model, the two predictions are as follows:

- "Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds."
- "Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight)."

Similarly, for the non-linear model:

- "Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds."
- "Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight)."

Both sets of questions (linear and non-linear) focus on the second research question about students' abilities to critique the reasoning of others in making predictions using rates of change.

### 5.3.1 Summary of critiquing question results from the surveys

The survey results showed much higher success rates with critiquing the linear model (69%) as compared to critiquing the non-linear one (16%). Students showed an understanding of slope as a constant rate of change that can be used to make predictions over any interval. They had large gaps, however, in their understanding of when it was appropriate to use a derivative to make predictions in the non-linear context.

A higher proportion of students used a ratio-of-totals approach in the derivative interpretation question (15.9%) than in the derivative critiquing questions (4.3%). However, there were 14.4% of the students who stated they needed another derivative, and it was not clear in the surveys which derivative they would need or what they would do with that derivative. The interview follow-up questions were designed to understand students' reasoning behind needing another derivative and what they would do with that derivative.

Also, 20% of the students made no distinction in critiquing the one-pound increase and the ten-pound increase, displaying incorrect understanding of the appropriateness of using a derivative to make predictions. The interview questions further explored why students had similar levels of confidence in both predictions.

### 5.3.2 Ideal knower responses to the critiquing questions

For Nurse Jodi's prediction of a one-pound increase assuming the linear model, Jarrod responded:

> <u>Jarrod:</u> *Umm…I'd say, that is exactly correct, based on what I just said. Um…if you just take the change in y (your dose, given as 2 mg) over change in x (141-140, so 1 pound). 2 over 1, or 2, so assuming it's linear, it's exactly right.*

Key Point: He again stressed the slope being the change in y over the change in x.

For Nurse Jodi's prediction of a twenty-pound increase, once again with the linear model, Jarrod responds:

*Jarrod: Umm…I believe it would be the same thing. Increasing by 40 mg, weight goes up by 20 pounds, so 40 over 20, or 2. If linear model is correct, then yes.*

*Interviewer: Same amount of confidence in both?*

*Jarrod: Yes.*

Key Point: He had equal confidence in both the one-pound and twenty-pound increase.

For the non-linear model, Jarrod responded to Nurse Jodi's prediction for a 1-pound increase:

*Jarrod: Umm, yeah, just like I just said. Because we know it's not linear, I don't have confidence in her statement, because that says if you go from 140 to 141, that it is following this line (tangent line), we are following that same slope. It's a good estimation, but not with a lot of certainty.*

*Interviewer: Would it be an over approximation or under approximation?*

*Jarrod: It would be an over in the case of this graph.*

Key Point: Jarrod stated he was not confident, but went on to explain that it would be a good estimate.

For Nurse Jodi's prediction for a 20-pound increase with the non-linear model:

*Jarrod: Ummm, the same thing. Just a different change in x. If you follow that same tangent line, assuming the slope is the same everywhere. It's a way over estimate.*

*Interviewer: What if we estimate the slope of 160 as ½. Could you do anything with this?*

*Jarrod: It would definitely tell you that you can't follow the linear pattern. It would tell you that you can't follow this type of model. Your slope has changed so much.*

Later, we go back to the 1-pound increase:

*Jarrod: Following the linear pattern, [the one-pound is] such a small change, so it might be negligible, but especially in the other [20-pound increase] case, the linear pattern won't exactly model the change.*

Key Point: Jarrod is able to demonstrate his understanding in Nurse Jodi's use of the tangent line ("follow the same tangent line") to approximate the change in the dosage. He shows visually on the graph that the error for the 1-pound increase is much smaller than the error on the 20-pound increase (Figure 5.2).

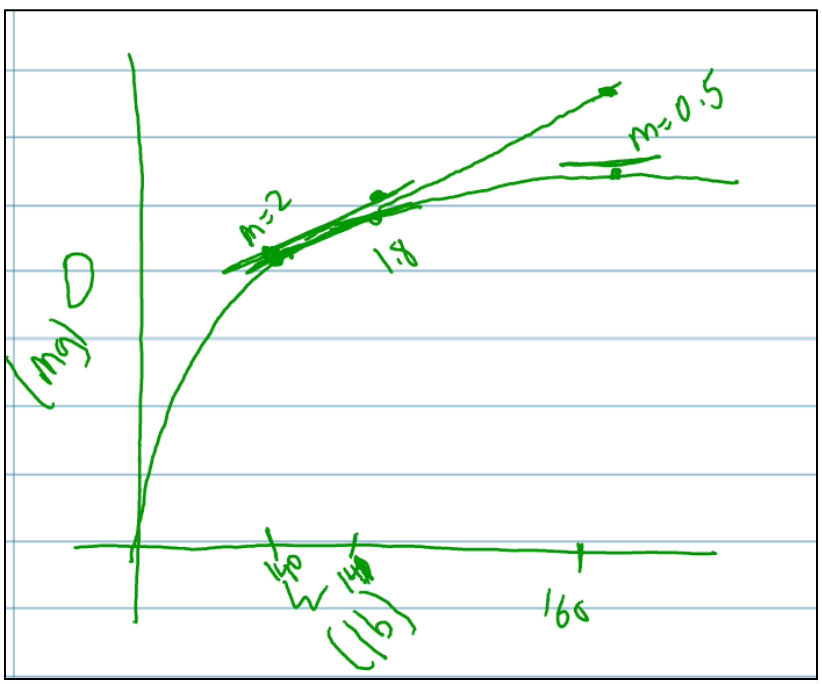



Figure 5.2. Jarrod's sketch of his non-linear graph and tangent line approximations

### 5.3.3 Interview responses to the critiquing questions for the linear context

Focusing on the linear predictions first, once again the success rates for the interviewees were higher than for the surveyed students (see Table 5.6).

| Surveys (N=69) | Interviews (N=13) |
|---|---|
| 48<br>69.6% | 12<br>92.3% |

Table 5.6. Survey and interview participants' success rates for critiquing questions with the linear context

Twelve of the thirteen interviewees confidently stated that Nurse Jodi was correct. I use an except from Chip to show sample reasoning, first for the 1-pound increase:

*Chip: Very confident, it follows my thinking a second ago. If you move over 1 on the x-axis, which is pounds, you go up 2 milligrams.*

For the 20-pound increase:

*Chip: Yes, that should be correct there too. I would say very confident again. Increasing by 20 pounds should be increasing the dosage by 40 milligrams. It's just a linear function there should not be other factors.*

*Interviewer: Equally confident for the two?*

*Chip: Yes.*

Brandon originally stated that he was not confident in Nurse Jodi's reasoning for the 1-pound increase, thinking that the slope of 2 meant that for every two pounds, your dosage would go up by 2 milligrams. But, he then decided to go back to thinking about slope as "rise over the run," and he changed his answer and said that Nurse Jodi was correct. For the 20-pound increase:

*Brandon: Yeah, I guess that makes sense.*

*Interviewer: OK, so somewhat? Very?*

*Brandon: Between somewhat and very.*

*Interviewer: OK, and for what reasoning?*

*Brandon: Well, it's a linear line. It doesn't change. So whatever the rise over the run is here will be the same there.*

*Interviewer: OK, so the rise over run is the same everywhere?*

*Brandon: Yes, the slope is the same everywhere.*

*Interviewer: Would there be any different in your reasoning for these two questions?*

*Brandon: Same reasoning for both.*

These twelve students reasoned that the slope, as a constant rate of change, held for any change in the independent variable. They were equally confident for both scenarios because of the rate being constant, in other words the rate could be applied for any change in the independent variable.

Dawn was the only interviewee who had different confidence levels in the two predictions for the linear model. For the one-pound increase:

*Dawn: I, umm…, I am… somewhat confident in this.*

*Interviewer: OK, and why somewhat?*

*Dawn: It kind of correlates to like, what I was saying about increasing by like 2 when I increase by 1 pound.*

*Interviewee: Yeah, that's what I thought I was saying up here. Umm…why are you not in the very confident category?*

*Dawn: I feel like I would have to plot it out specifically and visualize it better.*

*Interviewee: Ok, so it depends on what the graph looks like?*

*Dawn: Yes*

For the twenty-pound increase:

*Dawn: It has the same kind of setup, with 2 milligrams for each pound, so it makes sense. I feel like the way #4 is worded, I feel more OK about it. Ummm….Yeah, I think I would say very confident in this one.*

*Interviewer: Very confident. And what's different with the wording?*

*Dawn: I think it's that ummm…there is a little explanation. Like she increases by 40 (2 milligrams per pound of weight). Up there it just says she increases the dose when the patient's weight increases from here to here.*

*Interviewer: So there is more information down here?*

*Dawn: Yeah*

*Interviewer: And that gives you more confidence?*

*Dawn: Yeah.*

Even though Dawn was one of the students who interpreted the slope correctly, stating, "the dosage changes twice as much as the weight changes," she did not translate this to a "very confident" response for both of Nurse Jodi's predictions. Instead, she felt like there was more information given in the 20-pound increase problem that translated to more confidence on her part. The 20-pound increase problem specifically said that she would give "2 mg per each pound of weight," which defined the slope and gave her more confidence. Even though she interpreted the slope correctly on the previous question, her reasoning for these two questions show that her understanding is not solid.

### 5.3.4 Interview responses to the critiquing questions for the non-linear context

For the non-linear predictions, the success rates for the interviewees were once again higher than for the surveyed students (see Table 5.7).

| Surveys (N=69) | Interviews (N=13) |
|---|---|
| 11<br>15.9% | 4<br>30.8% |

Table 5.7. Survey and interview participants' success rates for critiquing questions with non-linear context

The success rates were also much lower than the predictions for the linear situation (30.8% vs. 92.3%). Only four of the 13 interviewees responded using correct reasoning for the predictions for the non-linear model. Incorrect reasoning fell into two categories: (1) students who used the derivative to calculate the change regardless of the input value, and (2) students who used the derivative to calculate the total dosage.

#### 5.3.4.1 Students who used correct reasoning for the critiquing questions with the non-linear context

The four interviewees who answered correctly talked about the derivative being a good approximation for small increases in the independent variable. Excerpts from two of these four are presented below.

Brandon used explicit knowledge of linear approximation, but was the only interviewee to do so. For the one-pound increase:

*<u>Brandon</u>: Yeah, I'm confident in that.*

*<u>Interviewer</u>: Confident. Because?*

*<u>Brandon</u>: Because it's just increasing by one pound. It's right here* [points to graph]. *So you could use that slope along the line, instead of the equation. Use the slope of two.*

*<u>Interviewer</u>: OK, would it be exact?*

*<u>Brandon</u>: No, it would be close to. But not exact. Very close.*

*<u>Interviewer</u>: Close a little bit high or a little bit low?*

*<u>Brandon</u>: Low.*

*<u>Interviewer</u>: Why did you say low?*

*<u>Brandon</u>: The shape of the graph. If it was the other way* [draws concave down graph] *it would be high.*

*<u>Interviewer</u>: OK, so it's where the tangent line falls in relation to the graph?*

*<u>Brandon</u>: Yeah*

*<u>Interviewer</u>: So you said very confident because you can use that slope to estimate.*

*<u>Brandon</u>: Yeah, you can use the formula, the…it's been a while.* $L(x) = f(a) + f'(a)$ ….

For the twenty-pound increase:

*<u>Brandon</u>: Oh, not confident.*

*<u>Interviewer</u>: And why not confident?*

*<u>Brandon</u>: Based on the slope, it would go out like that* [draws the tangent line]. *There would be a change. It wouldn't be accurate.*

*<u>Interviewer</u>: Under or over approximation?*

*<u>Brandon</u>: Under approximation by quite a bit.*

*<u>Interviewer</u>: And what makes 160 different from 141?*

*<u>Brandon</u>: How close it is from f(a).*

*<u>Interviewer</u>: Because 160 is farther away?*

*<u>Brandon</u>: Yeah.*

It is interesting to note that while Brandon gave very solid answers for the critiquing questions and the derivative interpretation question, he was the student who (earlier in the interview) was unable to verbalize the meaning of a slope of 2 milligrams per pound in the context of the problem.

Pam gave similar reasoning to Brandon, focusing on how close the independent value is from the input value of the derivative, but without mention of using the linear approximation to make the estimates. For the one-pound increase:

*<u>Pam</u>: Very confident.*

*<u>Interviewer</u>: And why?*

*<u>Pam</u>: Because 141 is very close…if there… you wouldn't be close. But it's close to 140 then that would be the same. So if you go down one, it would be a good estimate too.*

*<u>Interviewer</u>: OK, and ummm…would that be exact or close?*

*<u>Pam</u>: It wouldn't be exact, because the derivative is not exact. But it would be close. I'm not sure it would be close enough for medicine. But close.*

For the twenty-pound increase:

*<u>Pam</u>: Ummm….so, I'm not confident for this one. Because 160 is a long ways away from 140.*

*<u>Interviewer</u>: OK, so let's say 160 is right here* [draw 160 on graph]

*<u>Pam</u>: It's close, but not really close.*

Pam continues to draw the tangent line at 160 and estimated it to be a slope of 3 (Figure 5.3). I asked whether she could use that slope of 3 to figure out the change in dosage, but she said that one would need to know the derivative at a lot of points in between 140 pounds and 160 pounds to estimate the change in dosage:

<u>Pam</u>: *You'd have to take more measurements. Not with the information given. It wouldn't be accurate.*

<u>Interviewer</u>: *What kind of measurements?*

<u>Pam</u>: *You'd have to get the derivative of a bunch of points along the way* [she plots the points].

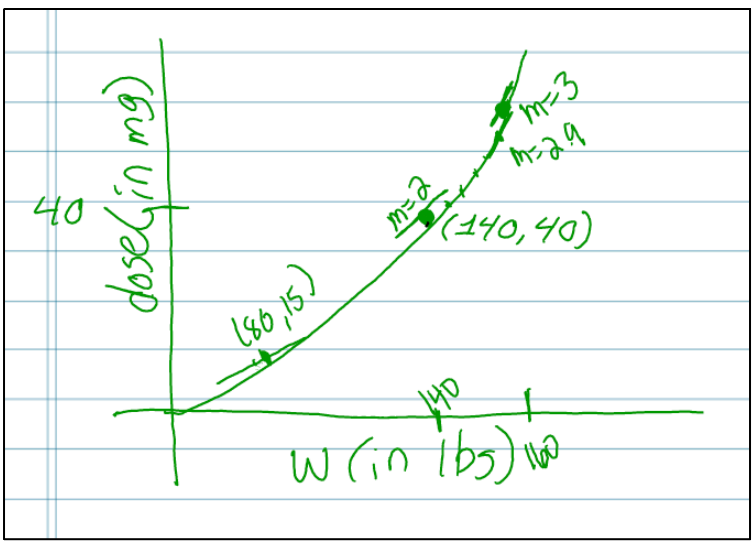



Figure 5.3. Pam's non-linear graph

**5.3.4.2 Students who used the derivative to calculate the change regardless of the input value**

Two of the nine remaining interviewees said that a derivative could be used to calculate the change, whether it was a one-pound or twenty-pound increase. One of these students, as seen below, requested another derivative (instead of the one provided) to calculate change. Both students expressed equal confidence in both predictions. Unlike an ideal knower, they did not understand that a derivative could be used to approximate change only in a small interval around the input value.

Jackie was equally confident in both of Nurse Jodi's responses, saying that the derivative could be used to approximate the change in dosage anywhere. Her reasoning was similar to that of the linear relationship (where slope is a constant rate of change that can be applied to any change in the input variable), but she did not have as much confidence as with her prediction for the linear relationship. She could not, however, verbalize why her confidence was lower, just that it was because it was a derivative instead of a slope. For the one-pound increase:

*Jackie: Ummm…..somewhat confident.*

*Interviewer: OK, somewhat confident. And what's your reasoning?*

*Jackie: It's like the first one* [linear]*, but the derivative is throwing me off.*

*Interviewer: So up here you were telling me for every pound, you get two milligrams per pound. Are you using the same reasoning here?*

*Jackie: Yes.*

*Interviewer: Somewhat confident instead of very because….*

*Jackie: I'm not as sure.*

For Jackie and the twenty-pound increase:

*Jackie: So, next patient is 20 pounds heavier, ….. somewhat confident, because, if, well, her patient was 140 and her next patient is 160, so that's 20 pounds different, and 20 times 2 is 40, so she would increase the dosage by 40.*

*Interviewer: OK, same confidence as the last problem?*

*Jackie: yeah.*

*Interviewer: No difference?*

*Jackie: No difference.*

John also agreed that the derivative could be used to calculate the change in both one-pound and twenty-pound increases, but explained that he needed a different derivative to calculate the change. He reasoned that the he needed the derivative at the ending weight (in this case 141 or 160) to calculate the change in dosage. For the twenty-pound increase:

*John: Still not confident. Because 160 is so much farther than 140.*

*Interviewer: OK, so say you have the derivative at 160, and say it's 4* [Figure 5.4]

*John: OK.*

*Interviewer: OK, so if the patient is 20 pounds greater, how much more would you give?*

*John: 80, I would multiply it by 4.*

*Interviewer: So you would give 4 milligrams for each additional pound of weight?*

*John: Yeah.*

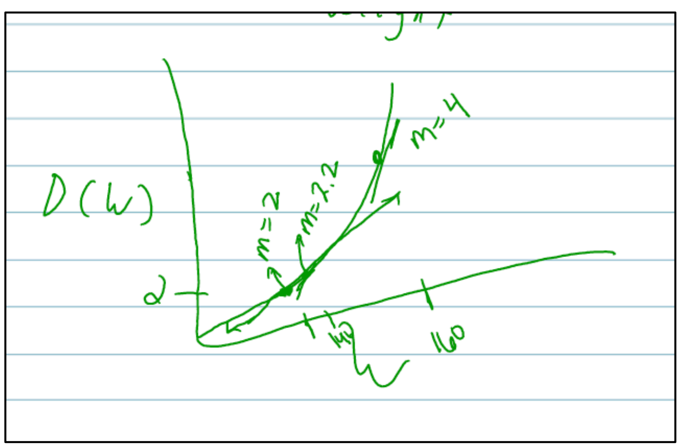



Figure 5.4. John's non-linear graph

John took a similar approach to Jackie, but used a different derivative (the derivative at the endpoint of the interval) to make the prediction. Both lacked the ideal knower's understanding that the derivative can only be used to approximate change for values close to the input value.

#### 5.3.4.3 Students who used the derivative to calculate total dosage

The seven remaining students stated that you needed another derivative (either at 141 or 160) to calculate the total dosage. Unlike John, who used another derivative to calculate the change in dosage (multiplying the derivative times the change in weight to yield the change in dosage), these students used another derivative to calculate the total weight (multiplying the derivative times the weight to yield total dosage). Excerpts from Harry, Emily, and Dawn are used to illustrate this reasoning.

Harry describes how he could find the total dosage for a 160-pound patient:

<u>Harry</u>: *And still not confident. Because it would be even more off because the slope is still increasing.*

<u>Interviewer</u>: *OK, so say the slope here is equal to, I don't know, three* [interviewer points to the tangent line at 160 pounds, Figure 5.5]. *So if she wanted to know how much more drug to give a patient who is 20 pounds greater, can she determine that from that derivative* [the derivative at 140] *or that derivative* [the derivative at 160] *or none of them?*

<u>Harry</u>: *That derivative* [points to the derivative at 160].

<u>Interviewer</u>: *She would take three milligrams and do what with it?*

<u>Harry</u>: *She would multiply 160 by 3. That would give her the right dosage.*

<u>Interviewer</u>: *OK. So multiply 160 by 3 and that would give her the total dosage?*

*Harry: Yes*

*Interviewer: Not the increase in dosage?*

*Harry: Yes, the total dosage.*

*Interviewer: So 160 pounds times 3 milligrams per pound….that will give her the total number of milligrams that she'll need.*

*Harry: Yes.*

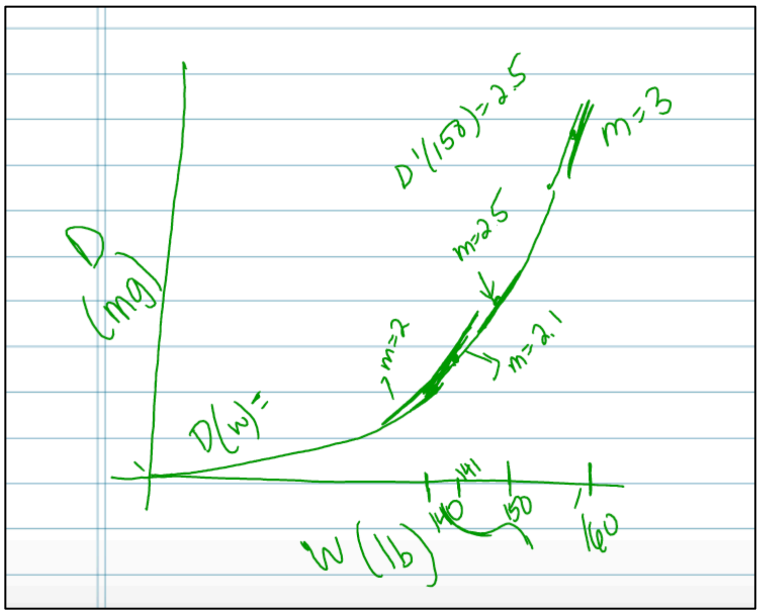



Figure 5.5. Harry's non-linear graph

Harry reasoned that the dependent variable can be calculated as the derivative times the input value (in other words, the total dosage is equal to the weight times the rate of change at that weight, or $D(n) = D'(n) * n$). Harry also used the incorrect ratio-of-totals approach to interpret slope, reasoning that the total dosage was two times the weight ($D = 2n$). This is very different than an ideal knower, who understands that the rate of change is only relevant at a specific input value, and can be used to approximate change at values very close to the input value.

At the end of the interview, I asked Harry if he had anything to add:

<u>Harry</u>: *Actually there is. With the 160 pounds time 3 mg per pound, one way to always know if I'm doing it right is if the units cancel out.*

<u>Interviewer</u>: *So the pounds cancel here, and I end up with mg* (Figure 5.6).

<u>Harry</u>: *Yes.*

<u>Interviewer</u>: *And that gives you confidence?*

<u>Harry</u>: *Yes*

<u>Interviewer</u>: *OK, so, you are not confident with these answers here* [interviewer points to Nurse Jodi's predictions], *but with your new way you are confident?*

<u>Harry</u>: *Yes*

<u>Interviewer</u>: *And equally confident for both of these?*

<u>Harry</u>: *Yes*

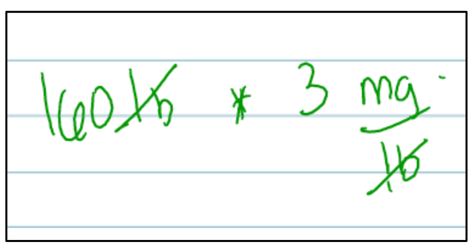



Figure 5.6. Harry's unit cancelation

Harry used the derivative to calculate the total dosage, and by showing the units canceling, he was able to justify to himself that his reasoning was correct. Even though his reasoning was incorrect, he was able to feel confident in his answers because of his unit analysis. He said he "knows he's correct" because he ends up with the correct units.

Emily had similar reasoning to Harry, but drew a graph for the non-linear relationship that was increasing for a while, and then it started decreasing (Figure 5.7). This introduced a potentially troubling situation for her because the slope at 160 pounds is negative on her graph. I asked her to explain her reasoning for drawing the graph:

*<u>Emily</u>: Umm…I drew it based on the drug's only going to have a certain effect up until you can't give someone any more, and then it won't have an effect on a patient and you can't give them any more….Like there is going to be a cap on the amount of dosage that you can physically give a patient. Not even depending on their weight. Depending on the potency of the drug itself.*

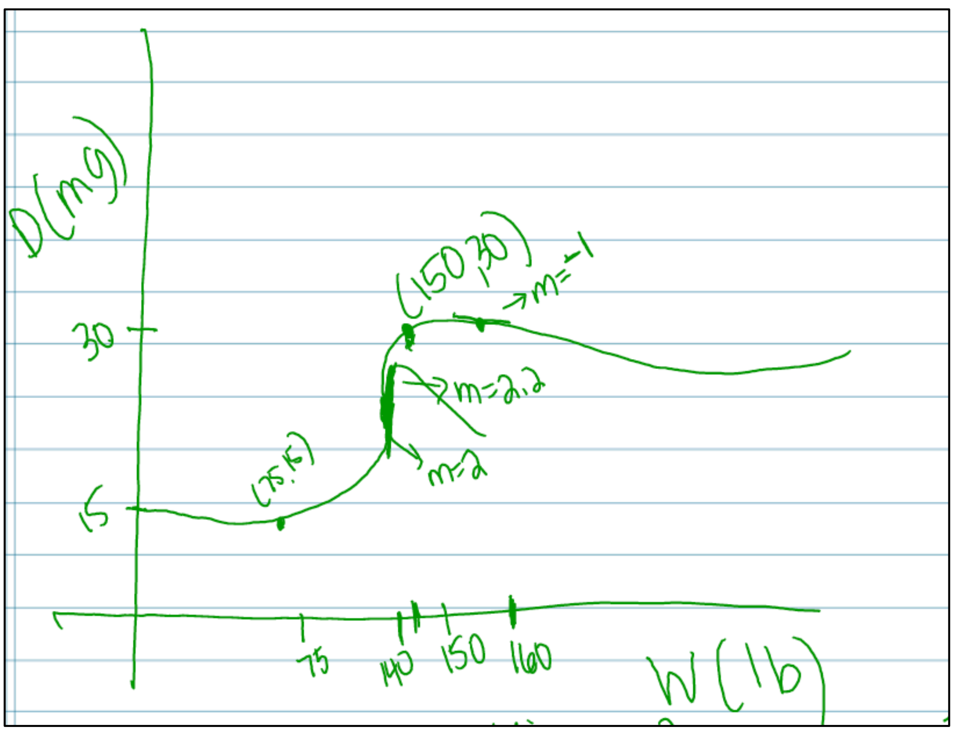



Figure 5.7. Emily's non-linear graph

For the one-pound increase, she gave similar reasoning to Harry and Kelly:

*<u>Interviewer</u>: What if we knew the slope at 141?*

*<u>Emily</u>: She would have to re-do her calculations for 141.*

*Interviewer: OK, say the patient was increasing in weight by a pound, and say we knew that* $D'(141) = 2.2$ *milligrams per pound.*

*Emily: Yes.*

*Interviewer: How much would she have to increase, or give, the patient?*

*Emily: She would have to give the patient 2.2 milligrams for every pound that they weigh. So they would have to increase it, they would have to basically multiply the weight by 2.2 instead of just 2.*

*Interviewer: So her total dosage would be that?*

*Emily: I think so.*

*Interviewer: 141 pounds times 2.2.*

*Emily: I think so.*

*Interviewer: And that would give her the total dosage?*

*Emily: I think so.*

For the 20-pound increase question, Emily was not able to use the same reasoning that she had previously, since multiplying a negative slope times the weight would yield a negative dosage. Estimating the derivative at 160 pounds to be –1 (as shown in Figure 5.7), she worked to change her reasoning to account for the negative slope:

*Emily: She would have to decrease the patient's dosage rather than increase it.*

*Interviewer: So if we knew the dosage of a 140-pound patient, we would decrease the dosage by how much? If she were 20 pounds heavier?*

*Emily: Umm….you kind of have the graph right here. It's capped off at 30 milligrams* [she points to the maximum point], *so she'd have to decrease it by however much off the patient weighs off the cap, by negative 1.*

*Interviewer: So if the max was 150, and she was 10 pounds heavier than that cap, you would have to decrease it?*

*Emily: Yeah. Because the slope is going down.*

*Interviewer: What would she have to decrease it by?*

*Emily: She's ten pounds over the cap, so she might have to decrease it 10 milligrams? Or so?*

*Interviewer: Because the slope is -1 there?*

*Emily: Yeah.*

*Interviewer: All right, so this slope gives you the decrease for a one-pound increase?*

*Emily: Yeah, I guess so.*

Emily was able to make meaning of her reasoning for the negative slope by adjusting her logic slightly and focusing on a maximum point. Instead of using the derivative to calculate the total dosage, she adjusted her reasoning and used the negative derivative to calculate the decrease in dosage from the maximum point. Described using Piaget's language of accommodation versus assimilation, when faced with something that caused her original reasoning to fail, instead of changing that understanding (accommodation), Emily found a way to make the anomalous piece of information fit her understanding (assimilation) (Piaget, 1970).

Like Emily, Dawn described using the derivative to calculate the total dosage, not the change in dosage. When she drew a graph similar in shape to Emily's (Figure 5.8), she was not able to apply her incorrect reasoning to the negative slope, and she started doubting her approach.

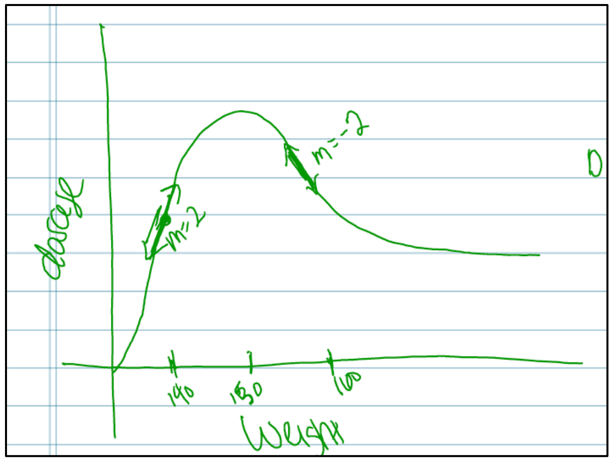



Figure 5.8. Dawn's non-linear graph

*Interviewer: How could I use that for my total dosage, or could I?* [interviewer points to the derivative of -2]

*Dawn: I think it's different.*

*Interviewer: Different because?*

*Dawn: There is a negative slope involved.*

*Interviewer: Is there something we could use it, or anything else, to determine the amount of drug to give?*

*Dawn: I don't think so. I'm started to re-think that what I said earlier about multiplying the 3 by the 160, or the 140 by 2, is not correct. It's probably not correct.*

*Interviewer: What's getting you to think it's not correct? It doesn't work with a negative?*

*Dawn: Yeah. Because that would make a negative dosage.*

*Interviewer: OK, let's try this question again. If I know the derivative at 140 is equal to 2. What does that tell me about a 140-pound patient?*

*<u>Dawn</u>: I don't know what it says about the patient. I know it says something about the dosage. Something about the rate of change. But that's an instantaneous rate of change. I'm not really sure.*

Unlike Emily, she was not able to make meaning when the slope was negative, and instead starting doubting her initial reasoning. Instead of taking Emily's approach of amending her reasoning to account for the negative slope (and needing to know the maximum value to do so), Emily instead reasoned that she knew the derivative had something to do with the instantaneous rate of change, but was unsure what exactly that meant or what one could do with it.

There were three students (Chip, Kelly, and Missy) who also took the total dosage approach, but for whom I probed a little more as to whether there were multiple ways to find the total dosage for a patient. All three concluded that there were two ways to find out the total dosage for a certain weight. One way was reading the dependent value from the graph (finding the y-value), and the other way was to multiply the derivative by the weight to yield the total dosage.

For example, during Chip's interview, from Figure 5.9 he reasoned that a 160-pound patient would require 160 * 4 or 640 mg of the drug, and a 140-pound patient would require 140 * 2 = 280 mg of the drug.

*<u>Chip</u>: So for a 160-pound patient, you would be administering 640 milligrams for a 160-pound patient, as opposed to 280-milligrams for a 140-pound patient. It seems like a steep change, but it's what the slopes are giving.*

*<u>Interviewer</u>: So, you are saying if I'm 140 pounds and my this is my slope, my derivative is 2, I can multiply 140 by 2 to get my total dosage?*

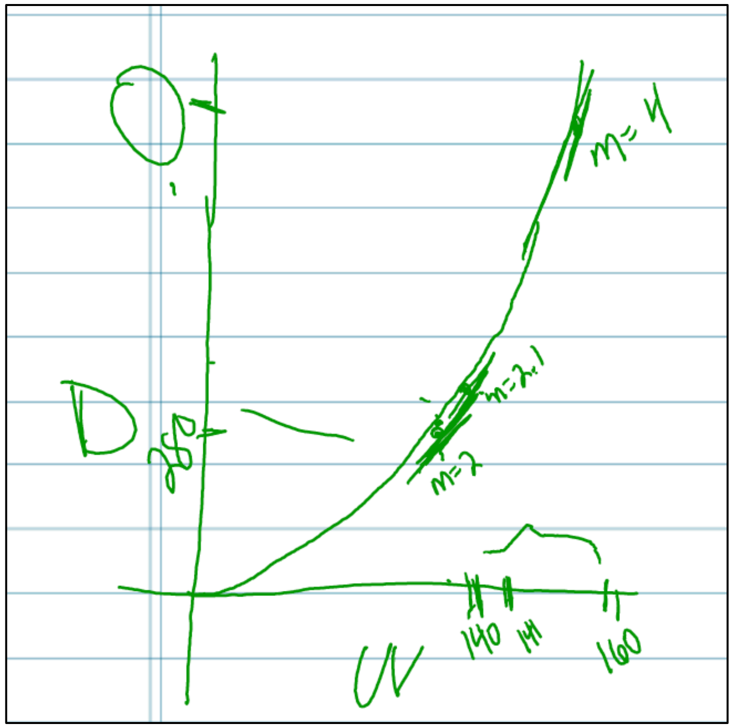



Figure 5.9. Chip's non-linear graph

*Chip: Yes.*

*Interviewer: OK. So 140 times 2. So you are telling me this dosage over here will be 280?* [I write 280 on the y-axis].

*Chip: Yeah*

*Interviewer: OK, and for 141, I would multiply 141 times 2.1, and that will give me that*? [I point to the y-axis corresponding to where w = 141] *And 160 times 4 will give me my total dosage here?* ? [I point to the y-axis corresponding to where w = 160]

*Chip: Yeah.*

*Interviewer: So does that mean there are two ways to find, or to calculate, a y-value? One directly from reading the graph, and one from taking the derivative and multiplying it times the weight?*

*Chip: Absolutely, yes.*

Chip reasoned that the y-value on the graph should correspond to the value that is calculated by multiplying the derivative at a weight by that specific weight.

Missy also stated that multiplying the derivative by the weight would give the total, but she goes on to use a point on the graph to calculate the slope of the tangent line. At the end of Missy's interview, I drew a second set of axes and plotted the point (140, 200) (Figure 5.10). I asked her to describe the point in the context of the problem, in order to see whether she understood that the function represents total dosage as a function of the weight. An ideal knower would understand that the point represents the dosage of 200 mg given to a 140-pound patient.

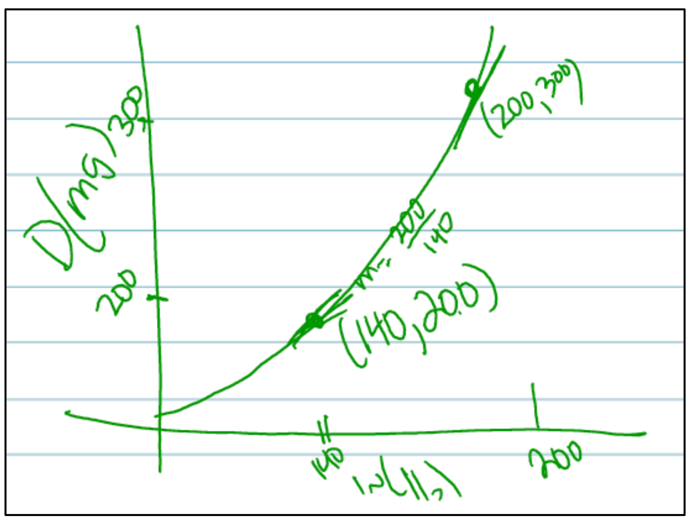



Figure 5.10. Graph used for Missy to explain the meaning of a point on the graph

*<u>Missy</u>: It means the ratio would be like 200 over 140. Or like the slope at that point.*

*<u>Interviewer</u>: So you can calculate the slope based on the y and the x?*

*<u>Missy</u>: Yes.*

And then later on I draw another point:

*Interviewer: OK, so this here point (200, 300) right here* [I draw the point (200, 300) on the graph]. *I can use that to calculate the slope at that point?*

*Missy: Yeah.*

Missy's reasoning for the interpretation of the point on the graph reveals many misunderstandings on her part. Not only does she not understand that a point on the graph represents the amount of drug needed for a specific weight, she believes that she can calculate $\frac{y}{x}$ to find the slope of the tangent line. In other words, the slope of the line between a point and the origin is equivalent to the instantaneous rate of change at a point. Missy's misunderstandings seem to be rooted in incorrect reasoning about functions and knowledge of tangent lines.

### 5.3.5 Summary of interviewees' successes and struggles with the critiquing questions

In summary, most interviewed students performed very well when critiquing the linear predictions, even more so than students had on the surveys. In particular 92.3% of interviewees gave correct answers whereas 69.6% of surveyed students responded correctly. In the interviews, students used correct reasoning of slope as a constant rate of change that can be used to estimate change for any increase in the independent variable, and they were able to express confidence in Nurse Jodi's use of the slope to make predictions. Dawn was the only interviewee who had differing confidence levels for the two questions. Surprisingly she had more confidence in the 20-pound increase prediction scenario than the 1-pound increase scenario.

It is evident from the interviews, however, that very few students appropriately use derivatives to make predictions. Instead, students used incorrect reasoning in one of two ways. Demonstrating one inappropriate line of reasoning, two students used derivatives (either the derivative at 140 in the case of Jackie who agreed with Nurse Jodi, or a derivative at another point, namely 141 or 160, in the case of John who did not have confidence in Nurse Jodi) to calculate the change in dosage. These students did not understand that the derivative was an instantaneous rate of change that could be used accurately only at the input value (or for a small region around the input value). Instead, they applied the derivative like the slope in the linear situation, as something that could be used to approximate change over any interval.

In the second incorrect line of reasoning, seven students took the approach where they used another derivative (the derivative at 160, for example) to calculate the total dosage, stating that the derivative times the weight gives the total dosage. Six of the seven had interpreted slope as a ratio-of-totals, thus extending their incorrect ratio-of-totals slope interpretation to derivatives and concluding $D(n) = D'(n) * n$. A few of these students tried to adjust their reasoning for a negative slope (Emily and Dawn), at least one successfully in her mind (Emily).

Students also failed to recognize when to apply their knowledge of linear approximation. Brandon was the only student to use linear approximation language in his answers.

Recall that on the written surveys, 18% of students mentioned needing another derivative to answer the question. I was expecting that students who said they needed another derivative in the interview would use the derivative to approximate the change over the interval (like Nurse Jodi, but using the derivative at 141 pounds or 160 pounds, instead of the derivative at 140 pounds). In other words, I was expecting them to reason that it was appropriate to use a derivative to estimate a change over any interval. Instead, 7 of the 8 interviewed students (87.5%) who used another derivative, did so to calculate the total dosage. Their reasoning was that a derivative (a rate of change) can be used to find the total dosage by multiplying the rate of change by the weight ($D(n) = D'(n) * n$), thus extending the incorrect knowledge of the slope being a rate of change that can be used to calculate total weight ($D(n) = m * n$).

### 5.4 Summary of interview results

Figure 5.11 compares the success rates on the survey and interview linear context questions. Figure 5.12 compares the same for the non-linear context questions. The success rates across the board were higher for the interviewees. One thing that stands out is that the success rates for the critiquing questions were much higher for linear than non-linear scenarios, indicating students' inability to understand how a derivative can (or can not) be used to make a prediction.

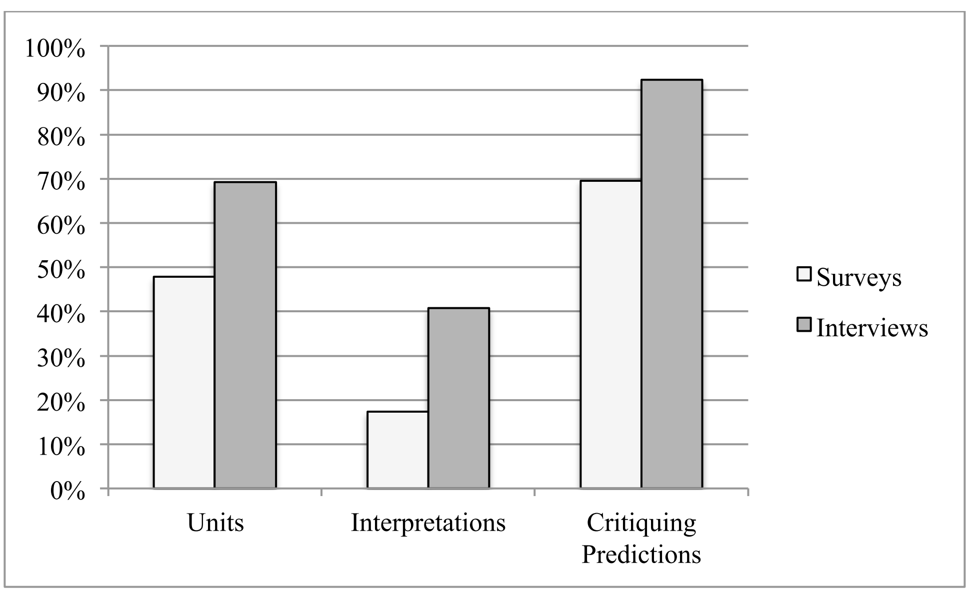



Figure 5.11. Survey and interview participants' success rates on the linear questions (units, interpretations, and critiquing)

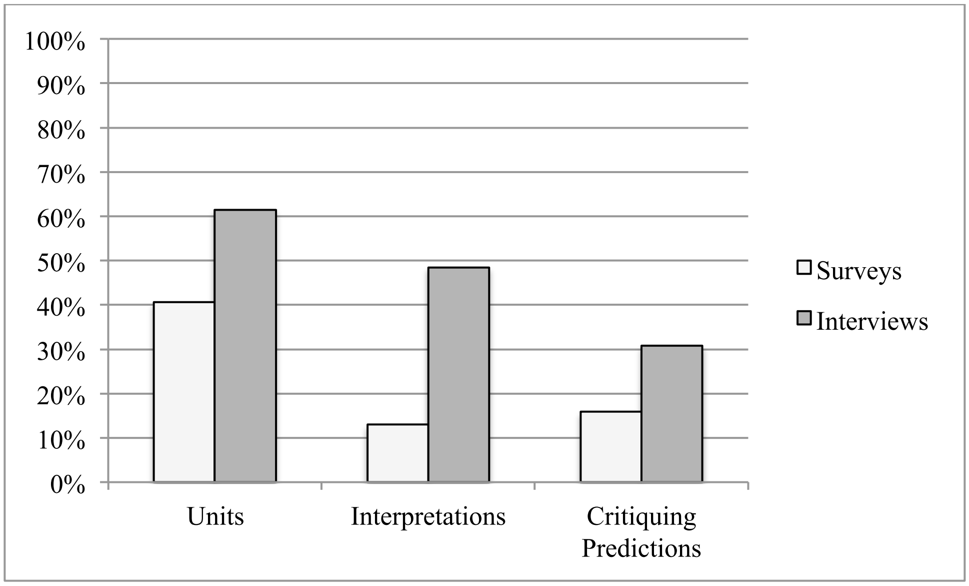



Figure 5.12. Survey and interview participants' success rates on the non-linear questions (units, interpretations, and critiquing)

There were four areas that I investigated through the interviews. First, whereas only 35% of the students identified correct units on both the slope and derivative in the surveys, there was a success rate of 61.5% in the interviews. Possible explanations could be that the context was easier (the survey questions introduced the "10-acre tract" and some students in turn used "acres" as part of the units) and also, I made sure students had their graphs labeled appropriately (including units) before moving to the unit question, which could have aided them in answering the unit questions. Successful interviewed students were able to identify the slope as the "change in y over change in x" in order to reason that the units on the slope would be the "units on y over the units on x." Even those students who used a ratio-of-totals approach to defining slope as "y over x" were able to apply that to the units and get the correct "units on y over units on x." Unsuccessful students were unable to apply knowledge of slope to the units, and instead changed the units (putting the independent variable in the numerator) or introduced other combinations.

For the interpretation questions on the surveys, 39% of students gave a slope explanation using an incorrect ratio-of-totals interpretation ($B(n) = 2n$), thus implying a directly proportional relationship, and 16% gave a similar derivative explanation ($B(n) = B'(n) * n$). In the interviews, a much higher percentage took the ratio-of-totals approach to the slope (61.5%) and the analogous derivative interpretation of $D(n) = D'(n) * n$ (53.8%). Students' misunderstandings of slope as a ratio-of-totals seems to be impacting their abilities to interpret derivative correctly, as six of the eight ratio-of-totals respondents on the slope question (75%) went on to interpret the derivative incorrectly as $B(n) = B'(n) * n$. Whereas at first glance one might reason that a ratio-of-totals approach to slope might be a small mistake on the part of the student (leaving out the word "additional" for example), the high percentage of students who go on to interpret the derivative incorrectly as $B(n) = B'(n) * n$ points to much more than leaving out a word. Instead, it seems like the ratio-of-totals approach to slope is carrying over to an analogous incorrect interpretation of derivative. Whereas an ideal knower would be able to interpret slope ratio-of-differences, and the derivative as an instantaneous rate of change at a specific input value, many with misunderstandings used an incorrect ratio-of-totals approach to the slope and a similar $B(n) = B'(n) * n$ approach to the derivative.

The survey responses for Farmer Jim's predictions in the non-linear model showed 22.1% of the students did not distinguish between the 1-pound and the 10-pound increases, giving similar confidence levels and reasoning. In the interviews, two of the 13 students (15.4%) gave similar confidence levels for both the 1-pound and 20-pound drug increases form Nurse Jodi's predictions. Both students used the derivatives to calculate the change over the interval, signaling that they reasoned similarly to how they did with the linear critiquing questions. Namely, they used both the slope of a linear relationship and a derivative of a non-linear relationship to estimate change over any interval. They failed to demonstrate the ideal knower's understanding that a derivative of a non-linear function can only be used to estimate change in a small interval around the input value.

Lastly, focusing on the critiquing questions, Table 5.8 summarizes the interviewees' responses for the interpretation questions and the non-linear scenario critiquing questions. I chose to focus on these questions for this table because most students performed well on the linear scenario critiquing questions. Also, there were some patterns in how students answered the interpretation questions, and then how they answered the critiquing questions.

| | | Critiquing of Nurse Jodi's Predictions with Non-linear Context | | |
|---|---|---|---|---|
| | | Correct | Use derivative to find the change over any interval | Used Derivative to find the total dosage $D(n) = D'(n) * n$ |
| Slope & Derivative Interpretation | Correct for both | Jarrod, Pam, Craig | John | |
| | Correct for slope and incorrect for derivative | | | Dawn, Chip |
| | Incorrect for slope and correct for derivative | Brandon | | |
| | Use $D = 2n$ for slope and $D(n) = D'(n) * n$ for derivative | | Jackie | Maddie, Missy, Harry, Emily, Kelly |

Table 5.8. Summary of slope and derivative interpretation vs. critiquing of non-linear context, N=13

Only three students answered the interpretation questions and both critiquing questions in the non-linear context correctly, Jarrod (our "ideal knower"), Pam (who answered everything in the interview correctly except for the unit questions), and Craig (who starting describing slope as the ratio-of-totals but changed to the ratio-of-differences when I drew a graph that did not go through the origin). Six answered both the interpretation and critiquing questions incorrectly (Jackie, Maddie, Missy, Harry, Emily, and Kelly). All six used a ratio-of-totals interpretation for slope and a $D(n) = D'(n) * n$ interpretation of the derivative. Five of the six went on to conclude that a derivative can be used to predict the total dosage by multiplying the derivative by the weight, thus continuing the $D(n) = D'(n) * n$ approach. The last (Jackie) went on to use the derivative to predict the change over any interval.

In summary, for the seven students who interpreted slope incorrectly, six went on to critique the non-linear predictions incorrectly. Five of them took the $D(n) = D'(n) * n$ approach, analogous to their ratio-of-totals approach to the slope. Students' underdeveloped views of slope seem to be related to their misunderstandings of the derivative. This highlights that there was one common incorrect way of thinking that students displayed, anchored in students' misunderstandings of slope as a ratio-of-totals. This dominant incorrect reasoning was displayed in a variety of different tasks; it was by far the most common unproductive way of thinking. This unproductive reasoning is similar to that seen in younger students; this will be addressed more in the next chapter.

# 6. CONCLUSIONS AND IMPLICATIONS

This study focused on students' understanding of slope and derivative, two fundamental rate of change concepts. Specifically, this investigation centered on students' interpretation and use of slope and derivative in the context of real life situations, and their ability to critique the reasoning of someone else's predictions made using a slope or derivative.

The findings provide a valuable look into students' understanding of slope and derivative; specific instructional implications are given in this chapter that can help guide the mathematical education community. By learning more about how students think about slope and derivative, including what misunderstandings many bring into our calculus classes, the mathematics education community can strengthen calculus teaching, and work toward improving students' conceptual knowledge and increasing retention rates.

## 6.1 Results and contributions to the literature for research question #1: Interpretation of the slope and derivative

The first research question focused on students' interpretations of slope and derivative in the context of the problem: *Research Question #1: Is there a relationship between calculus students' understanding of slope and their understanding of derivative? Specifically, do students' abilities to correctly interpret the slope as a constant rate of change make them more likely to be able to interpret the derivative as an instantaneous rate of change?*

Key findings from the study which bring insight into research question #1 can be summarized as follows; they are described in more details in the sections that follow:

- Students who interpreted the slope correctly were no more likely to interpret the derivative correctly than those who did not interpret slope correctly.
- The dominant incorrect way to interpret the slope has been to use a ratio-of-totals approach $\left(\frac{y}{x}\right)$, implying a directly proportional relationship of the form $y = mx$, with a y-intercept of zero.
- Most students who interpreted the slope incorrectly as a ratio-of-totals went on to interpret the derivative similarly, as something that could be used to find the value of the dependent variable, in other words $f(x) = f'(x) * x$.
- Students struggled to understand the units on the slope and derivative, a key component to correctly interpreting the meaning of the rate of change.

These findings support prior research that showed that rates of change are hard for students (Carlson, 1998; Hackworth, 1994); the findings also extend the research around high school students' difficulties with slope (Stump, 2001) to college students.

### 6.1.1 Students who correctly interpreted slope were not more likely to interpret the derivative correctly

Students were not more successful on the slope interpretation questions (e.g., "a slope of 2 bushels per pound means that for each additional pound of nitrogen, 2 more bushels of corn are yielded") than the derivative interpretation questions (e.g., "$B'(20) = 2$ means that when the amount of nitrogen is 20 pounds, the yield is increasing at a rate of 2 bushels per pound") in either the surveys or the interviews (see Table 6.1). Since slope is a concept covered multiple times since middle school, and derivative is a more advanced topic that is not only new, but builds off slope knowledge, I was expecting higher success rates for the slope question. This was not the case, and furthermore, students' success at interpreting slope did not seem to make them more likely to interpret the derivative accurately.

| | Slope Interpretation | Derivative Interpretation |
|---|---|---|
| Surveys (N=69) | 17% | 13% |
| Interviews (N=13) | 31% | 38% |

Table 6.1. Comparison of success rates for the interpretation questions for the surveys and interviews

This finding is surprising, as the ideas of slope and derivative are tightly connected, and therefore one would expect that success at understanding slope would translate to success at understanding derivative. This does not seem to be the case, and leads one to believe that students are not understanding the connections between these two rates of change.

The success rates for the interpretation questions were higher for the interview questions than for the survey questions. Two things help to explain these differences. First, many of the surveyed students did not answer the question fully, or in the context of the problem, which was not the case for the interviews where I was able to ask follow-up questions. Second, the survey questions had extraneous information (it mentioned that there was a 10-acre tract of land), and some students incorrectly introduced acres into their units, which contributed to their incorrect interpretation.

Even though the success rates were higher for the interview questions, fewer than 40% of the interviewed students were able to interpret these rates of change in context.

Both the interview and survey data provide evidence that not only do students' abilities to interpret slope not make them more able to interpret the derivative, but that students struggle across the board with understanding slope and derivative as rates of change.

### 6.1.2 A ratio-of-totals approach to slope interpretation was the dominant incorrect reasoning

The most common incorrect reasoning on the surveys for slope interpretation was the ratio-of-totals $\left(\frac{y}{x}\right)$ reasoning, which implies a directly proportional relationship of the form $y = mx$. Over one-third of the surveyed students (39%) answered with this reasoning (see Table 6.2).

| | Ratio-of-Totals Approach |
|---|---|
| Surveys (N=69) | 39% |
| Interviews (N=13) | 54% |

Table 6.2. Comparison of rates of using the incorrect ratio-of-totals reasoning for the slope interpretation questions for the surveys and interviews

The interview data were even more striking, with 54% of the students using a ratio-of-totals approach to interpret the slope. Probing interview questions shed light on the fact that many of these students think that for all linear relationships, the slope can be used to calculate the total (in other words all linear relationships are directly proportional relationships). It is not a matter of just forgetting the word "additional;" they instead think that slope is the ratio-of-totals, as opposed to the correct ratio-of-differences.

### 6.1.3 The students who used the ratio-of-totals approach for slope interpretation often went on to interpret the derivative similarly

In the surveys, only 13.0% of the students interpreted the derivative correctly; another 18.8% gave an accurate interpretation but did not give it within the context of the problem (for example, "it is the slope of the tangent line when n = 20"). The most common incorrect response (15.9%) was that the total yield could be determined by multiplying the derivative by the amount of nitrogen, in other words $f(x) = f'(x) * x$ (see Table 6.3). For those students, 73% of them had interpreted the slope using the ratio-of-totals approach ($f(x) = m * x$).

| | $f(x) = f'(x) * x$ Approach |
|---|---|
| Surveys (N=69) | 16% |
| Interviews (N=13) | 54% |

Table 6.3. Comparison of rates of using the incorrect $\boldsymbol{f(x) = f'(x) * x}$ reasoning for the derivative interpretation questions for the surveys and interviews

The interviews provided additional evidence of students' inabilities to interpret the slope as a ratio-of-differences (instead taking a ratio-of-totals approach) leading to struggles with interpreting the derivative. Seven of the 13 interviewed students (54%) took the $f(x) = f'(x) * x$ approach to interpreting the derivative. Six of the seven interviewed students (85.7%) who had taken the incorrect ratio-of-totals slope interpretation went on to take the analogous incorrect derivative interpretation ($f(x) = f'(x) * x$). Students expressed confidence in using the derivative to calculate the y-value of the function.

These unsuccessful students also did not convey understanding of the covarying relationship between the derivative and the function value, something Carlson (1998) found lacking in even the most talented second-semester calculus students, who exhibited difficulties with "demonstrating an awareness of the impact change in one variable has on the other" (p. 142). Instead of taking an across-time approach to the problem and showing understanding of the derivative as a function of the independent variable, they took a point-wise approach in which they used the derivative at a point to calculate something at that specific input value.

Unsuccessful students are not demonstrating knowledge of the changing derivative. Instead, they are applying their incorrect reasoning of a slope as a ratio-of-totals to their interpretation of the derivative.

### 6.1.4 Students struggled with the units on the slope and derivative

While not directly related to my research question, knowledge of the units on the slope and derivative are necessary to accurately interpret the meaning of the rates of change. In the surveys, there were no significant differences between students' success on the slope and derivative unit problems. In both cases, less than half of the students answered the question correctly (see Table 6.4), and only 35% answered both the slope and derivative units correctly. Without knowledge of the units on these rates of change, it is not surprising that the success rates on the interpretation questions were so low.

| | Slope Units | Derivative Units |
|---|---|---|
| Surveys (N=69) | 48% | 41% |
| Interviews (N=13) | 69% | 61% |

Table 6.4. Comparison of success rates for the unit questions for the surveys and interviews

Only 22% of the incorrect answers were given as a ratio; in other words, many of the students did not know that the units on the slope should be a ratio of the form $\frac{unit}{unit}$. Even though many of these students explained later on that the slope was “y over x,” they could not put this knowledge to use when answering the questions about the units of the slope. These findings are consistent with those of other researchers who focused on high school students’ difficulties with viewing slope as a ratio (Barr, 1980, 1981; Stump, 2001).

The success rates on the interview questions were better, perhaps signaling that the context in the interview questions or my follow-up questions affected their success. Still, one-third of the interviewees struggled with units.

#### 6.1.5 Summary of research question #1 and contributions to the literature

On the written surveys, only 17% of students interpreted the slope correctly in the context of the problem (e.g., “a slope of 2 bushels per pound means that for each additional pound of nitrogen, 2 more bushels of corn are yielded”), and only 13% interpreted the derivative correctly (e.g. “$B'(20) = 2$ means that when the amount of nitrogen is 20 pounds, the yield is increasing at a rate of 2 bushels per pound”). While slightly improved over Bezuidenhout’s (1992) findings, where only 2 of 100 participants were able to interpret the meaning of a derivative in the context of a problem, it is still discouraging. The results from the interview questions about slope and derivative are better (31% and 38% respectively), but as mentioned previously, these may be a result of being able to ask follow-up questions and having an easier context.

We know students must understand rates of change in general to succeed in calculus (Hackworth, 1994), and my research adds to the set of findings that show that rates of change are not well-understood by calculus students, many of whom may have fundamental misconceptions (Bezuidenhout, 1998). Over a third of the surveyed students and almost two-thirds of the interviewed students viewed slope as a ratio-of-totals, which is much higher than the 11% of interviewees in Hauger (1995). Hauger's study did not have students interpret the slope verbally; instead students had to estimate from a graph how fast a population was changing. While the correct approach was to estimate the rate of change using a slope $\left(\frac{\Delta y}{\Delta x}\right)$, 11% of the students calculated $\frac{y}{x}$ instead. In my study, six of the seven interviewees who used an incorrect ratio-of-totals interpretation of slope went on to interpret the derivative in a similar incorrect fashion, leading me to believe that it was not just a simple act of leaving out the word "additional."

We know that proportional reasoning is difficult for students (Hoffer & Hoffer, 1988; Lawton, 1993; Lesh et al., 1988; Tourniaire & Pulos, 1985), and perhaps students' misunderstandings of linear functions in general, and directly proportional relationships specifically (for which the slope *is* a ratio-of-totals), lead to an impoverished understanding of slope. Even with the slope interpretation questions, students performed poorly, and those who performed poorly on the slope interpretation question were likely to carry their misunderstandings on to their derivative interpretation (and later to the critiquing questions too).

We know students' difficulties with calculus concepts are often tied to their underdeveloped views of rates of change (Carlson, 1998; Hackworth, 1994; Orton, 1983; Thompson, 1994). This study adds to these findings, by highlighting the prevalent view of slope as a ratio-of-totals, as demonstrated by 39% of surveyed students and 54% of interviewed students. Findings also point to a connection between this ratio-of-totals view of slope and students' view of the derivative as something that can be used to calculate a y-value, in other words $f(x) = f'(x) * x$. In the interviews, of the seven students who took this incorrect approach to derivative interpretation, six had used a ratio-of-totals approach. The fact that students' impoverished view of slope is impacting their view of the derivative in these specific ways is not something that has been mentioned in the literature to date.

While my research questions did not deal directly with units, some useful findings on students' understanding of units come out of the study. Dorko and Speer (2015) found that calculus students had difficulty with units for area and volume, where only 26.6% of college students labeled correct units for all five area and volume tasks that were given to them. Students who struggled with the tasks seemed to not have knowledge of arrays or dimensionality. The types of unit questions that were given to the calculus students in my study were different, but in general, regardless of the context, it seems that units are difficult for students. Whereas arrays and dimensionality seemed to be lacking in the former study, in my study many students did not think of the units on the slope or derivative as a ratio, necessary knowledge to correctly find the units. These findings extend those at the high school level that found students difficulties viewing slope as a ratio (Barr, 1980, 1981; Stump, 2001).

On a broad level, understanding units is important because our national standards call on students to attend to precision, and one way to do so is by "specifying units of measure" (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP6, National Governors Association Center for Best Practices, 2010). In terms of this research, understanding units is important because without such understanding, it seems unlikely that students would be able to correctly interpret the meaning of the slope and derivative.

As pointed out in the Chapter 2, there has not been much research to date about college students' verbal interpretation of rate of change in real-life contexts, despite calls for calculus instruction to focus on "verbal approaches to applications on the derivative concept" (Maharaj, 2013, p. 15). This study's focus on verbal interpretation of the derivative in the context of real-life situations begins to fill that gap. Through these verbal interpretation questions, it was established that the ratio-of-totals view of slope is prevalent, and is impacting students' abilities to understanding the meaning of the derivative.

### 6.2 Results and contributions to the literature for research question #2: Critiquing the reasoning of others and recognizing appropriate uses of slope and derivative

The second research question focused on students' abilities to critique the reasoning of others, specifically students' abilities to recognize appropriate uses of slope and derivative to make predications. *Research Question #2: Given predictions based on slope and derivative, can students appropriately critique the reasoning?*

Key findings from the study which bring insight into research question #2 can be summarized as follows; they are described in more details in the sections that follow:

- Students struggled with the critiquing questions for the non-linear context, showing little understanding of the covarying nature of the derivative. In other words, they had difficulties with the idea that the derivative is an instantaneous rate of change whose value changes depending on its input value.

- The dominant incorrect ways to interpret the slope and derivative were applied to the critiquing questions, where many concluded that they could calculate the total by using the derivative ($f(x) = f'(x) * x$).

These findings support prior research that highlighted students' difficulties using calculus to analyze dynamic situations (Carlson, 1998), and students' struggles with use of the derivative to approximate the function near a point (Asiala et al., 1997). These finding extend the prior research by identifying students' common incorrect use of the derivative as a ratio-of-totals to be used to calculate the function value.

### 6.2.1 Students demonstrated little understanding of the covarying nature of the derivative

Considering the poor performance on the interpretation questions, it is not surprising that students were not able to critique the reasoning of someone else's predictions in the non-linear context (Table 6.5). In the surveys, only 16% of the students correctly critiqued the reasoning of Farmer Jim; just one student used the language of linear approximation. Almost as many (14%) stated that they needed another derivative (though none stated what they would do with that derivative). Many students did not give answers with complete reasoning (42%). The interviews allowed me to ask follow-up questions that allowed me to understand student thinking more completely than in the surveys.

| | Correct | Need another derivative |
|---|---|---|
| Surveys (N=69) | 16% | 14% |
| Interviews (N=13) | 31% | 62% |

Table 6.5. Comparison of the response rates for the critiquing questions in the non-linear context for the surveys and interviews

Success rates for the interviews were almost double that of the surveys (though only one student used language of linear approximation). What also increased substantially in the interviews was the percentage of students who said they needed another derivative (62%).

Two of the thirteen interviewed students used the derivative in the same manner as Nurse Jodi, to calculate the change in dosage for a certain change in weight. One agreed with Nurse Jodi, and one used the same approach as Nurse Jodi but requested that the derivative at the end weight be used to estimate the change. They made these predictions regardless of the magnitude of the change in weight. In other words, they did not recognize that the derivative, as an instantaneous rate of change, was only appropriate to use to estimate the change for values close to the input value.

These students did not express an understanding of confidence levels changing depending on the distance from the known value and the predicted value. Using the language of covariational reasoning, students did not attend to the ways the quantities of the weight and the rate of change of weight altered in relation to each other.

Seven interviewed students who said they needed another derivative used the derivative at the end weight to calculate the total dosage of the patient, by multiplying the derivative times the weight. These students also did not demonstrate understanding of the covarying nature of the derivative; instead of seeing the derivative as a function that changes as the input value changes (an across-time understanding), they instead saw the derivative as something that can be used to calculate the value of a function at a specific point (a point-wise understanding).

Of the nine interviewed students who incorrectly critiqued the reasoning of Nurse Jodi in the non-linear context, all demonstrated little knowledge of the covarying nature of the derivative. They thought of the derivative as something that could be used to calculate something at a point (either the total dosage or the change in dosage over an interval of any magnitude), instead of a function that changes depending on the input value.

### 6.2.2 The incorrect interpretation of slope and derivative as a ratio-of-totals was applied to the critiquing questions

Most of the students who interpreted the slope as a ratio of totals ($f(x) = m * x$) and the derivative as the analogous $f(x) = f'(x) * x$ went on to apply this incorrect interpretation to the critiquing questions. Specifically, seven of the 13 interviewed students (54%) disagreed with Nurse Jodi's use of the derivative to estimate the change in dosage, and instead used the derivative to calculate the total dosage ($D(x) = D'(x) * x$). Of these seven, five had used the ratio-of-totals approach to slope interpretation and the analogous $f(x) = m * x$ approach to derivative interpretation. The other two had correctly interpreted the slope but incorrectly interpreted the derivative.

These undesirable generalizations that students formed using their impoverished view of slope seem to be influencing their understanding the derivative as a continuously varying rate of change. One interviewed student, Missy, took her undesirable generalization one step further to say that the slope at a point could be determined by calculating $\frac{y}{x}$.

Based on the findings in this study, students' thinking of slope as a ratio-of-totals seems to be preventing them from making sense of the derivative as an instantaneous rate of change. Most troubling was the high percentage of interviewees (54%) who claimed that one could calculate the total dosage by multiplying the derivative by the weight ($f(x) = f'(x) * x$).

### 6.2.3 Summary of research question #2 and contributions to the literature

Many of the contributions to the literature mentioned in section 6.1.5 apply to this section as well, where students applied their impoverished view of slope as a ratio-of-totals to their critiquing of others' predictions.

New to this section is the focus on critiquing the reasoning of others. Noted in Chapter 2 is the call in our national standards to have students critique the reasoning of others, and "distinguish correct logic or reasoning from that which is flawed and – if there is a flaw in an argument – explain what that is" (Standards for Mathematical Practice section, CCSS.MATH.PRACTICE.MP3, National Governors Association Center for Best Practices, 2010). No literature exists probing student thinking about ideas related this important standard. This study begins to explore students' abilities to recognize a flawed argument and explain their reasoning. Unfortunately, in the case of this study, very few students were able to do this effectively. Many of their difficulties stemmed from, once again, their inability to interpret the slope and derivative correctly, which carried over to their inability to effectively critique the reasoning of others.

## 6.3 Limitations of the study

This study has limitations related to the sample size, sample population, questionnaire design, and data collection methods. A total of 69 students at one university were surveyed and 13 students were interviewed. Although there is no particular indication that these samples are not representative of the larger population of calculus students, a larger and more representative sample size would enable stronger generalizations about all differential calculus students.

Because only one context was used for the surveys (yield of a crop as a function of nitrogen level applied) and one for the interviews (dosage of a drug as a function of the weight of a patient), the ability to make conclusions about students' understandings across other contexts is limited. A larger study with more diverse contexts would enable stronger generalizations about all contexts.

In the surveys, the students often gave vague or incomplete written responses. This was avoided in the interviews because the interviewer asked follow-up questions and encouraged the participants to provide more details. Students might have had lower motivation and focus when they were filling out the written survey, which might have led to answers that did not accurately reflect their understandings. There is also the chance that the interviewer unintentionally influenced students' answers in the interviews. Additional interviews would provide more evidence of student understanding, but was beyond the scope and time available for this project.

### 6.4 Instructional implications

Expanding on previous findings that show students lack solid understanding of rates of change in general (Hackworth, 1994; Orton, 1984), findings suggest that students do not have a full understanding of what slope and derivative mean as a rate of change in the context of modeling situations, nor do they understand appropriate uses of slope and derivative to make predictions. Findings also suggest that students' incorrect interpretation of slope (as a ratio-of-totals which translates to a directly proportional relationship of the form $f(x) = m * x$) seems to be related to their incorrect interpretation and use of a derivative (as $f(x) = f'(x) * x$). These findings have instructional implications for both calculus instructors and middle school mathematics teachers.

### 6.4.1 Instructional implications for calculus instructors

Implications for calculus instructors center around three area: (1) identifying and addressing students' shortcomings around slope interpretation, (2) stressing limitations of derivatives and appropriateness of predictions, and (3) attending to units.

As college calculus instructors, we need to assess our students coming into our calculus courses. What is their understanding of slope, and their interpretation of slope in modeled contexts? By asking questions similar to those in this study early on in the calculus course, instructors can assess students' understandings coming into the course. When students answer slope interpretation questions using a ratio-of-totals approach, we cannot assume that they just "left out the word additional." We must address these shortcomings, taking the time to step back to the middle school concepts of slope, linear relationships, and directly proportional relationships (a subset of all linear relationships). Based on the results of this study, which showed a large percentage of the students having a ratio-of-totals approach, we need to take the time at the start of our calculus course so that all students understand the differences between directly proportional relationships and other linear relationships, and how the interpretation of slope is different in each. This would be best accomplished when ideas of average rates of change are introduced at the start of the course. Clicker questions would be an ideal way to get at the common misunderstandings around slope, and instructors could follow up with a worksheet or homework questions to reinforce these important concepts. When interpretations of the derivative are covered later on in the course, questions about linear equations and slope could be revisited, and distinctions made between the two.

Also, while students learn about the derivative in calculus, we have to provide enough opportunities for students to understand the differences between slope (a constant rate of change) and derivative (an instantaneous rate of change). I suggest combining questions about slope (linear relationships) with questions about derivative (non-linear relationships) so that students have opportunity to interpret both rates of change.

Second, we need to focus on not just what derivatives can be used for (linear approximation, marginal cost, etc.), but also stress their limitations in making predictions. Even in applied calculus texts, these limitations are not stressed. For example, in *Applied Calculus* (Hughes-Hallet et al., 2014) the section on interpretation of the derivative and tangent line approximation states “if the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit” (p. 104). Similarly in *Applied CALC* (Wilson, 2012), it states that you can interpret $f'(a) = c$ as “the value of the function f will increase (decrease) by about c units of output between a units of input and a+1 units of input” (p. 70). But never do they mention what “changing rapidly” means, nor do they explicitly state whether it is appropriate to use the derivative to approximate change further away than $f(a + 1)$. As an instructor, I would suggest giving students opportunities to compare the estimated values (from the derivative) with the actual value (from the original function), both by calculating the values and showing the error graphically. Discussions around how much error might be appropriate in different contexts would be beneficial. For example, in calculating the approximate change in a fish stock, one might allow for more error than in calculating the approximate change in the breaking distance of a newly designed vehicle.

Hughes-Hallet et al. (2014) also includes questions about using the derivative to estimate values of a function, and says “since the derivative tells us how fast the value of a function is changing, we can use the derivative at a point to estimate values of the function at nearby points” (p. 106). They do not define “nearby” points however, and in some of their exercises they have students estimate values that are as much as five or ten units away from original input value.

This study has shown that many calculus students do not know how to appropriately use a derivative to make an estimate. Textbooks are not giving students opportunities to discuss the appropriateness of predictions; instead students are asked to make estimates without being asked to consider how far away the input value is, or how rapidly the function is changing around the point of interest. We must address these ideas in our classroom, so students have opportunities to reflect on the appropriateness of their predictions.

Lastly, we must attend to units throughout the mathematics curriculum. While not a focus of the research study, it was clear that many of the calculus students in this study struggled with units on slope and derivative, and that those struggles impacted their ability to understand and appropriately use slope and derivative. While elementary students’ understanding of unit has been studied in detail, very little research has been done at the college level (Dorko & Speer, 2015), despite the importance of units in many calculus topics (differentiation, integration, differential equations). While mathematics instructors might not view units as important a topic as our physics and engineering colleagues, being aware of the difficulties students have around units, and attending to units when they arise in our curriculum, will be very beneficial to our students.

### 6.4.2 Instructional implications for middle school mathematics teachers

At the middle school level, instructional implications suggestions fall into two categories: (1) providing opportunities for students to understand that directly proportional relationships are a subset of all linear relationships, and how the interpretation of slope differs in both cases, and (2) providing more opportunities to understand and interpret the covariant aspects of functions.

I recommend providing more extensive opportunities for students to understand that directly proportional relationships are a subset of all linear relationships, and that not all linear relationships are of the form $f(x) = mx$. For example, students can calculate both $\frac{y}{x}$ and $\frac{\Delta y}{\Delta x}$ for linear relationships of the form $f(x) = mx$ and $f(x) = mx + b$ and make conclusions about the slope based on the results. Extending to slope interpretations, students need the opportunity to interpret slope as a rate of change (the functional approach to slope), similar to the questions in this study, but specifically focusing on interpreting slope of directly proportional relationships versus those that are not. By highlighting the differences between directly proportional relationships and other linear relationships, students will have the opportunity to understand how the interpretation of the slope differs. Focus must also be on units, as soon as students are introduced to slope. Oftentimes in the middle school curriculum, students calculate slope as a number, but the units are not mentioned or stressed. We know from research at the college level that students struggle with units; we must do better attending to units throughout the mathematics curriculum.

We also know that current curricula “provide little opportunity for developing the ability to: interpret and represent covariant aspects of functions, understand and interpret the language of functions, interpret information from functional events, etc.” (Carlson, 1998, p. 142). As a mathematics education community, we need to continue to improve our instruction and focus our attention on the topics (such as covariational reasoning and interpretation) that the research is pointing to as critical to the success of our students.

### 6.5 Future research

More research must be done on students’ interpretations of both slope and derivative as rates of change. It seems clear from this project that students’ most common incorrect interpretation of slope as a ratio-of-total (as $\frac{y}{x}$ instead of $\frac{\Delta y}{\Delta x}$ ) is influencing their ability to understand the derivative. How early do these ways of thinking form? Student thinking about slope as a rate of change can be researched at the middle school level to see whether the ratio-of-totals interpretation is prevalent in those early years. We know that “full concept development appears to evolve over a period of years” (Carlson, 1998. p.143). The concept of slope is first introduced in middle school, utilized throughout high school, and then expanded on in calculus with the introduction of the derivative. Research could be done to examine how students’ understanding of slope evolves over these key years.

At the college level, one could also focus on the effectiveness of specific instruction that is designed to address these misunderstandings. Differential calculus students could be pretested on their knowledge of slope as a rate of change at the start of the course, and then instruction on slope and interpretation of slope (for linear functions of the form $f(x) = mx + b$ and $f(x) = mx$) could occur prior to starting derivatives. After instruction on derivatives, a post-test could measure student gains on interpreting both slope and derivative, to see whether the targeted instruction on slope aids in students' understanding of derivative.

Also important would be to design surveys and interviews to probe the common incorrect response that a derivative can be used to calculate the dependent variable $(f(x) = f'(x) * x)$. Questions could be designed to challenge this idea, for example where $f'(x)$ is negative, to see whether students assimilate or accommodate their incorrect way of thinking.

Knowing more about students' understanding of slope and derivative as rates of change can help educators improve our instruction. The concept of slope is fundamental to the mathematics curriculum from the middle grades and on; we must do better to ensure that our students have solid conceptions of this rate of change. If we can help students develop these understandings when the concept is first introduced in middle school, and if necessary address any misunderstandings they bring forward to high school and beyond, we can ensure that students in our calculus course have the solid foundation necessary to understanding the complexities of the derivative. More success in calculus translates to more persistence in STEM fields, thus reaching our overarching goal of increasing the number of STEM graduates in our universities.

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## APPENDIX A: SURVEY INSTRUMENT WITHOUT GRAPHS PROVIDED

**Let B(n) be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen.**

A. Assume that B(n) is a linear function with a slope equal to 2 (m = 2)

0. On the graph to the right, give a rough sketch of what the function B(n) looks like. Label the axes, but no need to scale them.

1. What are the units on the slope, m = 2?

2. Explain what this slope (m = 2) means in the context of the problem.

3. Using the slope (m = 2), Farmer Jim predicts that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn. How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident | Somewhat Confident | Not Confident

   Explanation:

4. Farmer Jim purchases 20 pounds of nitrogen for his tract this season. At the last minute, he decides to invest more in nitrogen and increases the application to 30 pounds. Based on his model, he predicts that will get him 20 additional bushels (2 bushels for each additional pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident | Somewhat Confident | Not Confident

   Explanation:

B. Now, assume that B(n) is a non-linear function.

0. On the graph to the right, give a rough sketch of what the function B(n) looks like, assuming that the nitrogen is helpful to the crop up until a certain point and then too much is harmful. Label your axes, but no need to scale them.

1. What are the units on $\frac{dB}{dn}$? (also known as $B'(n)$)

2. Explain the meaning of the statement $B'(20) = 2$ in the context of the problem.

3. Using the fact that $B'(20) = 2$, Farmer Jim predicts that his corn yield will increase by 2 bushels when his nitrogen application increases from 20 pounds to 21 pounds. How much confidence do you have in his reasoning? (circle one and provide explanation)

   Very Confident          Somewhat Confident          Not Confident

   Explanation:

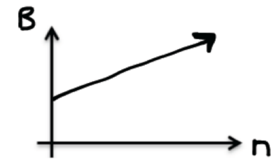
4. Farmer Jim purchases 20 pounds of nitrogen for his tract this year. Last minute, he decides to invest more in nitrogen and raises it to 30 pounds. Since $B'(20) = 2$, he predicts that the additional nitrogen will yield him 20 additional bushels (2 bushels for each pound of nitrogen) . How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident          Somewhat Confident          Not Confident

   Explanation:

## APPENDIX B: SURVEY INSTRUMENT WITH GRAPHS PROVIDED

**Let B(n) be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen.**




A. Assume that B(n) is a linear function with a slope equal to 2 (m = 2), shown in the graph.

1. What are the units on the slope, m = 2?

2. Explain what this slope (m = 2) means in the context of the problem.

3. Using the slope (m = 2), Farmer Jim predicts that by going from 20 pounds of nitrogen to 21 pounds of nitrogen, he will produce 2 more bushels of corn. How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident          Somewhat Confident          Not Confident

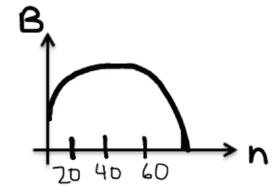
   Explanation:

4. Farmer Jim purchases 20 pounds of nitrogen for his tract this season. At the last minute, he decides to invest more in nitrogen and increases the application to 30 pounds. Based on his model, he predicts that will get him 20 additional bushels (2 bushels for each additional pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident          Somewhat Confident          Not Confident

   Explanation:

B. Now, assume that B(n) is a non-linear function, such that nitrogen is helpful to the crop up until a certain point and then too much is harmful, as show in the graph:




0. What are the units on $\frac{dB}{dn}$? (also known as $B'(n)$)

1. Explain the meaning of the statement $B'(20) = 2$ in the context of the problem.

2. Using the fact that $B'(20) = 2$, Farmer Jim predicts that his corn yield will increase by 2 bushels when his nitrogen application increases from 20 pounds to 21 pounds. How much confidence do you have in his reasoning? (circle one and provide explanation)

   Very Confident Somewhat Confident Not Confident

   Explanation:

3. Farmer Jim purchases 20 pounds of nitrogen for his tract this year. Last minute, he decides to invest more in nitrogen and raises it to 30 pounds. Since $B'(20) = 2$, he predicts that the additional nitrogen will yield him 20 additional bushels (2 bushels for each pound of nitrogen). How much confidence do you have in Jim's reasoning? (circle one and provide explanation)

   Very Confident Somewhat Confident Not Confident

   Explanation:

## APPENDIX C: INTERVIEW INSTRUMENT

**For certain drugs, the amount of dose given to a patient, D (in milligrams), depends on the weight of the patient, w (in pounds).**

A. Assume that D(w) is a linear function with a slope equal to 2 (m = 2).

0. On the graph below, give a rough sketch of what the function D(w) looks like. Label the axes, but no need to scale them.

1. What are the units on the slope, m = 2?

2. Explain what this slope (m = 2) means in the context of the problem.

3. Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)

   Very Confident Somewhat Confident Not Confident

   Explanation:

4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation)

   Very Confident Somewhat Confident Not Confident

   Explanation:

B. Now, assume D(w) is a non-linear function.

0. On the graph below, give a rough sketch of what the function D(w) might look like.

1. What are the units on $\frac{dD}{dw}$? (also known as $D'(w)$)

2. Explain the meaning of the statement $D'(140) = 2$ in the context of the problem.

3. Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)

   Very Confident    Somewhat Confident    Not Confident

   Explanation:

4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation).

   Very Confident    Somewhat Confident    Not Confident

   Explanation:

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Jennifer Tyne was raised in Troy, New York and graduated from Troy High School in 1988. She attended Boston College and graduated summa cum laude in 1992 with a Bachelor of Arts in Mathematics. After volunteering in Bethel, Alaska with the Jesuit Volunteer Corps, she received her Master of Science in Operations Research in 1995 from the University of North Carolina at Chapel Hill. She worked as a Biometrician in fisheries management for a tribal organization in the Pacific Northwest, and then returned to the Northeast and settled in coastal Maine. She has been a Lecturer in the Department of Mathematics and Statistics at the University of Maine since 2001. In addition to teaching undergraduate mathematics courses, she has co-authored the general education mathematics textbook *Algebraic Models in Our World*. She lives in Orono with her husband Andy and three children (Patrick, Eleanor, and Maureen). She is a candidate for the Master of Science in Teaching degree from the University of Maine in August 2016.