5-2012

On the Algebraic Reformulation of the Partition Function

Emily Igo

ON THE ALGEBRAIC REFORMULATION OF THE PARTITION FUNCTION

By

Emily Rose Igo

B.A. Boston College, 2010

A THESIS

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Master of Arts

(in Mathematics)

The Graduate School

The University of Maine

May, 2012

Advisory Committee:

Ali Ozluk, Professor of Mathematics, Advisor

Andrew Knightly, Associate Professor of Mathematics, Co-Advisor

William O. Bray, Professor of Mathematics

William Snyder, Professor of Mathematics

THESIS ACCEPTANCE STATEMENT

On behalf of the Graduate Committee for Emily Rose Igo I affirm that this manuscript is the final and accepted thesis. Signatures of all committee members are on file with the Graduate School at the University of Maine, 42 Stodder Hall, Orono, Maine.

Andrew Knightly, Associate Professor of Mathematics (date)

LIBRARY RIGHTS STATEMENT

ON THE ALGEBRAIC REFORMULATION OF

THE PARTITION

FUNCTION

By

Emily Rose Igo

Thesis Advisor: Dr. Ali Ozluk

Co-Advisor: Dr. Andrew Knightly

An Abstract of the Thesis Presented

in Partial Fulfillment of the Requirements for the

Degree of Master of Arts

(in Mathematics)

May, 2012

The partition function has long enchanted the minds of great mathematicians, dating from Euler's attempts at calculating the value of this function in the 1700's, to Hardy and Ramanujan's asymptotic approach in the early twentieth century, through to Rademacher's representation as an explicit infinite series mid-century. This thesis will explore the historical attempts at grasping the behavior of this function, with particular attention paid to Euler's Pentagonal Number Theorem and Rademacher's Infinite Sum. We will then explore two reformulations due to Ono et al., with sample calculations from the recent algebraic reformulation, announced January, 2011.

DEDICATION

In memory of my advisor, Dr. Ali Ozluk, for all of his kindness, patience and support in the writing of this thesis.

For my grandfather, who has always inspired and encouraged me.

ACKNOWLEDGEMENTS

I would like to thank Andy Knightly for all of his support, patience and assistance in the completion of this thesis; without his help and time, I would not have been able to write this page.

I would also like to thank Dr. Ali Ozluk, who directed my work and helped guide the majority of this thesis. He was an invaluable resource and huge support, and I wish he could see the finished product of all our hard hours of work together.

Thank you to Bill Bray and Chip Snyder for their willingness to join my committee, and especially Chip for stepping in at the last minute.

To my family and Damien, thank you for your support, encouragement, patience and meals- I would not have made it through this without you!

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

A partition of a non-negative integer n is any non-decreasing sequence of integers which sum to n; the partition function, $p(n)$, counts the number of partitions of an integer. The partition function has long enchanted the minds of great mathematicians, dating from Euler's attempts at calculating the value of this function in the 1700's, to Hardy and Ramanujan's asymptotic approach in the early twentieth century, through to Rademacher's representation as an explicit infinite series mid-century.

In the next chapter, we will explore more definitions and examples relating to the partition function, as well as some of the historical methods used to computer the number of partitions. This will include recursive formulas, formulas based on partitions having a specific largest value for each term, and the asymptotic formula.

The third chapter will examine Rademacher's explicit representation of the partition function as an infinite series, including a closer look at the reformulation of $A_k(n)$, a key inner sum from his definition of the partition function. There will also be several examples of computations of $A_k(n)$.

The fourth chapter focuses on Ono and Bringmann's reformulation of the partition function, which is the main result discussed in this thesis. This reformulation takes the explicit representation of $p(n)$ as an infinite series to a finite, closed sum relating to a Poincaré series. This is historically significant in the field of partition theory, as it is the first instance of a finite sum for $p(n)$.

Finally, in the fifth chapter we will discuss a further reformulation of $p(n)$ due to Ono and Bruinier; this additional reformulation allows an expansion of the Poincaré series, providing a computable estimate of $p(n)$ in terms of algebraic numbers. Based on this expansion and certain quadratic forms, the details of two examples of computing $p(n)$ will be included.

Chapter 2

AN INTRODUCTION TO PARTITION THEORY

It has been a long-standing problem of interest in number theory to consider the number of partitions of an integer. First we consider the distinction between ordered and unordered partitions; after handling the simpler case, we will overview some historical methods used to compute $p(n)$, including Euler's recursion, the Pentagonal Number Theorem, and Hardy and Ramanujan's asymptotic formula. Some special cases will be considered, such as restricting the largest part, and outlines of derivations will be given, particularly relating to formal power series.

2.1 Ordered and Unordered Partitions

Definition A *partition of a non-negative integer* n is any non-decreasing sequence of positive integers which sum to n.

Example

$$n = 4 = \begin{cases} 4 \\ 2+2 \\ 1+1+2 \\ 1+3 \\ 1+1+1+1 \end{cases} \tag{2.1}$$

Definition The *partition function of* n, denoted $p(n)$, counts the number of partitions of n (such as $p(4) = 5$). In particular, $p(n)$ counts the unordered partitions of n.

Note that $p(n)$ grows rapidly and irregularly, so this value is difficult to compute for large n.

Definition By removing the restriction to non-decreasing sequences from the definition of *partitions*, we arrive at the *ordered partitions.*

Notation We will denote the number of *ordered partitions of* n by $\tilde{p}(n)$.

Example

$$n = 4 = \begin{cases} 4 \\ 2+2 \\ 1+1+2 \\ 1+2+1 \\ 2+1+1 \\ 1+1+1+1 \\ 3+1 \\ 1+3 \end{cases} \tag{2.2}$$

Remark As demonstrated in Table (2.1), for each n, the ordered paritions are obtained by:

(i) adding 1 to the last term of each of the partitions of $n-1$ and

(ii) appending 1 to the end of each of the partitions of $n-1$.

n	Ordered Partitions	$\tilde{p}(n)$
1	1	1
2	2	2
	1 + 1	
3	3	4
	1 + 2	
	2 + 1	
	1 + 1 + 1	
4	4	8
	1 + 3	
	2 + 2	
	1 + 1 + 2	
	3 + 1	
	1 + 2 + 1	
	2 + 1 + 1	
	1 + 1 + 1 + 1	

Table 2.1. Ordered Partitions of n

Claim 2.1.1. $\tilde{p}(n) = 2^{n-1}$.

Proof. We have already shown that $\tilde{p}(1) = 1 = 2^{1-1}$; now assume for $n = 1, ..., k-1$ that $\tilde{p}(k-1) = 2^{k-2}$.

Then for $n = k$, (i) gives us $\tilde{p}(k-1)$ terms and (ii) gives us an additional $\tilde{p}(k-1)$ terms, so $\tilde{p}(k) = \tilde{p}(k-1) + \tilde{p}(k-1) = 2 \cdot \tilde{p}(k-1) = 2 \cdot 2^{k-2} = 2^{k-1}$, hence $\tilde{p}(n) = 2^{n-1}$.

Note that there will be no duplications by creating partitions in this manner, for suppose for a contradiction that two partitions of k obtained in this manner are identical. In particular, the last term of each of these two partitions will be equal to some $j \neq 0$; subtracting 1 from this term, we are left with two identical partitions of $k-1$. Now we know that one partition of $k-1$ will go to two different partitions of k via (i) and (ii), but we are reversing only (ii) with our process. This is a contradiction, as there are no identical partitions of $k-1$, hence no duplications arise in our method of creating new partitions. □

There is also a combinatorial proof of Claim (2.1.1).

Proof. The ordered partitions of n into k parts may be written as $x_1+x_2+...+x_k = n$, where $x_1, x_2, ..., x_k \geq 1$ are all integers. Then the number of ordered partitions of n into k parts is

$$\tilde{p}_k(n) = \binom{n-1}{k-1}$$

and hence the number of ordered partitions of n is

$$\begin{aligned} \tilde{p}(n) &= \sum_{k=0}^{n} \binom{n-1}{k-1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \\ &= 2^{n-1}. \end{aligned}$$

□

2.2 Historical Methods of Computing $p(n)$

2.2.1 Euler's Recursion

Definition Euler derived a recursive formula for the number of partitions of n, where each part does not exceed k, denoted $p_k(n)$.

$$\textit{(Euler's Recursion)} \qquad p_k(n) = p_{k-1}(n) + p_k(n-k), \qquad p_k(0) = 1 \tag{2.3}$$

He derived this directly from the following formal identity:

$$\frac{1}{(1-x)(1-x^2)...(1-x^k)} = \sum_{n=0}^{\infty} p_k(n)x^n, \qquad p_k(0) = 1 \tag{2.4}$$

Equation (2.4) is rooted in formal power series (see Appendix A), by taking a product of k geometric power series. See Rademacher [12] and the proof outline below for further discussion.

Proof. (sketch of ideas for (2.4))

Taking a product of geometric series, we may write

$$\begin{aligned}
\frac{1}{(1-x)(1-x^2)...(1-x^k)} &= \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}...\sum_{\nu_k=0}^{\infty} x^{\nu_1+2\nu_2+...+k\nu_k}, \ \nu_i \in \mathbb{Z} \text{ for all } i \\
&= 1 + \sum_{n=1}^{\infty}\Big(\sum_{\substack{\nu_1,\nu_2,...,\nu_k \geq 0 \\ \nu_1+2\nu_2+...+k\nu_k=n}} 1\Big)x^n \\
&= 1 + \sum_{n=1}^{\infty} p_k(n)x^n
\end{aligned}$$

We see that

$$\sum_{\substack{\nu_1,\nu_2,\ldots,\nu_k\geq 0\\ \nu_1+2\nu_2+\ldots+k\nu_k=n}} 1 = p_k(n)$$

as the sum is over $\underbrace{1+\ldots+1}_{\nu_1}+\underbrace{2+\ldots+2}_{\nu_2}+\ldots+\underbrace{k+\ldots+k}_{\nu_k}=n$, hence k is the largest part of n. □

Remark In (2.4), letting k tend to infinity gives us the unlimited partitions, or the case of $p(n)$, the number of partitions of n.

Proof. (Derivation of Euler's Recursion (2.3))

We now have

$$\frac{1}{\prod_{m=1}^{k-1}(1-x^m)} = (1-x^k)\sum_{n=0}^{\infty} p_k(n)x^n \quad \text{(from (2.4))}$$

hence

$$\sum_{n=0}^{\infty} p_{k-1}(n)x^n = (1-x^k)\sum_{n=0}^{\infty} p_k(n)x^n.$$

Taking only the terms containing x^n, we obtain from the left hand side

$$p_{k-1}(n)x^n$$

and from the right hand side

$$(1-x^k)p_k(n)x^n+(1-x^k)p_k(n-k)x^{n-k} = p_k(n)x^n-p_k(n)x^{n+k}+p_k(n-k)x^{n-k}-p_k(n-k)x^n$$

and finally, taking just the coefficients of x^n, we obtain

$$p_k(n) = p_{k-1}(n) + p_k(n-k).$$

$\square$

Example Again using $n = 4$ and, in this case, $k = 4$ (so we will obtain the count of all partitions of 4), we see by repeated applications of (2.3):

$$\begin{aligned} p_4(4) &= p_3(4) + p_4(0) \\ &= p_2(4) + p_3(1) + p_4(0) \\ &= p_1(4) + p_2(2) + p_3(1) + p_4(0) \\ &= p_1(4) + p_1(2) + p_2(1) + p_3(1) + p_4(0) \\ &= 1 + 1 + 1 + 1 + 1 \\ &= 5 \end{aligned}$$

as we calculated previously by force.

It is important to note that the number of terms will grow quickly for large n.

Example (*An exact formula for* $p_2(n)$)

From (2.4) we may expand the left hand side into partial fractions; for $k = 2$, this is

$$\frac{1}{(1-x)(1-x^2)} = \frac{1/2}{(1-x)^2} + \frac{1/4}{1-x} + \frac{1/4}{1+x}.$$

Note that $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, so

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=10}^{\infty} x^n$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... = \sum_{n=0}^{\infty}(n+1)x^n.$$

We have geometric series with terms alternating in the $\frac{1/4}{1+x}$ term but not in the $\frac{1/4}{1-x}$ term, hence the odd powers will cancel and we will be left with two copies of each even term. Hence,

$$\frac{1}{(1-x)(1-x^2)} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{2}\sum_{m=0}^{\infty} x^{2m}.$$

We can then see that

$$p_2(n) = \frac{1}{2}(n+1) + a = \begin{cases} \frac{1}{2}(n+1) + 0 & \text{when } n \text{ is odd} \\ \frac{1}{2}(n+1) + \frac{1}{2} & \text{when } n \text{ is even} \end{cases}$$

hence

$$\frac{1}{2}n + \frac{1}{2} + a = \begin{cases} \frac{1}{2}n + \frac{1}{2} & \text{when } n \text{ is odd} \\ \frac{1}{2}n + 1 & \text{when } n \text{ is even} \end{cases}$$

and finally

$$p_2(n) = \lfloor \frac{n}{2} \rfloor + 1.$$

Example (*An exact formula for $p_3(n)$*)

A similar method as with $k = 2$ may be used for the case of $k = 3$. We see, by partial fraction decomposition, that

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{\frac{1}{6}}{(1-x)^3} + \frac{\frac{1}{4}}{(1-x)^2} + \frac{\frac{17}{72}}{1-x} + \frac{\frac{1}{8}}{1+x} + \frac{\frac{1}{9}}{1-\frac{x}{\rho}} + \frac{\frac{1}{9}}{1-\frac{x}{\bar{\rho}}} \quad (2.5)$$

where $\rho^3 - 1 = 0$, $\rho \neq 1$ and $\bar{\rho}$ is the complex conjugate of ρ.

We can rearrange this into a sum of power series, recalling some basic facts, such as

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + -...$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=0}^{\infty}(n+1)x^n$$
$$\frac{1}{(1-x)^3} = 2 + 6x + 12x^2 + ... = \sum_{n=0}^{\infty}\binom{n+2}{2}x^n$$

(Generally, $\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n$; the coefficient here represents the number of non-negative solutions to $y_1 + y_2 + ... + y_k = n$.)

Combining these facts, we arrive at

$$\begin{aligned}
\frac{1}{(1-x)(1-x^2)(1-x^3)} =& \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n \\
&+ \frac{17}{72}\sum_{k=0}^{\infty}x^k + \frac{1}{8}\sum_{k=0}^{\infty}(-x)^k \\
&+ \frac{1}{9}\sum_{k=0}^{\infty}(\frac{x}{\rho})^k + \frac{1}{9}\sum_{k=0}^{\infty}(\frac{x}{\bar{\rho}})^k \\
=& \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n \\
&+ \frac{17}{72}\sum_{k=0}^{\infty}x^k + \frac{1}{8}\sum_{k=0}^{\infty}(-x)^k \\
&+ \frac{1}{9}(\sum_{k=0}^{\infty}(\bar{\rho}^k + \rho^k)x^k)
\end{aligned}$$

as $\rho = \frac{1}{\bar{\rho}}$. We also use the fact that

$$\bar{\rho}^k + \rho^k = \begin{cases} 2 & \text{when } k = 3j, j \in \mathbb{Z} \\ -1 & \text{elsewhere} \end{cases}$$

Then letting the last three terms be named as below

$$\mathcal{T} := \frac{17}{72}\sum_{k=0}^{\infty}x^k + \frac{1}{8}\sum_{k=0}^{\infty}(-x)^k + \frac{1}{9}(\sum_{k=0}^{\infty}(\bar{\rho}^k + \rho^k)x^k)$$

we see

$$
\begin{aligned}
\mathcal{T} = \frac{17}{72}\sum_{j=0}^{\infty} x^{3j} + \frac{1}{8}\sum_{j=0}^{\infty}(-1)^j x^{3j} + \frac{2}{9}\sum_{j=0}^{\infty} x^{3j} \\
+ \frac{17}{72}\sum_{j=0}^{\infty} x^{3j+1} - \frac{1}{8}\sum_{j=0}^{\infty}(-1)^j x^{3j+1} - \frac{1}{9}\sum_{j=0}^{\infty} x^{3j+1} \\
+ \frac{17}{72}\sum_{j=0}^{\infty} x^{3j+2} + \frac{1}{8}\sum_{j=0}^{\infty}(-1)^j x^{3j+2} 1\frac{1}{9}\sum_{j=0}^{\infty} x^{3j+2}
\end{aligned}
$$

Gathering terms for cases of j odd and j even, we then see that

$$
\begin{aligned}
\mathcal{T} &= (\frac{17}{72} + \frac{2}{9} - \frac{1}{8})\sum_{j=0}^{\infty} x^{3j} + \frac{2}{8}\sum_{j \text{ even}} x^{3j} \\
&+ (\frac{17}{72} - \frac{1}{9} - \frac{1}{8})\sum_{j \text{ even}} x^{3j+1} + (\frac{17}{72} - \frac{1}{9} + \frac{1}{8})\sum_{j \text{ odd}} x^{3j+1} \\
&+ (\frac{17}{72} - \frac{1}{9} + \frac{1}{8})\sum_{j \text{ even}} x^{3j+2} + (\frac{17}{72} - \frac{1}{9} - \frac{1}{8})\sum_{j \text{ odd}} x^{3j+2} \\
&= \frac{1}{3}\sum_{j=0}^{\infty} x^{3j} + \frac{1}{4}\sum_{j \text{ even}} x^{3j} + \frac{1}{4}\sum_{j \text{ odd}} x^{3j+1} + \frac{1}{4}\sum_{j \text{ even}} x^{3j+2} \\
&= \frac{1}{3}\sum_{m=0}^{\infty} x^{3m} + \frac{1}{4}\sum_{m=0}^{\infty} x^{2m}
\end{aligned}
$$

This last equality is obtained by reindexing. Finally, we arrive at

$$
\begin{aligned}
\frac{1}{(1-x)(1-x^2)(1-x^3)} =& \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n \\
&+ \frac{1}{3}\sum_{m=0}^{\infty} x^{3m} + \frac{1}{4}\sum_{m=0}^{\infty} x^{2m} \\
=& \sum_{n=0}^{\infty} p_3(n)x^n
\end{aligned}
$$

Taking $k = 3$, we have the coefficient $p_3(n) = \frac{1}{6}\binom{n+2}{2} + \frac{1}{4}(n+1) + a$ with

$$a = \begin{cases} \frac{1}{3} + \frac{1}{4} & \text{when } n \equiv 0 \mod 6 \\ 0 & \text{when } n \equiv 1 \text{ or } 5 \mod 6 \\ \frac{1}{4} & \text{when } n \equiv 2 \text{ or } 4 \mod 6 \\ \frac{1}{3} & \text{when } n \equiv 3 \mod 6 \end{cases}$$

With some rearrangement, we get

$$p_3(n) = \begin{cases} \frac{n(n+6)}{12} + 1 & \text{when } n \equiv 0 \mod 6 \\ \frac{n(n+6)}{12} + \frac{8}{12} & \text{when } n \equiv 2 \text{ or } 4 \mod 6 \\ \frac{n(n+6)}{12} + \frac{9}{12} & \text{when } n \equiv 3 \mod 6 \\ \frac{n(n+6)}{12} & \text{when } n \equiv 1 \text{ or } 5 \mod 6 \end{cases}$$

and finally

$$p_3(n) = \lfloor \frac{n(n+6)}{12} \rfloor + 1$$

Example (*Extending $p_3(n)$ to case $k = 4$*)

Combining

$$p_3(n) = \lfloor \frac{n(n+6)}{12} \rfloor + 1$$

and

$$p_k(n) = p_{k-1}(n) + p_k(n-k)\ , \qquad p_k(0) = 1$$

we may obtain

$$\begin{aligned} p_4(n) &= p_3(n) + p_4(n-4) \\ &= p_3(n) + p_3(n-4) + p_4(n-8) \\ &= p_3(n) + p_3(n-4) + p_3(n-8) + p_4(n-12) \\ &= \sum_{k \le \frac{n}{4}} p_3(n-4k). \end{aligned}$$

Example Let $n = 4$. Then we have

$$\begin{aligned} p_4(4) &= \sum_{k \le 1} (\lfloor \frac{(4-4k)(4-4k+6)}{12} \rfloor + 1) \\ &= \lfloor \frac{4(4+6)}{12} \rfloor + 1 + \lfloor \frac{(4-4)(4-4+6)}{12} \rfloor + 1 \\ &= 3 + 2 \\ &= 5. \end{aligned}$$

2.2.2 Pentagonal Number Theorem

Definition the *pentagonal numbers* are

$$\omega_\lambda = \frac{\lambda(3\lambda - 1)}{2}, \qquad \lambda = 0, \pm 1, \pm 2, ...$$

Example

$$\omega_0 = 0$$

$$\omega_1 = 1$$

$$\omega_{-1} = 2$$

$$\omega_2 = 5$$

$$\omega_{-2} = 7$$

$$\omega_3 = 12$$

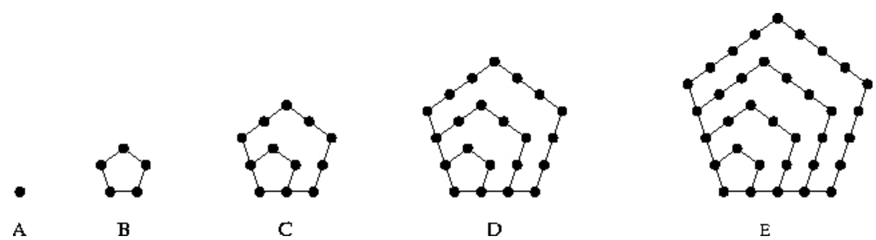

Figure 2.1. Pentagonal Numbers

This has a geometric interpretation, where $1, 5, 12, \ldots$ (corresponding to the positive valued lambdas) are the vertices of each set of nested pentagons. As one can see in Figure (2.1), 1 comes from the vertices in A, 5 comes from the vertices in B, 12 comes from counting each vertex in C once, and so on. These are commonly known as the pentagonal numbers. Our definition of the pentagonal numbers includes the *generalized pentagonal numbers*, $0, 2, 7, \ldots$ (corresponding to the non-positive valued lambdas). These are the interior vertices of the nested pentagons. Again from Figure (2.1), we retrieve no values from A as there is no interior, 0 from B, 2 from C, 7 from D, and so on.

Theorem 2.2.1. (Pentagonal Number Theorem)

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}x^{\frac{\lambda(3\lambda-1)}{2}}$$

Proof.

$$\begin{aligned}\prod_{m=1}^{\infty}(1-x^m) =&(1-x)(1-x^2)(1-x^3)...\\ =&1-x-(1-x)x^2-(1-x)(1-x^2)x^3-(1-x)(1-x^2)(1-x^3)x^4-...\end{aligned}$$

Here, the equality in the second line is obtainedby distributing with a shift, meaning the first term is distributed to the second, then the product of the first two terms is distributed to the third, and so on. Next, we will distribute the term $(1-x)$, and obtain:

$$\begin{aligned}=&1-x-x^2-(1-x^2)x^3-(1-x^2)(1-x^3)x^4-(1-x^2)(1-x^3)(1-x^4)x^5\\ &-...+x^3+(1-x^3)x^4+(1-x^2)(1-x^3)x^5+...\\ =&1-x-x^2-x^3+x^5-(1-x^2)x^4+(1-x^2)x^7-(1-x^2)(1-x^3)x^5\\ &+(1-x^2)(1-x^3)x^9-...+x^3+(1-x^2)x^4+(1-x^2)(1-x^3)x^5+...\end{aligned}$$

The equality in the second line is obtained by distrubting the term $(1-x^{m-1})$. Cancelling terms, we then see:

$$
\begin{aligned}
&=1-x-x^2+x^5+(1-x^2)x^7+(1-x^2)(1-x^3)x^9+...\\
&=1-x-x^2+x^5+x^7+(1-x^3)x^9+...-x^9-(1-x^3)x^{11}-...
\end{aligned}
$$

where the second equality is obtained by distributing $(1-x^2)$. Finally, by distributing $(1-x^3)$ and then making appropriate cancellations, we obtain:

$$
\begin{aligned}
&=1-x-x^2+x^5+x^7+x^9-x^{12}+-...-x^9-x^{11}+x^{14}-+...\\
&=1-x-x^2+x^5+x^7+-x^{12}-+...
\end{aligned}
$$

We may continue this process inductively; let $P := \prod_{m=1}^{\infty}(1-x^m)$, and assume we have

$$
\begin{aligned}
P = &\sum_{\lambda=-k+1}^{k}(-1)^{\lambda}x^{\omega_\lambda}+(-1)^k x^{\omega_k+k}\{(1-x^k)+(1-x^k)(1-x^{k+1})x^k\\
&\qquad +(1-x^k)(1-x^{k+1})(1-x^{k+2})x^{2k}+...\}\\
= &\sum_{\lambda=-k+1}^{k}(-1)^{\lambda}x^{\omega_\lambda}+(-1)^k x^{\omega_k+k}\{1-x^k+(1-x^{k+1})x^k-(1-x^{k+1})x^{2k}\\
&\qquad +(1-x^{k+1})(1-x^{k+2})x^{2k}-(1-x^{k+1})(1-x^{k+2})x^{3k}+...\}\\
= &\sum_{\lambda=-k+1}^{k}(-1)^{\lambda}x^{\omega_\lambda}+(-1)^k x^{\omega_k+k}\{1+(1-x^{k+1})x^k+(1-x^{k+1})(1-x^{k+2})x^{2k}+\\
&\qquad ...-x^k-(1-x^{k+1})x^{2k}-(1-x^{k+1})(1-x^{k+2})x^{3k}-...\}
\end{aligned}
$$

The above equality is obtained by rearrangement of terms. Next, we expand to obtain:

$$= \sum_{\lambda=-k+1}^{k} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^k x^{\omega_k+k}\{1 + x^k - x^{2k+1} + (1-x^{k+1})x^{2k} - x^{3k+2} +$$

$$x^{4k+2} + ... - x^k - (1-x^{k+1})x^{2k} - (1-x^{k+1})(1-x^{k+2})x^{3k} - ...\}$$

Finally, by cancelling and distributing to the first three terms, we obtain:

$$= \sum_{\lambda=-k+1}^{k} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^k x^{\omega_k+k} + (-1)^{k+1} x^{\omega_k+3k+1} + (-1)^{k+1} x^{\omega_k+4k+2}\cdot$$

$$\{1 + (1-x^{k+1})x^{k+1} + (1-x^{k+1})(1-x^{k+2})x^{2k+2} + ...\}$$

Recalling $\omega_\lambda = \frac{\lambda(3\lambda-1)}{2}$, we see that

$$\begin{aligned} \omega_k + k &= \frac{k(3k-1)}{2} + k = \frac{3k^2 - k + 2k}{2} = \frac{3k^2+k}{2} \\ &= \frac{k(3k+1)}{2} = \frac{-k(-3k-1)}{2} = -\omega_k \end{aligned}$$

and

$$\begin{aligned} \omega_k + 3k + 1 &= \frac{k(3k-1)}{2} + 3k + 1 = \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(3k+2)(k+1)}{2} = \frac{(k+1)(3(k+1)-1)}{2} = \omega_{k+1} \end{aligned}$$

Continuing from above, and using these identities, we get

$$P = \sum_{\lambda=-k+1}^{k} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^{-k} x^{\omega_{-k}} + (-1)^{k+1} x^{\omega_{k+1}} + (-1)^{k+1} x^{\omega_{k+1}+k+1}\{1 + ...\}$$

$$= \sum_{\lambda=-k}^{k+1} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^{k+1} x^{\omega_{k+1}+k+1}\{1 + (1 - x^{k+1})$$

$$+ (1 - x^{k+1})(1 - x^{k+2}) x^{2k} ...\}$$

By formal power series theory (see Appendix A), we may reduce $(modx^{\omega_{k+1}})$ and get

$$P = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\omega_\lambda} (modx^{\omega_{k+1}}).$$

□

Combining the Pentagonal Number Theorem (2.2.1) with equation (2.4) we get, by comparing coefficients, another recursive formula

$$p(n) = \sum_{0<\omega_\lambda \le n} (-1)^{\lambda-1} p(n - \omega_\lambda), \qquad p(0) = 1 \tag{2.6}$$

Remark It has previously been shown that the number of terms in the right hand side is approximately $(\frac{2}{3})\sqrt{6n}$.

Example

$$p(0) = 1 \text{ by definition}$$

Example

$$
\begin{aligned}
p(1) &= (-1)^{1-1} p(1-\omega_1) \\
&= p(1-1) = p(0) = 1
\end{aligned}
$$

Example

$$
\begin{aligned}
p(2) &= (-1)^{1-1} p(2-\omega_1) + (-1)^{-1-1} p(2-\omega_{-1}) \\
&= p(2-1) + p(2-2) = p(1) + p(0) = 1 + 1 = 2
\end{aligned}
$$

Example

$$
\begin{aligned}
p(3) &= (-1)^{1-1} p(3-\omega_1) + (-1)^{-1-1} p(3-\omega_{-1}) \\
&= p(3-1) + p(3-2) = p(2) + p(1) = 2 + 1 = 3
\end{aligned}
$$

Example

$$
\begin{aligned}
p(4) &= (-1)^{1-1} p(4-\omega_1) + (-1)^{-1-1} p(4-\omega_{-1}) \\
&= p(4-1) + p(4-2) = p(3) + p(2) = 3 + 2 = 5
\end{aligned}
$$

Example

$$p(5) = (-1)^{1-1}p(5-\omega_1) + (-1)^{-1-1}p(5-\omega_{-1}) + (-1)^{2-1}p(5-\omega_2)$$

$$= p(5-1) + p(5-2) - p(5-5) = p(4) + p(3) - p(0) = 5 + 3 - 1 = 7$$

Example

$$p(6) = (-1)^{1-1}p(6-\omega_1) + (-1)^{-1-1}p(6-\omega_{-1}) + (-1)^{2-1}p(6-\omega_2)$$

$$= p(6-1) + p(6-2) - p(6-5) = p(5) + p(4) - p(1) = 7 + 5 - 1 = 11$$

Example

$$\begin{aligned} p(7) &= (-1)^{1-1}p(7-\omega_1) + (-1)^{-1-1}p(7-\omega_{-1}) \\ &\quad + (-1)^{2-1}p(7-\omega_2) + (-1)^{-2-1}p(7-\omega_{-2}) \\ &= p(7-1) + p(7-2) - p(7-5) + p(7-7) \\ &= p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15 \end{aligned}$$

2.2.3 Asymptotic Formula

In 1918, Hardy and Ramanujan derived the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

with upper bound of the error

$$O(e^{\frac{5\pi(\frac{2}{3})^{\frac{1}{2}}(n-\frac{1}{24})^{\frac{1}{2}}}{8}})$$

There is an effective version of this bound due to Lehmer [8], where given a specific n one may compute the bound, and hence use this asymptotic formula for calculating an estimate of $p(n)$.

As n tends to infinity, the ratio of $p(n)$ over the right hand side will approach 1.

Example For $n = 200$,

$$\frac{p(200)}{\frac{1}{4\cdot 200\sqrt{3}}e^{\pi\sqrt{\frac{2\cdot 200}{3}}}} = \frac{3972999029388}{\frac{1}{800\sqrt{3}}e^{\pi\sqrt{\frac{400}{3}}}} \approx .968965$$

Remark There was much historical significance in this result, as Hardy and Ramanujan developed the circle method of proof specifically for this asymptotic formula; they represented $p(n)$ as an integral over the unit circle and computed the integral by dividing the unit circle into segments, computing some exactly and approximating others, giving us the error term.

Chapter 3

AN EXPLICIT REPRESENTATION OF $p(n)$

We will explore a reformulation of $A_k(n)$, used in the definition of the partition function $p(n)$, which will lead to a more usable form of $p(n)$. We will also compare computations of this inner sum using methods due to Rademacher and Whiteman.

3.1 Rademacher's Representation

An improvement on the asymptotic formula, due to Rademacher, was given in 1937:

$$p(n) = 2\pi(24n-1)^{\frac{-3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}(\frac{\pi\sqrt{24n-1}}{6k}) \tag{3.1}$$

where $A_k(n)$ is a Kloosterman-type sum,

$$A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{\frac{-2\pi inh}{k}+\pi is(h,k)}, \tag{3.2}$$

$s(h,k)$ a Dedekind sum,

$$s(h,k) = \sum_{\mu=1}^{k-1} (\frac{\mu}{k} - \lfloor \frac{\mu}{k} \rfloor - \frac{1}{2})(\frac{h\mu}{k} - \lfloor \frac{h\mu}{k} \rfloor - \frac{1}{2}),$$

and $I_{\frac{3}{2}}(\epsilon)$ is a modified Bessel function of the first kind, with $\epsilon = \frac{\pi\sqrt{24n-1}}{6k}$,

$$I_{\frac{3}{2}}(\epsilon) = \sum_{n=0}^{\infty} \frac{(\frac{\epsilon}{2})^{2n+\frac{3}{2}}}{n!\Gamma(\frac{3}{2}+n+1)}.$$

This is a solution to the modified Bessel equation

$$\epsilon^2 \frac{d^2y}{d\epsilon^2} + \epsilon \frac{dy}{d\epsilon} + (\epsilon^2 - (\frac{3}{2})^2)y = 0.$$

This was an improvement upon the asymptotic formula as it provided an explicit formula, though it is still difficult to calculate. Below we provide an outline of the reformulation of $A_k(n)$, to arrive at a new form. See Rademacher [12] for further details of the proof of the formula for $p(n)$.

Theorem 3.1.1. (Rademacher's reformulation of $A_k(n)$)

If n is a positive integer, then

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}(\frac{\pi\sqrt{24n-1}}{6k}) \quad (3.1)$$

Furthermore, $A_k(n)$ (3.2) is equal to

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x (mod 24k) \\ x^2 \equiv -24n+1 (mod 24k)}} (-1)^{\{\frac{x}{6}\}} e(\frac{x}{12k}) \tag{3.3}$$

In the sum, x runs through the residue classes modulo $24k$, and $\{\alpha\}$ denotes the integer nearest to α, rounding up if α is a half-integer.

Proof. (of (3.3))

Let us assume the following formulae due to Rademacher [12]:

$$\Psi(t,\alpha) = \frac{e^{-\pi\alpha^2 t}}{\sqrt{t}}\Psi(\frac{1}{t}, -\imath\alpha t), \text{ where } \Psi(t,\alpha) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 t} \quad \text{(R.36.2)}$$

$$\omega_{hk} = e^{\pi\imath s(h,k)} \quad \text{(R.118.42)}$$

$$A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} \omega_{hk} e^{\frac{-2\pi\imath hn}{k}} \quad \text{(R.120.5)}$$

$$f(e^{\frac{2\pi\imath h}{k}\frac{-2\pi z}{k}}) = e^{\frac{\pi\imath h}{12k}-\frac{\pi z}{12k}}\eta(\frac{h}{k}+\frac{\imath z}{k})^{-1}, \ f(x) := \prod_{m=1}^{\infty}(1-x^m)^{-1} \quad \text{(R.118.1)}$$

$$f(e^{\frac{2\pi\imath h}{k}\frac{-2\pi z}{k}}) = e^{\frac{\pi\imath h}{12k}-\frac{\pi z}{12k}}\epsilon(a,b,c,d)^{-1}\sqrt{z}\eta(\frac{h'}{k}+\frac{\imath}{zk})^{-1} \quad \text{(R.118.3)}$$

$$\epsilon(a,b,c,d) = e^{\pi\imath(\frac{h-h'}{12k})-\pi\imath s(h,k)} \quad \text{(R.118.4)}$$

We take $hh' \equiv -1(modk)$ and note that $(h,k) = 1$, and we also take the definition of $\eta(\tau) = e^{\frac{\pi\imath\tau}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi\imath n\tau})$ as Dedekind's Eta Function.

Now, combining (R.118.1) and (R.118.3), we have

$$\begin{aligned}
e^{\frac{\pi\imath h}{12k}-\frac{\pi z}{12k}}\eta(\frac{h}{k}+\frac{\imath z}{k})^{-1} &= e^{\frac{\pi\imath h}{12k}-\frac{\pi z}{12k}}\epsilon(a,b,c,d)^{-1}\sqrt{z}\eta(\frac{h'}{k}+\frac{\imath}{zk})^{-1} \\
\eta(\frac{h'}{k}+\frac{\imath}{zk}) &= \sqrt{z}\eta(\frac{h}{k}+\frac{\imath z}{k})e^{\frac{\pi\imath h-\pi z-\pi\imath h+\pi z}{12k}}e^{-\pi\imath(\frac{h-h'}{12k})+\pi\imath s(h,k)} \qquad (3.4) \\
&= \sqrt{z}\eta(\frac{h}{k}+\frac{\imath z}{k})e^{(\frac{\pi\imath}{h-h'})(h'-h)}e^{\pi\imath s(h,k)} \\
&= \omega_{hk}e^{(\frac{\pi\imath}{h-h'})(h'-h)}\sqrt{z}\eta(\frac{h}{k}+\frac{\imath z}{k}) \qquad (3.5)
\end{aligned}$$

The rearrangement in line (3.4) is found by application of (R.118.4), and the final equality in line (3.5) is found by application of (R.118.42).

Combining the definition of $\eta(\tau)$ and Euler's Pentagonal Number Theorem (2.2.1), we see

$$\begin{aligned}\eta(\tau) &= e^{\frac{\pi\imath\tau}{12}}\prod_{1}^{\infty}(1-e^{2\pi\imath m\tau}) \qquad \text{(from (R.65.2))}\\ &= e^{\frac{\pi\imath\tau}{12}}\sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}e^{\pi\imath\lambda(3\lambda-1)\tau}\\ &= \sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}e^{3\pi\imath(\lambda-\frac{1}{6})^2\tau}\end{aligned}$$

This last equality comes from the fact that

$$\lambda(3\lambda-1)+\frac{1}{12}=3(\lambda^2-\frac{1}{3}\lambda+\frac{1}{36})=3(\lambda-\frac{1}{6})^2$$

Letting $\tau=\frac{h}{k}+\frac{\imath z}{k}$, $\lambda=2kq+j$ and $Re(z)>0$, we get

$$\eta(\frac{h}{k}+\frac{\imath z}{k})=\sum_{j=0}^{2k-1}(-1)^j e^{\frac{3\pi\imath h}{k}(j-\frac{1}{6})^2}\sum_{q=-\infty}^{\infty}e^{-\frac{3\pi z}{k}(2kq+j-\frac{1}{6})^2} \tag{3.6}$$

From (R.36.2), we see

$$\sum_{n=-\infty}^{\infty}e^{-\pi(n+\alpha)^2t}=\Psi(t,\alpha)=\frac{e^{-\pi\alpha^2t}}{\sqrt{t}}\Psi(\frac{1}{t},-\imath\alpha t)$$

and taking $t=12kz$ and $\alpha=\frac{1}{2k}(j-\frac{1}{6})$ in (3.6) and the above, we see that

$$\eta(\frac{h}{k}+\frac{\imath z}{k})=\frac{1}{2\sqrt{3kz}}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{3\pi\imath h}{k}(j-\frac{1}{6})^2}\sum_{m=-\infty}^{\infty}e^{\frac{-\pi m^2}{12kz}+\frac{\pi\imath m}{k}(j-\frac{1}{6})} \tag{3.7}$$

Applying (3.7) to the right hand side of (3.5), and (3.6) to the left hand side, and making appropriate cancellations, we may compare the coefficients for the first power of $e^{\frac{-\pi}{12kz}}$ to see

$$1 = \omega_{hk}\frac{1}{2\sqrt{3k}}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{\pi\imath h}{k}j(3j-1)}(e^{\frac{\pi\imath}{k}(j-\frac{1}{6})}+e^{\frac{-\pi\imath}{k}(j-\frac{1}{6})})$$

which, rearranging, gives us our final form of ω_{hk}:

$$\omega_{hk} = \frac{1}{2\sqrt{3k}}\sum_{j(mod2k)}(-1)^j e^{\frac{-\pi\imath h}{k}j(3j-1)}(e^{\frac{\pi\imath}{k}(j-\frac{1}{6})}+e^{\frac{-\pi\imath}{k}(j-\frac{1}{6})}) \tag{3.8}$$

We will now apply (3.8) to our definition of $A_k(n)$, where we let $\sum'_{h(modk)}$ denote $\sum_{\substack{h \bmod k\\(h,k)=1}}$:

$$\begin{aligned}
A_k(n) = &\sum_{h(modk)}{}' \omega_{hk}e^{\frac{-2\pi\imath hn}{k}}\\
=&\frac{1}{2\sqrt{3k}}\Big(\sum_{h(modk)}{}' e^{\frac{-2\pi\imath hn}{k}}\sum_{j(mod2k)}(-1)^j e^{-\frac{\pi\imath h}{k}j(3j-1)+\frac{\pi\imath}{k}(j-\frac{1}{6})}\\
&\qquad+\sum_{h(modk)}{}' e^{-\frac{2\pi\imath hn}{k}}\sum_{j(mod2k)}(-1)^j e^{-\frac{\pi\imath h}{k}j(3j-1)-\frac{\pi\imath}{k}(j-\frac{1}{6})}\Big)\\
=&\frac{1}{2\sqrt{3k}}\Big(e^{\frac{-\pi\imath}{6k}}\sum_{j(mod2k)}(-1)^j e^{\frac{\pi\imath j}{k}}\sum_{h(modk)}{}' e^{\frac{-2\pi\imath h}{k}(n+\frac{j(3j-1)}{2})}\\
&\qquad+e^{\frac{\pi\imath}{6k}}\sum_{j(mod2k)}(-1)^j e^{\frac{-\pi\imath j}{k}}\sum_{h(modk)}{}' e^{\frac{-2\pi\imath h}{k}(n+\frac{j(3j-1)}{2})}\Big)\\
=&\frac{\sqrt{k}}{2\sqrt{3}}\Big(e^{\frac{-\pi\imath}{6k}}\sum_{\substack{j(mod2k)\\ \frac{j(3j-1)}{2}\equiv -n(modk)}}(-1)^j e^{\frac{\pi\imath j}{k}}+e^{\frac{\pi\imath}{6k}}\sum_{\substack{j(mod2k)\\ \frac{j(3j-1)}{2}\equiv -n(modk)}}(-1)^j e^{-\frac{\pi\imath j}{k}}\Big)
\end{aligned} \tag{3.9}$$

The first equality is found by distribution, the second by rearrangement, and the last equality follows from the observation that the term of the inner sum is equal to one for each h. The third equality also gives us a congruence condition, namely

$$\frac{1}{2}j(3j-1) \equiv -n(mod k).$$

If the congruence fails to hold, then the sum over h vanishes.

Next, we note that

$$\begin{aligned} e^{\frac{\pi\imath j}{k}-\frac{\pi\imath}{6k}} + e^{-(\frac{\pi\imath j}{k}-\frac{\pi\imath}{6k})} &= e^{\pi\imath(\frac{j}{k}-\frac{1}{6k})} + e^{-\pi\imath(\frac{j}{k}-\frac{1}{6k})} \\ &= 2cos((\frac{j}{k}-\frac{1}{6k})\pi) \\ &= 2cos(\frac{(6j-1)\pi}{6k}), \end{aligned}$$

so we finally have

$$A_k(n) = \sqrt{\frac{k}{3}} \sum_{\substack{j(mod 2k) \\ \frac{j(3j-1)}{2} \equiv -n(mod k)}} (-1)^j cos(\frac{(6j-1)\pi}{6k}).$$

We may further rewrite $A_k(n)$. First, note the following equivalences:

$$\frac{1}{2}j(3j-1) \equiv -n(mod k)$$

$$12j(3j-1) \equiv -24n(mod 24k)$$

$$36j^2 - 12j + 1 \equiv -24n + 1(mod 24k), \qquad \text{and finally,}$$

$$(6j-1)^2 \equiv -24n + 1(mod 24k)$$

Now let $\nu = -24n + 1$. Then in (3.9) we have

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}}\Big(\sum_{\substack{j(mod 2k)\\(6j-1)^2 \equiv \nu (mod 24k)}} (-1)^j e^{\frac{\pi\imath}{6k}(6j-1)} + \sum_{\substack{j(mod 2k)\\(6j+1)^2 \equiv \nu (mod 24k)}} (-1)^j e^{\frac{\pi\imath}{6k}(6j+1)} \Big)$$

The first term is immediate; for the second term, we must use the substitution $2k - j$ in the place of j; note that this may be done as $(-1)^j = (-1)^{j+2k}$, so we have not changed parity. Thus, for the second term, we have:

$$\begin{aligned} e^{\frac{\pi\imath}{6k} - \frac{\pi\imath j}{k}} &= e^{\frac{\pi\imath}{6k}(1-6j)} \\ &= e^{\frac{\pi\imath}{6k}(1-6(2k-j))} \qquad \text{(applying our substitution)} \\ &= e^{\frac{\pi\imath}{6k}(6j+1)} e^{-2\pi\imath} \\ &= e^{\frac{\pi\imath}{6k}(6j+1)} \end{aligned}$$

Noting that, as j runs through the integers $mod 2k$, $6j$ runs through the integers $mod 12k$, we may also write

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \sum_{\substack{6j\pm1(mod 12k)\\(6j\pm1)^2 \equiv \nu (mod 24k)}} (-1)^j e^{\frac{\pi\imath}{6k}(6j\pm1)}$$

Next let $l = 6j \pm 1$ with $(l, 6) = 1$. Then $j = \frac{l\mp1}{6} = \{\frac{l}{6}\}$, as $j \in \mathbb{Z}$. Observe also that

$$(-1)^{\{\frac{l+12k}{6}\}}e^{\frac{\pi\imath(l+12k)}{6k}} = (-1)^{\{\frac{l}{6}\}+\{2k\}}e^{\frac{\pi\imath l}{6k}}e^{\frac{\pi\imath 12k}{6k}}$$

$$= (-1)^{\{\frac{l}{6}\}}e^{\frac{\pi\imath l}{6k}}$$

hence

$$\sum_{\substack{l(mod 12k)\\ l^2\equiv\nu(mod 24k)}} (-1)^{\{\frac{l}{6}\}}e^{\frac{\pi\imath l}{6k}} = \frac{1}{2}\sum_{\substack{l(mod 24k)\\ l^2\equiv\nu(mod 24k)}} (-1)^{\{\frac{l}{6}\}}e^{\frac{\pi\imath l}{6k}}$$

so we may finally write our final form of $A_k(n)$:

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}}\sum_{\substack{l(mod 24k)\\ l^2\equiv\nu(mod 24k)}} (-1)^{\{\frac{l}{6}\}}e^{\frac{\pi\imath l}{6k}}$$

□

3.2 Calculations of $A_k(n)$

3.2.1 Formulas for Calculating $A_k(n)$

Here we will compare the values obtained for $A_k(1)$ by using Rademacher's formula (3.3) and the equations found in Whiteman's paper [14], for specific values of k.

From Rademacher's formula (3.3)

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{12}}\sum_{\substack{x(mod 24k)\\ x^2\equiv -24n+1(mod 24k)}} (-1)^{\{\frac{x}{6}\}}e(\frac{x}{12k})$$

we will define

$$A^*_k(n) := \sum_{\substack{x(mod 24k)\\ x^2\equiv -24n+1(mod 24k)}} (-1)^{\{\frac{x}{6}\}} e(\frac{x}{12k})$$

We have also, due to Whiteman [14], the following (where $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol for $a, b \in \mathbb{Z}$):

For $k = p^\alpha$, $\alpha \geq 1$, $p > 3$ prime, and $\nu = 1 - 24n$,

$$A_k(n) = \begin{cases} 0 & \nu \text{ a non-residue of } k \\ 2\left(\frac{3}{k}\right) k^{\frac{1}{2}} cos(\frac{4\pi m}{k}) & \nu \equiv (24m)^2 (mod k) \\ 0 & \nu \equiv 0(mod p),\ \alpha > 1 \\ \left(\frac{3}{k}\right) k^{\frac{1}{2}} & \nu \equiv 0(mod p),\ \alpha = 1 \end{cases} \tag{3.10}$$

For $k = 3^\beta$, $\beta \geq 1$,

$$A_k(n) = 2(-1)^{\beta+1}\left(\frac{m}{3}\right)(\frac{k}{3})^{\frac{1}{2}} sin(\frac{4\pi m}{3k}) \tag{3.11}$$

for $m \in (Z)$ such that $(8m)^2 \equiv 1 - 24n(mod 3k)$

For $k = 2^\lambda$, $\lambda \geq 0$,

$$A_k(n) = (-1)^\lambda \left(\frac{-1}{m}\right) k^{\frac{1}{2}} sin(\frac{4\pi m}{8k}) \tag{3.12}$$

for $m \in (Z)$ such that $(3m)^2 \equiv 1 - 24n(mod 8k)$.

3.2.2 Examples

Example We have by Rademacher, for $k = 1$, we have $x = 1, 5, 7, 11, 13, 17, 19, 23$ satisfying $x^2 \equiv 1(mod24)$. Then

$$A_1^*(1) = \sum_{\substack{x(mod24)\\ x^2\equiv 1(mod24)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{6})} = 2e^{\frac{-\pi\imath}{6}} + 2e^{\frac{\pi\imath}{6}} - 2e^{\frac{-5\pi\imath}{6}} - 2e^{\frac{5\pi\imath}{6}} = 4\sqrt{3}$$

So

$$A_1(1) = \frac{1}{4}\sqrt{\frac{1}{3}}(4\sqrt{3}) = 1$$

And by Whiteman's equation (3.12), we have $\lambda = 0$ and $(3m)^2 \equiv -23mod8 \Rightarrow m = 3$, hence

$$A_1(1) = (-1)^0 \left(\frac{-1}{3}\right) (1)^{\frac{1}{2}} sin(\frac{12\pi}{8}) = (-1)sin(\frac{3\pi}{2}) = (-1)(-1) = 1$$

Example We have by Rademacher, for $k = 2$, $x = 5, 11, 13, 19, 29, 35, 37, 43$ satisfying $x^2 \equiv 25(mod48)$. Then

$$A_2^*(1) = \sum_{\substack{x(mod48)\\ x^2\equiv 25(mod48)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{12})} = -2e^{\frac{-5\pi\imath}{12}} - 2e^{\frac{5\pi\imath}{12}} + 2e^{\frac{-11\pi\imath}{12}} + 2e^{\frac{11\pi\imath}{12}} = -2\sqrt{6}$$

So

$$A_2(1) = \frac{1}{4}\sqrt{\frac{2}{3}}(-2\sqrt{6}) = -1.$$

By Whiteman's equation (3.12), we have $\lambda = 1$ and $(3m)^2 \equiv -23(mod16) \Rightarrow m = 1$, so

$$A_2(1) = (-1)^1 \left(\frac{-1}{1}\right) 2^{\frac{1}{2}} sin(\frac{4\pi}{16}) = (-1)(-1)^0\sqrt{2}\frac{1}{\sqrt{2}} = -1$$

Example We have by Rademacher, for $k = 3$, $x = 7, 11, 25, 29, 43, 47, 61, 65$ satisfying $x^2 \equiv 49(mod72)$. Then

$$A_3^*(1) = \sum_{\substack{x(mod72) \\ x^2 \equiv 49(mod72)}} (-1)^{\{\frac{x}{6}\}} e^{\pi \imath (\frac{x}{18})} = -2e^{\frac{-7\pi\imath}{18}} - 2e^{\frac{7\pi\imath}{18}} + 2e^{\frac{-11\pi\imath}{18}} + 2e^{\frac{11\pi\imath}{18}} = -8sin(\frac{\pi}{9})$$

So

$$A_3(1) = \frac{1}{4}\sqrt{\frac{3}{3}}(-8sin(\frac{\pi}{9})) = -2sin(\frac{\pi}{9})$$

By Whiteman's equation (3.11), we have $\beta = 1$ and $(8m)^2 \equiv 4(mod9) \Rightarrow m = 2$, so

$$A_3(1) = 2(-1)^2 \left(\frac{2}{3}\right) (\frac{3}{3})^{\frac{1}{2}} sin(\frac{8\pi}{9}) = 2(2(mod3))sin(\frac{8\pi}{9}) = -2sin(\frac{\pi}{9})$$

Example For the case of $k = 5$, there are no solutions to the congruence $x^2 \equiv 97(mod120)$ since there are no solutions to $x^2 \equiv 97 \equiv 2(mod5)$, hence $A_5(1) = 0$ by Rademacher. Similarly by equation (3.10) of Whiteman, as 97 is a non-residue of 5, $A_5(1) = 0$.

Example For the case of $k = 7$, there are no solutions to the congruence $x^2 \equiv 145(mod168)$ since there are no solutions to $x^2 \equiv 145 \equiv 5(mod7)$, hence $A_7(1) = 0$ by Rademacher. Similarly by equation (3.10) of Whiteman, as 145 is a non-residue of 7, $A_7(1) = 0$.

Example For the case of $k = 11$, there are no solutions to the congruence $x^2 \equiv 241(mod 264)$ since there are no solutions to $x^2 \equiv 241 \equiv 5(mod 11)$, hence $A_{11}(1) = 0$ by Rademacher. Similarly by equation (3.10) of Whiteman, as 241 is a non-residue of 11, $A_{11}(1) = 0$.

Example We have by Rademacher, for $k = 13$, $x = 17, 35, 43, 61, 95, 113, 121, 139, 173, 191, 199, 217, 251, 269, 277, 295$ satisfying $x^2 \equiv 289(mod 312)$. Then

$$\begin{aligned} A^*_{13}(1) = & \sum_{\substack{x(mod 312) \\ x^2 \equiv 289(mod 312)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{78})} \\ = & -2e^{\frac{-17\pi\imath}{78}} - 2e^{\frac{17\pi\imath}{78}} + 2e^{\frac{-35\pi\imath}{78}} + 2e^{\frac{35\pi\imath}{78}} - 2e^{\frac{-43\pi\imath}{78}} - 2e^{\frac{43\pi\imath}{78}} + 2e^{\frac{-61\pi\imath}{78}} + 2e^{\frac{61\pi\imath}{78}} \\ = & 8sin(\frac{2\pi}{39}) - 8cos(\frac{17\pi}{78}) \end{aligned}$$

So

$$A_{13}(1) = \sqrt{\frac{13}{3}}(2sin(\frac{2\pi}{39}) - 2cos(\frac{17\pi}{78}))$$

By Whiteman's equation (3.10), we have $\alpha = 1$, $k = 13$, $n = 1$, and $\nu = -23$, along with $(24m)^2 \equiv 3(mod 13) \Rightarrow m = 1$, so

$$A_{13}(1) = 2\left(\frac{3}{13}\right) 13^{\frac{1}{2}} cos(\frac{44\pi}{13}) = 2\sqrt{13}(3^6(mod 13))cos(\frac{44\pi}{13}) = 2\sqrt{13}cos(\frac{44\pi}{13})$$

It is easy to check on a computer system, such as Mathematica, that these two are equivalent.

Example For the case of $k = 17$, there are no solutions to the congruence $x^2 \equiv 385(mod 408)$ since there are no solutions to $x^2 \equiv 385 \equiv 11(mod 17)$, hence $A_{17}(1) = 0$

by Rademacher. Similarly by equation (3.10) of Whiteman, as 385 is a non-residue of 17, $A_{17}(1) = 0$.

Example For the case of $k = 19$, there are no solutions to the congruence $x^2 \equiv 433(mod456)$ since there are no solutions to $x^2 \equiv 433 \equiv 15(mod19)$, hence $A_{19}(1) = 0$ by Rademacher. Similarly by equation (3.10) of Whiteman, as 433 is a non-residue of 19, $A_{19}(1) = 0$.

Example We have by Rademacher, for $k = 23$, $x = 23, 115, 161, 253, 299, 391, 437, 529$ satisfying $x^2 \equiv 529(mod552)$. Then

$$A^*_{23}(1) = \sum_{\substack{x(mod552)\\ x^2 \equiv 529(mod552)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{138})} = 2e^{\frac{-\pi\imath}{6}} + 2e^{\frac{\pi\imath}{6}} - 2e^{\frac{-5\pi\imath}{6}} - 2e^{\frac{5\pi\imath}{6}} = 4\sqrt{3}$$

So

$$A_{23}(1) = \frac{1}{4}\sqrt{\frac{23}{3}}(4\sqrt{3}) = \sqrt{23}$$

By Whiteman's equation (3.10), we have $\alpha = 1$, $k = 23$, $n = 1$, and $\nu = -23 \equiv 0(mod23)$, so

$$A_{23}(1) = \left(\frac{3}{23}\right) 23^{\frac{1}{2}} = (3^{11}(mod23))\sqrt{23} = \sqrt{23}$$

Example We have by Rademacher, for $k = 29$, $x = 37, 79, 95, 137, 211, 253, 269, 311, 385, 427, 443, 485, 559, 601, 617, 659$ satisfying $x^2 \equiv 673(mod696)$. Then

$$A^*_{29}(1) = \sum_{\substack{x(mod696)\\ x^2\equiv 673(mod696)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{174})}$$

$$=2e^{\frac{-37\pi\imath}{174}} + 2e^{\frac{37\pi\imath}{6=174}} - 2e^{\frac{-79\pi\imath}{174}} - 2e^{\frac{79\pi\imath}{174}} + 2e^{\frac{-95\pi\imath}{174}} + 2e^{\frac{95\pi\imath}{174}} - 2e^{\frac{-137\pi\imath}{174}} - 2e^{\frac{137\pi\imath}{174}}$$

$$=8cos(\frac{37\pi}{174}) - 8sin(\frac{4\pi}{87})$$

So

$$A_{29}(1) = \sqrt{\frac{29}{3}}(2cos(\frac{37\pi}{174}) - 2sin(\frac{4\pi}{87}))$$

By Whiteman's equation (3.10), we have $\alpha = 1$, $k = 29$, $n = 1$, and $\nu = -23$, along with $(24m)^2 \equiv 6(mod29) \Rightarrow m = 10$, so

$$A_{29}(1) = 2\left(\frac{3}{29}\right) 29^{\frac{1}{2}} cos(\frac{40\pi}{29}) = 2\sqrt{29}(3^{14}(mod29))cos(\frac{40\pi}{29}) = -2\sqrt{29}cos(\frac{40\pi}{29})$$

It is easy to check that these are equivalent using a program such as Mathematica.

Example We have by Rademacher, for $k = 31$, $x = 47, 77, 109, 139, 233, 263, 295, 325, 419, 449, 481, 511, 605, 635, 667, 696$ satisfying $x^2 \equiv 721(mod744)$. Then

$$A^*_{31}(1) = \sum_{\substack{x(mod744)\\ x^2\equiv 721(mod744)}} (-1)^{\{\frac{x}{6}\}} e^{\pi\imath(\frac{x}{186})}$$

$$=2e^{\frac{-47\pi\imath}{186}} + 2e^{47\frac{\pi\imath}{186}} - 2e^{\frac{-77\pi\imath}{186}} - 2e^{\frac{77\pi\imath}{186}} + 2e^{\frac{-109\pi\imath}{186}} + 2e^{\frac{109\pi\imath}{186}} - 2e^{\frac{-139\pi\imath}{186}} - 2e^{139\frac{\pi\imath}{186}}$$

$$=8sin(\frac{23\pi}{93}) - 8sin(\frac{8\pi}{93})$$

So

$$A_{31}(1) = \sqrt{\frac{31}{3}}(2sin(\frac{23\pi}{93}) - 2sin(\frac{8\pi}{93}))$$

By Whiteman's equation (3.10), we have $\alpha = 1$, $k = 31$, $n = 1$, and $\nu = -23$, along with $(24m)^2 \equiv 8(mod 31) \Rightarrow m = 20$, so

$$A_{31}(1) = 2\left(\frac{3}{31}\right)\sqrt{31}cos(\frac{80\pi}{31}) = 2\sqrt{31}(3^{15}(mod 31))cos(\frac{80\pi}{31}) = -2\sqrt{31}cos(\frac{80\pi}{31})$$

It is easy to check that these two are equivalent using a program such as Mathematica.

Chapter 4

ARITHMETIC REFORMULATION OF $p(n)$

In this chapter we will explore the process used in the arithmetic reformulation of $p(n)$ from an infinite sum into a finite sum, which will be useful for more practical comutational purposes.

4.1 Definitions

Before reformulating, we will take note of the following definitions and notation.

Notation We let $\{x\}$ denote the nearest integer to x.

Notation We let $e(x)$ denote $e^{2\pi\imath x}$.

Notation We let $\left(\frac{r}{s}\right)$ denote the Jacobi symbol for $r, s \in \mathbb{Z}$.

Definition Let $\mathcal{Q}_d^{(p)}$ denote the set of positive definite integral binary quadratic forms of discriminant $-d = b^2 - 4ac$, with $d > 0$ and where $6|a$.

Definition Let $Q(x, y) := ax^2 + bxy + cy^2 \in \mathcal{Q}_d^{(p)}$. We denote $Q(x, y) = [a, b, c](x, y)$. For every such Q, there is a unique complex number τ_Q in the upper half plane such that $Q(\tau_Q, 1) = 0$. We call this τ_Q a *complex multiplication point*, or *CM point*.

Definition We will let $\Gamma_0(6)$ be the group of matrices of the form

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \; 6|\gamma, \; det(M) = 1, \; \alpha, \beta, \gamma, \delta \in \mathbb{Z}$$

Definition The subgroup $\Gamma_\infty \subset \Gamma_0(6)$ is the set of matrices of the form

$$\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \; n \in \mathbb{Z}$$

Definition Let $M \in \Gamma_0(6)$ and $Q \in \mathcal{Q}_d^{(p)}$. We define the right group action of $\Gamma_0(6)$ on $\mathcal{Q}_d^{(p)}$ by

$$Q \circ M := [a, b, c](\alpha x + \beta y, \gamma x + \delta y) =: [a', b', c']$$

Remark $SL_2(\mathbb{Z})$ acts on the upper-half plane by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \; z \in \text{upper-half plane }, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

It is easily verified that this is a group action, namely (for $A, B \in SL_2(\mathbb{Z})$),

$$A(Bz) = (AB)z$$

$$Iz = z \text{ where } I \text{ is the identity matrix}$$

For example, the fact that Az belongs to the upper-half plane is a consequence of Claim (4.2.1) below.

Definition Γ_{τ_Q} is the *isotropy subgroup* fixing the CM point τ_Q.

Definition We define $\chi_{12}([a, b, c]) := \left(\frac{12}{b}\right)$.

4.2 Proofs of Useful Claims

Here we will prove some preliminary claims that will be of use in reformulating $p(n)$.

Claim 4.2.1. *The group* $\Gamma_0(6)$ *keeps the CM point* τ *in the upper half plane.*

Proof. We assume $Im(\tau) > 0$. Then

$$\begin{aligned} Im(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) &= \frac{Im(\alpha\tau+\beta)\overline{(\gamma\tau+\delta)}}{|\gamma\tau+\delta|^2} \\ &= \frac{Im(\alpha\tau+\beta)(\overline{\gamma\tau}+\overline{\delta})}{|\gamma\tau+\delta|^2} \\ &= \frac{Im(\alpha\overline{\gamma}\tau\overline{\tau}+\alpha\overline{\delta}\tau+\beta\overline{\gamma\tau}+\beta\overline{\delta})}{|\gamma\tau+\delta|^2} \end{aligned}$$

Note that as $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, we have $Im(\alpha\overline{\gamma}\tau\overline{\tau}) = 0$ and $Im(\beta\overline{\delta}) = 0$. Then in the numerator we are left with

$$\begin{aligned} Im(\alpha\overline{\delta}\tau+\beta\overline{\gamma\tau}) &= Im(\alpha\delta\tau-\beta\gamma\tau) \\ &= \alpha\delta Im(\tau) - \beta\gamma Im(\tau) \\ &= (\alpha\delta-\beta\gamma)Im(\tau) \\ &= Im(\tau), \qquad (\text{as } \alpha\delta-\beta\gamma = 1). \end{aligned}$$

Hence $Im(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) = \frac{Im(\tau)}{|\gamma\tau+\delta|^2} > 0$. □

Claim 4.2.2. *The action of* $\Gamma_0(6)$ *on* $Q(x,y) = [a,b,c](x,y)$ *preserves the residue class of* $b(mod12)$*, i.e.* $b \equiv b'(mod12)$ *in* $Q \circ M := [a,b,c](\alpha x+\beta y, \gamma x+\delta y) := [a',b',c']$.

Proof. Recall that $6|a$, $6|\gamma$, and that $Q = ax^2 + bxy + cy^2$ as defined. Then

$$\begin{aligned} Q \circ M &= a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\gamma x + \delta y) + c(\gamma x + \delta y)^2 \\ &= (a\alpha^2 + b\alpha\gamma + c\gamma^2)x^2 + (2a\alpha\beta + b\alpha\delta + b\beta\gamma + 2c\gamma\delta)xy + (a\beta^2 + b\beta\delta + c\delta^2)y^2 \end{aligned}$$

so then we see

$$\begin{aligned} b' &= 2a\alpha\beta + b\alpha\delta + b\beta\gamma + 2c\gamma\delta \\ &\equiv b(\alpha\delta + \beta\gamma)(mod 12) \\ &\equiv b(1 + 2\beta\gamma)(mod 12) \qquad \text{(note: } \alpha\delta - \beta\gamma = 1) \\ &= b(mod 12) \end{aligned}$$

□

Claim 4.2.3. *Let $Q(x, y) = ax^2 + bxy + cy^2$. Then the imaginary part of the CM point is preserved under the action of $M \in \Gamma_\infty$.*

Proof. Let τ_Q be the CM point for $Q(x, y)$. Then $Q(x, 1) = ax^2 + bx + c = 0$ gives us $\tau_Q = \frac{-b+\sqrt{b^2-4ac}}{2a}$, thus $Im(\tau_Q) = \frac{\sqrt{b^2-4ac}}{2a}$. Note that $\gamma = 0$ in Γ_∞, hence

$$\begin{aligned} Q \circ M &= [a, b, c](\alpha x + \beta y, \gamma x + \delta y) \\ &= (a\alpha^2)x^2 + (2a\alpha\beta + b\alpha\delta)xy + (a\beta^2 + b\beta\delta + c\delta^2)y^2. \end{aligned}$$

We let τ_{QM} be the CM point of $Q \circ M$, so it is the upper half plane solution to

$$Q \circ M(x,1) = (a\alpha^2)x^2 + (2a\alpha\beta)x + (a\beta^2 + b\beta\delta + c\delta^2) = 0.$$

We then see that

$$\tau_{QM} = \frac{-(2a\alpha\beta + b\alpha\delta) + \sqrt{-d}}{2(a\alpha^2)}$$

where

$$-d = (2a\alpha\beta + b\alpha\delta)^2 - 4(a\alpha^2)(a\beta^2 + b\beta\delta + c\delta^2)$$

Then $-d$ reduces to $b^2(\alpha\delta)^2 - 4ac(\alpha\delta)^2 = b^2 - 4ac$ as $det(M) = 1$ in Γ_∞. Further, the denominator reduces to $2a\alpha^2 = 2a$ as $\alpha = \pm 1$, hence $\alpha^2 = 1$.

Thus we have $Im(\tau_{QM}) = \frac{\sqrt{b^2-4ac}}{2a} = Im(\tau_Q)$ as desired. □

Claim 4.2.4. *The coefficient b of the integral binary quadratic form $Q(x,y) = 6kx^2 + bxy + cy^2$ with discriminant $-24n+1$ satisfies the congruence $b^2 \equiv -24n+1 (mod 24k)$.*

Proof. We know $-24n+1 = b^2 - 4ac$, hence $b^2 = -24n+1+4(6k)(c) = -24n+1+24kc$, which is equivalent to $b^2 \equiv -24n+1(mod 24k)$. □

Claim 4.2.5. *The group Γ_∞ preserves $b^2 \equiv -24n+1(mod 24k)$ for $Q(x,y) = 6kx^2 + bxy + cy^2$.*

Proof. Assume $Q(x,y) = 6kx^2 + bxy + cy^2$ with $b^2 \equiv -24n + 1(mod24k)$, and let $M \in \Gamma_\infty$. Then

$$\begin{aligned} Q \circ M &= [6k, b, c](\alpha x + \beta y, \delta y) \\ &= 6k(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\delta y) + c(\delta y)^2 \\ &= (6k\alpha^2)x^2 + (6k(2\alpha\beta) + b\alpha\delta)xy + (6k\beta^2 + b\beta\delta + c\delta)y^2. \end{aligned}$$

Consider $b' = 12k\alpha\beta + b\alpha\delta$. Then

$$\begin{aligned} (b')^2 &= 24k(6k\alpha^2\beta^2) + 24k(b\alpha^2\beta\delta) + b^2(\alpha^2\delta^2) \\ &\equiv b^2(\alpha^2\delta^2)(mod24k) \\ &\equiv b^2(mod24k) \\ &\equiv -24n + 1(mod24k). \end{aligned}$$

□

Claim 4.2.6. *The group Γ_∞ does not preserve $b(mod24k)$ but does identify the congruence classes $b, b + 12k(mod24k)$.*

Proof. As above, let $M \in \Gamma_\infty$. Then the middle coefficient of $Q \circ M$ is $b' = 12k\alpha\beta + b\alpha\delta$. Recall that $\alpha = \delta = \pm 1$ as $\alpha\delta = 1$ in Γ_∞, hence $b' = \pm 12k\beta + b$.

Then if $2|\beta$, we have $b' \equiv b(mod24k)$.

If $2 \not| \beta$, then β is odd and we may write $\beta = 2l+1, l \in \mathbb{Z}$. Then

$$\begin{aligned}\pm 12k(2l+1) + b &= \pm(24kl + 12k) + b \\ &\equiv \pm 12k + b (mod 24k) \not\equiv b(mod 24k).\end{aligned}$$

□

Claim 4.2.7. *Given* $l \in \mathbb{Z}$ *where* $(l,6) = 1$, $\left(\frac{3}{l}\right) = (-1)^{\{\frac{l}{6}\}}$.

Proof. Given $(-1)^{\{\frac{l}{6}\}}, (l,6) = 1$, we have Table (4.1),

l	1	5	7	11
$\{\frac{l}{6}\}$	0	1	1	2
$(-1)^{\{\frac{l}{6}\}}$	1	-1	-1	1

Table 4.1. $-1^{\{\frac{l}{6}\}}$ for given values of l

Then since

$$\begin{aligned}(-1)^{\{\frac{l+12}{6}\}} &= (-1)^{\{\frac{l}{6}\}+\{\frac{12}{6}\}} \\ &= (-1)^{\{\frac{l}{6}\}}(-1)^2 \\ &= (-1)^{\{\frac{l}{6}\}}\end{aligned}$$

it is clear from Table (4.1) that $(-1)^{\{\frac{l}{6}\}}$ is periodic, with period 12.

Now note that by quadratic reciprocity,

$$\begin{aligned}\left(\frac{3}{l}\right) &= \left(\frac{l}{3}\right)(-1)^{\frac{l-1}{2}\frac{3-1}{2}} \\ &= \left(\frac{l}{3}\right)(-1)^{\frac{l-1}{2}} = \left(\frac{l}{3}\right)(\frac{-1}{l})\end{aligned}$$

Taking in this case $(l, 6) = 1$, we have Table (4.2).

l	1	5	7	11
$\left(\frac{l}{3}\right)$	1	-1	1	-1
$\left(\frac{-1}{l}\right)$	1	1	-1	-1
$\left(\frac{3}{l}\right)$	1	-1	-1	1

Table 4.2. Legendre values for l

Since

$$\begin{aligned}\left(\frac{3}{l+12}\right) &= \left(\frac{l+12}{3}\right)(-1)^{(\frac{3-1}{2})(\frac{(l+12)-1}{2})} \quad \text{(where } l = 4k+1,\ k \in \mathbb{Z} \text{ since } (l,6) = 1) \\ &= \left(\frac{l+12}{3}\right)(-1)^{2k} \\ &= \left(\frac{l}{3}\right) \qquad \text{(since } l + 12 \equiv l(mod 3)) \\ &= \left(\frac{3}{l}\right)(-1)^{(\frac{3-1}{2})(\frac{l-1}{2})} \\ &= \left(\frac{3}{l}\right)(-1)^{2k} \qquad \text{(again by } l = 4k+1,\ k \in \mathbb{Z}) \\ &= \left(\frac{3}{l}\right)\end{aligned}$$

we see that $\left(\frac{3}{l}\right)$ has period 12, hence for $(l, 6) = 1$, we have $(-1)^{\{\frac{l}{6}\}} = \left(\frac{3}{l}\right)$. □

Remark We may thus write, from (3.3),

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\substack{l(mod 24k) \\ l^2 \equiv v(mod 24k)}} (-1)^{\{\frac{l}{6}\}} e^{\frac{\pi i l}{6k}}$$

$$= \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\substack{l(mod 24k) \\ l^2 \equiv v(mod 24k)}} \left(\frac{3}{l}\right) e^{\frac{\pi i l}{6k}}.$$

Claim 4.2.8. *$\chi_{12}(Q)$ is fixed under $\Gamma_0(6)$.*

Proof. Let $M \in \Gamma_0(6)$, and $Q(x,y) = ax^2 + bxy + cy^2$. Then

$$Q \circ M(x,y) = (a\alpha^2 + b\alpha\gamma + c\gamma^2)x^2 + (2a\alpha\beta + b(\gamma\beta + \delta\alpha) + 2c\gamma\delta)xy + (a\beta^2 + b\beta\delta + c\delta^2)y^2$$

so, recalling $a = 6k$ and $\gamma = 6l$ for $k, l \in \mathbb{Z}$, along with $\alpha\delta - \beta\gamma = 1$, we have

$$b' = 2a\alpha\beta + b(\gamma\beta + \delta\alpha) + 2c\gamma\delta$$

$$= 2(6k)\alpha\beta + b(1 + 2\beta(6l)) + 2c(6l)\delta$$

$$= 12(k\alpha\beta + cl\delta) + b + 12bl\delta$$

$$= 12(k\alpha\beta + cl\delta + bl\beta) + b$$

$$= b + 12q, \qquad q = k\alpha\beta + cl\delta_b l\beta \in \mathbb{Z}.$$

Now, since $b' = b + 12q$ and as we have shown above that $\chi_{12}(Q) = (-1)^{\{\frac{b}{6}\}}$, and the latter is of period 12, it follows that $\chi_{12}(Q \circ M) = (-1)^{\{\frac{b'}{6}\}} = (-1)^{\{\frac{b}{6}\}}$, and is thus fixed under the action of $\Gamma_0(6)$. □

Lemma 4.2.9. *We parametrize the solutions of* $b^2 - 24kc = -24n + 1$ *(for* $k, b, c \in \mathbb{Z}$ *and* $k, c > 0$*) by the tuples* (b, c, k)*. Then there is a one-to-one mapping between the tuples* (b, c, k) *and points*

$$z \in \bigcup_{Q \in \mathcal{Q}^{(p)}_{24n-1}/\Gamma_0(6)} \{A\tau_Q : A \in \Gamma_0(6)/\Gamma_{\tau_Q}\} =: \mathcal{A}.$$

Proof. We define a map $\Phi(b, c, k) = \tau_Q$ where $Q = 6kx^2 + bxy + cy^2$. Observe that the discriminant of Q is $b^2 - 24kc = -24n + 1$, so $Q \in \mathcal{Q}^{(p)}_{24n-1}$, hence $\tau_Q \in \mathcal{A}$.

Now, given $z \in \mathcal{A}$, there is a $Q \in \mathcal{Q}^{(p)}_{24n-1}$ and $A \in \Gamma_0(6)$ such that $z = A\tau_Q$, by the definition of $\mathcal{A}$. We have previously shown $A\tau_Q$ is in the upper half plane by Claim (4.2.1) and have $A\tau_Q = \tau_{Q\circ A^{-1}} = \tau_{Q'}$ from linear algebra, where $Q \circ A^{-1} = Q' = 6kx^2 + bxy + cy^2$ for some $b, c, k \in \mathbb{Z}$, and without loss of generality, $k, c > 0$. We define $\Phi^{-1}(\tau_{Q'}) = (b, c, k)$. Then

$$\Phi \circ \Phi^{-1}(A\tau_Q) = \Phi \circ \Phi^{-1}(\tau_{Q'}) = \Phi(b, c, k) = \tau_{Q'} = A\tau_Q$$

and

$$\Phi^{-1} \circ \Phi(b, c, k) = \Phi^{-1}(\tau_Q) = (b, c, k).$$

Hence Φ is a bijective mapping from (b, c, k) to $\mathcal{A}$ as desired. □

4.3 Arithmetic Reformulation

Proposition 4.3.1. (Proposition 4.2 from [11])

If k *and* n *are positive integers, then we have*

$$A_k(n) = \sqrt{\frac{k}{3}} \sum_{Q \in \mathcal{Q}^{(p)}_{24n-1}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} \sum_{\substack{A \in \Gamma_\infty \backslash \Gamma_0(6) \\ Im(A\tau_Q) = \frac{\sqrt{24n-1}}{12k}}} e(-Re(A\tau_Q)).$$

Proof. Recall our earlier definition of $A_k(n)$, due to Rademacher:

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\substack{x(mod24k) \\ x^2\equiv(-24n+1)(mod24k)}} (-1)^{\{\frac{x}{6}\}} e(\frac{x}{12k}) \quad (3.1)$$

We have previously shown in (4.2.3) that $Im(A\tau_Q) = \frac{\sqrt{-b^2+4ac}}{2a} = \frac{\sqrt{24n-1}}{12k}$, and in (4.2.7) that $\chi_{12}(Q) = \left(\frac{12}{b}\right) = \left(\frac{3}{b}\right)\left(\frac{2}{b}\right)^2 = \left(\frac{3}{b}\right) = (-1)^{\{\frac{x}{6}\}}$. We have $-Re(A\tau_Q) = -(\frac{-b}{12k}) = \frac{b}{12k}$ immediately from (4.2.2).

Note that by taking $A \in \Gamma_\infty \backslash \Gamma_0(6)$, we are only considering those matrices that preserve both the imaginary part of our CM point and $b(mod24k)$, thus fixing τ_Q $\#\Gamma_{\tau_Q}$ times. However, $\#\Gamma_{\tau_Q}$ (the *stabilizer for* τ_Q) also includes $\pm I$ for b, and also considers $b + 12k$ with $\pm I$ for $b + 12k$ as well. Hence we are dividing by an extra factor of four, since $\pm I \in \Gamma_\infty$ (which we are modding out by). □

Remark This is called the *arithmetic reformulation* as we are removing the congruence conditions, and instead considering group actions.

Definition Given $f(z)$ a $\Gamma_0(6)$ invariant function, we define the *trace* $Tr^{(p)}(f;n)$ to be

$$Tr^{(p)}(f;n) := \sum_{Q\in\mathcal{Q}^{(p)}_{24n-1}/\Gamma_0(6)} \frac{\chi_{12}(Q)f(\tau_Q)}{\#\Gamma_{\tau_Q}}. \quad (4.1)$$

Observe that this is well-defined as by (4.2.2), we have shown $b \equiv b'(mod12)$.

Definition We define a *Poincaré series* $P(z)$ by

$$P(z) := 4\pi \sum_{A\in\Gamma_\infty\backslash\Gamma_0(6)} Im(Az)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi Im(Az))e(-Re(Az)). \tag{4.2}$$

Remark $P(z)$ is absolutely convergent and holomorphic on the upper half of the complex plane.

Claim 4.3.2. *Let $M \in \Gamma_0(6)$. Then $P(Mz) = P(z)$ for all z in the upper half of the complex plane, meaning $P(z)$ is $\Gamma_0(6)$ invariant.*

Proof. Let $f(z) = Im(Az)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi Im(Az))e(-Re(Az))$. By definition,

$$P(Mz) = 4\pi \sum_{A\in\Gamma_\infty\backslash\Gamma_0(6)} f(A(Mz))$$

Let $A' = AM$, where as A ranges over $\Gamma_\infty\backslash\Gamma_0(6)$, so does A'. Then

$$= 4\pi \sum_{A\in\Gamma_\infty\backslash\Gamma_0(6)} f(A'z)$$

$$= P(z)$$

□

Theorem 4.3.3. (Theorem 1.2 from [11])

If n is a positive integer, then

$$p(n) = \frac{Tr^{(p)}(P;n)}{24n-1}.$$

In particular, we have that

$$p(n) \equiv -Tr^{(p)}(P;n)(mod 24)$$

Proof. If we combine Theorem (3.1) with Proposition (4.3.1), we see that

$$\begin{aligned} p(n) =& 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{1}{k}\sqrt{\frac{k}{3}} \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} \\ & \sum_{\substack{A \in \Gamma_\infty \backslash \Gamma_0(6) \\ Im(A\tau_Q) = \frac{\sqrt{24n-1}}{12k}}} e(-Re(A\tau_Q)) I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \\ =& \frac{2\pi}{\sqrt{3}}(24n-1)^{-\frac{3}{4}} \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \\ & \sum_{\substack{A \in \Gamma_\infty \backslash \Gamma_0(6) \\ Im(A\tau_Q) = \frac{\sqrt{24n-1}}{12k}}} e(-Re(A\tau_Q)) I_{\frac{3}{2}}(2\pi Im(A\tau_Q)) \end{aligned}$$

Now we let $Im(A\tau_Q)^{\frac{1}{2}} = \frac{(24n-1)^{\frac{1}{4}}}{2\sqrt{3k}}$, so that we have $\frac{1}{\sqrt{k}} = \frac{2\sqrt{3}Im(A\tau_Q)^{\frac{1}{2}}}{(24n-1)^{\frac{1}{4}}}$. This will remove our dependence on k, and we may now rewrite

$$\begin{aligned} p(n) =& \frac{2\pi}{\sqrt{3}}(24n-1)^{-\frac{3}{4}} \frac{2\sqrt{3}}{(24n-1)^{\frac{1}{4}}} \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} \\ & \sum_{A \in \Gamma_\infty \backslash \Gamma_0(6)} e(-Re(A\tau_Q)) I_{\frac{3}{2}}(2\pi Im(A\tau_Q) Im(A\tau_Q)^{\frac{1}{2}} \\ =& \frac{4\pi}{24n-1} \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} \\ & \sum_{A \in \Gamma_\infty \backslash \Gamma_0(6)} e(-Re(A\tau_Q)) I_{\frac{3}{2}}(2\pi Im(A\tau_Q) Im(A\tau_Q)^{\frac{1}{2}} \end{aligned}$$

$$=(24n-1)^{-1} \sum_{Q \in \mathcal{Q}^{(p)}_{24n-1}/\Gamma_0(6)} \frac{\chi_{12}(Q)}{\#\Gamma_{\tau_Q}} P(\tau_Q) \qquad \text{(by (4.2))}$$

$$=(24n-1)^{-1} Tr^{(p)}(P;n) \qquad \text{(by (4.1))}$$

$\square$

Chapter 5

ALGEBRAIC REFORMULATION OF $p(n)$

Ono and Bruinier further reformulated $p(n)$ ([10]); we will briefly discuss this algebraic reformulation and use it to compute cases of $p(n)$, but must first discuss the formulation of the Poincaré Series used in defining $p(n)$. We will also see a brief discussion of some applications of the partition function.

5.1 Reformulation of the Poincaré Series

We have now shown that

$$p(n) = \frac{Tr^{(p)}(P;n)}{24n-1}, \qquad Tr^{(p)}(P;n) = \sum_{Q \in \mathcal{Q}^{(p)}_{24n-1}} P(\tau_Q)$$

where

$$P(z) := 4\pi \sum_{A \in \Gamma_\infty \backslash \Gamma_0(6)} Im(Az)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi Im(Az)) e(-Re(Az)). \tag{5.1}$$

Bruinier and Ono reformulated $P(z)$ (see [10]) by letting $q = e^{2\pi \imath z}$ and taking Dedekind's Eta Function

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{5.2}$$

and the quasimodular Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \tag{5.3}$$

where $\sigma(n)$ denotes as usual the sum of the divisors of n. Then we define

$$F(z) := \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} \tag{5.4}$$

Expanding the numerator of (5.4) and dividing by two, we get:

$$\begin{aligned}&\frac{1}{2}(E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z))\\ &= 1 - 12(\sum_{n=1}^{\infty}\sigma(n)q^n - 2\sum_{n=1}^{\infty}\sigma(n)q^{2n} - 3\sum_{n=1}^{\infty}\sigma(n)q^{3n} + 6\sum_{n=1}^{\infty}\sigma(n)q^{6n})\end{aligned}$$

Expanding the denominator of (5.4), we get

$$\begin{aligned}&\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2\\ =&(q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n))^2(q^{\frac{2}{24}}\prod_{n=1}^{\infty}(1-q^{2n}))^2(q^{\frac{3}{24}}\prod_{n=1}^{\infty}(1-q^{3n}))^2\cdot\\ &\qquad (q^{\frac{6}{24}}\prod_{n=1}^{\infty}(1-q^{6n}))^2\\ =&q\big((\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}})(\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)})(\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{3k(3k-1)}{2}})\cdot\\ &\qquad (\sum_{k=-\infty}^{\infty}(-1)^k q^{3k(3k-1)})\big)^2\end{aligned}$$

where this last equivalence is due to the Pentagonal Number Theorem (2.2.1).

We may then see that

$$F(z) = \frac{1 - 12q - 12q^2 - 12q^3 - 12q^4 - 72q^5 - 12q^6 - 96q^7 - 12q^8 - 12q^9 - ...}{q(1 - 2q - 3q^2 + 4q^3 + 6q^4 + 6q^5 - 16q^6 - 8q^7 + 9q^8 - 12q^9 + ...)}$$

and hence, by long division,

$$F(z) = \frac{1}{q} - 10 - 29q - 104q^2 - 273q^3 - 760q^4 - 1685q^5 - 4008q^6 - 8334q^7 - 17560q^8 - 34563q^9 - ... - 12082000967090q^{55} - ... - 249876940038754420 0q^{100} - ... \tag{5.5}$$

This was extended to the one hundredth power for computational purposes (see Appendix B for the full expansion).

Theorem 5.1.1. *Given $z = x + \imath y$, $x, y \in \mathbb{R}$, $P(z)$ from equation (5.1) equals*

$$P(z) = -(\frac{1}{2\pi\imath}\frac{d}{dz} + \frac{1}{2\pi y})F(z).$$

Remark See [4] and [11] for further discussion of the reformulation of $P(z)$.

Now by our expansion of $F(z)$, we see

$$P(z) = (1-\frac{1}{2\pi y})\frac{1}{q}+\frac{5}{\pi y}+(29+\frac{29}{2\pi y})q+(208+\frac{52}{\pi y})q^2+(819+\frac{273}{2\pi y})q^3+(3040+\frac{380}{\pi y})q^4+...$$

We may calculate $p(n)$ using this algebraic reformulation, called thus as it is now described by taking a finite sum of algebraic numbers. The partition function is still described as before, but now taking our reformulation of the Poincaré series in

$$p(n) = \frac{1}{24n-1}Tr^{(p)}(P;n) = \frac{1}{24n-1}\sum_{Q \in \mathcal{Q}^{(p)}_{24n-1}} P(\tau_Q)$$

This is stated in [10] as follows:

Theorem 5.1.2. Theorem 1.1

If n is a positive integer, then we have that

$$p(n) = \frac{1}{24n-1} Tr(n).$$

Moreover, the numbers $P(\tau_Q)$, as Q varies over $Q_{24n-1}^{(p)}$, form a multiset of algebraic numbers which is the union of Galois orbits in the discriminant $-24+1$ Hilbert class field.

This result is a key breakthrough in the field of partitions, as we now have not only a closed, finite sum, but it is also expressed in terms of algebraic numbers. This makes it possible to compute $p(n)$ in a finite amount of time.

Remark As $p(n)$ is an integer and $P(\tau_Q)$ algebraic numbers, we may instead consider $\sum_{Q\in\mathcal{Q}_{24n-1}^{(p)}} Re(P(\tau_Q))$.

5.2 Finding $\Gamma_0(6)$ Representatives

For each n, we must also find a set of representativesfor $\mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)$. Gross et al. describe a process for this in [6], by obtaining such representatives from a set of representatives for $SL_2(\mathbb{Z})$-equivalence classes of quadratic forms, with the given discriminant. These may be found in tables, as in Davenport [5].

Let $\{P_1, ..., P_l\}$ be a set of representatives for $\mathcal{P}_d^0/SL_2(\mathbb{Z})$, where $\mathcal{P}_d^0$ represents the set of primitive positive definite forms of discriminant $-d$.

We let $A_j = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ for $j = 1, ..., l$ satisfy

$$\begin{pmatrix} \tilde{a} & \frac{1}{2}(\tilde{b} - \tilde{\rho}) \\ \frac{1}{2}(\tilde{b} + \tilde{\rho}) & \tilde{c} \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6) \tag{5.6}$$

Writing $A^T P_j A = \begin{pmatrix} 6a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, let $Q_j = [6a, b, c]$.

Then $\{Q_1, ..., Q_l\}$ is a set of representatives for for $\mathcal{Q}_d^{(p),0}/\Gamma_0(6)$ by [6], where $\mathcal{Q}_d^{(p),0}$ represents the primitive forms; when $n = 1, 2, 3$, d is squarefree, so $\mathcal{Q}_d^{(p),0} = \mathcal{Q}_d^{(p)}$ by equation (1) of Gross et al [6].

5.3 Examples

Example (*The case of* $n = 1$)

Let us consider the case of $n = 1$. We wish to show that $p(1) = 1$ by the above methods.

Observe that $24n - 1 = 23$ for $n = 1$. From Davenport [5], we have for discriminant -23 the set of reduced positive definite forms $\mathcal{P}$:

$$\{P_1, P_2, P_3\} = \{[1, 1, 6], [2, 1, 3], [2, -1, 3]\}$$

To find the desired $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A \in SL_2(\mathbb{Z})$ for P_1, we must solve the congruence (5.6):

$$\begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6)$$

or

$$\begin{pmatrix} \alpha \\ \alpha + 6\gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6).$$

Now $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \delta \end{pmatrix}$ is a solution, so we need $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ such that $\begin{pmatrix} 0 & \beta \\ 1 & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$; $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is such a matrix.

Then

$$A^T P_1 A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 6 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

hence $\tilde{Q_1} = [6, -1, 1]$ and we have

$$Q_1 = 6x^2 - xy + y^2.$$

To find the desired $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A \in SL_2(\mathbb{Z})$ for P_2, we must solve the congruence (5.6):

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6)$$

or

$$\begin{pmatrix} 2\alpha \\ \alpha + 3\gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6).$$

Now $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \delta \end{pmatrix}$ is a solution, so we need $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ such that $\begin{pmatrix} 3 & \beta \\ 1 & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$; $\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$ is such a matrix.

Then

$$\begin{aligned} A^T P_2 A &= \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 24 & -\frac{13}{2} \\ -\frac{13}{2} & 2 \end{pmatrix} \end{aligned}$$

hence $\tilde{Q_2} = [24, 13, 2]$ and we have

$$Q_2 = 24x^2 - 13xy + 2y^2.$$

To find the desired $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A \in SL_2(\mathbb{Z})$ for P_3, we must solve the congruence (5.6):

$$\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6)$$

or

$$\begin{pmatrix} 2\alpha \\ -\alpha + 3\gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} (mod 6).$$

As $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \delta \end{pmatrix}$ is a solution, so we need $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ such that $\begin{pmatrix} 3 & \beta \\ 1 & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$; $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ is such a matrix.

Then

$$\begin{aligned} A^T P_3 A &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 18 & \frac{25}{2} \\ \frac{25}{2} & 9 \end{pmatrix} \end{aligned}$$

hence $\tilde{Q_3} = [18, 25, 9]$ and we have

$$Q_3 = 18x^2 + 25xy + 9y^2.$$

Thus we have that the $\Gamma_0(6)$ representatives are

$$\begin{aligned}\mathcal{Q}_{-23}^{(p)} &= \{Q_1, Q_2, Q_3\} \\ &= \{6x^2 - xy + y^2, 24x^2 - 13xy + 2y^2, 18x^2 + 25xy + 9y^2\}\end{aligned}$$

Therefore the complex multiplication points (CM points) are given by:

$Q_1(x,1) = 6x^2 - x + 1 = 0$ gives $x = \frac{1 \pm \sqrt{1-4\cdot 6}}{12}$, hence $\frac{1}{12} + \frac{1}{12}\sqrt{-23} = \tau_{Q_1}$.

$Q_2(x,1) = 24x^2 - 13x + 2 = 0$ gives $x = \frac{13 \pm \sqrt{(-13)^2 - 4\cdot 24\cdot 2}}{48}$, hence $\frac{13}{48} + \frac{1}{48}\sqrt{-23} = \tau_{Q_2}$.

$Q_3(x,1) = 18x^2 + 25x + 9 = 0$ gives $x = \frac{-25 \pm \sqrt{25^2 - 4\cdot 9\cdot 18}}{2\cdot 18}$, hence $-\frac{25}{36} + \frac{1}{36}\sqrt{-23} = \tau_{Q_3}$.

Then we see that

$P(\frac{1}{12} + \frac{1}{12}\sqrt{-23}) \approx 13.96548628$

$P(\frac{13}{48} + \frac{1}{48}\sqrt{-23}) \approx 4.53757041 - 3.09939725\imath$

$P(-\frac{25}{36} + \frac{1}{36}\sqrt{-23}) \approx 4.51725686 + 3.09789059\imath$

hence

$$Re(\sum_{Q \in \mathcal{Q}_{-23}^{(p)}} P(\tau_{Q_i})) \approx 23.0203137$$

and finally

$$p(1) = \frac{23.0203137}{23} = 1.00088 \approx 1.$$

Note that aas we do not know the error term for our expansion of the modular form $F(z)$, we are unable to verify the accuracy of these approximations. As such, these examples should be taken as illustrations rather than rigorous proof.

Example (*The case of* $n = 2$)

Let us consider the case of $n = 2$. We wish to show that $p(2) = 2$ by the above methods.

Observe that $24n - 1 = 47$ for $n = 2$, and from [4] the $\Gamma_0(6)$ representatives are

$$\begin{aligned}\mathcal{Q}_{-47}^{(p)} =& Q_1, Q_2, Q_3, Q_4, Q_5\\ =&\{6x^2 + xy + 2y^2, 12x^2 + xy + y^2, 18x^2 + 13xy + 3y^2,\\ & 24x^2 + 25xy + 7y^2, 36x^2 + 49xy + 17y^2\}\end{aligned}$$

Therefore the CM points are given by:

$Q_1(x, 1) = 6x^2 + x + 2 = 0$ gives $x = \frac{-1\pm\sqrt{1-24\cdot 2}}{12}$, hence $-\frac{1}{12} + \frac{\sqrt{-47}}{12} = \tau_{Q_1}$.

$Q_2(x, 1) = 12x^2 + x + 1 = 0$ gives $x = \frac{-1\pm\sqrt{1-24}}{2\cdot 12}$, hence $-\frac{1}{24} + \frac{\sqrt{-47}}{24} = \tau_{Q_2}$.

$Q_3(x, 1) = 18x^2 + 13x + 3 = 0$ gives $x = \frac{-13\pm\sqrt{13^2-4\cdot 18\cdot 3}}{2\cdot 18}$, hence $-\frac{13}{36} + \frac{\sqrt{-47}}{36} = \tau_{Q_3}$.

$Q_4(x, 1) = 24x^2 + 25x + 7 = 0$ gives $x = \frac{-25\pm\sqrt{25^2-4\cdot 24\cdot 7}}{2\cdot 24}$, hence $-\frac{25}{48} + \frac{\sqrt{-47}}{48} = \tau_{Q_4}$.

$Q_5(x, 1) = 36x^2 + 49x + 17 = 0$ gives $x = \frac{-49\pm\sqrt{49^2-4\cdot 17\cdot 36}}{2\cdot 36}$, hence $-\frac{49}{72} + \frac{\sqrt{-47}}{72} = \tau_{Q_5}$.

Then we see, by using the expansion of $P(z)$, that

$P(-\frac{1}{12} + \frac{\sqrt{-47}}{12}) \approx 26.390682 + 12.376065\imath$

$P(-\frac{1}{24} + \frac{\sqrt{-47}}{24}) \approx 26.390682 - 12.376065\imath$

$P(-\frac{13}{36} + \frac{\sqrt{-47}}{36}) \approx -.062017987$

$P(-\frac{25}{48} + \frac{\sqrt{-47}}{48}) \approx 20.640327 - 12.692206\imath$

$P(-\frac{49}{72} + \frac{\sqrt{-47}}{72}) \approx 20.693650 + 12.479125\imath$

hence

$$Re\Big(\sum_{Q\in\mathcal{Q}^{(p)}_{-47}} P(\tau_{Q_i})\Big) = Re(94.053323 - .213081\imath) = 94.053323$$

and finally

$$p(2) = \frac{94.053323}{47} = 2.00113 \approx 2.$$

5.4 Applications

Partitions are used in various fields of physics. In quantum physics, they apply to the angular momentum of fermions; see Benjamin and Quinn [3] for further details. Partitions have been used in statistical mechanics to represent the form of crystal structures, and also in a discussion of the hard hexagon model; see Andrews [2] for additional details.

Partition theory is also used in group representation theory, with the Young Tableaux, and the partition function is used in the expression of the number of non-isomorphic abelian groups. See Kavassalis [7] for further information and sources on details.

Ono's new result is particularly interesting in a purely mathematical sense, as relating the number of partitions of an integer to the trace of certain quadratic roots was a major breakthrough in relating different mathematical concepts to each other. Moreover, it is the first time that a definite closed formula has been found for $p(n)$ in terms of algebraic numbers, thus there is potential for major computational improvement over previous methods (such as approximations or infinite series). This in turn may lead to the discovery of further uses of the partition function itself.

BIBLIOGRAPHY

[1] Andrews, George E. "Partitions." *History of Combinatorics*. Web. [http://www.math.psu.edu/vstein/alg/antheory/preprint/andrews/chapter.pdf].

[2] Andrews, George E. "The Reasonable and Unreasonable Effectiveness of Number Theory in Statistical Mechanics." *Proceedings of Symposia in Applied Mathematics* 46 (1992): 21-34. Print.

[3] Benjamin, Arthur T., and Jennifer J. Quinn. "Composite Fermions and Integer Partitions." *Journal of Combinatorial Theory* A 95 (2001): 390-97. Print.

[4] Bruinier, Jan H., and Ken Ono. "Algebraic Formulas for the Coefficients of Half-Integral Weight Harmonic Weak Maass Forms." American Institute of Mathematics, 20 Jan. 2011. Web. 15 Apr. 2011.

[5] Davenport, Harold. "Quadratic Forms." *The Higher Arithmetic: An Introduction to the Theory of Numbers*. 7th ed. Cambridge, UK: Cambridge UP, 2008. 144. Google Books. Web. 15 Feb. 2012.

[6] Gross, B., W. Kohnen, and D. Zagier. "Heegner Points and Derivatives of L-Series II." *Inventiones Mathematicae* 84 (1986): 225-320. William A Stein. Web.

[7] Kavassalis, S. C. "Finite Formula Found for Partition Numbers." *PLoS Blogs Network*. PLOS Blogs, 20 Jan. 2011. Web. 12 Oct. 2011. [http://blogs.plos.org/badphysics/2011/01/20/ono/].

[8] Lehmer, D. H. 'On the Remainders and Convergence of the Series for the Partition Function." *Proceedings of the London Mathematical Society* (1939): 362-73. Print.

[9] Niven, Ivan, Herbert S. Zuckerman, and Hugh L. Montgomery. *An Introduction to the Theory of Numbers*. New York: Wiley, 1991. Print.

[10] Ono, Ken, and Jan Hendrik Bruinier. "An Algebraic Formula for the Partition Function." *American Institute of Mathematics*. Web. [http://www.aimath.org/news/partition/brunier-ono].

[11] Ono, Ken, and Kathrin Bringmann. "An Arithmetic Formula for the Partition Function." *Mathematics Subject Classification* (2006). Print.

[12] Rademacher, Hans. *Topics in Analytic Number Theory*. Berlin: Springer-Verlag, 1973. Print.

[13] Whiteman, Albert L. "A Note on Kloosterman Sums." *Bulletin of the American Mathematical Society* 51.6 (1945): 373-77. Whiteman: A Note on Kloosterman Sums. Project Euclid. Web.

[14] Whiteman, Albert L. "A Sum Connected with the Partition Function." *Bulletin of the American Mathematical Society* 53.6 (1947): 598-603. Whiteman: A Sum Connected with the Partition Function. Project Euclid. Web.

Appendix A
FORMAL POWER SERIES

Here we provide an outline of facts from the theory of formal power series, as discussed in Rademacher [12].

Definition A *formal power series* is defined to be

$$A = a_0 + a_1x + a_2x^2 + ... = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n \in R$$

where R is a commutative ring with unit element, and has no zero divisors.

Recall that integral domains are defined to be commutative rings with unit element and no zero divisors.

Claim .0.1. *Formal power series form a domain, $\mathcal{D}$, with the following actions:*

$$A + B = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ... \textit{(addition)}$$

$$AB = d_0 + d_1x + d_2x^2 + ...,\ d_n = a_0b_n + a_1b_{n-1} + ... + a_nb_0 \ \textit{(multiplication)}$$

$$I = 1 + 0x + 0x^2 + ... \textit{(unit element)}$$

$$0 = 0 + 0x + 0x^2 + ... \textit{(zero element)}$$

Claim .0.2. *Units such as*

$$D = 1 + a_1x + a_2x^2 + ...$$

have unique inverses

$$D^{-1} = 1 + b_1x + b_2x^2 + ...$$

where the b_n *are found from*

$$(1 + a_1x + a_2x^2 + ...)(1 + b_1x + b_2x^2 + ...) = 1,$$

which gives

$$a_1 + b_1 = 0,\ a_2 + a_1b_1 + b_2 = 0,\ a_3 + a_2b_1 + a_1b_2 + b_3 = 0,\ ...$$

Note that

$$(1 - x)(1 + b_1x + b_2x^2 + ...) = 1 \text{ gives us } -1 + b_1 = 0,\ -b_{n-1} + b_n = 0$$

hence

$$b_n = 1,\ n = 1, 2, 3, ...$$

and then we have the identity:

$$(1 - x)(1 + x + x^2 + ...) = 1 \text{ or } (1 - x)^{-1} = 1 + x + x^2 + ... \tag{7}$$

We also have $D_1^{-1}D_2^{-1} = (D_1D_2)^{-1}$ given the leading coefficient of D_1, D_2 is equal to one.

Definition We may define an *infinite sum of power series*

$$A_1 + A_2 + A_3 + ... = S \in \mathcal{D}$$

given that

$$A_1 + A_2 + A_3 + ... \equiv S(modx^N) \text{ for all } N \in \mathbb{N}$$

Remark Loosely speaking, this says for any $N \in \mathbb{N}$ there are only finitely many terms containing x^n for $n < N$.

Definition Similarly, we may define an *infinite product*

$$B_1 B_2 B_3 .... = M$$

given that

$$B_1 B_2 B_3 ... \equiv M (mod x^N) \text{ for all } N \in \mathbb{N}$$

In general, the indeterminate x may be replaced by any polynomial, given the polynomial has no constant term. This is because, $(mod x^N)$, we have polynomials rather than infinite formal power series. When polynomials with no constant term are substituted in to x^n, the degree of each term will be larger than x^n, and we may reduce again $(mod x^N)$.

Claim .0.3. *Given* $G = 1 + x + x^2 + ...$ *and* $P = (1+x)(1+x^2)(1+x^4)(1+x^8)...$, *we have* $G = P$.

Proof. (idea)

$$\begin{aligned}(1-x)P &= (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)... \\ &= (1-x^2)(1+x^2)(1+x^4)(1+x^8)... \\ &= (1-x^4)(1+x^4)(1+x^8)...\end{aligned}$$

Then we see that $(1-x)P \equiv 1 (mod x^{2^k}),\ k \in \mathbb{Z}^{\geq}$, hence $(1-x)P = 1$ by infinite products. Since we saw earlier that $(1-x)G = 1$, it follows that $(1-x)P = (1-x)G$ and hence $P = G$. □

Appendix B
EXPANSION OF $F(z)$

Given equation (5.4)

$$F(z) := \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z))}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2}$$

we may rewrite the numerator as

$$\text{num} = 1 - 12(\sum_{n=1}^{\infty}\sigma(n)q^n - 2\sum_{n=1}^{\infty}\sigma(n)q^{2n} - 3\sum_{n=1}^{\infty}\sigma(n)q^{3n} + 6\sum_{n=1}^{\infty}\sigma(n)q^{6n}),$$

which we may then expand to:

num$= 1-12q-12q^2-12q^3-12q^4-72q^5-12q^6-96q^7-12q^8-12q^9-72q^{10}-144q^{11}-12q^{12}-168q^{13}-96q^{14}-72q^{15}-12q^{16}-216q^{17}-12q^{18}-240q^{19}-72q^{20}-96q^{21}-144q^{22}-288q^{23}-12q^{24}-372q^{25}-168q^{26}-12q^{27}-96q^{28}-360q^{29}-72q^{30}-384q^{31}-12q^{32}-144q^{33}-216q^{34}-576q^{35}-12q^{36}-456q^{37}-240q^{38}-168q^{39}-72q^{40}-504q^{41}-96q^{42}-528q^{43}-144q^{44}-72q^{45}-288q^{46}-576q^{47}-12q^{48}-684q^{49}-372q^{50}-216q^{51}-168q^{52}-648q^{53}-12q^{54}-864q^{55}-96q^{56}-240q^{57}-360q^{58}-720q^{59}-72q^{60}-744q^{61}-384q^{62}-96q^{63}-12q^{64}-1008q^{65}-144q^{66}-816q^{67}-216q^{68}-288q^{69}-576q^{70}-864q^{71}-12q^{72}-888q^{73}-456q^{74}-372q^{75}-240q^{76}-1152q^{77}-168q^{78}-960q^{79}-72q^{80}-12q^{81}-504q^{82}-1008q^{83}-96q^{84}-1296q^{85}-528q^{86}-360q^{87}-144q^{88}-1080q^{89}-72q^{90}-1344q^{91}-288q^{92}-384q^{93}-576q^{94}-1440q^{95}-12q^{96}-1176q^{97}-684q^{98}-144q^9 9-372q^{100}-1224q^{101}$

We may also rewrite the denominator as

denom =

$$q\big((\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}})(\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)})(\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{3k(3k-1)}{2}})(\sum_{k=-\infty}^{\infty}(-1)^k q^{3k(3k-1)})\big)^2$$

which we may then expand to:

$$\text{denom}= q(1 - 2q - 3q^2 + 4q^3 + 6q^4 + 6q^5 - 16q^6 - 8q^7 + 9q^8 - 12q^9 + 12q^{10} - 12q^{11} + 38q^{12} + 32q^{13} - 18q^{14} + 16q^{15} - 126q^{16} - 18q^{17} + 20q^{18} + 24q^{19} + 48q^{20} - 24q^{21} + 168q^{22} + 24q^{23} - 89q^{24} - 76q^{25} - 27q^{26} - 64q^{27} + 30q^{28} + 36q^{29} - 88q^{30} - 32q^{31} - 36q^{32} + 252q^{33} - 96q^{34} + 36q^{35} + 254q^{36} - 40q^{37} - 114q^{38} - 48q^{39} + 42q^{40} - 96q^{41} - 52q^{42} + 48q^{43} + 54q^{44} - 336q^{45} - 96q^{46} - 48q^{47} - 87q^{48} + 178q^{49} + 378q^{50} + 152q^{51} + 198q^{52} + 54q^{53} + 72q^{54} + 128q^{55} - 60q^{56} - 60q^{57} - 660q^{58} - 72q^{59} - 538q^{60} + 176q^{61} - 144q^{62} + 64q^{63} + 228q^{64} + 72q^{65} + 884q^{66} - 504q^{67} - 504q^{68} + 192q^{69} + 792q^{70} - 72q^{71} + 218q^{72} - 508q^{73} + 267q^{74} + 80q^{75} - 192q^{76} + 228q^{77} - 520q^{78} + 96q^{79} + 81q^{80} - 84q^{81} - 492q^{82} + 192q^{83} - 756q^{84} + 104q^{85} - 90q^{86} - 96q^{87} + 810q^{88} - 108q^{89} - 608q^{90} + 672q^{91} + 264q^{92} + 192q^{93} + 120q^{94} + 96q^{95} + 1154q^{96} + 174q^{97} + 108q^{98} - 356q^{99} - 618q^{100} - 756q^{101})$$

Hence by long division, we obtain $F(z)$ up to the one hundredth power:

$$F(z) = q^{-1}(1 - 10q - 29q^2 - 104q^3 - 273q^4 - 760q^5 - 1685q^6 - 4008q^7 - 8334q^8 - 17560q^9 - 34563q^{10} - 68080q^{11} - 127210q^{12} - 238008q^{13} - 428579q^{14} - 767808q^{15} - 1339605q^{16} - 2322136q^{17} - 3938840q^{18} - 6641256q^{19} - 11004164q^{20} - 18110800q^{21} - 29396445q^{22} - 47399776q^{23} - 75525219q^{24} - 119602776q^{25} - 187488685q^{26} - 292150064q^{27} - 451293015q^{28} - 693184192q^{29} - 1056544104q^{30} - 1601892720q^{31} - 2412131000q^{32} - 3614038360q^{33} - 5381800272q^{34} - 7976577872q^{35} - 11756874290q^{36} - 17252498424q^{37} - 25189071067q^{38} - 36623872928q^{39} -$$

$$53003394771q^{40} - 76408202000q^{41} - 109677267110q^{42} - 156852011328q^{43} - 223429216288q^{44} - 317156668960q^{45} - 448544078280q^{46} - 632270338752q^{47} - 888181387116q^{48} - 1243781326488q^{49} - 1736124840335q^{50} - 2416187501080q^{51} - 3352424782032q^{52} - 4638365241360q^{53} - 6399176115010q^{54} - 8804829942696q^{55} - 12082000967090q^{56} - 16536729551296q^{57} - 22575800446782q^{58} - 30745455057584q^{59} - 41769310564388q^{60} - 56614256352720q^{61} - 76556876713300q^{62} - 103294913638400q^{63} - 139062739596981q^{64} - 186818663148120q^{65} - 250443716686960q^{66} - 335055123586656q^{67} - 447345291602292q^{68} - 596103464293232q^{69} - 792789882544071q^{70} - 1052398901680000q^{71} - 1394423876594491q^{72} - 1844279102053272q^{73} - 2434909054371082q^{74} - 3209118949925872q^{75} - 4222242370939305q^{76} - 5545941359639264q^{77} - 7272599701179653q^{78} - 9521492524088496q^{79} - 12445975806404298q^{80} - 16243455726657680q^{81} - 21167090259308427q^{82} - 27541822855695888q^{83} - 35783321104813198q^{84} - 46423656323060928q^{85} - 60141788333392025q^{86} - 77804514688350304q^{87} - 100515301158310128q^{88} - 129679298702725536q^{89} - 167081174675333524q^{90} - 214987933920656880q^{91} - 276272380475290780q^{92} - 354575066653720128q^{93} - 454499271791079477q^{94} - 581865065544717184q^{95} - 744015876274890395q^{96} - 950215568901119256q^{97} - 1212128405221148640q^{98} - 1544437308222115880q^{99} - 1965592258777124496q^{100} - 2498769400387544200q^{101})$$

BIOGRAPHY OF THE AUTHOR

Emily Rose Igo was born May 27, 1988 to Arthur and Deborah Igo. She grew up in South Portland, Maine and graduated as Valedictorian from South Portland High School in 2006. In 2010 she graduated Cum Laude and with Departmental Honors with a Bachelor of Arts in Mathematics from Boston College in Chestnut Hill, MA. She is a member of Pi Mu Epsilon and is nominee for Excellence in Graduate Teaching Assistantship. She is a candidate for the Masters of Arts degree in Mathematics from The University of Maine in May, 2012.