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From Points to Potlucks: An Exploration of Fixed Point Theorems with Applications to Game Theory Models of Successful Integration Practices

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FROM POINTS TO POTLUCKS: AN EXPLORATION OF FIXED POINT THEOREMS WITH APPLICATIONS TO GAME THEORY MODELS OF SUCCESSFUL INTEGRATION PRACTICES

by

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Abstract:

Potlucks have many names: shared community dinners, faith suppers, “bring-a-dish” dinners, etc. They represent the desire to share food with other people and make new friends, sometimes learning about other cultures in the process. Not only does one have to decide what dish to bring, but one must also decide how large of a dish, if there will be a theme, and what course it will fit. For instance, if everyone brings side dishes, there will not be enough food for everyone, and if someone brings food that most of the group cannot eat, then feelings will be hurt on all sides. And in a way, having a potluck is similar to creating integration policies. Successful integration policies are fair to all people and take a “two-street” approach, while simultaneously being a collaborative affair.

This paper will first explore fixed point theory, including the Kakutani Fixed Point Theorem and Brouwer Fixed Point Theorem; fixed point theorems are a significant field of mathematics and have many well-known applications. One of these applications is game theory, which is the study of how rational actors make decisions in everyday situations. Building upon the mathematical aspects of the first few chapters and the basics of game theory, this paper aims to build its own game theory model called the “Potluck Metaphor” that will model several methods of integration in the European Union; context for the model will be provided by critiquing three primary integration models and a brief literature review of the related field. Starting off with a simple game theory model for a dinner party, this paper will then slowly expand these models to show their applicability to European integration policy on an organizational level and on a member-specific level.

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Also a big thank you to my committee members Mark Haggerty and Stefano Tijerina for joining my project in its final stages and providing me with invaluable advice on the economic and international aspects of the thesis. Also a huge thank you to everyone in the Honors College, especially Melissa Ladenheim, for always having my back and helping me accomplish an almost impossible deadline.

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INTRODUCTION

"Give me your tired, your poor, Your huddled masses yearning to breathe free, The wretched refuse of your teeming shore. Send these, the homeless, tempest-tost to me, I lift my lamp beside the golden door!" This quote is inscribed on the Statue of Liberty, the large copper-cast woman who greets all those who enter into the United States. The United States is known as a country of immigrants, and the integration of migrants into standard American culture has found many forms, such as expansive Chinatowns, cultural festivals, and seasonal labor. Whether a melting pot or a multicultural society, the United States has an incredible number of residents who were foreign-born, and the United States in 2017 had almost 20% of the world's migrant population.[1]

Yet in the European Union, the environment is very different. Many EU members have never been known as the receiving countries, and with the increasing migrant numbers, integration policy has suddenly come to the forefront of all European Union political debates. Views on the impacts of migrants on crime, their willingness to integrate, and their benefits to society are heavily mixed between EU members, and due to disagreements on definitions of integration and human rights, outdated models, and societal differences, the future of migrants in the European Union is unclear.[2]

Therefore, many challenges exist in deciding what is best for all EU members with regards to integration policy. On one hand, the European Union wants to make sure that its ideals are being upheld in the new population, but on the other, these member countries need to make sure that they are respecting basic human rights. New policies and

[1] Gonzalez-Barrera, A., & Connor, P. (2019). Around the world, more say immigrants are a strength than a burden.
[2] Ibid.

new ideas are desperately needed in the years ahead as the numbers of refugees and migrants seeking protection grow in response to climate change, political conflict, and plummeting economies. The European Union is currently in a transitional stage as it attempts to decide the new path towards integration, and whatever it decides will heavily impact the future of European migrants for decades to come.

This paper has three primary objectives: to mathematically analyze several fixed point theorems, to explore the concept of integration in the modern European Union, and to suggest a new framework for understanding integration that is built upon foundational game theory.

Through incorporating higher level mathematics, this paper aspires to provide an interdisciplinary approach to a complicated issue. Starting with three primary fixed point theorems, the first chapter will provide an overview of the development of fixed point theory, which is a field with many applications and uses in modern mathematics; this will include discussion of theorems such as Banach's Fixed Point Theorem, Kakutani's Fixed Point Theorem, and von Neumann's Intersection Theorem. Many of these fixed point theorems were used to prove the existence of solutions in game theory models, which leads us to the next section of the paper.

Game theory is a unique field that manages to combine common rationality, decades-old mathematical theorems, and modern policy trends, and it serves as the bridge between International Affairs and Mathematics. In Chapter 2, the foundations of game theory are discussed through the lens of a common dinner game, as are the differences between pure and mixed strategy equilibria. Furthermore, the connection between mathematics and game theory is expanded upon in the section on Nash Equilibria; the existence of Nash Equilibria was directly proven through use of Kakutani's Fixed Point Theorem (and later Brouwer's Fixed Point Theorem). The full adapted proof is also presented in this chapter.

Then, in Chapter 3, we focus on the definition of integration and the analysis of several traditional models. Around the world, many countries have discriminatory or unfair integration practices that border on assimilation; several governments do not treat integration as a "two-way" process. All of the demands and requirements are put onto the immigrant families, leading to social divides and desperate measures. If, however, citizenship were easier to obtain, or translators were provided on a larger scale, integration could become easier and more beneficial to all. Furthermore, more people might learn the proper way to handle integration policies if the policies are made easier to understand; think about how much the "melting pot" or "fruit salad" metaphor has impacted the common understanding of immigration. By introducing a new way of understanding integration policies, the "Potluck Metaphor", perhaps the dialogue around integration can also change.

The final two chapters are centered on the Potluck Metaphor, a two-person game theory model that was built for this paper. For this game, the two players are a general EU member country and the migrants that reside in the member country, and the Potluck Metaphor illustrates the exchange between the country's government and the migrant population when determining integration policy. Firstly, the Potluck Metaphor uses potluck dishes to represent integration policies and investigates how governments can transition from assimilationist to exclusionist, or from ethnic enclaves to multicultural. It also suggests several optimal exchanges between the member country and its migrant population, emphasizing the importance of the two-way process and open dialogue. If the two players are not always acting rationally, or the Potluck Metaphor is played on a more local level, the outcomes might change; these potential changes are expanded in a few examples later in Chapter 4. In one case study, the Potluck Metaphor even breaks due to a different structure of government. Overall, it is the author's hope that by introducing a new way of understanding integration policies, the "potluck", perhaps the dialogue around integration can also change.

CHAPTER 1

FIXED POINT THEOREMS AND RELATED DEFINITIONS

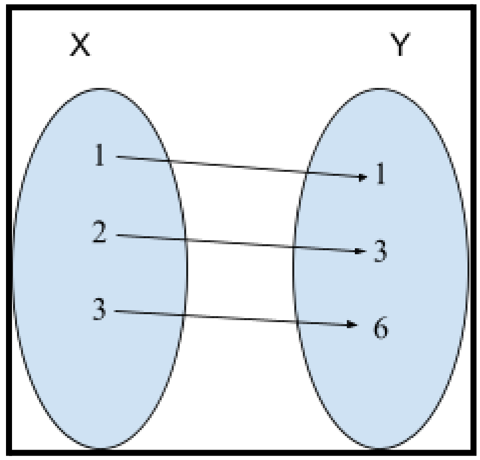
Developed throughout the past two hundred years, fixed point theorems have appeared from all around the world: France, the Netherlands, Japan, Latvia, America, etc.[3] Fixed point theory contributes to the foundations of several modern fields of mathematics, such as topology and nonlinear analysis, and they often guarantee the existence of solutions, rather than how to find them. In this section, we will be discussing the development of three major fixed point theorems: Banach's Fixed Point Theorem, Brouwer's Fixed Point Theorem, and Kakutani's Fixed Point Theorem. Banach's Fixed Point Theorem is one of the primary metric fixed point theorems, while Brouwer's Fixed Point Theorem and Kakutani's Fixed Point Theorem are both more topological in nature. Then from Kakutani's Fixed Point Theorem, we will move to one of its most popular applications: game theory.

Background Definitions

Before we establish the definition of a fixed point, we must first establish what we mean by "map". Let x and $f(x)$ be the input and output, respectively. By definition, a map f assigns each input x to an output $f(x)$, and each input value receives an output value when being mapped. Then f is a map from the input to the output; the map f can also be referred to as a function. One example of a single-valued map f from set X to set Y could be defined by $f(1) = 1, f(2)=3, f(3)=6,$ etc. In a single-valued map, a single input element gets mapped to a single output element. Visually, the map would look like this:

[3] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 391-412.

Figure One: Visual Example of a Map

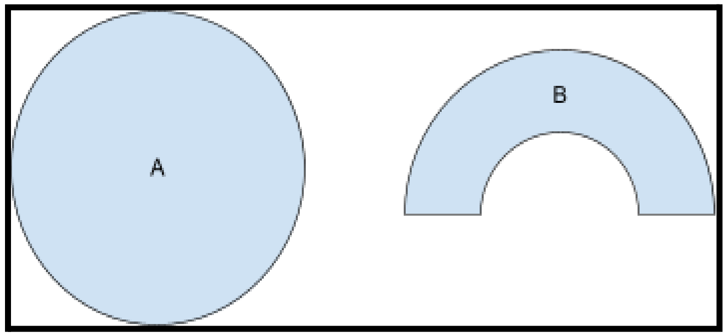


Note that there exist many mappings between X and Y; we could have just as easily defined $f: X \rightarrow Y$ as $f(1) = 2, f(2)=3,$ and $f(3)=1$.

Another type of map is a set-valued map, in which points are mapped to subsets of X. The main difference between a single-valued map and a set-valued map is the codomain of the map; in a single-valued map, points are mapped to points, while in a set-valued map, points are mapped to sets. For instance, a set-valued map f of $X= \{ 1, 2, 3\}$ could be written as $f(1) = \{1,3\}, f(2)=\{2\},$ and $f(3)=\{1,3\}$. A second example of a set-valued map is Clarke's generalized derivative.[4] To define this, first we will need the definition of a convex set. Let $\boldsymbol{R}$ be defined as the real numbers, and $\boldsymbol{R}^n$ be defined as n-dimensional Euclidean space. By definition, a subset X of $\boldsymbol{R}^n$ is convex if for any $x_1, x_2 \in X$ and any k such that $0 \leq k \leq 1, (kx_1+(1- k)x_2) \in X$; in the convex subset X, we can draw a line from any point $x_1 \in X$ to any $x_2 \in X$ and still remain within X.[5] In the image below, the set A on the left is convex, while the set B on the right is not convex.

[4] Clarke, F. H. (1990). Optimization and nonsmooth analysis, 70-75.
[5] Munkres, J. (2014). Topology: Pearson New International Edition (2nd edition), 15.

Figure Two: Convex Subset vs. Non-convex Subset



Then Clarke's generalized derivative map f is written as [6]

$\partial f(x) = co\{z \in \mathbf{R}$: *there exists a sequence* $x_n \to x$ *such that* $x_i \in X \backslash Z_f$ *for all* i *and* $f'(x_n) \to z\}$,

where Z_f is the set of points where f is nondifferentiable and co is convex closure. In the generalized derivative, the convex closure of the set is the smallest set that is convex while still containing the original set, which in this case is the set of nearby derivatives at a point of interest. This map is defined as a set-valued map because it maps each $x \in X$ to a convex set $\partial f(x)$.

For instance, let $f(x)=|x|$. For simplicity's sake, first choose $x=0$. Then we can pick the sequence $x_n = 1/n$ to get $f'(x_n)=1$ for all n, so $f'(x_n) \to 1 = z$. On the other hand, if we pick $x_n = -1/n$, we get $f'(x_n)=-1$ for all n, so $f'(x_n) \to -1 = z$. Therefore, the generalized derivative at $x=0$ is

$$\partial f(0) = co\{-1,1\} = [-1,1].$$

Now choose $x=1$. In this case, we can pick a sequence $x_n = 1+(1/n)$ and get $f'(x_n)=1$ for all n, so $f'(x_n) \to 1 = z$. Any other sequence approaching $x=1$ would also give $z=1$. Therefore, the generalized derivative at $x=1$ is

$$\partial f(1) = co\{1\} = \{1\}.$$

[6] Clarke, F. H. (1990). Optimization and nonsmooth analysis, 70-75.

This last calculation makes sense because if $f(x)$ is a differentiable function at x, then $\partial f(x)=\{f'(x)\}$. After further mathematical calculations and more test points, we can conclude that in general

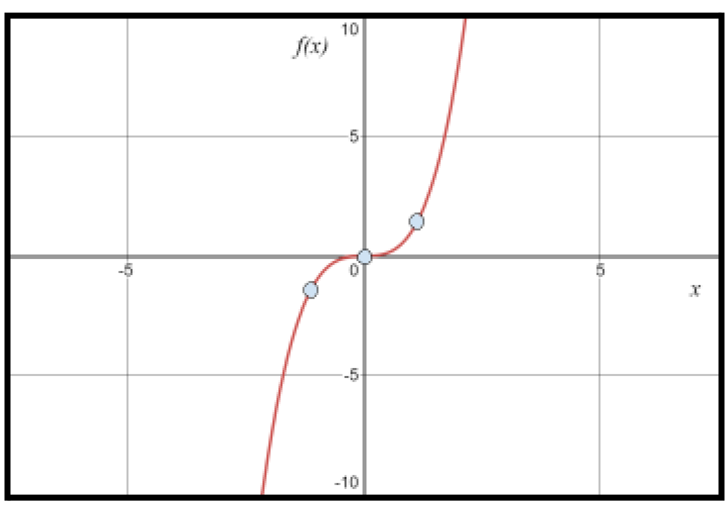
$$\partial f(x)= \{ \{1\} \text{ if } x>0$$

$$[-1,1] \text{ if } x=0$$

$$\{-1\} \text{ if } x<0 \}.$$

Returning now to a single-valued map, we can define a fixed point as a point that does not change when a map is applied; in other words, a fixed point of a particular function $f(x)$ is a point x^* such that $f(x^*)= x^*$. The point is fixed in place for a certain mapping, and the input will be the same as the output. For example, consider the map $f(x)=x^3$. The fixed points of $f(x)= x^3$ are -1, 1, and 0 since $(-1)^3= -1$, $(1)^3=1$, and $(0)^3=0$ because in each case, $x^*= f(x^*)= (x^*)^3$. The graph of this function with its fixed points in blue is shown below.

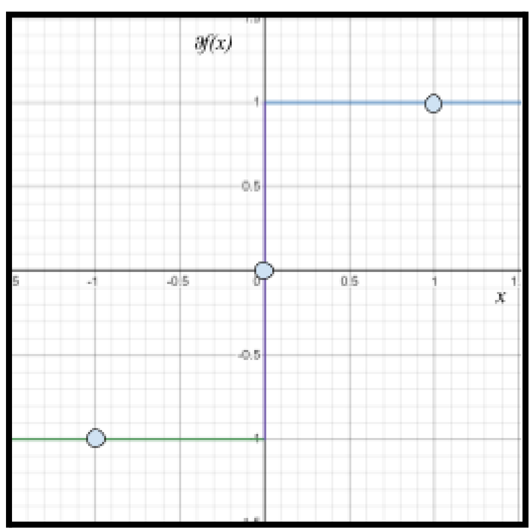
Figure Three A: Fixed Points of $f(x)=x^3$



Another interesting type of single-valued map is a differential operator, which we will denote by d/dx. The differential operator assigns a continuously differentiable function to another function, i.e. $d/dx: X \to Y$ where X is the set of continuously differentiable functions on the interval $[a, b]$ and Y is the set of corresponding continuous functions on the interval $[a, b]$. For instance, $d/dx(x^3) = 3x^2 \neq x^3$, so $f(x) = x^3 \in X$ is not a fixed point of the derivative operator. On the other hand, $d/dx(e^x) = e^x$, so $f(x) = e^x \in X$ is a fixed point of the map.

On the other hand, the fixed points of a set-valued map are found somewhat differently. A fixed point of a set-valued map is a point that satisfies $x^* \in f(x^*)$. Instead of the input being the same as the output, the input now has to be an element of the output (i.e. an element of a subset). In the generalized derivative of $f(x)=|x|$, we have three inputs for which the input is an element of the output, so the map f on $X=\boldsymbol{R}$ has three fixed points: $x^* = 1, -1, 0$. This is expressed graphically below.

Figure Four: Fixed Points of the Generalized Derivative of the Absolute Value Function



Analysis of Three Famous Fixed Point Theorems

Now that we know the meaning of a fixed point, we can move on to three major fixed point theorems. The primary metric space fixed point theorem is known as Banach's Fixed Point Theorem, but there are a few more definitions that are important to know beforehand: metric spaces, complete, Lipschitz continuous, and contractions.

Let us first establish the definition of a metric space. Let X be a non-empty set. Then d is a distance function (or metric) in X if for all $x, y, z,$ in X:

I. $d(x,y) \geq 0;$

II. $d(x,y)=0$ *if and only if* $x=y;$

III. $d(x,y) = d(y,x);$

IV. $d(x,z) \leq d(x, y) + d(y,z).$

Then (X, d) is a metric space, often written as just the metric space X.[7] A common example of a metric space is the set of all integers with the standard metric $d(x,y)= | x- y |$ as demonstrated below.

Example: Let $\boldsymbol{Z}$ be the set of all integers, and let d be the metric $d(x,y)= | x- y|$, $x, y \in \boldsymbol{Z}$. Then $(\boldsymbol{Z}, d)$ is a metric space.To prove that $(\boldsymbol{Z}, d)$ is a metric space, it has to satisfy four conditions of being a metric space:

I. $d(x,y) = | x- y | \geq 0.$

II. $d(x,y)=0$ *if and only if* $x=y$: *if* $x=y$, *then* $| x- y |= 0 \Rightarrow d(x,y)=0$; *if* $d(x,y)=0$, *then* $| x- y |= 0 \Rightarrow x=y$. *Therefore,* $d(x,y)=0$ *if and only if* $x=y$.

III. $d(x,y) =| x- y | = | y- x | = d(y,x).$

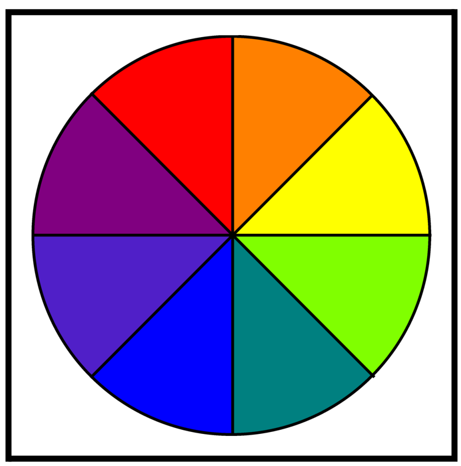
[7] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 229.

IV. $d(x,z) = |x- z| = |x- y+y- z|$. Then by the triangle inequality, we can conclude that $|x- y+y- z| \leq |x- y|+|y- z| = d(x, y) +d(y,z)$.

Therefore, $(\mathbf{Z},d)$ is a metric space.

Another example of a metric space is the color wheel with a distance function *d(x,y)*= minimum number of color changes between color *x* and color *y*. For example, the distance between red and yellow (i.e. *d(red, yellow)*) is 2, while the distance between red and blue (i.e. *d(red, blue)*) is 3.

Figure Five: Color Wheel as a Metric Space [8]



Example: Let *X* be the set of colors on the color wheel, and let *d* be the metric *d(x,y)*= minimum number of color changes between color *x* and color *y*. Then *(X,d)* is a metric space. To prove that *(X, d)* is a metric space, it has to satisfy the four conditions of being a metric space:

I. $d(x,y) \geq 0$: distance between two color changes cannot be negative

[8] Clipart from Google Images

II. *$d(x,y)=0$ if and only if $x=y$: if $x=y$, then there are 0 color changes between color x and color y, so $d(x,y)=0$; if $d(x,y)=0$, then there are 0 color changes between color x and color y, so $x=y$. Therefore, $d(x,y)=0$ if and only if $x=y$.*

III. *$d(x,y)$ = number of color changes between color x and color y= number of color changes between color y and color x= $d(y,x)$.*

IV. *$d(x,z)$ = number of color changes between color x and color z . There are only two scenarios: color y is the same as color x or color z, or color y is neither color x or color z. If color y is the same as color x or color z, then without loss of generality, let $x=y$. Then the inequality holds because $d(x,z)\leq d(x, y) +d(y,z)= d(x, x) +d(x,z)= d(x,z)$. On the other hand, if color y is a distinct color that is not color x or color z, then there are two cases: color y is one of the colors in the minimum number of color changes between color x and color z, or color y is one of the colors in the maximum number of color changes between color x and color z. If the former, then $d(x,z)= d(x, y) +d(y,z)$, and if the latter, then $d(x,z)\leq d(x, y) +d(y,z)$. Therefore, $d(x,z) \leq d(x, y) +d(y,z)$ for all colors x, y, z in X.*

Therefore, (X,d) is a metric space.

Note that the color wheel is not a metric space for all distance functions. Let g represent the distance function $g(x,y)$= number of color changes between color x and color y when going in a clockwise direction. The first two axioms in the proof remain unchanged, but the third axiom fails: $g(x,y)= g(y,x)$ for all x, y. For example, the number of color changes in a clockwise direction between orange and yellow is not equal to the number of color changes in a clockwise direction between yellow and orange, so $g(x,y) \neq g(y,x)$ for x=orange and y=yellow. Therefore, (X,g) is not a metric space. This goes to show that the distance function is a crucial part of establishing a metric space, and choosing different distance functions can impact the classification.

Now to establish the definition of a complete metric space. A sequence x_n is convergent in the metric space X to a point p if for every $\varepsilon > 0$, there exists an $N \in \boldsymbol{N}$ ($\boldsymbol{N}$ as the set of natural numbers) such that $d(x_n, p) < \varepsilon$ for all $n \geq N$; the point p can also be referred to as the limit of the sequence, and we can write $\lim x_n = p$ or $x_n \rightarrow p$.[9] Then X is a complete metric space if every Cauchy sequence in X is convergent in X (aka has a limit within X). A Cauchy sequence is a sequence x_n that has arbitrarily close terms; in other words, for every $\varepsilon > 0$, there exists an $N \in \boldsymbol{N}$ such that for all $n \geq N$ and all $k \geq N$, we have $d(x_n, x_k) < \varepsilon$. If every Cauchy sequence is convergent in a metric space X, then we say that X is Cauchy-complete (or just complete).[10] On a more intuitive level, a metric space is complete if there are no “gaps” in the metric space that a Cauchy sequence of X could converge to. Common examples of complete metric spaces include the real numbers with a standard metric $d(x,y) = |x - y|$, while an example of an incomplete metric space is the set of rationals with the standard metric.[11]

Moving on to continuity, a function $f: X \rightarrow Y$ with X and Y as metric spaces is continuous at $x_2 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x_1 \in X$ and $d(x_1, x_2) < \delta$, $d(f(x_1), f(x_2)) < \varepsilon$. This implies the following property of a continuous function: a function f is continuous at $x_2 \in X$ if the limit of $f(x)$ as x approaches x_2 is equal

[9] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 246.
[10] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 77.
[11] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 251.

to $f(x_2)$.[12] A map $f: X \rightarrow X$ is Lipschitz continuous if there exists a $k \geq 0$ such that $d(f(x_1), f(x_2)) \leq k\, d(x_1, x_2)$, $\forall\ x_1, x_2 \in X$. If $0 \leq k < 1$, then f is said to be a contraction, i.e. the map is creating less distance between the two outputs $f(x_1)$ and $f(x_2)$ (aka "contracting").[13] Furthermore, all maps that are contractions are continuous, shown as follows (adapted from lecture notes by Mark Walker)[14]: Let (X,d) be a metric space, and let $f: X \rightarrow X$ be a contraction. Let $\varepsilon > 0$ be arbitrary, and choose $\delta = \varepsilon$. Then $d(x_1, x_2) < \delta$ implies that $d(f(x_1), f(x_2)) \leq kd(x_1, x_2) < k\delta < \varepsilon$. Therefore, $d(f(x_1), f(x_2)) < \varepsilon$. Because x_1 and x_2 are arbitrarily chosen, f must be continuous at any $x \in X$, so f must be continuous on X.

Banach's Fixed Point Theorem (otherwise known as Banach's Contraction Principle) was first stated by Stefen Banach in 1922, and it was used to prove the existence of an integral equation solution.[15] It then became extremely popular to use due to its simplicity and applicability to many different fields of mathematics; through generalizations of his theorem, various maps can be proven to have fixed points (as discussed later). Using the above definitions, we can now state Banach's Fixed Point Theorem: [16]

Theorem 1 (Banach's Fixed Point Theorem): Let X be a complete metric space, and $f: X \rightarrow X$ be a contraction. Then there exists a unique fixed point x^* of f in X.

It is absolutely necessary for X to be a complete metric space in the above theorem because contractions on an incomplete metric space are not required to have fixed points

[12] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 111-112.
[13] Pata, V. (2019). Fixed point theorems and applications (Vol. 116).
[14] Walker, M. (2017). Fixed Point Theorems [Lecture Notes]. University of Arizona.
[15] Jleli, M., & Samet, B. (2014). A new generalization of the Banach contraction principle, 1-8.
[16] Pata, V. (2019). Fixed point theorems and applications (Vol. 116), 3-5.

(although it is possible that an incomplete metric space X has a fixed point for every contraction f as shown in Suzuki and Takahashi's paper on fixed point theorems).[17] Additionally, using intuitive reasoning, Banach's Fixed Point Theorem is relatively simple to understand. If the repeated application of the map causes the distance between two outputs to become smaller and smaller, the outputs are converging towards a unique point, and because the sequence is Cauchy and the metric space is complete, we know that the unique point has to exist within X. The standard proof for Banach's Fixed Point Theorem is below, which is adapted from "Fixed Point Theorems and Applications" by Vittorino Pata: [18]

Proof (Banach's Fixed Point Theorem): Let $x_0 \in X,$ and let X be a complete metric space. Then let $\{x_n\}$ be an iterative sequence defined by $x_{n+1} = f(x_n)$, $n \in \boldsymbol{N}$. Noting that $f(x_n) \in X$ for all $n \in \boldsymbol{N}$ since $x_n \in X$ for all n, f being a contraction implies

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq k\, d(x_n, x_{n-1}),\ \forall\ n \in \boldsymbol{N}$$

for $0 \leq k < 1$. Then we can conclude that

$$d(x_{n+1}, x_n) \leq k\, d(x_n, x_{n-1}) \leq k^2 d(x_{n-1}, x_{n-2}) \leq \ldots \leq k^n d(x_1, x_0) = k^n d(f(x_0,)\, x_0).$$

Let $m \geq 1$, $m \in \boldsymbol{N}$. Then by the triangle inequality,

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}. x_n)$$

with $x_{n+m-1} \in X$. Using an extension of the triangle inequality, we can repeat this step again, concluding that

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n)$$

[17] Suzuki, T., & Takahashi, W. (1996). Fixed point theorems and characterizations of metric completeness, 371-382.
[18] Pata, V. (2019). Fixed point theorems and applications (Vol. 116), 5-6.

$$\leq (k^{n+m} + \cdots + k^n)\ d(f(x_0), x_0)$$

$$= k^n(1+k+\ldots+k^m)\ d(f(x_0), x_0).$$

Using the geometric series formula for the first m-terms, we then have

$$k^n(1+k+\ldots+k^m)\ d(f(x_0), x_0) = k^n((1-k^{m+1})/(1-k))\ d(f(x_0), x_0)$$

$$\leq ((k^n)/(1-k))\ d(f(x_0), x_0),$$

where the last inequality holds because $0 \leq k < 1$ and $d(f(x_0), x_0) \geq 0$. To prove that $\{x_n\}$ is a Cauchy sequence, let $\varepsilon > 0$, and choose N such that $((k^N)/(1-k))d(f(x_0), x_0) < \varepsilon$; we can choose such an N because $0 \leq k < 1$ and $d(f(x_0), x_0) \geq 0$. Then there exists an $N \in \mathbf{N}$ such that for all $n \geq N$, we have

$$d(x_{n+m}, x_n) \leq ((k^n)/(1-k))d(f(x_0), x_0) \leq ((k^N)/(1-k))d(f(x_0), x_0) < \varepsilon$$

because $0 \leq k < 1$. Letting $j = n+m$, we have $d(x_j, x_n) < \varepsilon$ for all $j, n \geq N$. Therefore, $\{x_n\}$ is a Cauchy sequence, and because X is a complete metric space, every Cauchy sequence must have a limit within X. Let x^* be the limit of $\{x_n\}$. Using the fact every contraction is continuous, we know that f is continuous, and based upon a property of continuity,

$$f(x^*) = \text{limit as } x_n \rightarrow x^* \text{ of } f(x_n) = \text{limit as } x_n \rightarrow x^* \text{ of } x_{n+1} = x^* .$$

Therefore, there exists a fixed point x^* of f in X. To prove that x^* is unique, let a^* be another fixed point of the contraction f in X. Then

$$d(x^*, a^*) = d(f(x^*), f(a^*)) \leq kd(x^*, a^*).$$

So $x^* = a^*$ because k must be less than 1, and x^* is a unique fixed point of f in X.

Banach's Fixed Point Theorem has several applications, including the ability to find the existence of solutions to integral equations,[19] and several famous generalizations,

[19] Jleli, M., & Samet, B. (2014). A new generalization of the Banach contraction principle, 1-8.

such as the Boyd-Wong Fixed Point Theorem, the Carisiti Theorem, and the Ciric Fixed Point Theorem.[20] For instance, in the Ciric Fixed Point Theorem, the function *f* does not have to be a contraction or continuous due to its usage of a lower semi-continuous function in its distance metric.[21]

The next theorem of focus is Brouwer's Fixed Point Theorem. Several decades in the making, Brouwer's Fixed Point Theorem has both predecessors and successors; the theorem was even named after Brouwer before he had published it publicly. The first theorem related to Brouwer's Fixed Point Theorem was developed by Henri Poincare, a renowned French mathematician and astronomer, in the late 19th century. Poincare's Theorem used the Intermediate Value Theorem (and later the Implicit Function Theorem) to study periodic trajectories and solutions in a three-body system, and the theorem was steps away from concluding Brouwer's main point. Yet the connection between the two was only seen in the mid 20th century.[22]

And how was Brouwer's Fixed Point Theorem named after him before the paper was even published? The answer is good old-fashioned letters. Brouwer had sent letters in the beginning of 1910 to Jacques Hadamard, a well-known French mathematician, and in the letters, he spoke of his new theorem concerning mapping fixed points of a sphere/ball onto itself. Hadamard was fascinated by both Poincare's and Brouwer's work, and during his work in 1910, he built upon the Kronecker Index, generalizing it until it was almost the same as the Brouwer degree.[23] Therefore, in his note at the end of "Introduction à la théorie des fonctions de variables réelles", Hadamard referenced the

[20] Pata, V. (2019). Fixed point theorems and applications (Vol. 116), 3-5.
[21] Ibid.
[22] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 391-412.
[23] Ibid.

unpublished work of Brouwer, giving credit where credit was due. Brouwer later published his own work a year later.[24]

The first development of Brouwer's topological work was in relation to the hairy ball theorem, which stated that "any continuous vector field on a sphere having even dimension has at least one singular point".[25] It was given its odd name due to the inability to create a perfectly smooth system of hairs when brushing a coconut or similarly circular hairy object. When one envisions hairs as vector fields, it becomes much easier to understand the theorem; assuming that the vector field is continuous, there will always be at least one point that "stands up" on the sphere. The hairy ball theorem also has implications for weather patterns and the fundamental theorem of algebra.[26]

This theorem then developed into Brouwer's Fixed Point Theorem on a sphere, which concluded that "any continuous mapping of an n-dimensional sphere onto itself, having a topological degree different from $(-1)^{n+1}$ has a fixed point".[27] He then generalized the fixed point theorem on a sphere to several other cases.

This paper will be focusing on two other versions of Brouwer's Fixed Point Theorem. The first is the closed unit ball version; by definition, a closed unit ball in the metric space (X, d) is defined as $B^n := \{ x \in X : d(0, x) \leq 1 \}$.[28] The topological unit ball version can be found below. [29]

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[24] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 391-412.
[25] Ibid.
[26] Stewart, I. (1995). *Concepts of modern mathematics*.
[27] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 391-412.
[28] Lebl, J. (2018). Introduction to Real Analysis, Volume I.
[29] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 391-412.

Theorem 2a (Brouwer's Fixed Point Theorem for a Unit Ball): Let B^n be a closed unit ball of $\boldsymbol{R}^n$. If $f: B^n \rightarrow B^n$ is a continuous map, then there exists a fixed point x^* of f in B^n.

Therefore, according to Brouwer's Fixed Point Theorem for a Unit Ball, there must exist a fixed point x^* for the map f on B^n if f is a continuous map on B^n. To understand Brouwer's Fixed Point Theorem for a Unit Ball in a more visual sense, think of a globe (a closed ball in $\boldsymbol{R}^3$). When you spin a traditional globe, the North Pole always stays in the same place, no matter how fast you spin it; even if the globe rotates a little, there is always bound to be at least one point on the globe for which you will start and end at the same place. If B^3 is a globe in $\boldsymbol{R}^3$, the continuous map f is the rotation around the poles of the globe from the "places where you begin" on the globe to the "places where you end" on the globe. No matter how large the rotation is, the North Pole will always stay the same, and the point that stays the same is the fixed point of f on the globe. The South Pole is also a fixed point in this scenario.

The theorem can also be generalized to a non-sphere setting involving convex compact sets. A non-empty compact convex subset X of $\boldsymbol{R}^n$ is a subset that is closed, bounded, and convex and has at least one element; in $\boldsymbol{R}^n$, compact is equivalent to closed and bounded.[30] Firstly, a subset X of a metric space is closed if it contains all of its limit points (otherwise known as cluster points or accumulation points); by definition, a point $p \in X$ is a limit point of X if for every $\varepsilon > 0$, the set $B^{\varepsilon}(p) \cap X \backslash \{p\}$ is not empty, i.e. the point p is a limit point if the neighborhood around p has some other point that is not p.[31] Next, a subset X of a metric space is bounded if there is some constant C such that $d(x_1,$

[30] Munkres, J. (2014). Topology: Pearson New International Edition (2nd edition).
[31] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 263-264.

$x_2) \leq C$ for all $x_1, x_2 \in X$; visually, a bounded set is a set whose elements are all close enough that we could draw a boundary around them.[32] For example, the set $X = (1,2) = \{ x \in \boldsymbol{R}: 1 < x < 2\}$ is bounded with $C=1$, while the set of all real numbers is not bounded (because it goes off into infinity). Finally, the definition of a convex subset was briefly mentioned earlier during the generalized derivative example; in a convex subset X, we can draw a line from any point $x_1 \in X$ to any $x_2 \in X$ and still remain within X.

Therefore, now equipped with the definition of a convex compact set, we can understand the next setting of Brouwer's Fixed Point Theorem: [33]

Theorem 2b (Brouwer's Fixed Point Theorem for a Convex Compact Set): Let X be a non-empty compact convex subset of $\boldsymbol{R}^n$ for any given $n \in N$. If $f: X \rightarrow X$ is continuous, then there exists a fixed point x^* of f in X.

In the one-dimensional case, Brouwer's Fixed Point Theorem recovers the IVT (Intermediate Value Theorem), which states that a continuous function f in the interval $[a,b]$ will always attain every value between $f(a)$ and $f(b)$.[34]

Brouwer's Fixed Point Theorem naturally leads to the next theorem of interest, Kakutani's Fixed Point Theorem. In a poll conducted by J. Franklin in his 1983 article titled "Mathematical methods of economics", over 95% of mathematicians could state the Brouwer Fixed Point Theorem, but only 7% could state Kakutani's conclusion.[35]Although Kakutani's Fixed Point Theorem is much less known compared to Brouwer's Theorem, it

[32] Munkres, J. (2014). Topology: Pearson New International Edition (2nd edition), 20.
[33] Dinca, G., & Mawhin, J. (2021). Brouwer Degree, 62.
[34] Carlson, S. (2016). Brouwer's fixed point theorem.
[35] Park, S. (1999). Ninety years of the Brouwer fixed point theorem, 187-222.

draws upon many similar concepts. First, define $P(X)$ as the power set of X, i.e. the set that contains all subsets of X. Then the theorem follows below.[36]

Theorem 3a (Kakutani's Fixed Point Theorem): Let X be a non-empty compact convex subset of $\boldsymbol{R}^n$. Define f as a set-valued map $f: X \rightarrow P(X)$ such that

I. For all $x \in X$, the set $f(x)$ is non-empty and convex;

II. The graph of f is closed, i.e. $Gr(f) = \{(x, f(x)): x \in X\}$;

Then there exists a fixed point x^* of f in X.

The above version of Kakutani's Fixed Point Theorem is the most standard, and it uses many of the definitions that were covered earlier in this paper. Note that the graph f is a subset of the Cartesian product $X \times P(X)$; by definition, the Cartesian product is the product of two sets that contains all of the ordered pairs (x,y) with $x \in X$ and $y \in P(X)$. Consequently, a closed graph is a closed subset of the cartesian product of its domain and codomain.[37] Therefore, the second condition in Kakutani's Theorem can be rewritten to state that the graph f must be a closed subset of $X \times P(X)$.

Let us return to the generalized derivative in order to illustrate Kakutani's Theorem. First let X be a non-empty compact convex subset of $\boldsymbol{R}^n$ for any given $n \in N$, and let $x \in X$. Then recall the definition of Clarke's generalized derivative of a locally Lipschitz continuous f as a set-valued map:

$$\partial f(x) = co\{z \in \boldsymbol{R}: \text{there exists a sequence } x_n \rightarrow x \text{ such that } x_i \in X \backslash Z_f \text{ for all } i \text{ and } f'(x_n) \rightarrow z\}.$$

[36] Osborne, M. J., & Rubinstein, A. (1994). A course in game theory.
[37] Lebl, J. (2018). Introduction to Real Analysis, Volume I, 14.

The first condition is satisfied by the definition of the generalized derivative; the generalized derivative has to be a convex set for all $x \in X$, and the generalized derivative of any $x \in X$ is always non-empty because f is locally Lipschitz continuous.[38] The second condition is satisfied by the definition of the generalized derivative. Because the map f is a subset of the cartesian product $X \times P(X)$, it is sufficient to prove that graph f is a closed subset of $X \times P(X)$. We already know that f is a subset of $X \times P(X)$, and as we proved earlier, a set is closed if and only if it contains all of its limit points. By Clarke's Proposition 2.6.2(a), which utilizes the Bolzano-Weierstrass Theorem, the graph ∂f is closed.[39] Therefore, because the two conditions are satisfied, there must exist a fixed point x^* of ∂f if $\partial f: X \rightarrow P(X)$ also holds.[40] This is true for numerous functions, one of which is $f(x) = |x|$ with $X = [-1, 1]$.

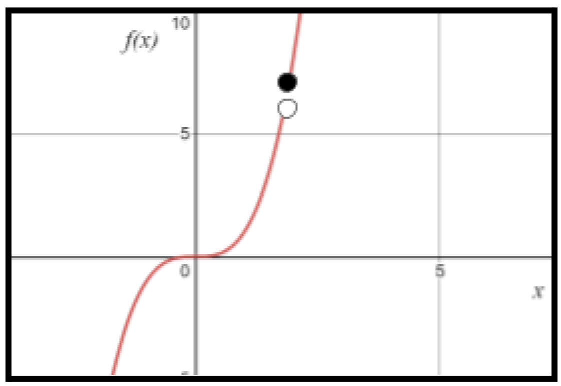
When Kakutani originally wrote his theorem in his 1941 paper titled "A Generalization of Brouwer's Fixed Point Theorem", he stated it in a different way. Before continuing on to the original version of the theorem, two terms need to be defined: simplex and upper semi-continuity. First, an n-dimensional simplex S in $\boldsymbol{R}^m$ with $m \geq n + 1$ is the convex closure of $n + 1$ distinct points $v_0, v_1, v_2 \ldots, v_n$ (i.e. the vertices of S); examples of simplexes include a line segment in $\boldsymbol{R}^2$ (1-simplex) and a triangle in $\boldsymbol{R}^3$ (2-simplex).[41] For $n>0$, an n-dimensional simplex also has a definite boundary, implying that it is bounded; because an n-simplex, $n>0$, is closed and bounded, it easily follows that an n-simplex, $n>0$, is compact.[42] Furthermore, let K be a subset of X, and let f be the set-valued map from K to some other set. Then f is an upper semi-continuous map if for

[38] Clarke, F. H. (1990). Optimization and nonsmooth analysis, 70-75.
[39] Ibid.
[40] Ibid.
[41] Maliwal, A. (2016). Sperner's Lemma, The Brouwer Fixed Point Theorem, the Kakutani Fixed Point Theorem, and Their Applications in Social Sciences.
[42] Seifert, H., & Threlfall, W. (1980). A textbook of topology, 39-41.

every $x \in K$ and every open set $U \supset f(x)$, there exists a neighborhood V of x such that if $y \in V$ then $f(y) \subset U$.[43] Note that a set U is open if for every $x \in U$, there exists a $\delta > 0$ such that there exists an open ball of radius δ such that $B(x,\delta) \subset U$; if U is an open set and $x \in U$, then U can also be defined as a neighborhood around X.[44]

Semi-continuity is weaker than the aforementioned property of being continuous because a map f is continuous if f is both upper semi-continuous and lower semi-continuous. For instance, the graph of f below is upper semi-continuous, but not lower semi-continuous or continuous:

Figure Six: Upper Semi-Continuous Function



On the other hand, the set-valued graph of the generalized derivative of the absolute value function (see Figure Four) is upper semi-continuous. This brings us to the next version of Kakutani's Fixed Point Theorem below.[45]

[43] Pata, V. (2019). Fixed point theorems and applications (Vol. 116).
[44] Lebl, J. (2014). Basic analysis: Introduction to real analysis, 237.
[45] Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem, 457-459.

Theorem 3b (Original Kakutani's Fixed Point Theorem): Let S be an n-dimensional simplex, and let f be an upper semi-continuous set-valued map $f: S \to P(S)$. Then there exists a fixed point x^* of f in S.

Kakutani's Fixed Point Theorem also connected Brouwer's Fixed Point Theorem to Von Neumann's Topological Intersection Theorem (which was at the time was part of a very well-known and published treatise by von Neumann). [46]

Theorem 4 (Von Neumann Intersection Theorem): Let X and Y be two non-empty compact convex sets in the Euclidean spaces R^n and R^m with $n,m \in N$. Let U and V be two closed subsets of X x Y. If for all $x \in X$, $U(x) = \{ y' \in Y \mid (x, y') \in U \}$ is non-empty and convex, and for all $y \in Y$, $V(y) = \{ x' \in X \mid (x', y) \in V \}$ is non-empty and convex, U and V must have a common point.

When von Neumann developed his intersection theorem, he did not apply fixed point theory at all; the connection was only made by Kakutani after he proved that both Von Neumann's Intersection Theorem and Von Neumann's Minimax Theorem (discussed later) could be deduced from his own fixed point theorem. For instance, if $X=Y$ and $U(x) = \{ (x, x) \in U$ such that $x \in X\}$ in the above theorem, then Von Neumann's Intersection Theorem simplifies to a version of Kakutani's Fixed Point Theorem.[47] To demonstrate, substitute $X=Y$ and $U(x) = \{ (x, x) \in U$ such that $x \in X\}$. Next, change U to f, and rewrite

[46] Chang, S. Y. (1991). Separation and von Neumann intersection theorems.
[47] Ibid.

other terms using the definition of a closed graph as the Cartesian product of the domain and codomain.

This method of writing Kakutani's Fixed Point Theorem through von Neumann's theorems is particularly useful in several applications, especially in game theory and zero-sum games. As discussed in the next section, Kakutani's Fixed Point Theorem provided the primary proof for Nash's Equilibrium, which revolutionized the method of solving traditional games.

CHAPTER 2

APPLICATIONS IN GAME THEORY

To play a game, one needs a strategy, perhaps even a group of strategies. By definition, a game is "a description of strategic interaction that includes the constraints on the actions that the players can take and the players' interests, but does not specify the actions that the players do take" [48], while a strategy is a "complete contingent plan for a player in a game".[49] As such, games consist of several strategies and plans with several players involved, and the number of players can range from two to infinity. An example of a two player game is flipping a coin, while an example of a game that can be played with an arbitrary large number of players is Jenga.

Let Player A be a player in a game, and let S_a represent the set of strategies (i.e. the strategy space) that Player A can play:

$$S_a = \{s_a^1, \ldots, s_a^m\}, \text{ with m total strategies.}$$

A single pure strategy chosen by Player A would be referred to as s_a^i, such that i is between 1 and m, and several pure strategies are grouped together into a strategy set for each player. In more mathematical terms, strategy sets can also be referred to as m-tuples. By definition, an m-tuple of elements of a set S_a is a function $f: \{1, \ldots, m\} \rightarrow S_a$, and it is commonly written $\{s_a^1, \ldots, s_a^m\}$.[50] Furthermore, let n be the number of players, and let $s = (s_1, \ldots, s_n)$ be the strategy profile.[51] By definition, the strategy profile is the outcome of the game, i.e. the combination of each player's strategy. [52] Then $s_{-a} = (s_1, \ldots, s_{a-1}, s_{a+1}, \ldots s_n)$ is

[48] Osborne, M. J., & Rubinstein, A. (1994). A course in game theory.
[49] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book], 22.
[50] Munkres, J. (2014). Topology: Pearson New International Edition (2nd edition).
[51] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book].
[52] Katz, A., Williams, C., & Strandberg, A. (n.d.). Nash Equilibrium.

the strategy profile of everyone in the game except for Player A, so we can effectively write $s = (s_a , s_{-a})$.[53]

For simplicity's sake, all games within this paper will be finite standard games. These games have a finite number of turns and a finite number of players, and in a finite standard game, a player is often trying to win before the end of all turns.[54] Some examples of finite standard games are chess, curling, and deciding where to go for dinner, while some examples of infinite games are US-Mexico relations and the relationship between humans and nature. In an infinite game, there are an unlimited number of turns and an unlimited number of players; to easily understand infinite games, one could break down infinite games into a series of finite games (like analyzing trade deals during the 21st century between the United States and Mexico instead of analyzing the full relationship between the United States and Mexico as an infinite game).[55]

As an example, imagine you are a frequent member of a potluck. You have the choice of either bringing a dessert or bringing an entree each time, and to keep things simple, assume that there is going to only be one other participant. In this scenario, you always bring dessert, so you bring dessert (Player A); the other participant always brings an entree, so he/she brings an entree (Player B). Therefore,

$$n = 2$$

$$S_a = \{ Entree, Dessert \}$$

$$S_b = \{ Entree, Dessert \}$$

$$s_a = Dessert$$

$$s_b = Entree$$

[53] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book].
[54] Carse, J. (2011). Finite and infinite games, 21.
[55] Ibid.

$$s = (Dessert, Entree)$$

$$s_{-a} = (Entree).$$

So there are two players, two strategies available to Player A, two strategies available to Player B, and two strategies in the strategy profile. Yet what happens if you always bring dessert and the other person always brings dessert? This creates a new strategy profile (or outcome). Therefore, the overall set of all possible strategy profiles is defined as *S*, and in the above example, *S = {(Entree, Entree), (Dessert, Entree), (Entree, Dessert), (Dessert, Dessert)}* is the set of 4 possible outcomes for the potluck. The set of strategy profiles *S* can also be defined as the Cartesian Product of all of the strategy sets for the *n* players:

$$S = S_1 \times S_2 \times \ldots \times S_n = \Pi_j\, S_j\,,\ 1 \leq j \leq n$$

If someone switches their strategy and chooses a different option to always pick, then the strategy profile changes, and a different outcome occurs. All potential outcomes of the game are elements of *S*, however, and by the nature of the game, all players are aware of all of the potential outcomes. Player A and Player B may not know what the other person is bringing before the dinner occurs, but they know that the other player will either be bringing a dessert or entree. In other words, both players have the same information set, i.e. the same information is available to both players at the decision-making stage.[56]

Furthermore, we are also going to assume that both of the players are rational in the standard sense. Mathematically, a rational decision by a player would be to choose the outcome that results in the highest payoff for that player; the standard definition of rationality means maximizing one's perceived satisfaction or highest number of utils. Yet there are many different definitions of rationality based upon the players' motivations. If

[56] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book], 22.

a player is focused on equality, he/she would choose the rational option of trying to make everyone equally happy, while if a player is completely altruistic, he/she would only be focused on making whichever choice made others the happiest (with no regards to his/her own payoff).[57] For the purposes of the dinner game, we will assume that each player is selfish and playing by the typical definition of rationality.

We will also assume that the players in the game are acting simultaneously. In a simultaneous game, players make their decisions at the same time; for instance, if bringing a dish to a dinner party, one wouldn't know whether someone else is bringing a dessert or entree until arriving at the dinner table. On the other hand, if the two guests texted about what they were bringing the day before the event, the game could become sequential, i.e. players are playing in response to each other's previous move. One example of a simultaneous game would be rock-paper-scissors, while one example of a sequential game would be chess.

So you made your decision, and you chose to bring dessert, and Player B made the decision to bring an entree. To determine your satisfaction from bringing dessert, you would evaluate the outcome of your chosen strategy in the game, i.e. the payoff function. Mathematically, the payoff to Player A when evaluated at a strategy s is written as $f_a(s)$, and the payoff is heavily dependent on the strategy of other players.[58] Let us say that the only thing that matters to each player in the potluck game is that they each get to eat some entree and some dessert; if they get both, then each player has a payoff of 1, and if they don't get both, each player has a payoff of -1. If the strategy profile is $s = (Dessert, Entree)$, then all of the players are happy, and $f_a(s) = 1$. On the other hand, if the strategy

[57] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book].
[58] Katz, A., Williams, C., & Strandberg, A. (n.d.). Nash Equilibrium.

profile is s = *(Dessert, Dessert)*, then none of the players are happy, and $f_a(s)$ = - 1. See the payoff matrix below in Table 1 for a summary of $(f_a(s),f_b(s))$ given a particular outcome.

Table One: Simple Payoff Matrix of the Dinner Game

	Entree (Player B)	Dessert (Player B)
Entree (Player A)	(-1, -1)	(1, 1)
Dessert (Player A)	(1,1)	(-1, -1)

When looking at the table above, it appears that both players would be equally happy with s= *(Dessert, Entree)* or s= *(Entree, Dessert)*. Those are the two best solutions for both players, and this can be confirmed by the concept of a Nash Equilibrium.

Pure Nash Equilibrium

In general game theory practice, one can use a Nash Equilibrium to find the most optimal and rational solution. By definition, a Nash Equilibrium is "a set of strategies, one for each of the n players of a game, that has the property that each player's choice is his best response to the choices of the n-1 other players".[59] In other words, it is the best strategy for each player given the knowledge of the other players' strategies. The formal statement of Nash's Equilibrium Theorem can be found below.[60]

Theorem 5 (Nash Equilibrium Theorem): Every finite standard game with a finite number of players has at least one Nash Equilibrium.

[59] Holt, C. A., & Roth, A. E. (2004). The Nash equilibrium: A perspective.
[60] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book].

Mathematically, a Nash Equilibrium can be written as a strategy profile $s = (s_1, \ldots, s_n)$ that has the property that $f_i(s) \geq f_i(s_1, \ldots, s_i', \ldots, s_n)$ for all $i = 1, \ldots, n$.[61] If $f_i(s) > f_i(s_1, \ldots s_i', \ldots, s_n)$ for all n, then s is a strict Nash equilibrium, and if $f_i(s) \geq f_i(s_1, \ldots s_i', \ldots, s_n)$ for each i-th player, then s is a weak Nash Equilibrium.[62] A strict Nash Equilibrium is always better than the other options, while a weak Nash Equilibrium is always equal to or better than the other options.

Nash Equilibria were first developed by John F. Nash in January 1950; at the time, his research focused on finding the equilibrium point of several n-tuples in order to discover the optimal solutions to standard n-person games. In the original paper introducing his equilibrium concept, he used the aforementioned Kakutani's Fixed Point Theorem to prove the existence of an equilibrium point between strategy sets.[63] This involved proving that strategies of the set Y of probability distributions on S_i for each payoff of the i-th player could be mapped to subsets of Y (thus being a set-valued map), and this map would be closed. The set Y is composed of the mixed strategies, a concept that will be covered in the next section. If those conditions were satisfied, then Kakutani's Fixed Point Theorem could conclude that there existed a fixed point of the set-valued map, and this fixed point is what we now call Nash Equilibrium. The extended proof of this concept is found in the next section.

A few years later, Nash released a new version of his proof in his paper titled "Non-Cooperative Games" that was entirely based upon Brouwer's Fixed Point Theorem, specifically for the 6-dimensional closed ball.[64] In his new game, there were three players

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[61] Katz, A., Williams, C., & Strandberg, A. (n.d.). Nash Equilibrium. Brilliant.
[62] Ibid.
[63] Nash Jr, J. F.(1950). Equilibrium points in n-person games. Proceedings of the national academy of sciences, 36(1), 48-49.
[64] Nash Jr, J. F.(1951). Non-cooperative games.

in a non-cooperative poker game, each with their own payoff function and strategy set; in this case, *S* is defined as the set of all strategy profiles of the three players. By perturbing the strategies around the strategy profile, *f* maps the set of all strategy profiles *S* to itself, and since *S* is homeomorphic to a 6-dimensional closed ball, Brouwer's Fixed Point Theorem applies. Therefore, there must exist a fixed point x^* in *S*, and this fixed point is defined as a Nash Equilibrium.[65]

Returning to the potluck game, it appears that the game has two Nash Equilibria, as seen in Table 2.

Table Two: Nash Equilibria of the Normal Dinner Game

	Entree (Player B)	Dessert (Player B)
Entree (Player A)	(-1, -1)	(1, 1)
Dessert (Player A)	(1,1)	(-1, -1)

To find Nash Equilibria in a finite simple game when using a payoff matrix like the one above, look at each column and determine what the best choice for Player A is given Player B's decision. Then look at each row and determine what the best choice for Player B is given Player A's decision (it is completely acceptable to use one's hands to block out rows and columns while completing this). The best choices in the table above are highlighted in pink, and whichever cells of the table have both "best choices" are the Nash Equilibria. In the table above, the Nash Equilibria are *s= (Dessert, Entree)* and *s= (Entree, Dessert)*.

[65] Maliwal, A. (2016). Sperner's Lemma, The Brouwer Fixed Point Theorem, the Kakutani Fixed Point Theorem, and Their Applications in Social Sciences.Ayesha Maliwal - Sperner's Lemma, Etc.pdf

In some games, there is only one Nash Equilibrium (like in the famous Prisoner's Dilemma), while in others such as the dinner game, there can be multiple Nash Equilibria. For example's sake, let us say you (Player A) loves dessert and would be ecstatic if there was nothing but dessert at the dinner party; your dessert-loving heart considers *s= (Dessert, Dessert)* to have a payoff of 2 instead of -1. Then the payoff matrix changes, as do the Nash Equilibria.

Table Three: Nash Equilibria of the Dinner Game With A Love of Dessert

	Entree (Player B)	Dessert (Player B)
Entree (Player A)	(-1, -1)	(1, 1)
Dessert (Player A)	(1,1)	(2, -1)

In Table 3, there is now only one Nash Equilibrium, *s= (Dessert, Entree)*, due to the change in perceived payoff. Furthermore, based upon the above table, one can assume that Player A would always choose to bring dessert, as the payoffs and outcomes for Player A choosing to bring dessert are always better or equal than bringing an entree. In response to this strategy, Player B will likely always bring an entree because he/she knows that Player A will always bring dessert. This gives us an "equilibrium" where both players make the most rational choice that leads to the best payoff while taking into account the actions of the other player; the strategy profile *s= (Dessert, Entree)* is the singular optimal outcome with this particular payoff matrix.

Mixed Strategy Nash Equilibrium

In the above example, we assumed that each player is always going to bring the same thing to each potluck; this is called a “pure strategy” because each player chooses the same option each time.[66] Yet that is not always the case, and to account for different collections of rational choices or randomization, mixed strategies are needed. A mixed strategy is a probability distribution on the set of available pure strategies to a certain player, and it is how one calculates the probability of picking each option each round.[67] For instance, in the most recent table, we could predict that Player A would usually bring dessert and Player B would usually bring an entree in response. Mathematically, a single mixed strategy for the i-th player in the dinner game would be written as

$$\sigma_i = (p_i^{Entree}\, s_i^{Entree}\, ,\, p_i^{Dessert}\, s_i^{Dessert}),$$

$$\text{with } p_i^{Entree} + p_i^{Dessert} = 1 \text{ and } 0 \leq p_i^{Entree},\, p_i^{Dessert} \leq 1.$$

Here the notation above is understood to be equivalent to $\sigma_i = (p_i^{Entree}, p_i^{Dessert}),$ i.e. with only the probabilities of the corresponding pure strategies rather than explicitly writing the strategies. For consistency’s sake and ease of reading, the primary notation will be to include the probability with the pure strategy, i.e. the first notation. Furthermore, the other general formulas for the i-th player follow naturally from the pure strategy formulas referenced in the previous section:

$$\sigma = (\sigma_1, \ldots, \sigma_n)$$

$$\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$$

$$\sigma = (\sigma_i, \sigma_{-i})$$

$$Y_i = \{\sigma_i \mid 0 \leq p_i^m \leq 1 \text{ and } \textstyle\sum_m p_i^m = 1,\; m = m_i = \text{pure strategies of the i-th player}\}$$

[66] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book].
[67] Katz, A., Williams, C., & Strandberg, A. (n.d.). Nash Equilibrium.

$$Y = \Pi_j \, Y_j = Y_1 \; x \; Y_2 \; x \; \ldots \; x \; Y_n, \; n = \text{number of players.}$$

Note that the probability distribution applied to the *i*-th player's set of pure strategies must be equal to 1; in the dinner game, this could be a ½ chance of picking Entree and a ½ chance of picking Dessert. Furthermore, because pure strategies are the building blocks for mixed strategies, we can also express the *i*-th player's set of mixed strategies Y_i as a simplex of dimension $m-1$, with m representing the number of pure strategies of the player (for a refresher on the definition of a simplex, see page 22).[68] As such, the pure strategies are the vertices of the simplex, and a mixed strategy σ_i of the *i*-th player is a point in the set formed by the convex closure of all the *i*-th player's pure strategies. Additionally, because the probability distribution assigned to each pure strategy of the *i*-th player can be any value equal to or between 0 and 1, we can then conclude that each player has an infinite number of probability combinations, i.e. an infinite number of mixed strategies.

We can take this one step further to conclude that $Y = Y_1 \; x \; Y_2 \; x \; \ldots \; x \; Y_n$ must be a subset and exist in the Euclidean space of dimension u, with u= the sum of all of the players' numbers of pure strategies. For instance, in the dinner game, Y would be a subset of $\boldsymbol{R}^4$ because each player has two strategies, implying that Y_a and Y_b both exist in $\boldsymbol{R}^2$.

The next important concept for mixed strategies is the concept of expected payoffs. In the case of mixed strategies, the payoff functions of the form $f(\sigma)$ are "the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies".[69] In other words, using mixed strategies, a player can attach probabilities to all of their pure strategies, and these

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[68] Maliwal, A. (2016). Sperner's Lemma, The Brouwer Fixed Point Theorem, the Kakutani Fixed Point Theorem, and Their Applications in Social Sciences.
[69] Nash Jr, J. F. (1950). Equilibrium points in n-person games.

probabilities will determine the chances of players getting certain payoffs. Imagine you are at a casino, and you have already made some money. You can either keep playing rounds in the hopes of winning, or you can step away with your earnings. Which strategy you choose would be based upon what you expected to earn from each choice; therefore, mixed strategies are commonly used to determine chances and “risks”.

Now return to the dinner with more considerations in mind. Let Player A have the probability of bringing an entree be represented by p, and let Player B have the probability of bringing an entree be represented by q in the normal dinner game ($0 \leq p, q \leq 1$). Because there are only two strategies for each player, we can assume that the probability of bringing dessert for each player is (1-p) and (1-q), respectively. Then in this example,

$$n = 2$$

$$\sigma = (\sigma_a, \sigma_b) = ((ps_a^{Entree}, (1-p)\, s_a^{Dessert}), (qs_b^{Entree}, (1-q)s_b^{Dessert}))$$

$$Y_a = \{\sigma_a \mid 0 \leq p \leq 1\} = \{(ps_a^{Entree}, (1-p)\, s_a^{Dessert}) \mid 0 \leq p \leq 1\}$$

$$Y_b = \{\sigma_b \mid 0 \leq q \leq 1\} = \{(qs_b^{Entree}, (1-q)s_b^{Dessert}) \mid 0 \leq q \leq 1\}$$

$$\sigma_{-a} = (qs_b^{Entree}, (1-q)s_b^{Dessert})$$

$$Y = Y_a \times Y_b\,.$$

As such, the probabilities and associated payoffs become what are known as the expected payoffs, as seen below in Table 4.

Table Four: Mixed Strategies of the Normal Dinner Game

	Entree (Player B)	Dessert (Player B)	Player A E. Payoff
Entree (Player A)	(-1, -1)	(1, 1)	-q + (1-q) = 1-2q
Dessert (Player A)	(1,1)	(-1, -1)	q - (1-q) = -1+2q
Player B E. Payoff	-p + (1-p) = 1 - 2p	p - (1-p) = -1+2p	

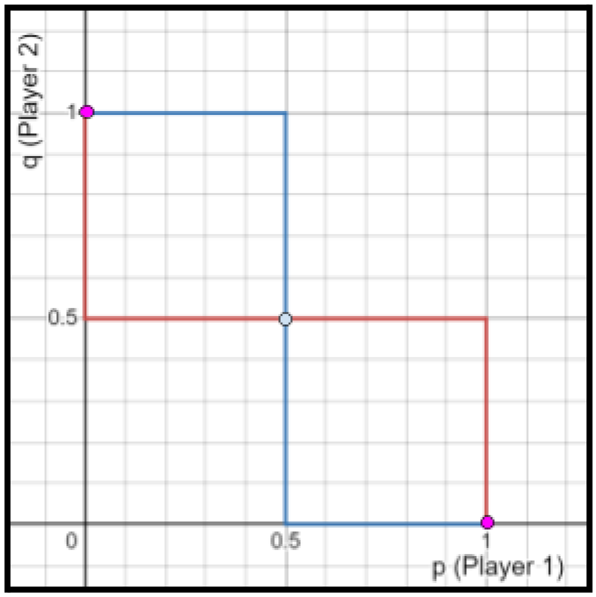
As seen in the table above, Player A should choose to bring an entree when $q < ½$ and choose to bring a dessert when $q > ½$ in order to maximize his/her payoff; if $q = ½$, then either strategy is optimal. Similarly, Player B should choose to bring an entree when $p < ½$ and choose to bring a dessert when $p > ½;$ if $p = ½$, then either strategy is optimal as well. These sets of inequalities are referred to as mixed strategy profiles.

Mixed strategies also have Nash Equilibria, i.e. the values of p and q where both players are indifferent between each of their pure strategies.[70] In other words, a mixed strategy Nash Equilibrium is the profile in which neither player can benefit from choosing a different strategy. A mixed strategy profile σ^* can also be defined as a Nash Equilibrium if $f_i(\sigma^*_i, \sigma^*_{-i}) \geq f_i(\sigma_i, \sigma^*_{-i})$ for all $\sigma_i \in Y_i$ for the i-th player (this follows from the previously mentioned definition of a Nash Equilibrium of pure strategies on page 31).[71] For instance, in the above mixed strategy profiles, it is easy to see that at $p=½$ and $q=½$, both players will "randomize" their options; when the other player has an equal chance of bringing a dessert or an entree, you will also have an equal chance of bringing a dessert or an entree. Therefore, at $(p=½, q=½)$, there is a mixed strategy Nash Equilibrium $\sigma = ((½\ s_a^{Entree},\ ½\ s_a^{Dessert}),\ (½\ s_b^{Entree},\ ½\ s_b^{Dessert}))$. Furthermore, using these

[70] Tadelis, S. (2013). Game theory: an introduction,101-123.
[71] Ozdaglar, A. (2010, February 18). Game Theory with Engineering Applications, Lecture 5: Existence of a Nash Equilibrium [Slides].

inequalities, we can now graph the mixed strategy Nash Equilibrium (for the calculations, please see Appendix A).

Figure Seven: Mixed Strategies of the Normal Dinner Game Graph



In the graph above, the red line is Player 1's best responses to different values of q, and the blue line is Player 2's best responses to different values of p. For example, Player A should choose p=1 when q < ½ and choose p=0 when q > ½ in order to maximize his/her payoff; if q = ½ , then p can be any value between 0 and 1. The two lines intersect at three points: (1,0), (0,1), and (½, ½). The points (1,0) and (0,1) (the two pink circles) represent the two pure Nash equilibria discovered earlier, while (½, ½) (the blue circle) represents the mixed strategy Nash equilibrium.

We can also calculate expected payoff from mixed strategy profiles. By definition, a player's expected payoff is the average payoff a player would get based upon his/her own choice and the probability of the other player playing certain strategies.[72] Let p=½

[72] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book], 38.

and q=½. Focusing on Player A, we can see that Player B has a 50% probability of bringing a dessert and a 50% probability of bringing an entree; therefore, Player A's expected payoff of bringing a dessert is ½ (1) + ½ (-1) = 0, and Player A's expected payoff of bringing an entree is ½ (-1) + ½ (1) = 0. In this case, each player's expected payoff for each strategy is ½ (1) + ½ (-1) = 0 due to the symmetry of the game.[73]

Yet what if Player A develops a love of desserts again? Let Player A have the probability of bringing an entree be represented by p and a payoff of 2 for *s = (Dessert, Dessert)*, and let Player B have the probability of bringing an entree be represented by q in the dinner game. Just as the Nash Equilibria changed in Table 3, the expected payoff matrix will change as well in Table 5.

Table Five: Mixed Strategies of the Dinner Game with a Love of Dessert

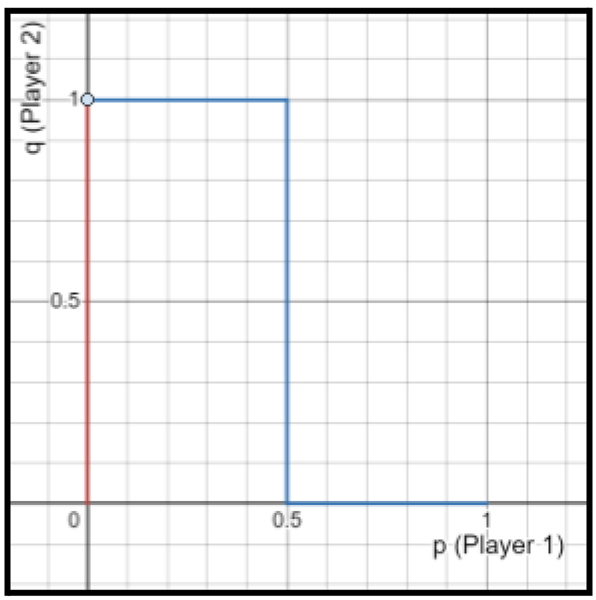
	Entree (Player B)	Dessert (Player B)	Player A E. Payoff
Entree (Player A)	(-1, -1)	(1, 1)	-q + (1-q) = 1-2q
Dessert (Player A)	(1,1)	(2, -1)	q + 2(1-q) = 2-q
Player B E. Payoff	-p + (1-p) = 1 - 2p	p - (1-p) = -1+2p	

Starting again with Player A in the table above, observe that Player A should only choose to bring an entree if q > -1; because p and q represent positive probability values, we can then assume that Player A should always choose to bring dessert (i.e. never bring an entree), regardless of the other player's choice. On the other hand, Player B should choose to bring an entree when p < ½ and choose to bring a dessert when p > ½; if p = ½ , then either strategy is a good choice. As for expected payoffs, Player A has ½ (-1) + ½

[73] Watson, J. (2013). Strategy: An Introduction to Game Theory (Third ed.) [E-book], 38.

(1) = 0 as expected payoff if he/she brings an entree, and ½ (1) + ½ (2) = 1.5 as expected payoff if he/she brings dessert. Therefore, as we previously assumed, Player A should always bring a dessert because $f_a(\sigma_a^{Dessert}) > f_a(\sigma_a^{Entree})$. Player B, however, has 1 (1) + 0 (-1) = 1 as expected payoff with an entree, and 0 (1) + 1 (-1) = -1 as expected payoff with a dessert. Given Player A's strategy, Player B should always bring an entree in order to avoid a negative payoff, i.e. $f_b(\sigma_b^{Entree}) > f_b(\sigma_b^{Dessert})$. The graph of the mixed strategy equilibrium confirms this (see Figure Eight).

Figure Eight: Mixed Strategies With A Love of Dessert Graph



In contrast to the mixed strategy graph of the normal dinner game, this one has only one intersection at (0,1), and this corresponds to Player A always bringing a dessert and Player B always bringing an entree. Therefore, the Nash Equilibrium is $\sigma = ((0, s_a^{Dessert}), (s_b^{Entree},0))$, and in this scenario, the Nash Equilibrium of the pure strategy is the same as the Nash Equilibrium of the mixed strategies. Note that there is always a mixed strategy Nash Equilibrium because the best response strategies of each player must always have a best mixed strategy outcome; every finite standard game with a finite number of players must have at least one "best response mixed strategy outcome", i.e. a Nash equilibrium.[74]

Now equipped with the definition and applications of mixed strategies, we can understand how Nash deduced the existence of equilibrium points. See below for a proof of Nash's Equilibrium using Kakutani's Fixed Point Theorem, adapted from "Game Theory" by Drew Fudenberg and Jean Tirole, as outlined by Ozdaglar in a lecture from 2010: [75] [76]

Theorem 5 (Nash Equilibrium Theorem): Every finite standard game with a finite number of players has at least one Nash Equilibrium.

Proof: Consider a game with *n* players. Let $S_i = \{s_i^1,\ldots,s_i^m\}$, with $m=m_i$ total pure strategies, be the strategy set of the *i*-th player, and let $S=\Pi_j S_j$ with $1 \le j \le n$ be defined as the overall set of all strategy profiles. Let Y represent the overall set of mixed strategy outcomes σ, that is $Y= \Pi_j\ Y_j = Y_1 \times Y_2 \times \ldots \times Y_n$, where each Y_i for the *i*-th player is defined as the set of mixed strategies available to this player and is a simplex with dimension m_i-1. Then let $P(Y)$ be defined as the power set of Y (i.e. a set of all of the subsets of Y). Finally, let g be defined as the set-valued map $g\colon Y \rightarrow P(Y)$ that represents the best response functions of the players; for instance, g_i maps a mixed strategy profile σ to the *i*-th player's set of mixed

[74] Ozdaglar, A. (2010, February 18). Game Theory with Engineering Applications, Lecture 5: Existence of a Nash Equilibrium [Slides].
[75] Fudenberg, D., & Tirole, J. (1991). Game theory.
[76] Ozdaglar, A. (2010, February 18). Game Theory with Engineering Applications, Lecture 5: Existence of a Nash Equilibrium [Slides].

strategies with the highest payoff when the other players play σ_{-i}. Therefore, we can define $g(\sigma)= g_i(\sigma_{-i})$ for all i, where $g_i(\sigma_{-i})$ is the best response of the *i-th* player in response to the other strategies σ_{-i} that are played by the other players. In order to meet the requirements of Kakutani's Fixed Point Theorem, we must first prove that Y is a non-empty compact convex subset of $\boldsymbol{R}$^u, with $u=$ $m_1+\ldots+m_n$ (i.e. the sum of all m_i) :

I. Non-empty: Each strategy of the i-th player has a probability distribution by the definition of a mixed strategy, so Y must be non-empty.

II. Compact: We concluded earlier that all n-simplexes, $n>0$, are compact because they are closed and bounded, and so, each Y_i is a compact set. This holds even if each player has a different number of pure strategies than the others (i.e. in a two player game, Player A has m pure strategies, and Player B has n pure strategies, with $m\neq n$); all of the players will have strategy sets that are represented by an n-simplex, $n>0$, regardless. Therefore, as Y is a Cartesian product of each Y_i , Y must also be compact.

III. Convex: Each Y_i can be defined as an m-simplex with dimension $m-1$, and because all simplexes are convex, all Y_i of all players must be convex. Therefore, as the Cartesian products of convex sets must be convex, Y must also be convex.

Therefore, we can conclude that Y is a non-empty compact convex subset of $\boldsymbol{R}^{n*m}$ for any given $n\in N$ and $m\in N$. Then in order to satisfy Kakutani's Fixed Point Theorem, g has to meet these conditions:

I. For all $\sigma \in Y$, the set $g(\sigma)$ is convex: Note that $g(\sigma)$ is convex if and only if $g_i(\sigma_{-i})$ is convex for all i because $g(\sigma) = g_i(\sigma_{-i})$ for all i. Let $\sigma_i^1, \sigma_i^2 \in g_i(\sigma_{-i})$, and let t_i be defined as all of the other mixed strategy profiles of the i-th player that aren't σ_i. Assume for sake of contradiction that $g_i(\sigma_{-i})$ is not convex for some i. As mentioned earlier, a set $g_i(\sigma_{-i})$ is convex if for any σ_i^1, σ_i^2 and any k such that $0 \leq k \leq 1$, $(k\sigma_i^1 + (1-k)\sigma_i^2) \in g_i(\sigma_{-i})$. Therefore, if $g_i(\sigma_{-i})$ is not convex, there must exist σ_i^1, σ_i^2 and k such that $(k\sigma_i^1 + (1-k)\sigma_i^2) \notin g_i(\sigma_{-i})$. Yet for all k such that $0 \leq k \leq 1$, $k f_i(\sigma_i^1, \sigma_{-i}) + (1-k) f_i(\sigma_i^2, \sigma_{-i}) \geq f_i(t_i, \sigma_{-i})$ because σ_i^1 and σ_i^2 are the best responses of the i-th player and would thereby have a greater than or equal to expected payoff than all of the other mixed strategies. Furthermore, due to the fact that the payoff function of the i-th player is linear,[77] we can also write $f_i(k\sigma_i^1 + (1-k)\sigma_i^2, \sigma_{-i}) \geq f_i(t_i, \sigma_{-i})$. This implies that $k\sigma_i^1 + (1-k)\sigma_i^2 \in g_i(\sigma_{-i})$ because it can be written as the best response for all σ_{-i}. Therefore, $g_i(\sigma_{-i})$ is convex for all i, which implies that $g(\sigma)$ is convex.

II. For all $\sigma \in Y$, the set $g(\sigma)$ is non-empty: As the best response function, $g(\sigma)$ is composed of the maxima of each player's mixed strategy payoff functions; because each player has to have payoffs from mixed strategies, there has to be a mixed strategy that has the highest payoff. That is, each continuous payoff function for the i-th player on Y_i must have a maxima because continuous functions on compact sets (e.g Y_i) have maxima by the Extreme Value Theorem, and we know that the payoff functions are continuous because they are linear.

[77] Fudenberg, D. & Levine, D. K. (1998). The theory of learning in games (Vol. 2), 30.

III. The graph of g is closed: Assume for sake of contradiction that the graph g is not closed. Then there exists a sequence that does not have a limit point in $g(\sigma)$, i.e. a sequence $(\sigma^n,\sigma^{*n}) \to (\sigma,\sigma^*)$, $\sigma^{*n} \in g(\sigma^n)$ such that $\sigma^* \notin g(\sigma)$. Therefore, $\sigma_i^* \notin g_i(\sigma)$ for some i-th player, so the mixed strategy σ_i^* is not one of the best responses for an i-th player to play given certain payoffs. This goes against the criteria of the best response function, so the sequence cannot exist. Therefore, the graph g must be closed.

Hence, by Kakutani's Fixed Point Theorem, there exists a fixed point σ^* of g in Y, and the fixed point σ^* is the mixed strategy profile that is now referred to as Nash Equilibrium.

Therefore, every finite standard game with a finite number of players has at least one mixed strategy Nash Equilibrium, and through using Kakutani's Fixed Point Theorem, abstract mathematics can prove that in a game with n players, there will be an outcome for which all n players will settle. This aligns with the existence of a mixed strategy Nash Equilibrium in the previous two dinner examples.

For a simpler explanation, Nash's Equilibrium Theorem is also supported by von Neumann's Topological Intersection Theorem (**Theorem 4**). As was mentioned before, von Neumann's Topological Intersection Theorem can be simplified to Kakutani's Fixed Point Theorem, so this proof follows naturally from that connection. Let Player A and Player B be two players in a game. Then define each player's set of strategies by S_a and S_b respectively. Let the Cartesian product of S_a and S_b be represented by the overall set of strategy profiles S, and let U and V be two closed subsets of S (we are assuming they are closed for brevity's sake).Then let U be defined as the strategy profile(s) with the maximum payoff for Player A in response to each of Player B's strategies, and let V be defined as the strategy profile(s) with the maximum payoff for Player B in response to each of Player A's strategies. If U and V have common point(s), these point(s) are Nash Equilibria, and this explanation can easily be generalized to fit when n is the number of players with $s = (s_1, \ldots, s_n)$.

Nash's Equilibrium Theorem is also connected to von Neumann's other famous theorem, the Minimax Theorem, in the study of zero-sum games. When Player A and Player B are competing in a zero-sum game, the loss in payoff of one player is equal to the win of payoff of the other player. In this type of game, the Nash equilibrium point can be represented by a saddle point; by definition, a saddle point of a map $f\colon X_1 x X_2 \to \boldsymbol{R}$ is a point $(x_1{*}, x_2{*}) \in X_1 x X_2$ such that $f(x_1, x_2{*}) \leq f(x_1{*}, x_2{*}) \leq f(x_1{*}, x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$.[78] If we take X_1 to be Player A's strategy space and X_2 to be Player B's strategy space, then the strategy profile $(x_1{*}, x_2{*})$ is a Nash Equilibrium.[79] This also follows from von Neumann's Minimax Theorem: every finite two-person zero-sum game has at least one Nash equilibrium of mixed strategies, and they are the maximin *[mixed strategy that gives the largest expected payoff]* mixed strategies.[80] Von Neumann is incredibly well-known in both mathematics and game theory, and his maximin theorem was easily applied to zero-sum models when they were created. Although none of the games featured in this paper are zero-sum games, it is important to recognize an alternative

[78] Gharesifard, B., & Cortés, J. (2013). Distributed convergence to Nash equilibria in two-network zero-sum games.
[79] Ibid.
[80] Prisner, E. (2014). Game theory through examples (Vol. 46), 201-203.

method of determining the existence of a Nash Equilibrium that is not the matrix method or Kakutani's method.

In conclusion, game theory is straightforward to understand. A group of players make rational (and sometimes selfish) choices with the information available, and a variety of outcomes are achieved. For example, in the dinner game, two dinner party guests had to choose between bringing an entree or bringing a dessert; as seen in the last few pages, expected payoffs can quickly change based upon editing a single preference. Additionally, the connection between fixed point theory and game theory is very strong, especially when considering the theorems referenced in this paper. Like a chain reaction, each mathematician influenced the next and/or built upon the mathematicians before him, culminating in the concept we know today as Nash Equilibria. And now equipped with the basics of game theory, it is time to move onto the background behind the Potluck Metaphor.

CHAPTER 3

ANALYSIS OF INTEGRATION IN THE EUROPEAN UNION

The most important aspect of integration policy is defining what integration is. Everyone and every country in the European Union has a different definition for it, and this makes it difficult to judge what is the best example of integration. For instance, as seen in the French hijab and kippah controversy of 2016, some saw religious pluralism (i.e. multiculturalism) as a threat to the traditionally Christian French identity. Those that were encouraging the ban believed that integration was only achieved when someone gave up their old way of life, even if that included religion and the right to wear religious headgear. This was met with international outcry, however, as many around the world believed that integration should not force a particular religious belief; becoming French does not mean becoming Christian and adhering to Christian standards of dress.[81] Other potential areas of disagreement over integration include dietary restrictions, second languages, and ethnic enclaves. Discrepancies such as these lead to difficulties in implementation of integration policy, regardless of integration rhetoric. One early document that describes the European ideal of integration is the 1999 Tampere European Council release. In clause 18, it states:

> "The European Union must ensure fair treatment of third country nationals who reside legally on the territory of its Member States. A more vigorous integration policy should aim at granting them rights and obligations comparable to those of

[81] Katz, E. B. (2018). Where Do the Hijab and the Kippah Belong? On Being Publicly Jewish or Muslim in Post-Hebdo France, 99-114.

EU citizens. It should also enhance non-discrimination in economic, social and cultural life and develop measures against racism and xenophobia." [82]

A few years later, in 2003, the Commission of the European Committees published a report on integration and immigration in the European Union, and it noted that many European countries felt that "the policies they have conducted so far had not been sufficiently effective".[83]

Therefore, the 2003 report tried to establish new guidelines for what integration policies should look like in the modern era, especially with policies concerning potential assimilation. It also gave a formal definition for what the modern standard of integration policy should be: "a two-way process based on mutual rights and corresponding obligations of legally resident third country nationals and the host society which provides for full participation of the immigrant".[84] The new definition blended the assimilationist model with the multiculturalism model (discussed below) by ensuring that the policies respect both sides of the agreement, the migrant and the receiving country. The country must guarantee the migrant basic rights, including economic and cultural rights, and the migrant must in turn respect the fundamental norms and values of the receiving country.[85] Neither side must sacrifice their identity. Although this definition was not clearly implemented in real life policy, we will see later on that this is the ideal form of integration due to the mutual respect for values.

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[82] European Council. (1999). Presidency Conclusions. Tampere European Council. *Bulletin of the European Union*, 10-1999.

[83] Commission of the European Communities (CEC). (2003). Communication from the Commission to the Council, the European Parliament, the European Economic and Social Committee and the Committee of the Regions on Immigration, Integration and Employment. COM (2003) 336 Final, 8.

[84] Commission of the European Communities (CEC). (2003). Communication from the Commission to the Council, the European Parliament, the European Economic and Social Committee and the Committee of the Regions on Immigration, Integration and Employment. COM (2003) 336 Final, 17.

[85] Ibid.

The Three Traditional Models of Integration and Their Critiques

There are three traditional models of integration in the European Union: assimilationist, multiculturalism, and exclusionist. In the assimilationist model, it is expected that migrants assimilate into the native culture, giving up their old lifestyles, beliefs, and cultural practices.[86] Through this, a migrant would lose all individuality and blend into what is seen as the "norm" for the receiving country (the "melting pot").

The second model, multiculturalism, emphasizes equality, human rights, minority rights, and diversity. This model recognizes that many countries are not homogenous, so there is often no standard culture that must be adhered to. It is similar to a fruit salad; each fruit keeps its own shape and identity, but they come together to form the salad, regardless of differences. Examples of policy used in this model include permitting dual citizenship, celebrating diversity through cultural festivals, and funding for bilingual education.[87] Multiculturalism is seen, however, as a threat to many traditional-leaning societies.

The final model, exclusionist, focuses on the idea of temporary migration, i.e. the "takeout box". It completely ignores the earlier models' assumption that the migrants are permanent and insists that there is no need to assimilate because the migrants are only in the country for a short term. Through this model, the idea of national identity and belonging to a certain country excludes those who have migrated there, leading to discriminatory policies and restrictive immigration channels.[88]

[86] Castles, S. (1995). How nation‐states respond to immigration and ethnic diversity, 293-308.
[87] Koopmans, R. (2013). Multiculturalism and immigration: A contested field in cross-national comparison, 151.
[88] Castles, S. (1995). How nation‐states respond to immigration and ethnic diversity, 293-308.

Every model, however, has its flaws. Perhaps the most obvious flaw is in the assimilationist model, which is accused of forcing migrants to give up their identities. When a country is following an assimilationist model, more of the burden is on the newer groups attempting to integrate into the majority society, and although a minority group might not give up all of its defining traits when assimilating, a significant portion of cultural diversity is lost.[89] This structure is commonly seen in the traditional image of American immigration and more recently the French ideal.[90]

On the other hand, the multiculturalism model focuses too much on the cultural aspects of human rights, ignoring other societal issues such as unemployment, social isolation, and/or discrimination.[91] As Kymlicka discusses in his 2012 report on the multicultural model, there are several factors that must be considered when trying to implement multiculturalism, including desecuritization of ethnic relations, human rights, border control, diversity of immigrant groups, and economic contributions. For example, he states that "multiculturalism works best when it is genuinely *multi*cultural" [92]; there must be several source countries instead of just one in order to avoid a clear divide between the majority ethnic group and minority ethnic group. The multicultural model itself does not distinguish between countries with one source country vs. multiple source countries. In addition, the objective of celebrating the "authentic" culture of a single ethnic group could lead to divides in that group or the commodification of traditional culture.[93]

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[89] Brown, S. K., & Bean, F. D. (2006). Assimilation models, old and new: Explaining a long-term process. Migration Policy Institute.
[90] Dora Kostakopoulou (2010) Matters of Control: Integration Tests, Naturalisation Reform and Probationary Citizenship in the United Kingdom, 831.
[91] Kymlicka, W. (2012). Multiculturalism: Success, failure, and the future, 1.
[92] Kymlicka, W. (2012). Multiculturalism: Success, failure, and the future, 2.
[93] Kymlicka, W. (2012). Multiculturalism: Success, failure, and the future, 5.

Thirdly, the exclusionist model finds its flaw in the inaccurate assumption that all migration is temporary. Nowadays, many refugees and migrants find themselves living for many years in the host country, unable to return to their home countries because of ongoing political strife or uninhabitable conditions. For instance, the European Union granted around 566,100 citizenships to citizens of non-member countries in just 2018, and this does not even include children that were born in the EU or those who are on long-term visas.[94] As of 2021, 5.3% of the people living in the EU were non-EU citizens, and in 2020, despite pandemic effects, the European Union granted around 620,600 citizenships to citizens of non-member countries.[95] These statistics suggest that many migrants are living in the EU long enough to study for and obtain a member's country citizenship, which contradicts the exclusionist model's main principle.

In conclusion, the three primary models are all valid models through which one can analyze integration policy. None of them are perfect in addressing the demands of all parties, but all of them provide a lens through which to view integration efforts. The assimilationist model is a "melting pot", the multicultural model is a "fruit salad", and the exclusionist model is a "takeout box", and each represents a potential future for European Union members.

Individual Characteristics of Migrants: Age, Gender, Etc.

Before getting into further analysis, another factor that is important to address is the ages and gender of migrants involved. In earlier decades, the stereotypical migrants were individuals who were looking for work and sending remittances home. They were

[94] Eurostat (2020). Migration and migrant population statistics.
[95] Eurostat (2022). Migration and migrant population statistics.

temporary and had no plans of settling down in the receiving countries, so European countries did not have to worry about assimilation.[96] The aforementioned exclusionist model worked because migrants and refugees were mostly temporary and could easily move back to the home country. Nowadays, however, with family reunification and large refugee populations, many migrants travel as families and/or have intentions of settling down in areas with better opportunities and safer lives.[97] This means that European countries must at least consider education for the children, doctors' visits, citizenship paths, and permanent social benefits for migrants. In terms of gender, there were slightly more men that settled in the EU member countries during 2017 than women (54% vs. 46%), but this percentage varies widely based upon the receiving country.[98] For example, Greece's arrivals were mixed in age and gender, while Italy and Spain had over 70% of their arrivals identify as male.[99] There are also many unaccompanied minors who migrate to Europe, sometimes fleeing from war or in search of better opportunities.[100] Integrating a young child would require very different policies than integrating a whole family, such as adoption programs and education, which makes the challenge of European integration even more difficult.

Core Domains of Integration: A Framework

All of these critiques lead to one obvious answer: there is no perfect concept and/or model that will apply to all situations. Instead, the integration policies should be

[96] Dustmann, C. (2000). Temporary migration and economic assimilation (No. 186). IZA Discussion Papers, 2-8.
[97] UNICEF. (2017). Latest statistics and graphics on refugee and migrant children: latest information on children arriving in Europe.
[98] Eurostat (2020). Migration and migrant population statistics.
[99] UNICEF. (2017). Latest statistics and graphics on refugee and migrant children: latest information on children arriving in Europe.
[100] Ibid.

created on a country-to-country basis and generally applied with some basic guidelines. For example, the UK already has a large settled minority population due to colonial resettlement, so policies there should be very different from the more exclusive policies of Germany. The aforementioned three models all have different goals, but there needs to be a general consensus on what is acceptable in terms of integration policy and what falls under the core domains of integration. Every country should try to remain equally committed to human rights and diversity, as well as committed to mutual respect between all actors, while maintaining their different forms of integration policy.

A framework established by Ager and Strang (2008) provides an interesting and comprehensive outline of the main themes of integration, including which are foundational and which have more of a social aspect:[101]

I. **Markers and Means**: Employment, Housing, Education, Health

II. **Social Connection**: Social Bridges Social Bonds, Social Links

III. **Facilitators**: Language and Cultural Knowledge, Safety and Stability

IV. **Foundation:** Rights and Citizenship

As seen above, there are four categories of integration domains: markers and means, social connection, facilitators, and foundation. The four sections in the markers and means category are what someone should expect to receive as part of a society, while the social connection focuses primarily on an individual perception of belonging.[102]

Through this basic framework, European governments can address specific areas of concern, targeting their reform policies to a single issue instead of trying to focus on broader integration reform. For example, an aging European country can choose to

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[101] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.
[102] Ibid

streamline the path to employment at the expense of advanced language acquisition; not only does this increase the workforce, but it also allows migrants and refugees to support themselves at an earlier point. Furthermore, clearer policies are more easily implemented than those that are vague, especially in the technicalities of the immigration and refugee law sphere. Although there are many controversial issues in integration policy that lead to implementation challenges, three of these issues are particularly salient: employment, language and cultural knowledge, and social bridges.[103]

Barriers to Integration: Employment

Employment is one of the most important factors when considering integration because it provides a way for migrants to support themselves and their families, as well as facilitates social ties and improves language skills. Employment gives someone the power to be economically independent and self-reliant, strong and filled with purpose.[104] There are many difficulties, however, in establishing employment as a migrant in a new country and new community, three of which will be discussed below.

First of all, there is often a language barrier between the migrant and the employers, leading the migrant to take jobs that they are overqualified for, but require less experience with the native tongue. A potential solution is the development of career-specific language courses, as will be discussed later on.

Secondly, qualifications that a migrant might have obtained in the home country may not be recognized in the receiving country. For instance, many refugees cannot provide proof of previous qualifications/employment when applying for jobs, and even if

[103] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.
[104] Ibid.

they have the required documents, the certificate/papers may not be recognized.[105] These discrepancies, in addition to the effects of discrimination and language barriers, can make it more difficult for migrants to find jobs. When comparing the employment rate of working age non-EU migrants in the EU to EU nationals, the average employment rate of the non-EU migrants was 55% in 2017, significantly lower than the 68% reported by EU nationals.[106]

Thirdly, some migrants that are attempting to integrate into a European country may have a large amount of mental trauma from their past experiences and/or difficulty getting used to their new environments. This can make it difficult to find careers that support them; in one study conducted on the Vietnamese boat refugees in Norway, the rate of unemployment was very high, and the results of their study indicated that war trauma may have an impact on career choice and integration into the Norwegian labor market. To help alleviate the difficulties posed by the trauma, the study proposed that strong social networks should be established in order to support adjusting to life in Norway.[107] The development of these networks and how to develop them in migrant communities will be discussed in Chapter 5.

Barriers to Integration: Language and Cultural Knowledge

At what level of fluency can someone be considered "integrated"? Do they have to be able to read complex government documents, or is basic day-to-day language enough? And should someone have to know who established the first fish-and-chips shop

[105] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.
[106] European Commission. (2017). Inclusion of non-EU migrants.
[107] Hauff, E., & Vaglum, P. (1993). Integration of Vietnamese refugees into the Norwegian labor market: the impact of war trauma, 388-405.

in Britain in order to be considered British?[108] Is it fair to demand that a third country national know facts that even natural born citizens do not know? These are all questions that are important when thinking about the language and culture requirements of integration policy. Fortunately, the British government and other European governments have attempted to help with native language acquisition through two main ways: classes and interpreters. For instance, with the help of many English classes around the United Kingdom, migrants can improve their English as soon as possible. Student volunteers at universities, Catholic organizations, and general English courses: whatever is best, there is sure to be a class out there.[109] There are many barriers to attending these classes, however, such as lack of transportation, lack of funds, and high demand for limited spots.

The UK is also responsible for the more controversial "Life in the United Kingdom" Citizenship Test, which is summarized by professor Dr. Thom Brooks as "impractical", "inconsistent", "already outdated", "ineffective at judging English proficiency", and desperately in "need for reform".[110] Furthermore, the British test is not unique in its critiques; several European Union members (such as Denmark and the Netherlands) are facing similar backlash for difficult and/or generalized questions about culture. In some cases, the language level required for the cultural exam was higher than the official language level needed for the actual language exam.[111] Through creating these unnecessarily difficult exams, the European governments are showing their lack of commitment towards obtaining citizenship, a notable marker in the aforementioned integration framework.[112]

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[108] Brooks, T. (2013). The 'life in the United Kingdom' citizenship test: is it unfit for purpose?
[109] British Council. (2022). Study English in the UK.
[110] Brooks, T. (2013). The 'life in the United Kingdom' citizenship test: is it unfit for purpose?.
[111] Pochon-Berger, E., & Lenz, P. (2014). *Language requirements and language testing for immigration and integration purposes*, 10.
[112] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.

Barriers to Integration: Social Bridges

One of the most crucial aspects of integrating into a society is being accepted as a normal member of that society. By definition, social bridges are social connections that bind members of a community together and allow for open, friendly communication.[113] If a European Union member country is polarized along ethnic or racial lines, then it becomes an unwelcome environment, losing necessary open dialogue between the separate cultural groups. Most people, including refugees and non-refugees, feel more welcome and "at home" based upon the perceived level of friendliness of their neighbors and community. [114]

In general, the more negative the public perception of migrants is, the more restrictive the integration policies are. As seen in the "general threat vs. 2007 MIPEX (Migrant Integration Policy Index)" graph produced by Lambert et. al. in the paper "Attitudes towards immigration in Europe: myths and realities"[115], the "score on the total MIPEX" is a measure of how well migrants have integrated into general society, and the perceived general threat evaluates the public perception of migrants. In the graph, when the perceived general threat is high, the MIPEX score is low, and when the MIPEX score is high, the perceived general threat is low. In other words, countries with a high level of perceived threat from migrants will have less integration present in their societies. For example, EU members such as Portugal and Sweden have low perceived threat and a

[113] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.
[114] Ibid.
[115] Lambert, J., Fitzgerald, R., Callens, M., Dennison, J., Ford, R., & Hangartner, D. (2017). Attitudes towards immigration in Europe: myths and realities. *Brussels, Belgium: European Parliament*.

high MIPEX score, while Cyprus and Austria have high perceived threat and a low MIPEX score. [116]

Unfortunately, this leads to a stronger divide between the general public and incoming migrants and becomes a self-fulfilling prophecy. When the public perceives migrants as a threat, more restrictive policies arise in order to placate the public, and as more restrictive policies develop, the rhetoric delivered to the public is more and more negative; this is called the "policy-opinion circle". For instance, in European countries with a high level of threat perceived, the public and the migrants will slowly become more hostile towards the other side.

The second social barrier is the development of the "other". It is easy to talk about community integration and friendliness, but it is a completely different matter to see it implemented. Therefore, integration may not only be resisted by the government and general population; integration may also be resisted by the migrants. As mentioned above, the critique of the multiculturalism model is that cultural pluralism can increase the chances of ethnic isolation, the phenomenon where members of an ethnic community withdraw from the larger society and only interact with members of their own community. This is beneficial to the adjustment of the migrant into the receiving country, but it decreases the amount of connection between the majority and minority, leading to inaccurate perceptions of the "other" and a decrease in open communication.[117] For example, in the United Kingdom in the summer of 2001, there were several riots in which it was made clear how separate the two groups had become. There was "an almost

[116] Lambert, J., Fitzgerald, R., Callens, M., Dennison, J., Ford, R., & Hangartner, D. (2017). Attitudes towards immigration in Europe: myths and realities. *Brussels, Belgium: European Parliament*.
[117] Kostakopoulou, D. (2010). Matters of control: Integration tests, naturalisation reform and probationary citizenship in the United Kingdom, 832.

complete segregation based on race" during these riots, causing many to question the benefits of ethnic enclaves. [118] On one hand, ethnic enclaves increase the diversity of an area and create a large pull factor for future migrants, but on the other hand, they can divide an already polarized population if there is no dialogue between the enclaves and the rest of the city.

In conclusion, one can see that there are already many pre-existing barriers to integration. Integration comprises many forms of social, economic, and political memberships, and the two primary actors are the European governments and the immigrants. The Potluck Metaphor in the following chapter will attempt to add clarity to this complicated dynamic by reframing it as a game and suggesting future policy innovations through potential game solutions.

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[118] Cantle, T., Kaur, D., Athar, M., Dallison, C., Wiggans, A., & Harris, J. (2006). Challenging local communities to change Oldham.

CHAPTER 4:

THE POTLUCK METAPHOR

Everyone has heard of the phrase “bringing a lot to the table”, and in the Potluck Metaphor, that’s the objective. In a previous chapter, the dinner game was fairly uncomplicated; there were two options and two players. The payoffs were symmetric and equal, so when both players lost, they lost equally. In the Potluck Metaphor however, each of the two players has four options of a dish to bring to the potluck: chips, appetizer, entree, and dessert. Each dish also has its own consequences for the potluck, and each player knows them. Furthermore, in discussing payoffs for the Potluck Metaphor, we will be using a new term: utils. By definition, a util is a measure of satisfaction used in economics to describe the perceived benefit or loss of an individual; it is an abbreviation for a measure of utility. If a player has a high payoff, then they have positive utils, while if a player has a negative payoff, they have negative utils.

In the Potluck Metaphor, the implications are as follows:

I. **Chips:** Bare minimum; not accommodating; expects the other person to bring everything for the potluck.

II. **Appetizer:** Still mostly selfish; considers the other person a little bit; prioritizes their own needs above most things; expects the other person to bring more.

III. **Dessert:** Ready to start trying; effort is there; put some thought into their contributions; still has some considerations that hold them back.

IV. **Entree:** Brought a lot to the table; ready to integrate/accommodate; ready to contribute to the discussion; not afraid to spend what is needed.

When a player brings a chips/appetizer, they are implying that the larger burden is on the other player; in integration contexts, this means that the player is expecting the other to be more accommodating and put in more effort towards integration. On the other hand, if a player brings a dessert/entree, they are showing that they are more willing to contribute resources and time to successful integration practices; these efforts could include language classes, job training programs, or community festivals. In the next two chapters, the Potluck Metaphor hopes to create a comprehensive way of studying several primary integration models and how the members of the European Union and their migrant populations can move between them.

The players in the Potluck Metaphor are the migrants (Player A) and the general EU member country (Player B). As discussed in the last chapter, the migrants are coming from a variety of countries, and they generally care about creating good lives for themselves and their families. Their objective is to live in a welcoming country and retain a high amount of cultural independence. In some situations, the migrants are fleeing from a type of economic/political/ethnic turmoil, so they are searching for a peaceful, safe environment for which to live.

On the other hand, the EU member country is established and powerful. It is not particularly used to having multicultural societies that aren't mostly European, but it does not want to face internal pressure from its citizens or opposing member countries. For simplicity's sake, we are assuming that this general member is representative of the general ideals and policies of the European Union and that the general member country is not undergoing any type of extreme resource stress (i.e. Greece, which is also discussed in the case studies section). Overall, neither side wants to completely sacrifice their sense of identity (i.e. their culture, their way of life, etc.), and neither want to be taken advantage of by the other player. The potluck may stay the same each round, or the dishes might change as policies change; together, the two players can achieve multiple different outcomes: appetizer/entree, entree/entree, dessert/chips, etc.

In this game, the two players act simultaneously, but the game is played many times. The "dish" that each player brings will influence the future choices of the other player, and players are allowed to switch dishes in between turns. Additionally, each turn of the game is determined by a change in policy or situation. An example of a change in dish would be demonstrated anti-migrant sentiment. If there was a large public rally against migrants in response to the member government's welcoming of migrants, migrants would likely feel more discouraged and less welcome (thus moving from Dessert to Appetizer).

There are also some fixed rules of the game. Both players have to bring something to the potluck each turn; neither of them have the option of abstaining from the potluck or refusing to bring anything. Another fixed rule is that the government is not as affected by migrants' selfishness as the migrants are affected by the government's selfishness. If a group of migrants decide to not fully integrate in a country that is completely open and welcoming to integration, very little will happen. On the other hand, if a government decides to be unwelcoming while a group of migrants are attempting to fit in, the migrants would have negative util. A third fixed rule is that each player knows the implications and outcomes of bringing a certain dish. For instance, the government will not show up with strict integration rules and expect the migrants to believe that they are being incredibly accommodating; both of the players know that by bringing “Chips”, they are demonstrating a low commitment to improving integration.

Furthermore, both the migrants and the EU member country have the same amount of information at the beginning of the game, and all of the players know the payoffs and actions available to each. Both are rational, yet they still have to pay some attention to the best interests of the other player. For example, if the migrants make the member country unhappy, the government’s policy may become stricter, and if the member country makes the migrants unhappy, there may be some international pressure (as seen with the infamous hijab ban in France). That brings us to our final piece of background: the presence of actors who are not receiving any payoffs but can still exert influence on the game. Such actors include the international community, the general public, non-profits, etc, and these actors’ influences are accounted for in the implications and payoffs of the game. For instance, if the EU member country decides to bring “Entree” and spend a lot of its resources on integration, there are less resources being given to other programs, which could anger the public, which lowers the country’s utils received from choosing “Entree”. In conclusion, these separate actors are a component of the “environment” surrounding the two main players.

Now, using all of this information, we can understand the Potluck Metaphor:

Table Six: The Potluck Metaphor

	Chips (EU)	Appetizer (EU)	Dessert (EU	Entree (EU)
Chips (Migrants)	(-2, -2)	(-1, -1)	(1, 0)	(0, -1)
Appetizer (Migrants)	(-1, -1)	(0, 0)	(1, 1)	(1, 0)
Dessert (Migrants)	(-2, 1)	(-1, 1)	(2, 2)	(3, 2)
Entree (Migrants)	(-3, 1)	(-2, 2)	(2, 3)	(2, 2)

In the dinner game, the payoff trends were very simple; each player had the same payoff for each outcome with the exception of the dessert-lover iteration. In contrast, the Potluck Metaphor's payoff trends are much more varied due to the complexity of the situation. A few general assumptions for the payoffs are listed below (with Player A and Player B being arbitrary):

I. When Player A brings a larger dish than Player B, Player B has a higher payoff than Player A because Player B is contributing less resources.

II. When Player A and Player B bring the same dish, they both have the same payoff because they are sharing the burden of integration equally.

III. The payoffs for Player A generally increase as Player B brings a larger dish because Player B is putting more effort into integration.

IV. The payoffs for Player A generally decrease as the dishes for Player B become smaller because Player B is putting in less effort into integration.

V. When the environment is hostile (i.e. the two players are both contributing very little), the payoff will trend negative.

VI. When the environment is friendly (i.e. the two players are both contributing larger amounts of resources), the payoff will trend positive.

Next are the exceptions or unique payoffs in the above model and payoffs. One of the more noticeable exceptions is how the migrants decrease in payoff when bringing chips; it is the only occasion where the opposing player bringing a larger dish does not equal or raise the corresponding payoff. This decrease in payoff in *(Chips, Entree)* is due to the assumption that if the government adapts an incredibly open integration process, and migrants have absolutely no interest in integrating, the migrants would have a neutral opinion regarding the situation. They would just exist in strong ethnic enclaves in a welcoming country. On the other hand, the payoff is a little higher in *(Chips, Dessert)* because it represents the desire of the government to change and the movement towards new policies. The outcome could easily move from *(Chips, Dessert)* to *(Dessert, Dessert)* if the government and migrants successfully establish an open dialogue. Another interesting outcome is *(Appetizer, Appetizer)*. In this strategy profile, both players are acting a little less selfish, but the effort towards integration is not fully present. Therefore, players have close to neutral payoffs; they simply wait for the other person to bring a different perspective and/or dish.

Returning again to the model, each player in the potluck will eventually begin to fall into a pattern of bringing the same dish they brought during their previous turns, and this is described as the Nash Equilibria. As discussed in an earlier chapter, the easiest way to find Nash Equilibria is to choose the best response (i.e. highest payoff) of each player given the other player's strategy. As before, the best responses are highlighted in pink in the table below, and the pure Nash Equilibria are the cells that have both payoffs as best responses:

Table Seven: Pure Nash Equilibria of Potluck Metaphor

	Chips (EU)	Appetizer (EU)	Dessert (EU)	Entree (EU)
Chips (Migrants)	(-2, -2)	(-1, -1)	(1, 0)	(0, -1)
Appetizer (Migrants)	(-1, -1)	(0, 0)	(1, 1)	(1, 0)
Dessert (Migrants)	(-2, 1)	(-1, 1)	(2, 2)	(3, 2)
Entree (Migrants)	(-3, 1)	(-2, 2)	(2, 3)	(2, 2)

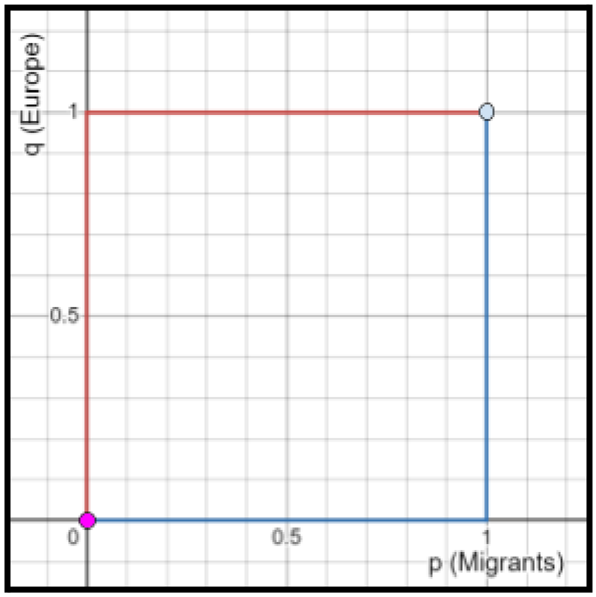
As seen above, there are three pure Nash Equilibria for this game: *(Dessert, Dessert)*, *(Entree, Dessert)*, and *(Dessert, Entree)*. This agrees with the formal definition of what the modern standard of integration policy should be: "a two-way process based on mutual rights and corresponding obligations of legally resident third country nationals and the host society which provides for full participation of the immigrant".[119] The outcomes that have both players working towards successful integration generally have better payoffs because they encourage open dialogue, decrease hostility, and show the potential for future cooperation, while the outcomes that have both players working against each other have divided or poor payoffs. The Potluck Metaphor's three pure Nash Equilibria also suggest that there are multiple options for integration policies to follow depending on the level of effort demonstrated by the EU member country or migrants.

There are also mixed strategy Nash Equilibria present in the Potluck Metaphor (for details of the mixed strategy calculations, see Appendix A). Let p and q represent the probability of Player A and Player B bringing a dessert respectively. Then after several

[119] Commission of the European Communities (CEC). (2003). Communication from the Commission to the Council, the European Parliament, the European Economic and Social Committee and the Committee of the Regions on Immigration, Integration and Employment. COM (2003) 336 Final, 17.

dominated strategy deletions, we can see that there is only one mixed strategy Nash Equilibrium:

Figure Nine: Mixed Strategies of the Potluck Metaphor Graph



As seen before, the pink point (0,0) represents one of the pure Nash Equilibrium, and the blue dot (1,1) represents the mixed strategy Nash Equilibrium. When both p and q are equal to 1, the payoffs of Player A (Blue Line) and Player B (Red Line) from picking either entree or dessert are equal, but by selecting p=1 and q=1, both players are deciding on the pure strategies of always choosing dessert. Therefore, the game has a mixed strategy Nash Equilibrium at $\sigma = ((0,\ s_{Migrants}{}^{Dessert}),\ (0,\ s_{Government}{}^{Dessert}))$. So it looks like we are back to a dessert lovers game, and this is supported by the standard definition of integration.[120] Neither player wants to be taken advantage of, but they both want to seem open and welcoming to the process. By not fully committing to the idea of a

[120] Commission of the European Communities (CEC). (2003). Communication from the Commission to the Council, the European Parliament, the European Economic and Social Committee and the Committee of the Regions on Immigration, Integration and Employment. COM (2003) 336 Final, 17.

completely multicultural society, both sides are preserving their independence and are poised to bring more or less based upon their opponent's actions; it represents the uncertainty of the modern world combined with the hope of future change.

Through implementing the Potluck Model, rhetoric around integration in the European Union and its member countries could shift. It represents the flow of the relationship between the European governments that are used to the old ways and the migrants that have recently begun demanding a voice. The Potluck Model never claims a perfect solution to integration (after all of those critiques of other models in the background chapter, it would be foolish to claim that). Instead, it suggests multiple routes towards the modern definition of integration and ways of interpreting both parties' feelings in a certain political environment. Both the pure and mixed strategy Nash Equilibria suggest interesting insight into the way forward for members of the European Union as they address modern integration.

Applications of the Potluck Model to European Case Studies

In this section, we will apply the Potluck Metaphor to four different countries in the EU: Malta, France, Hungary, and Greece. Each one is representative of different types of member countries in the European Union, from small to large or democratic to illiberal, and as a result, the Potluck Metaphor may change. Remember in the previous section that the Potluck Metaphor was between a general EU member country and the migrants residing within the country. In this section, a policy or trend from four specific countries will be separately analyzed through the lens of the model, and for each case study, the cultures and integration policies will affect the model in a different way. For example, in some cases, a specific policy will change the move of a player, and in others, it will change the payoffs; one case study even leads to the breakdown of the Potluck Metaphor.

The first case study is Malta's integration policies. As mentioned in the next chapter in the "Innovations in Social Bridges" section, Malta used to have a very tense situation between locals and migrants, but with the rollout of several new programs, Malta changed the rhetoric and policy surrounding incoming migrants.[121] The government's and public's perspective on migrants in Malta shifted to become more inclusive and more involved. This would necessitate a new turn because Malta would now be bringing more to the potluck than before, moving their strategy from appetizer to dessert. This can be demonstrated using the original model with a shift in strategy:

Table Eight: Maltese Integration Policies

	Chips (EU)	Appetizer (EU)	Dessert (EU	Entree (EU)
Chips (Migrants)	(-2, -2)	(-1, -1)	(1, 0)	(0, -1)
Appetizer (Migrants)	(-1, -1)	(0, 0)	(1, 1)	(1, 0)
Dessert (Migrants)	(-2, 1)	(-1, 1)	(2, 2)	(3, 2)
Entree (Migrants)	(-3, 1)	(-2, 2)	(2, 3)	(2, 2)

As seen in Table Eight, by changing their policies towards integration, Malta moved from bringing an appetizer to bringing a dessert. This changes the payoffs for all players involved and as we will discuss later on in the next chapter, moves Malta towards a more multicultural model. Furthermore, the positive util increase for all players is

[121] European Commission. (2019). Governance of Migrant Integration in Malta.

supported by the results of the “Integration= Belonging” program; in 2019, there were over 200 students that graduated from the program, and Malta confirmed that it is looking forward to expanding the program even further in the future, with a focus on expanding outreach and collaborating with several NGOs. Malta was traditionally an unfriendly place for migrants to live, so the inclusion of such programs suggest good future outcomes.[122] The perceived benefit from the change in policy is high, and Malta could even move to bringing an Entree within the next few decades.

The second case study we are focusing on is French integration policies. Widely known in France is the concept of laicite, which is the “national political tradition of universalism and secularism”[123], and throughout the past few decades, laicite has become much more than keeping church and state separate. Laicite has become an idea that can impact the control of the government on the private lives of the individual, the most famous case being the Islamic face-covering ban in 2010. The word became severely politicized as France became a multi-religious society, and laicite was used to argue against multiculturalism and all of its implications.[124]

Furthermore, in an analysis of word usage trends in France, integration was primarily used to refer to the mathematical concept of integrating equations until the mid-1980s. Then, as France began to move away from the exclusionist model in the 1980s, the primary meaning of integration shifted; it became a word that represented immigrants and their place in French society. Migrants were not temporary and young men; instead, they were families that looked to establish residency. Then when France

[122] Maltese Government. (2018, August 1). I Belong, Malta’s National Integration Programme.
[123] Bertossi, C. (2020). How the French Understand Immigrant Integration and Citizenship, 5.
[124] Bertossi, C. (2020). How the French Understand Immigrant Integration and Citizenship, 21-24.

began promoting family reunification, integration shifted from being an economic and political concept to a concept of identity and belonging.[125]

As such, integration became a component and opponent of laicite. Some 21st century French integration policies were very accommodating and granted many exceptions, while others were more punitive and targeted. On the other hand, migrants with more religious ties fluctuated in their responses to French policies; there was certainly a desire to fit into French culture, but not the type of French culture that required giving up components of their religions.[126] This ideological disagreement has made creating integration policy difficult, especially with France's struggles with laicite.

So where does France fall in the Potluck Metaphor? Based on the above observations, it seems that France is stuck in the "transitional phase"; this phase is a set of four outcomes in the center of Table 9.

Table Nine: French Integration Policies

	Chips (France)	Appetizer (France)	Dessert (France)	Entree (France)
Chips (Migrants)	(-2, -2)	(-1, -1)	(1, 0)	(0, -1)
Appetizer (Migrants)	(-1, -1)	(0, 0)	(1, 1)	(1, 0)
Dessert (Migrants)	(-2, 1)	(-1, 1)	(2, 2)	(3, 2)
Entree (Migrants)	(-3, 1)	(-2, 2)	(2, 3)	(2, 2)

In the transitional stage, the outcomes can vary as different policies emerge. For instance, the hijab-ban in 2010 pulled the French government towards the left of the

[125] Bertossi, C. (2020). How the French Understand Immigrant Integration and Citizenship, 13-16.
[126] Arslan, B. Z., & Açimuz, B. (2021). Reforming Laïcité or Reforming Islam?: Secularism, Islam, and the Regulation of Religion in France.

transitional stage, i.e. bringing an appetizer. On the other hand, when France created new policies in 2015-2020 that addressed migrant health and arrival support, then the French government moved towards the right of the transitional stage, i.e. bringing a dessert.[127]

Overall, French integration opinions are in a tumultuous state as France tries to determine what corner of the payoff matrix it will settle in, so whichever political party leads France in the future will have a strong impact on the future of French integration. Will both players agree on a definition of secularism as it relates to integration efforts and bring desserts and entrees? Or will they forever be caught up in French vocabulary and exceptions? Table Nine shows us where France is caught, but the information provided in the Potluck Metaphor suggests that France should settle down to the *(Dessert, Dessert)* Nash Equilibrium as long as France follows the same ideals as the European Union. If France deviates from European ideals (as has happened with Hungary), the settling point might change, and the Potluck Metaphor would no longer be applicable. Furthermore, the MIPEX analysis of France in 2020 indicated that France still encourages the public to view migrants as temporary; this would suggest that the French government (on its current policy trajectory) would settle somewhere in the Appetizer column.[128] This is one example of the Potluck Metaphor struggling to match real human behavior.

The third case study is Greece. Greece is an interesting case study because of the large migrant crisis on its borders in the last decade; in a recent collection of studies completed by Pew Research Center, around 82% of Greeks wanted "fewer or no additional migrants to move to their country, the highest share of any country surveyed" in the 2018 study.[129] Unlike many other EU members, Greece is taking the brunt of the

[127] Migrant Integration Policy Index. (2020). France. MIPEX 2020.
[128] Ibid.
[129] Connor, P. (2020). Fast facts on how Greeks see migrants as Greece-Turkey border crisis deepens.

migrant crisis, and other EU members are providing very little ongoing support.[130] This hostility is also reflected in integration policy, where migration and integration are referred to as security and protection issues, and many human rights NGOs are consistently undermined by high-ranking Greek government officials. By fostering fear and discriminatory practices against the new migrants, positive action in integration policy is difficult. [131] Therefore, in a country that is on the frontlines of the migrant crisis like Greece, the Potluck Metaphor would be adjusted as follows.

Table Ten: Greek Integration Policies

	Chips (Greece)	Appetizer (Greece)	Dessert (Greece)	Entree (Greece)
Chips (Migrants)	(-2, -2)	(-1, -1)	(1, 0)	(0, -2)
Appetizer (Migrants)	(-1, -1)	(0, 0)	(1, 1)	(1, -1)
Dessert (Migrants)	(-3, 2)	(-2, 2)	(2, 2)	(3, 1)
Entree (Migrants)	(-4, 2)	(-3, 3)	(2, 3)	(2, 1)

As we can see in Table 10, all of the payoffs where Greece was bringing a dessert or entree decreased by one, as compared to the original Potluck Metaphor; this is to represent the decrease in utils of Greece contributing resources to an already overwhelming economic and political issue. Contributing large dishes like desserts or entrees would exacerbate Greek resources because Greece is already stretched thin.[132] The perceived benefits of bringing such large dishes are very low, so the outcomes

[130] Office of the High Commissioner of Human Rights. (2022, June 22). Greece: Migration policy having "suffocating effect" on human rights defenders says UN expert [Press release].
[131] Ibid.
[132] Office of the High Commissioner of Human Rights. (2022, June 22). Greece: Migration policy having "suffocating effect" on human rights defenders says UN expert [Press release].

wouldn't be as high as in the traditional Potluck Metaphor. The other change is the payoffs of the migrants when the migrants bring a substantially larger dish than the Greek government. In the table above, each of these payoffs also decreased by one in order to represent the larger impact of strict Greek policy; because there are more migrants with fewer resources, the lack of two-way process affects the situation on a deeper level. The Greek government also gained an additional util for these four payoffs due to the level of power maintained and the benefit of contributing less resources.

This gives us two pure Nash Equilibria: *(Dessert, Dessert)* and *(Entree, Dessert)*. Note that the other pure Nash Equilibrium *(Dessert, Entree)* that was present in the original model is gone due to the shift in priorities and perceived benefit of the Greek government; the Greek government does not have the extra resources or the desire to spend resources more than necessary. As compared to the original Potluck Metaphor, the Greek version is harsher, with outcomes that are more negative and payoffs with larger differences. In a country that considers migration a security threat rather than a social issue, such adjustment of the model is needed.

The final case study is Hungarian integration policies. In contrast to Malta and France, Hungary is currently "democratically backsliding", i.e. the process by which a country slides backwards from democracy and Europism. One of the most recent tensions over Hungary democratically backsliding occurred in a European Court of Justice case between Poland, Hungary, and the EU.[133] In this case, Hungary and Poland wanted to block an EU mechanism that withholds funds if any EU laws are violated, and the Hungarian government was concerned about this clause because of recent democratic back-sliding and illiberal democracy trends. In the end, the European Court of Justice

[133] Al Jazeera. (2022, February 16). Poland, Hungary lose legal challenge against EU rule-of-law tool.

(ECJ) ruled that Poland and Hungary did not have a case; the ECJ determined that democratic backsliding had a political impact and a budgetary impact on the European Union. As a result, European countries that deviate from the traditional democratic model (i.e. began to democratically backslide) could face loss of funding.[134] The European Union, through this case, demonstrated that they wouldn't tolerate backsliding, especially from current EU members. A general EU member country would not backslide from democracy, therefore Hungary is slowly moving away from this archetype.

In addition to the democratic backsliding present in Hungary, there is also some controversy surrounding its integration practices; recent changes in integration and immigration policies have led to a less than welcoming environment. Some migrants receive preferential treatment while others stay in camps, and many non-governmental organizations have struggled due to a bill that criminalizes assisting irregular migrants.[135] Furthermore, according to the MIPEX 2020 report on Hungary, the Hungarian government's policies are below average for the EU, with many migrants in Hungary not receiving equal opportunities or access to public services.[136] Therefore, the Hungarian government's perspective on integration is fairly negative and behind the times, and with its current movement away from European ideals, the perceived benefit of strengthening integration policy will likely become smaller as other policies become more paramount. This trend paints a dark picture for the future of integration and the outcomes facing Hungary's migrant population.

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[134] Al Jazeera. (2022, February 16). Poland, Hungary lose legal challenge against EU rule-of-law tool.
[135] Amnesty International. (2022). Amnesty International Report 2021/22: The State of the World's Human Rights.
[136] Migrant Integration Policy Index. (2020). Hungary. MIPEX 2020.

This is where the Potluck Metaphor breaks. So far, we have assumed that the other player is bringing a dish to the table, and the only difference is the scale and effort put in. The closest analogy to what Hungary is doing, however, is bringing chips with meat flavoring to a potluck composed of mostly vegetarians. The treatment of NGOs and the backsliding from democratic values give Hungary a very different behavior that is not compatible with this model. We have also assumed that the other player is a standard member of the EU, yet as mentioned above, Hungary is slowly distancing itself from the European ideal. As such, the Potluck Metaphor is not currently applicable to this case because it fails to represent Hungary accurately.

In conclusion, the Potluck Metaphor proves itself to be useful in most general scenarios. Changes in policies and/or changes in attitude can be reflected in the model, and many different cultures and perspectives can utilize the general framework of the Potluck Metaphor. There are, however, some situations that can cause it to break. For instance, EU members that deviate from the European Union ideals would require a brand new model with different dishes and different payoffs, or in the case of France, the Potluck Metaphor could break due to a devotion to following the exclusionist model. Integration policy is incredibly important, but it is also incredibly complex. There are numerous factors that go into creating integration policy, and not all of them can be accurately represented by a mathematical model. These struggles are discussed at length in the next chapter, as well as an additional interpretation of the Potluck Metaphor.

CHAPTER 5

RECOMMENDATIONS AND CONCLUSION

Table Eleven: Potluck Metaphor Results

	Chips (EU)	Appetizer (EU)	Dessert (EU)	Entree (EU)
Chips (Migrants)	Exclusionist	Exclusionist	Ethnic Enclave	Ethnic Enclave
Appetizer (Migrants)	Exclusionist	Transitional	Transitional	Ethnic Enclave
Dessert (Migrants)	Assimilationist	Transitional	Transitional	Multicultural
Entree (Migrants)	Assimilationist	Assimilationist	Multicultural	Multicultural

The potluck model can also be interpreted through the lens of the popular models provided in Chapter 3. The first thing you may notice in Table 8 is the occupation of each “corner” of the payoff matrix by a certain model. Exclusionist governments become assimilationist governments when the migrants attempt to integrate without help from the government (e.g. the EU member country keeps bringing chips to the table), while ethnic enclaves become multicultural societies when migrants start to work towards full integration. Referring back to Table 7, note that assimilationist outcomes have a much higher payoff to the government, while multicultural models have a more similar positive payoff for both parties. Exclusionist models all tend to have negative payoffs, while ethnic enclaves are rather neutral. Finally, transitional stages’ payoffs tend to vary upon which traditional models they are closest to.

One might ask then how a European government or a group of migrants might move from one potluck to another. If a two-way process is the ideal, how does one get from bringing a bag of chips to bringing a show-stopping main course? For this point, this author would like to offer three main innovations: innovation in employment, innovation in language learning, and innovation in social bridges.

Innovation in Employment

As mentioned in the "Barriers to Integration" section, employment is one of the most crucial factors when developing integration policy. The following two innovations in employment directly address the barriers mentioned there: wider application of the EU Skills Profile for Third Country Nationals and promotion of migrant entrepreneur efforts.

With recent policy changes and the addition of several new tools, many European countries are already attempting to decrease the gap between EU nationals and non-EU migrants' employability. In 2017, the EU Skills Profile for Third Country Nationals was launched; this tool helps translate skills and qualifications of third country nationals to the European equivalent, as well as offer recommendations for future plans (including skills validation, recognition of diplomas, and other support services).[137] Therefore, this tool has the potential to reduce the number of mismatched jobs and increase the speed at which someone can return to their old occupation. When paired with an employment-specific language program (as discussed in the next section), this combination could give many migrants in the EU the capability for long-term fulfilling employment.

[137] European Commission. (2017). EU Skills Profile Tool for Third Country Nationals.

Furthermore, several European countries are beginning to encourage migrants to become entrepreneurs and follow their own business ideas in order to facilitate faster integration through employment. Through creating their own businesses, perfect language acquisition can be delayed, and strong ethnic social networks can be developed to support incoming waves.[138] In one study completed with Belgium's migrant population, the main motive behind becoming an entrepreneur was to quicken the integration process.[139] Examples of "migrant entrepreneur" policies and proposed actions in Europe are found below:

1. The success of Chinese restaurant entrepreneurs in the German restaurant business [140]
2. The promotion of PartecipAzione in Italy, which is an initiative aimed at supporting refugee-led businesses through four primary pillars[141]
3. Inclusion of migrants in self-employment policies and benefits in Germany[142]
4. The Greek "Love Welcomes" workshop which focuses on empowering women and other business owners in a refugee camp in Athens[143]
5. Proposed encouragement of "commercial gentrification" through establishing more migrant-led businesses in migrant neighborhoods in Amsterdam[144]

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[138] Leung, M. W. (2003). Beyond Chinese, beyond food: unpacking the regulated Chinese restaurant business in Germany, 103-118.
[139] Wauters, B., & Lambrecht, J. (2006). Refugee entrepreneurship in Belgium: Potential and practice, 509-525.
[140] Leung, M. W. (2003). Beyond Chinese, beyond food: unpacking the regulated Chinese restaurant business in Germany, 103-118.
[141] Intersos. (2018, January 1). PartecipAzione: Promoting refugee-led organizations. European Website on Integration.
[142] Kontos, M. (2003). Self-employment policies and migrants' entrepreneurship in Germany. Entrepreneurship & Regional Development, 15(2), 119-135.
[143] Fokschaner, S. (2020, September 23). 'We're giving people a reason to wake up': crafting a new life for refugees around the world. The Guardian.
[144] Kloosterman, R. C., & Van Der Leun, J. P. (1999). Just for starters: Commercial gentrification by immigrant entrepreneurs in Amsterdam and Rotterdam neighborhoods, 659-677.

Another innovative option with migrant entrepreneurship is to encourage settled migrants to hire the incoming wave. Not only does this reduce the effects of the preexisting language barrier, but it can also serve as an inspiration to new arrivals. This type of hiring could create a sense of belonging in the new community and show the opportunities the receiving country has to offer. This option could also be available to companies in the area that are trying to adopt social enterprise practices or governments looking to sponsor business with similar missions. For instance, the restaurant Sunhee's Kitchen in Troy, New York was developed by a college graduate with a mission to help the local immigrant community in honor of her own Korean heritage. In addition to hiring local immigrants, the restaurant acts as a community center for English classes and computer literacy programs. The owner, Jinah Kim, has even recently bought two additional spaces to expand her fairly successful mission.[145] Although it is difficult to apply business models that are successful in the United States to Europe, social enterprises and business models like these may be successful in the right European environment.

In conclusion, these types of social enterprises are slowly making their way into the public eye and conscience, and with them come employment opportunities for new immigrants and refugees. Governments could encourage participation in such programs by offering tax credits or pairing businesses with resettlement agencies [146], or as mentioned above, governments could also promote the self-sufficiency aspect of integration through encouraging and supporting migrant entrepreneurs.

[145] *From Farm to Restaurant* (2020). Sunhee's Farm And Kitchen.
[146] Lutheran Immigration and Refugee Services. (2018). *U.S. Employers' Guide to Hiring Refugees*.

Innovation in Language Learning

Although there may never be a general consensus on the level of fluency needed, receiving European countries can do many things in order to facilitate easy language learning, including but not limited to offering translation services, developing career-specific language courses, and promoting hybrid job-language programs.

To begin, offering translation services in the beginning is very beneficial because it helps ease the transition between the country of origin and the European host country. For example, the British government has begun to offer translation and interpreting services at important places, which helps ease the burden of suddenly having to become fluent in English.[147] This is an important step in the "two-way process" as it recognizes that the government has an equal duty in providing interpretation services when compared to the burden on the migrants' shoulders of learning English. Even if someone is fairly good at English, these translation services are still invaluable because it is difficult to navigate social services or health care in a foreign language; this requires a very technical vocabulary. In one healthcare study, both quality of healthcare and satisfaction of providers and patients decreased when there was a significant language barrier.[148] As such, supplying translators and interpreters in more areas around the EU could greatly improve satisfaction and quality of services.

Furthermore, EU members could also focus on developing career-specific language courses. As stated in the Migration Policy Institute's 2011 report on improving immigrants' employment prospects, "implementing effective employment-focused language systems is difficult, as policymakers must find ways to design cost-effective

[147] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework.
[148] Al Shamsi, H., Almutairi, A. G., Al Mashrafi, S., & Al Kalbani, T. (2020). Implications of language barriers for healthcare: a systematic review.

programs that are sufficiently tailored to the needs of a wide range of occupations and that take account of...literacy skills and financial and family circumstances" [149]; although difficult, it is not impossible to offer such courses if funding exists. Language programs can perhaps get around the funding barrier by sorting students into broad job categories such as "hospitality" or "technology"; the classes could also give broad lectures that then narrow into individual work focused on job-specific scenarios.

Moreover, language programs around the world tend to often ignore migrants' individual characteristics in favor of quick crash courses in the native tongue. For instance, broad courses that try to teach English to an elderly man the same way as a young rising professional are doomed to fail. For instance, the English needed by someone entering the workforce is wildly different compared to the English needed by a retiree who will only use English when shopping. In recent years, however, many European countries have picked up on this trend and decided to offer hybrid courses, with Portugal as a particularly interesting example. The Portuguese government recognized a shortage of labor in particular fields, such as hospitality and construction, so it began to offer additional language training with mid-level technical courses in the chosen fields. This not only helped younger migrants integrate faster by giving them employment, but it also provided many trained workers to the previously decreasing sectors. These jobs may not have been the most desirable, but language learning is more beneficial in immersion than in a classroom.[150] Sweden has a similar program in which the language classes group by education level and desired occupation.[151] If these programs could be developed all

[149] McHugh, M., & Challinor, A. E. (2011). Improving immigrants' employment prospects through work-focused language instruction.
[150] McHugh, M., & Challinor, A. E. (2011). Improving immigrants' employment prospects through work-focused language instruction, 3.
[151] Ibid.

over the EU and tailored for each country's employment deficiencies, it could help decrease the effects of the aging European population and provide faster integration for newer migrants.

Innovation in Social Bridges

The final innovation is innovation in social bridges. As discussed earlier, social belonging is incredibly important in a society with multiple different cultures, and in the potluck metaphor, the need to feel welcomed and understood is a crucial influence on payoffs, especially the migrants'. Social bridges represent the connections that tie groups of a society together, and they are also paramount in the two-way integration process.[152]

One of the approaches to encouraging positive social interaction between different groups is the "letting-be" approach suggested by Kostakopolou.[153] This approach "shifts the emphasis away from national identification and towards participation in practices of cooperation" [154], encouraging those in the society to participate in common social activities and become co-citizens *(*e.g. *s= (Dessert, Entree) or s= (Entree, Dessert))*. Examples of this approach in Europe are the Notting Hill Carnival in the UK, Karneval der Kulturen in Berlin, and BogerRio in Antwerp, Belgium.[155]

Creating more shared events and activities like these could immediately address biases and allow for social bridges to form between multiple groups, and through the inclusion of cultural leaders in the planning of such events and a shared sense of learning

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[152] Ager, A., & Strang, A. (2008). Understanding integration: A conceptual framework. Journal of refugee studies, 21(2), 166-191.
[153] Kostakopoulou, D. (2010). Matters of control: Integration tests, naturalisation reform and probationary citizenship in the United Kingdom, 839.
[154] Ibid.
[155] Saeys, A. (2021). Urban Multicultural Festivals: Spectacles of Diversity or Emancipatory Events?, 627-642.

between the two communities, these methods can avoid the commodification critique mentioned in the multiculturalism model. One European Union study even found that multicultural festivals could be “emancipatory events” for migrants, rather than just “feel-good celebrations of diversity”, demonstrating that “living together in diversity is possible, despite political discourses that frame ethnic differences as a cause of social conflicts and tensions”.[156] In other words, some people may regard festivals as places where one can eat different types of food and party, but these celebrations are also a good example of how to create social bridges in a multicultural community.

Furthermore, European governments cannot place all of the responsibility on migrants to blend into a standard norm when trying to integrate because it creates an unproductive power dynamic between the majority and the minority.[157] This power dynamic, when combined with lack of exposure, breeds inaccurate beliefs and generalized racism, but if dialogue and interaction is encouraged, this approach will destroy the perception of the “other”. Governments can also make efforts to combat the perception of the “other” by slowing down the policy-opinion circle. Take Malta for example. As seen in Lambert et. al.’s graph, it used to be the country with the highest perceived general threat across all MIPEX scores. In recent years, however, Malta has implemented new integration programs such as “Integration=Belonging” in December 2017, “I Belong Programme” in July 2018, and a person-to-person approach of integration.[158] Through focusing on belonging and developing strong social bridges

[156] Saeys, A. (2021). Urban Multicultural Festivals: Spectacles of Diversity or Emancipatory Events?, 627-642.
[157] Kostakopoulou, D. (2010). Matters of control: Integration tests, naturalisation reform and probationary citizenship in the United Kingdom, 839.
[158] European Commission. (2019). Governance of Migrant Integration in Malta.

across the different cultural groups, Malta has attempted to change the public rhetoric surrounding migrants.

In conclusion, all of these innovations only begin to scratch the surface of potential changes EU members could make. From degree transferral tools to multicultural festivals, there are a wide range of possible innovations, and each EU member country can choose whichever innovations work best with its ideals. Innovations are drastically needed, however, due to the average 2020 MIPEX score of the countries in the EU-28 being 49/100; this is wildly below the top ten countries' average score of 75/100. Furthermore, many EU members fall into the "halfway favorable" [equal rights, but lack of a secure future] and "halfway unfavorable" [not guaranteed equality and lack of a secure future] policy categories, which indicate that there is much more improvement to be made in the future of European integration policy.[159]

Further Research and Potential Critiques

As with any model, it is important to acknowledge the Potluck Metaphor's limitations. The Potluck Metaphor took a complex real world issue and then reduced it to a dinner party; rather than trying to come to the closest approximation of all aspects of integration policy across the European Union, the model instead attempts to give a simple metaphor for integration using two players: a general EU member and its migrant population. Additionally, applying this type of mathematical model to a real scenario required several assumptions: the simplicity of the model, the lack of outside influence, the attribution of payoffs, the fairness in information, and the human condition.

[159] Migrant Integration Policy Index. (2020). Main Policy Findings. MIPEX 2020.

Due to the interdisciplinary nature and wide focus of this paper, the Potluck Metaphor is a simpler model that lends itself to quick understanding for a wide range of readers. There are likely more complicated models in game theory and mathematics that would lend themselves to this problem, and this author encourages further exploration and/or research of immigration in a game theoretical framework (for more in-depth discussion of the differences between regional migrant/government dynamics, see "Refugee Negotiations from a Game-Theoretic Perspective" by Zeager, L. A., Ericson, R. E., & Williams, J. H.). [160] Other future research could be the adaptation of the Potluck Metaphor for the United States and other powerful countries like China, or the Potluck Metaphor could also be scaled down for local interactions in a multicultural neighborhood.

Furthermore, the model is also simplistic in that this type of model is generally used for individual behaviors, rather than group behaviors. Through utilizing the 2-person normal form game for two groups instead, the Potluck Metaphor is assuming the behavior of the individual is the behavior of the whole. In real life, there are often multiple parties in governments that want different outcomes, just like how there are migrants that have different priorities and different feelings towards integrating into their new country. By assuming that the migrants and the governments make decisions as a whole, the model ignores the complexity of human behavior and group opinions, creating a space where the migrant population can only have one opinion and the government can only have one opinion. Future work can attempt to "average" dishes in order to create well-rounded strategies; for instance, a country with a group of migrants that have no interest in

[160] Zeager, L. A., Ericson, R. E., & Williams, J. H. (2013). Refugee Negotiations from a Game-Theoretic Perspective.

integrating and a group of migrants that want to completely integrate can "average" the migrant opinion to bringing an appetizer or dessert.

Secondly, for simplicity's sake, the Potluck Metaphor limits the number of players to 2, but there could be many more. There are numerous other actors in integration policy, such as the general public, the international community, and sponsors, and the model definitely limits its scope and applicability by only having two players. Taking outside influence into account would also require many more assumptions, such as the extent of external international pressure and the strength of the government over public opinion, all of which would have their own implications. Further research could expand the model to include more players.

Thirdly, the Potluck Metaphor is a social science model with payoffs that were created in an ideal environment. As mentioned before the introduction of the model, the payoffs are somewhat subjective, and there is no doubt that some of the author's biases went into the model during development. Further research could improve the rules and create stricter boundaries for the payoffs, but attributing specific numbers is inherently difficult, especially given the wide and complex motivations of the players involved. The payoffs used in the Potluck Metaphor are measures of utility or "utils", which in and of itself is a completely economic concept. When applying mathematical models to real world scenarios, it is often easier if the payoffs are pre-existing, i.e. monetary amounts or numbers of objects. The implication of using "utils" is that the payoffs are undefined in real life; you cannot measure utils in your hand or on a piece of paper. This thereby implies that the payoffs themselves are assumptions, which is why all of the Potluck Metaphor's conclusions are and need to be robust and general in nature. More specific conclusions about the future of integration policy would require more specific payoffs, so future research could seek to add a more quantifiable form of payoff to the model.

Fourthly, like all models, the Potluck Metaphor exists in a simple and beautiful space. The innovations it describes and the motivations it gives to the two players are simple and fair. In the structure of the game, one of the fixed rules is that each player has the same amount of information regarding the game and players, yet this implies that there is a fairness of information in the real world, which is often not the case. Governments have a lot of power, and without media and international support, the migrants have very little. The implementation of multicultural models through using the Potluck Metaphor could suggest incredibly high benefits, but this would mean nothing to a government that does not care about the other player. The worst that happened in the Potluck Metaphor was an assimilationist society where the government contributed very little; there is not anything in this model that can begin to account for or describe the refugee crisis and treatment of migrants on Europe's border, let alone anywhere else. International pressure and donations have made the situation better, but there is still a crisis that is made worse by COVID-19. For ways to help, one can bring attention to "World Refugee Day" on June 20th, donate to an aid organization such as "Doctors without Borders", or get involved in an Erasmus program in one's local community.[161]

The Potluck Metaphor, therefore, fails to account for the human condition. Players aren't always rational, and players aren't always fair. As seen in the Hungary case study in which the Potluck Metaphor broke, governments can criminalize assistance, and migrants can receive unequal treatment. On the other hand, France still swears by a model that perceives migration as temporary, even when the statistics and

[161] European Youth Portal (2021, September 20). The situation of migrants and refugees in Europe.

non-governmental organizations say otherwise. Both of these countries show the flaws of the Potluck Metaphor, but as concluded at the end of Chapter 4, this doesn't mean the Potluck Metaphor is useless. As a general easy-to-understand model of European integration, the Potluck Metaphor works; on a more specific country-by-country basis, the model can fail.

In conclusion, integration is hard to define, which makes it even harder to implement. Due to the diversity of countries in Europe, it is very difficult to recommend generic integration policies that apply to all of the countries, but if there is no unity and accountability, then human rights can be ignored in favor of discriminatory policies and exams. The burden rests on both the migrant and the government in modern integration, which is a far cry from the lack of help given in earlier decades. Although integration is a two-way street, there are many factors that influence how much each side respects the other, including something as variable as government parties.

Yet there are tiny policies that could change everything, like the inclusion of age-based language courses or cultural orientations that focus on going to the library and enrolling a child in school. From this author's experience as a student ESL teacher in northern Scotland, the biggest barrier towards the students attending class was having to make the choice between a two hour bus ride and asking a friend for a favor; most of the students did not have a reliable form of transportation. Additionally, many of the mothers could never attend class because they could not afford childcare or did not want to leave their children alone. This led to a very poor class turnout, especially because the same students would not be able to come every week. Furthermore, employment programs can utilize the EU Skills Profile for Third Country Nationals in a more effective manner, allowing skills and qualifications to easily transfer between countries, while social enterprises and older settled migrants can provide meaningful employment opportunities that help newer migrants adjust to life in the new country.

We all now live in a globalized world, constantly surrounded by other cultures and people with different backgrounds. Integration is now more important than ever, and it is time for the European Union and all of its members to decide their new futures. The Potluck Metaphor was an attempt to address a very complex issue, but it was created in an ideal mathematical, rational, and fair space. Much like many of the mathematical theorems discussed earlier in this paper, there are many assumptions made, but it is this author's hope that the Potluck Metaphor can be used as a general framework for the ideal methods of integration in the European Union.

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APPENDIX

Figure Seven: Mixed Strategies of the Normal Dinner Game Graph

The following methods are adapted from Tadelis' book on Game Theory and Silz-Carson's video entitled "Mixed Strategy Nash Equilibrium".[162] [163] Let Player A have the probability of bringing an entree be represented by p, and let Player B have the probability of bringing an entree be represented by q in the normal dinner game ($0 \leq p, q \leq 1$). Define Player A's best response correspondence as a correspondence between σ_a and $f_a \sigma_b$, where σ_a is the mixed strategy of Player A and $f_a \sigma_b$ is the expected payoff of Player A given the mixed strategy of Player B. In other words, the input is a mixed strategy of Player A, and the output is the expected payoff of Player A (which is based upon Player B's choice). Then Player A's best response correspondence can be determined as follows:

$$[\sigma_a, f_a \sigma_b] = p(1-2q) + (1-p)(-1+2q)$$

$$= p - 2qp - 1 + p - 2qp + 2q$$

$$=(2q-1) + p(-4q+2)$$

Derivative (or Slope of the Line): $[\sigma_a, f_a \sigma_b]' = -4q + 2$

The best response correspondence is in a linear form with respect to p, so by determining the slope's value, we can determine the endpoint that maximizes the correspondence, i.e. determine the values of p that maximize the best response function. If the derivative is negative, then $p=0$ is the maximizer. In other words, if $q > ½$, the derivative is negative, so Player A's best response in order to get the highest payoff is to bring a dessert and

[162] Tadelis, S. (2013). Game theory: An Introduction,101-123.
[163] Silz-Carson, K. (2020, September 24). Mixed Strategy Nash Equilibrium [Video]. YouTube. https://www.youtube.com/watch?v=0w_r7lkmEvc

choose $p=0$. On the other hand, if $q < ½$, the derivative is positive, so $p=1$ is the maximizer. This is illustrated below.

BR (Red Line): $-4q + 2 > 0 \rightarrow q < ½ \rightarrow$ *maximized at* $p=1$

$-4q + 2 < 0 \rightarrow q > ½ \rightarrow$ *maximized at* $p=0$

$-4q + 2 = 0 \rightarrow q = ½ \rightarrow$ *maximized at* $p \in [0, 1]$

Now to determine Player B's best response correspondence:

$[\sigma_b, f_b \sigma_a] = q(1-2p) + (1-q)(-1+2p)$

$= q - 2qp - 1 + q - 2qp + 2p$

$=(2p-1) + q(-4p+2)$

Derivative (or Slope of the Line): $[\sigma_b, f_b \sigma_a]' = -4p + 2$

Because the function's derivative is only in terms of p, we can determine the value of q that maximizes the best response correspondence. If the derivative is negative, then $q=0$ is the maximizer, while if the derivative is positive, then $q=1$ is the maximizer.

BR (Blue Line): $-4p + 2 > 0 \rightarrow p < ½ \rightarrow$ *maximized at* $q=1$

$-4p + 2 < 0 \rightarrow p > ½ \rightarrow$ *maximized at* $q=0$

$-4p + 2 = 0 \rightarrow p = ½ \rightarrow$ *maximized at* $q \in [0, 1]$

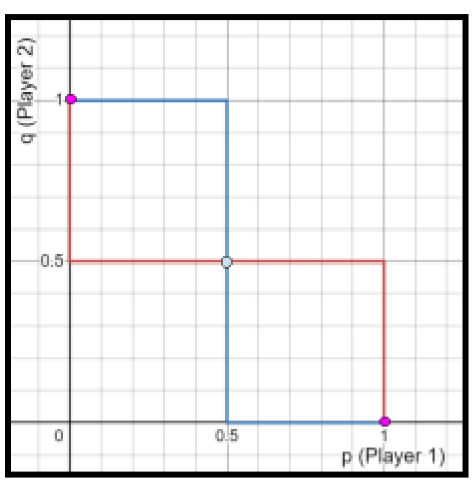


Figure Eight: Mixed Strategies With A Love of Dessert Graph

Let Player A have the probability of bringing an entree be represented by p, and let Player B have the probability of bringing an entree be represented by q in the normal dinner game ($0 \leq p, q \leq 1$). Then

$[\sigma_a, f_a \sigma_b] = p(1-2q) + (1-p)(2-q)$

$= p - 2qp + 2 - q + qp - 2p$

$=(-q+2) + p(-2q-2)$

BR (Red Line): $-2q-2 < 0 \rightarrow q \geq -1 \rightarrow q \geq 0 \rightarrow$ *maximized at* $p=0$

$[\sigma_b, f_b \sigma_a] = q(1-2p) + (1-q)(-1+2p)$

$= q - 2qp - 1 + q - 2qp + 2p$

$=(2p-1) + q(-4p+2)$

BR (Blue Line): $-4p + 2 > 0 \rightarrow p < ½ \rightarrow$ *maximized at* $q=1$

$-4p + 2 < 0 \rightarrow p > ½ \rightarrow$ *maximized at* $q=0$

$-4p + 2 = 0 \rightarrow p = ½ \rightarrow$ *maximized at* $q \in [0, 1]$

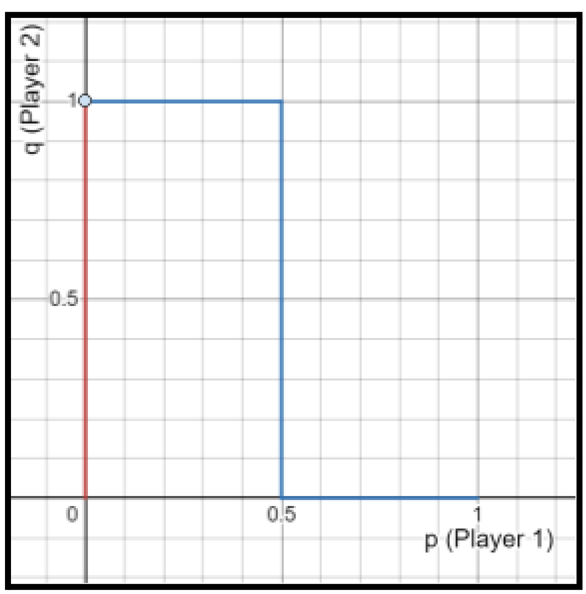


Figure Twelve: Mixed Strategies of the Potluck Metaphor Graph

The Potluck Metaphor had several strategies, so it was necessary to remove the strictly dominated strategies before finding the mixed strategy Nash Equilibrium. By definition, a pure strategy s_i is strictly dominated for the i-th player if there exists another pure strategy that has a higher payoff for all $s_{-i} \in S_{-i}$.[164] In other words, we can eliminate strategies that would never be a good option for a player to take, regardless of what the other players decided to play; this is common practice in finding mixed strategy Nash Equilibria because it reduces the number of strategies and decreases the amount of calculations needed. When eliminating strictly dominated strategies, one goes back and forth between players until no one has any strictly dominated strategies left. Below are the steps taken for the Potluck Metaphor:

1. Chips (EU Member) was strongly dominated by Dessert (EU Member)
2. Appetizer (EU Member) was strongly dominated by Dessert (EU Member)
3. Chips (Migrants) was strongly dominated by Dessert (Migrants)
4. Appetizer (Migrants) was strongly dominated by Dessert (Migrants)

The payoff matrix has also been updated to show the elimination, as seen below:

	Dessert (EU)	Entree (EU)	E. Payoff (Migrants)
Dessert (Migrants)	(2, 2)	(3, 2)	2q + 3(1-q)= 3-q
Entree (Migrants)	(2, 3)	(2, 2)	2q + 2(1-q)= 2
E. Payoff (EU)	2p + 3(1-p)= 3-p	2p + 2(1-p)= 2	

[164] Fudenberg, D., & Tirole, J. (1991). Game theory, 7-8.

Using the updated payoff matrix, we can now construct a graph. Let Player A have the probability of bringing an entree be represented by p, and let Player B have the probability of bringing an entree be represented by q in the normal dinner game ($0 \leq p, q \leq 1$). Then

$[\sigma_a, f_a \sigma_b] = p(3-q) + (1-p)(2)$

$= 3p - 3q + 2 - 2p$

$= (-3q+2) + p$

BR (Blue Line): $q < 1 \rightarrow$ *maximized at* $p=1$

$q = 1 \rightarrow$ *maximized at* $p \in [0, 1]$

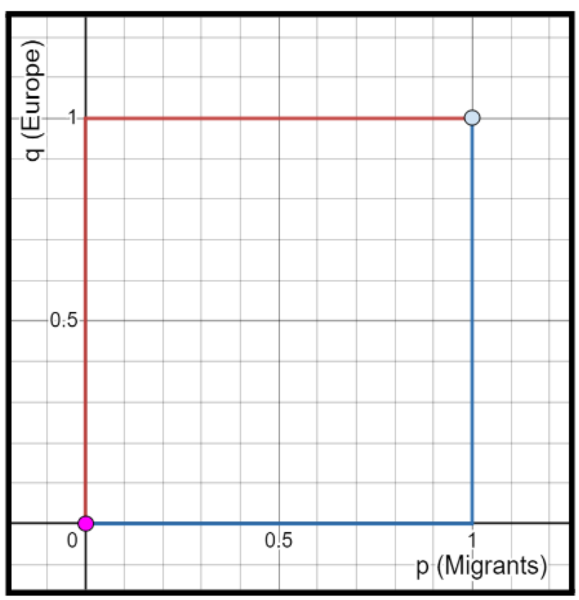
$[\sigma_b, f_b \sigma_a] = q(3-p) + (1-q)(2)$

$= 3q - 3p + 2 - 2q$

$= (-3p+2) + q$

BR (Red Line): $p < 1 \rightarrow$ *maximized at* $q=1$

$p = 1 \rightarrow$ *maximized at* $q \in [0, 1]$



AUTHOR'S BIOGRAPHY

Llewellyn A. Searing was born in Texas on May 29th, 2000. She was raised all over the country, from Georgia to New York, and she graduated from Guilderland High School as a National Merit Scholar in 2018. Majoring in international affairs and mathematics with a concentration in anthropology, Llewellyn graduated from the University of Maine summa cum laude with two Bachelors of Arts in 2022. During her time there, she spent a year abroad in Scotland, became a member of Phi Kappa Phi, played in the University of Maine orchestra, and led tours as a part of Team Maine for all four years. Her biggest accomplishment at the University of Maine was finding a new normal after breaking her neck during her junior year, but she never let it stop her from reaching the stars.

Furthermore, this thesis satisfies the requirements for the Mathematics and International Affairs Capstones at the University of Maine. Although the two fields may seem to have nothing in common, there are in fact many similarities, including but not limited to the usage of the word “integration”, the importance of history, and their relevance to economic applications such as game theory. Therefore, this thesis seeks to bridge the two disciplines through this unique method.

Upon graduation, Llewellyn is attending the Pennsylvania State Dickinson School of Law on a full merit scholarship with stipend. She hopes to become an immigration lawyer and one day change the world for the better.