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The Mathematics Behind Sudoku and How to Create Magic Squares

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THE MATHEMATICS BEHIND SUDOKU AND HOW TO CREATE MAGIC SQUARES

by

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Abstract

Sudoku puzzles date back to the 1800s in France and were introduced to America in the late 1970s. Since then, the puzzle has become a worldwide phenomenon. This thesis will be of expositional nature including the works of books and mathematical papers such as [2], [14], and [15], among others. The pages ahead will contain answers to some common questions about Sudoku such as, what is the minimum number of starting clues that will produce a unique solution? On the other side of the spectrum, what is the maximum number of starting clues that won't produce a unique solution?

Taking Sudoku one step farther, this paper will talk about Magic Squares and the algorithm used in making them. Even more interesting is that the algorithm provides the makings of a multimagic square, where every entry in a magic square is squared, with the rows, columns, and diagonals still adding to the same number [15]. See the Appendix for the computer code in the program MATLAB that creates multimagic squares.

Dedicated to my Dad, the person who instilled in me a love for math. Thanks for being my biggest fan and the person I look up to most.

Acknowledgements

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Contents

Author's Biography

Addison "Addie" LaBonte was born and raised in Maine. Her first love other than mathematics is soccer and she competed for 4 years as a Division I athlete at the University of Maine. Having grown up in York, Maine, Addie has a fond love for the ocean and hopes to raise a family one day in her hometown.

Being an avid and religious Shark Tank watcher, Addie hopes to use her math skills to find a job in the business world someday.

Introduction

Over the past decade or so, Sudoku puzzles have become a household name. Included in almost every newspaper and in many magazines, this popular mathematics puzzle has become a hit with math and non-math lovers at nearly every age group.

The earliest Sudoku related puzzles date back to the late 1800s where a newspaper in France published Latin squares (see Section 1 for examples) [1], which are Sudoku puzzles without the block condition. Then in 1979 in Indiana, Howard Garns, a freelance puzzle inventor published "Number Place" which included placing individual numbers into an empty 9 by 9 grid [1]. Just five years later in 1984 the game first appeared in Japan and was given the name Sudoku, which is short for "Suji wa dokushin ni kagiru", which means "the digits are limited to one occurence." [1] An American named Wayne Gould was visiting Japan and fell in love with the puzzle and brought it back to the United States [1]. Since then, it has become a worldwide sensation.

Why would a simple math game become so popular all over the world? Sudoku puzzles are a fun way to pass time. Their orderliness and organized nature make them particularly intriguing. People of every age and education background can learn how to complete these puzzles fairly easily. Lastly, Sudoku can be a mental challenge and a fulfilling accomplishment when completed.

A seemingly straightforward puzzle, the game of Sudoku can be seen as a gateway into other fields in math. Dissecting the various parts of Sudoku has allowed researchers to connect Sudoku with Latin squares, graph coloring, and even magic squares, among other topics. This thesis will tell the story of how Sudoku puzzles relate to a variety of different math topics that even the most experienced Sudoku player isn't aware of. The following pages will include some of the most intriguing questions about Sudoku and how to answer them; some of the topics even include questions that have yet to be answered and have even eluded many mathematicians.

1 Latin Squares

Definition 1. *A* Latin square *of order n is an $n \times n$ array in which each row and column contains each of the n symbols exactly once. The number n is the* order *of the Latin square.*

Sudoku is a Latin square of order 9, but additionally includes the condition that each block must also contain the numbers 1 through 9. An initial starting clue in any Sudoku puzzle says something about 20 other cells: 8 in its row, 8 in its column, and 4 other cells in its block but not its row or column.

1.1 How to Produce a Latin Square

For a Latin square of order n, the first line has its entries in numerical order, increasing from left to right. Each subsequent line is then produced by being shifted one place to the left. The rows don't pose a potential problem because the first row contains all the n digits and the following rows are just a numerical shift of 1 from the first. The columns also aren't a problem because the k^{th} row is the same as the kth column, as long as the order and numerical shift are relatively prime [2].

Theorem 2. *Let n be a positive integer and let L be the $n \times n$ matrix whose kth row for $1 \leq k \leq n$ is $(k, k+1, ..., n, 1, 2, ..., k-1)$, as read from left to right. Then, L is a Latin square of order n. [2, Chapter 2, Page 32]*

A detailed proof of this can be found in [2, Chapter 2, Page 33].

The j^{th} column from top to bottom is read: $(j, j+1, ..., n, 1, 2, ..., j-1)$. In creating Latin squares, shifting by 2 works for Latin squares of order 3 and 5 but it does not work for orders of 4 or 6. [2, Chapter 2, Page 34]

Lemma 3. *If n is a given order of a Latin square and d is an integer that divides n, then shifting by d will result in repeated rows and an invalid Latin square.*

If $n = d * s$ then shifting the row by d a total of s times will return to the original row. If k is any integer between 1 and $\frac{n}{d}$ inclusive, then row k will be the exact same as row $\frac{n}{d} + k$ (i.e. rows 1 and 4, 2 and 5, 3 and 6). [2, Chapter 2, Page 34]

Lemma 4. *If d is a divisor of n then a Latin square of order n shifted by d cannot be produced.*

If the shift has a divisor (besides 1) in common with the order then a Latin square cannot be produced. So, n and d must be relatively prime.

Definition 5. *The* greatest common divisor*, denoted by gcd is the largest number that divides two numbers. For instance, the $gcd(d,n)$ would be the greatest number that divides both d and n.*

If two numbers, say d and n are relatively prime, then $gcd(d,n) = 1$.

Theorem 6. *Let n and d be positive integers and let L be the $n \times n$ matrix whose k^{th} row is obtained by shifting the first row by d a total of $(k-1)$ times. So the k^{th} row reads: $[(k-1)d+1,(k-1)d+2,...,n,1,2,...,(k-1)d]$. Then L is a valid Latin square if and only if $gcd(d,n) = 1$.*

[2, Chapter 2, Page 35]

This theorem is very technical and can be found in [2, Chapter 2, Pages 35-38].

If $d = 1$ then the kth row is: $[(k-1)(1)+1,(k-1)(1)+2,...,n,1,2,...,(k-1)(1)] = (k,k+1,...,n,1,2,...,k-1)$. When $d = 1$, then for every value of n, $gcd(1,n) = 1$. [2, Chapter 2, Page 36]

In proving this theorem, the rows are all set because each row consists of the numbers from 1 to $(k-1)d+1$ to n, followed by the numbers from 1 to $(k-1)d$ for some value of k. The entries of the column are: $[1,(2-1)d+1,(3-1)d+1,...,(n-1)d+1]$ where each entry is considered modulo n.

An example of a Latin square of order 3 is:

1	2	3
2	3	1
3	1	2

As mentioned previously, this was easily constructed by shifting the row entries one place to the left.

2 Greco-Latin Squares

Definition 7. *Two Latin Squares of order n are* orthogonal *if the square obtained by superimposing them contains each possible ordered pair only once. This new square is referred to as a* Greco-Latin square.

An example of a Greco-Latin square is the following:

1A	2B	3C
2C	3A	1B
3B	1C	2A

This Greco-Latin square was produced by superimposing the following 2 Latin squares:

1	2	3		A	B	C
2	3	1		C	A	B
3	1	2		B	C	A

With a simple proof, it becomes evident that there is no valid 2×2 Greco-Latin square (there would be repeated pairs). Previous research has shown that there are no Greco-Latin square of order 6. A Greco-Latin square of order 3 and 5 can be created with shifting the numbers by one place and the alphabet letters by two places. This previously stated method only works for an order n that is odd because the shift by 2 produces a Latin square only when n is an odd number. This is because an even order and shift of 2 aren't relatively prime. [3]

Research has found that Greco-Latin squares exist for all values of n except 2 and 6. [2, Chapter 3, Page 43] This begs the question: for some n, how many pairwise orthogonal Latin squares can there be at the same time?

Definition 8. *A set of Latin squares is called* mutually orthogonal *if any two of them are orthogonal to each other.*

Lemma 9. *For $n \geq 2$, it is not possible to have more than $n-1$ mutually orthogonal Latin squares.*

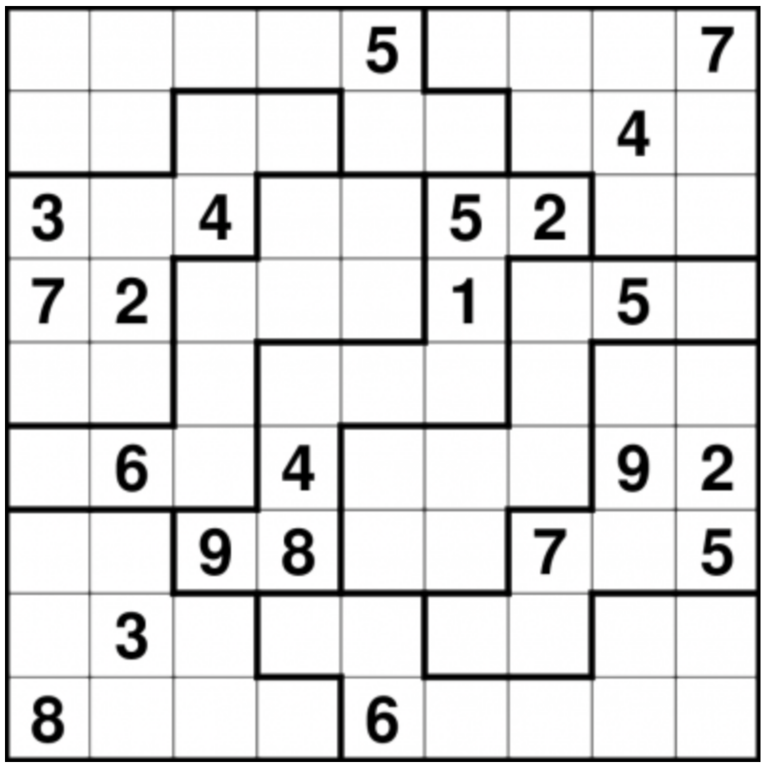
For example, for a 5×5 Latin square, there can be up to 4 mutually orthogonal Latin squares. Values of n that are either primes or a power of a prime attain that upper bound of $n-1$ mutually orthogonal Latin squares, and this is proved in [4]. This is because primes are relatively prime to all numbers less than said prime number [4].

In his 1776 paper titled "De Quadratic Magicis", Euler considered how Greco-Latin squares could be turned into magic squares [3]. This will be touched upon in

Section 9.

Definition 10. *A* Gerechte design *is an $n \times n$ Latin square that has been further subdivided into any n additional regions of size n. These regions don't have to be symmetric.*

This is an example of a Gerechte design.



The numbers 1 through n must then appear once in the row and column, as well as the additional region. Sudoku is an example of a 9×9 Gerechte design because the blocks in the puzzle act as the additional region. For a particular $n \times n$ Gerechte design, the number of mutually orthogonal Latin squares cannot exceed $n - d$. In this case, d is the largest overlap between one of the additional regions and some row or column in the given square. [2, Chapter 3, Page 48]

There exists Greco-Latin 4×4 Gerechte designs with 2×2 blocks. Since the largest overlap between the blocks and the rows and columns is 2, then $d = 2$. So, the maximum number of possibly mutually orthogonal Latin squares would be $4 - 2 = 2$.

Extending this to a 9×9 Greco-Latin Sudoku, every row, column, and block must contain A-I exactly once as well as 1-9 once. Each letter-number combination must appear exactly once in order for it to be a valid Greco-Latin Sudoku. Given the 3×3 block condition of a Sudoku puzzle, $d = 3$ because that is the number of squares that overlap with the rows and columns. Thus, the maximum number of mutually orthogonal Latin squares is $n - d = 9 - 3 = 6$.

3 Counting

Given n objects, there are $n!$ ways of arranging the objects in a straight line. The first object can be any of the n objects, and then the remaining $n - 1$ objects can appear second, etc.

Definition 11. *A* Shidoku *is a* 4×4 *Sudoku board with* 2×2 *blocks. Each row, column, and block contains the numbers 1 through 4.*

In a Shidoku, there are $4 * 3 * 2 * 1 = 24$ ways to fill in the first block. Let x be the number of ways to fill in the remaining 3 blocks; this gives the total number of Shidoku squares as $24x$.

Definition 12. *A Sudoku or Shidoku puzzle has an* ordered *first block if its entries are in numerical order.*

In a Shidoku there are $2 * 2 = 4$ ways of completing the first row and column given that the first block is ordered. Now in order to get the total number of Shidoku boards, we must count the Shidoku squares whose first row, column, and block appear as above and multiply that number by $24 * 2 * 2 = 96$. As seen below, there are only 3 different ways of completing the ordered Shidoku. Given that there are 96 Shidoku squares represented by each ordered square, the number of Shidoku squares is: $96 * 3 = 288$.

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Now, how does this apply to counting Sudoku puzzles? Label the 3×3 blocks of a Sudoku as: $B1$ through $B9$, in numerical order as seen below.

B1	B2	B3
B4	B5	B6
B7	B8	B9

Here, each small square represents a 3×3 block in a Sudoku grid.

Let $B1$ be in standard, or ordered form. So it would read: 1, 2, 3, 4, 5, 6, 7, 8, 9. $B1$ is the following:

1	2	3
4	5	6
7	8	9

There are $9! = 362,880$ other Sudoku squares obtainable from the standard form square simply by relabelling. Once $B1$ is filled in in ordered form, there are 3 possibilities for the top row of block $B2$:

Case 1a: $4,5,6$ in some order,

Case 1b: 7,8,9 in some order,

Case 2: some combination of 4,5,6,7,8,9 in some order.

In each of the 6 rows of blocks $B2$ and $B3$, we can choose any of the 3! permutations of the 3 entries. In case 1a and 1b, filling in the rows of $B2$ and $B3$ would become a 6-step process where each of the 6 mini rows can be completed in 3! ways. This totals $(3!)^6$ ways of completing the rows in each of the case 1 scenarios. So there are $2 * (3!)^6 = 93{,}312$ ways of filling in blocks $B2$ and $B3$ for case 1.

1	2	3	(4)	(5)	(6)	(7)	(8)	(9)
4	5	6	(7)	(8)	(9)	(1)	(2)	(3)
7	8	9	(1)	(2)	(3)	(4)	(5)	(6)

Here, the parenthesis around the numbers mean that any of the 3 choices in that block's row can be chosen for that cell. For example, the uppermost row in $B2$ reads: (4), (5), (6), but the numbers can be permuted.

The entries in the top row in blocks $B2$ and $B3$ are a mix of the numbers from the second and third rows from block $B1$. We can conclude there are $3 * (3!)^6$ ways of filling in the first 3 rows in the scenario. In case 2, we must fill in the first row of $B2$ with 3 digits from 4 through 9 but not 4, 5, 6 or 7, 8, 9. There are 18 ways of choosing 3 digits for case 2 for the first row of $B2$. There are a total of $18 * 3 * (3!)^6 = 2{,}519{,}424$ ways of filling in blocks $B2$ and $B3$ for case 2.

Thus, combining both cases, there are $93,312 + 2,519,424 = 2,612,736$ ways of filling in the first 3 rows of a Sudoku puzzle whose $B1$ block is in standard form. Considering there are 9! ways of filling in the first block, there are $2,612,736 * 9! = 948,109,639,680$ ways of filling in the first 3 rows of a Sudoku board. There are $(9!)^9$ different ways to fill in a 9×9 Sudoku with each number appearing once in each block. The probability that a Sudoku with block-compliancy is additionally row-compliant is:

$$k = \frac{(948,109,639,680)}{(9!)^9} \tag{1}$$

which is also the probability that it's column-compliant.

Lemma 13. *The probability of a puzzle being row-compliant and column-compliant is k^2, if the two probabilities are independent.*

[2, Chapter 4, Page 67]

Compliant means that the Sudoku is valid; for example, row-compliant means that each row contains 1 through 9. Column-compliant means each column contains 1 through 9, and the same holds for block-compliance.

Randomly choose a Sudoku puzzle that is block-compliant. Then the probability that it is additionally row-compliant is k. The probability that this Sudoku is also column-compliant is also called k.

Denoting the probability that a randomly chosen Sudoku puzzle is row-compliant by k and doing the same for column-compliance shows why the probability of both row- and column-compliance probability is k^2. The two probabilities were simply multiplied together.

Assuming independence, we could compute the number of Sudoku squares by multiplying the total number of block-compliant squares by the probability that a randomly selected square is both row- and column-compliant. This computation would be:

$$(9!)^9 * k^2 = (9!)^9 * ((948{,}109{,}639{,}680)^3/(9!)^9)^2 \approx 6.6571 * 10^{21} \tag{2}$$

[2, Chapter 4, Page 68]

Swapping the columns within a block, or swapping the blocks themselves ends up in a new configuration with the same number of completions. Each configuration of the first 3 rows can be turned into a configuration whose first row is ordered by following:

a) if necessary, permute the columns within $B2$ and $B3$ so that the entries in the first row of each respective blocks are increasing;

b) if necessary, exchange $B2$ and $B3$ so that the upper left entry of $B2$ is smaller than said entry in $B3$.

Following this procedure for case 1a) and 1b), there is only one possible way to order the first row. Now both case 1) scenarios have been reduced from $2 * (3!)^6 = 93{,}312$ possible ways to $(3!)^4 = 1296$ configurations, which is a reduction of 72 times. Since there are 6 ways of permuting the columns in $B2$ and 6 ways of permuting the columns in $B3$, step a) above implies there are 36 completions.

Researchers have found using computers that there are

6,670,903,752,021,072,936,960

possible Sudoku squares [5]. But, how many of these are truly different from one another?

4 Equivalence Classes

Two things are said to be equivalent if they possess a common property which allows us to treat them as identical for some purpose. For Sudoku, equivalence can be seen as taking one puzzle and building another one from it via transformations, which will be further discussed.

In working with Sudoku puzzles, it is imperative to remember that two puzzles may not look alike, but a simple switching of a row or column of one puzzle can produce the other.

Definition 14. *Three horizontally adjacent blocks are known as a* band*. Three vertically adjacent blocks are called a* pillar*.*

A valid transformation of a Sudoku means that something in the puzzle was switched to form another Sudoku that is row-, column-, and block-compliant. Switching 2 rows or columns in the same band or pillar would be an example of a valid transformation. On the other hand, switching rows and columns from different bands or pillars does not produce a valid Sudoku transformation. We can however permute the rows within a band or the columns within a pillar.

Any rotation or reflection through an axis of symmetry will lead to another valid Sudoku puzzle. The following reflections lead to another valid Sudoku: 0°, 90°, 180°, and 270°. There are 4 axes to choose: 1 vertical, 1 horizontal, and 2 diagonal.

A list of valid transformations include:

- Type 1 transformation: relabeling

- Type 2 transformation: permuting the rows in a band or the columns in a pillar
- Type 3 transformation: permuting the bands or pillars themselves
- Type 4 transformation: any rotation or reflection
- Type 5 transformation: composing any previous transformation

There are no other types of transformations that we will consider equivalent; this list is complete.

Lemma 15. *If 2 puzzles are fundamentally different, then one is not obtained by the other via the list of transformations.*

Thus, there is no way for one Sudoku puzzle to be transformed into another via the list of transformations if they are considered fundamentally different.

Let a random Sudoku square be s which is equivalent to x other squares. Unfortunately we cannot just divide the number of Sudoku by x because x varies as s does.

Lemma 16. *Two squares are* equivalent *if they differ only by a Type 1 transformation (i.e. relabeling of the digits).*

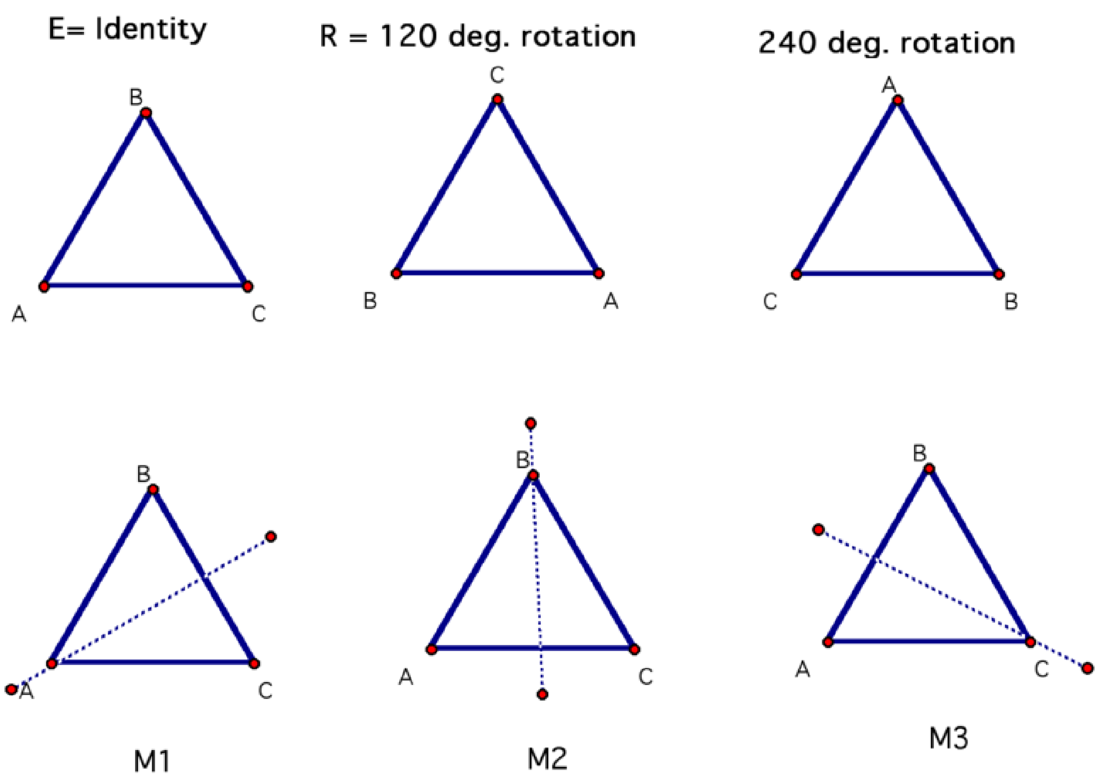
In other words, just a simple relabeling of a Sudoku that leads to another Sudoku means they are not truly fundamentally different.

Transformation types 2 through 5 are permutations of the Sudoku cells. Several consecutive permutations lead to another valid transformation. At least two transformations is known as a composition.

Some different types of transformations are: (these are for use with a triangle)

- Identity = i = doing nothing
- Rotate = r = rotate 120°
- Rotate = R = rotate 240°
- Reflect = a = reflect vertically
- Reflect = b = reflect along right diagonal
- Reflect = c = reflect along left diagonal

This picture shows the different transformations.



[6]

A composition of symmetries is like a product; in this case, $r * a = b$. The composition of any two of these transformations will return us to one of the original 6 symmetries. For instance, rotating by r and then R will return us back to the original, which is the same as i, or doing nothing. When a, b, or c is composed with itself, it returns the puzzle back to its original (which is equivalent to performing i). Composing symmetries is not commutative however. So, performing r then a is not the same thing as performing a then r.

Definition 17. *Starting with a specific Sudoku puzzle and performing symmetry on it until there are several puzzles are that not fundamentally different will form a group, called an* orbit.

In order to find the number of orbits in a given example, the use of Burnside's Lemma is needed. Burnside's Lemma uses some notation: define G as a finite group that acts on a set of elements X. For every g in the group G, let X^g denote the set of elements in X that are fixed by g. Hence, the number of orbits is equal to the average number of points that are fixed by an element of the group G. Here, G_x denotes the elements in G that fix specific x and X/G is the set of orbits.

Lemma 18 (Burnside's Lemma)**.**

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Proof.

$$\sum_{g \in G} |X^g| = |\{(g,x) \in G \times X : g * x = x\}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|G * x|} =$$

$$|G| \sum_{x \in X} \frac{1}{|G_x|} = |G| \sum_{A \in X/G} \sum_{x \in A} \frac{1}{|A|} = G \sum_{A \in X/G} 1 = |G||X/G|$$

[7]

□

This lemma is important because it can help determine how many different orbits there are. This will come in handy when deciphering how many Shidoku exist that are fundamentally different.

Lemma 19. *In order to calculate the number of orbits, the following process is used:*

1. Make a thorough list of all the group's elements; 2. For each one, count the number of elements they fix; 3. Compute the mean of all the symmetries of the number of fixed elements.

For showing how Burnside's Lemma works, consider the following example. Given one triangle, color each side either blue or red. All three sides can be one color, or both red and blue can make up the triangle. Since each side of the triangle has two color possibilities, there are $2 * 2 * 2 = 2^3 = 8$ different triangles that can be made. However, some of these triangles can be represented by a single orbit because their sides are colored the same, differing only by a rotation. The 4 possibilities are:

1. All sides are blue 2. One side is red, two sides are blue 3. One side is blue, two sides are red 4. All sides are red

So, there are only 4 orbits. Using Burnside's Lemma will help us arrive at this same answer. Looking at certain triangles shows that some are fixed under a certain symmetry. For example, an all blue or all red triangle remains the same no matter the symmetry performed on it. A triangle with a blue base and the other two sides red will remain fixed when the symmetry a (or vertical rotation) is performed on it. Going through each symmetry: i fixes all 8, r fixes 2, R fixes 2, and a, b, and c all fix 4. So, the average number of colors fixed by the different symmetries is found by:

$$\frac{8+2+2+4+4+4}{6} = \frac{24}{6} = 4 \tag{3}$$

.

Hence, 4 is the number of orbits, which was found above when the 4 possibilities were stated. This is Burnside's Lemma. [2, Chapter 5, Pages 88-90]

The number of fundamentally different Sudoku puzzles is equal to the number of different orbits. Burnside's Lemma can be used to help count mathematical objects, and which are truly different from each other. Consider using Burnside's Lemma with Shidoku, as seen below.

Lemma 20. *There are only 2 fundamentally different Shidoku puzzles.*

These are seen below.

Table 1: Type 1 Shidoku followed by Type 2 Shidoku:

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

Type 1 Shidoku

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Type 2 Shidoku

Working with computers, Arnold and Lucas [10] showed with that a group of 128 Shidoku symmetries and given a first fixed block, there are 56 symmetries that fix no Shidoku, 48 that fix 2, 9 that fix 4, 4 that fix 6, 6 that fix 8, 4 that fix 10, and 1 symmetry that fixes all 12 up to permutation. So, using Burnside's Lemma [see Lemma 18], the number of fundamentally different Shidoku squares is:

$$\frac{56*0+48*2+9*4+4*6+6*8+4*10+1*12}{128}=\frac{256}{128}=2.$$

There are 96 Shidoku that are equivalent to type 1 and 192 equivalent to type 2. [2, Chapter 5, Page 80]

Using computers, Russell and Jarvis found that there are 5,472,730,538 fundamentally different Sudoku squares. [11]

5 Graphs

Every Sudoku puzzle can be solved using graph coloring. For a 9×9 Sudoku with 81 cells, think of each of the 81 cells as a vertex. Each vertex creates a zone that consists of eight other vertices in its row, eight in its column, and four vertices in its block, but not in either the row or the column. For each vertex, add connecting edges to each of the other vertices in its zone. When their corresponding cells share a row, column, or block, two vertices share an edge. In Sudoku, each vertex is connected by an edge to eight other vertices in its row, eight in its column, and four in its block. Thus, each vertex is connected to twenty other vertices. The total number of edges in a 9×9 Sudoku is $\frac{81 * 20}{2} = 810$ edges. When coloring the graphs, assign colors to the vertices such that connected vertices are given different colors. If all 81 cells are assigned a color and no two cells that share an edge have the same color, then the Sudoku is solved.

Definition 21. *If connected by an edge, two vertices are said to be* adjacent*.*

Definition 22. *If adjacent vertices are always given different colors, then the graph is said to be* properly colored*. A proper coloring that uses n colors is called a* proper n-coloring*. So, a* proper coloring *means that a graph is fully colored, with each vertex being colored differently than its adjacent vertices.*

There is a perfect correlation between a proper 9-coloring Sudoku and a finished Sudoku.

Definition 23. *The minimal number of colors required for a proper coloring is known as the* chromatic number*.*

Example 24. *The chromatic number for a Sudoku puzzle is 9.*

Example 25. *A graph where every vertex is connected to all other vertices is called a* complete graph*, denoted by K_n on n vertices. Each vertex in a complete graph is connected to n-1 other vertices, so a proper coloring uses at least $n - 1$ colors. A Sudoku puzzle must have at least eight starting clues in order to be completed.*

If a Sudoku puzzle only contains starting clues with seven numbers, swapping the remaining two numbers will also provide a sound solution. Thus, starting with seven colors produces two solutions whereas starting with eight colors is needed to produce a complete Sudoku.

Corollary 26. *An $n \times n$ Sudoku must have at least $n - 1$ starting colors.*

Suppose there's a 9×9 Sudoku that only has the numbers 1 through 7 as part of its initial starting clues. Starting with only $n - 2$ clues means starting with only $n - 2 = 9 - 2 = 7$ colors. These starting clues give no information about the numbers 8 or 9 in the puzzle. So, after completing the puzzle, relabeling every 8 as a 9 and vice versa would produce another sound puzzle. Starting with at least 8 numbers for initial starting clues will not allow for switching 2 numbers at the end because these starting clues give information about all 9 numbers. Thus, there must be $n - 1$ starting clues for an $n \times n$ Sudoku.

Theorem 27. *Let G be a graph with its chromatic number $X(G)$ and let C be a partial coloring of G using $X(G) - 2$ colors. If the partial coloring can be finished to a proper* *coloring of G, then there must be at least two different ways of completing the coloring [14].*

The two colors that aren't used in the starting partial coloring can be switched in the proper coloring to produce another solution, as stated previously.

6 Chromatic Polynomials

In an original mathematical exploration of Sudoku, Herzberg and Murty [14, Theorem 1] related Sudoku to a graph and stated the following theorem.

Theorem 28. *Let G be a finite graph that has v vertices, and let C be a partial proper coloring of t vertices of G which uses d_o colors. Let $P_{G,C}(\lambda)$ denote the number of ways to complete the coloring using λ colors to obtain a proper coloring of G. Then, $P_{G,C}(\lambda)$ is a polynomial with coefficients of degree $v - t$.*

Sudoku puzzles are colored graphs in disguise. Each of the 81 cells in a Sudoku can be called a vertex. A colored graph is one where each vertex is given a color. An edge between two vertices means that the vertices are in the same row, column, and/or block. Using graph coloring for Sudoku means that each number is represented by a color. Thus, a colored vertex is connected to other vertices which are colored differently.

Definition 29. *A* proper coloring *is one where all adjacent vertices have different colors.*

Definition 30. *A* partial coloring *is one where only some vertices are colored.*

Once a Sudoku is properly colored (meaning that no 2 adjacent vertices share the same color) for all 81 cells, it is a completed and valid puzzle. When less than 81 cells are colored, the Sudoku has a partial coloring.

Before moving on, there are a couple definitions that are important.

Definition 31. *A* multiplicative function *is one where* $f(nm) = f(n)f(m)$ *if* $gcd(n,m) = 1$ *and* $f(1) = 1$.

Proposition 32. *The* principle of inclusion-exclusion *gives an organized way to find the number of elements in a union of a group of sets, the size of every set, and the size of all the possible intersections among said sets [8].*

For example, if we are trying to find the size of X, where X is made up of the overlapping sets A, B, C, and D, then the following equation would be used:

$$| X |=| A | + | B | + | C | + | D | - | A \cap B | - | A \cap C | ...+ | A \cap B \cap C | +...$$

Continue this until you have subtracted out $| A \cap B \cap C \cap D |$. This equation is finding the number of elements in X, which is made up of overlapping sets A through D.

Definition 33. *Define the Möbius function as*

$$\mu(n) = \begin{cases} 0; \text{ if there's a square number greater than 1 that divides } n \\ (-1)^{\omega(n)}; \ \omega(n) \text{ is the number of distinct primes dividing } n \end{cases}$$

Theorem 34. *If f or g is multiplicative then*

$$f(n) = \sum_{d|n} f(d) \iff g(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$$

Using an example of

$$n = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

where $f(d)$ =positive integers less than or equal to d, that are coprime to d, on the set of integers.

Proving Möbius inversion:

$$\sum_{y \le x} \mu(\frac{y}{x}) g(y) = \sum_{y \le x} \mu(\frac{y}{x}) * (\sum_{z \le y} f(z))$$

$$= \sum_{z \le y \le x} \mu(\frac{y}{x}) f(z)$$

$$= \sum_{z \le x} f(z) * (\sum_{z \le y \le x} \mu(\frac{y}{x}))$$

$$= f(x).$$

[13]

Proving Theorem 28 involves the use of Möbius inversion on a poset of graphs. A partially ordered set (or poset) is a set P in conjunction with a partial ordering denoted by $\le$ which satisfies the following:

1. x$\le x$ for all x that exists in P;

2. x$\leq y$ and y$\leq x$ implies that x must equal y

3. x$\leq y$ and y$\leq x$ implies that x$\leq z$.

Using the function where $f(d)$ =positive integers less than or equal to d, that are coprime to d, on the set of integers, $f(6) = 2$ (because 1 and 5 are coprime to 6). Another way to find this is by using the principle of inclusion-exclusion and the equation: $\sum_{d|n} \mu(d)\frac{n}{d}$ as defined by our Möbius function. So,

$$f(6) = (1 * \frac{6}{1}) + (-1 * \frac{6}{2}) + (-1 * \frac{6}{3}) + (1 * \frac{6}{6}) = 6 - 3 - 2 + 1$$

Doing the same process for f(12) is as follows:

$$f(12) = (1 * \frac{12}{1}) + (-1 * \frac{12}{2}) + (-1 * \frac{12}{3}) + (0 * \frac{12}{4}) + (1 * \frac{12}{6}) + (0 * \frac{12}{12})$$

$$f(12) = 12 - 6 - 4 + 0 + 2 + 0 = 4$$

To make sense of this number, our equation says that there are 4 numbers that are coprime to 12. This is correct because only the numbers less than or equal to 12 that are coprime to it are: 1, 5, 7, 11.

Next it is important to look at partially ordered sets (posets for short). Partial ordering can occur in divisibility. For instance, $n \geq m$ if m divides n. So, it can easily be seen that $4 \geq 2$ because 2 divides 4. But, how are 9 and 6 related? We cannot say that $9 \geq 6$ nor that $9 \leq 6$ because one does not divide the other according to our definition. This is an example of a partial ordered set.

Posets can be seen in graph theory as well. Define $G \geq H$ if connected vertices can be contracted on G and edges ending on vertices and get H. In the images below, call the first graph G and the second one H. If the lower two vertices are morphed into one (since they are connected by an edge), thus removing an edge, then this graph with one less edge is less than or equal to our original graph. Thus, $G \geq H$. This is an example of a poset within graph theory.

Given a finite poset P with partial ordering, $\leq$, we define the Möbius function $\mu : P \times P \to \mathbb{Z}$ recursively by setting

$$\mu(x,x) = 1, \sum_{x \leq y \leq z} \mu(x,y) = 0,$$

if $x \neq z$ [14].

The main theorem used in Möbius functions is the following. If $f : P \to \mathbb{C}$ is any complex function, then we define:

$$g(y) = \sum_{x \leq y} f(x)$$

then

$$f(y) = \sum_{x \leq y} \mu(x,y) g(x)$$

and conversely [14].

The classic Möbius on graphs is as following. In the proof of 28, G is a graph and C is a proper coloring of some of the vertices. Say that (G', C) is a reduction of (G, C) if G' is formed by contracting some edges in G. For each (G', C) of the poset, let $P_{G',C}(\lambda)$ denote the number of ways to properly complete the coloring G' with λ colors. Next, let $Q_{G',C}(\lambda)$ be the number of ways to color (not necessarily properly) the vertices of G' using λ colors. So, $Q_{G',C}(\lambda) = \lambda^{(v'-t)}$ where v' is the number of vertices of G' and t is the number of vertices on C. Contract away adjacent nodes of same color until it's proper of expand out from proper coloring of the subgraph.

Then:

$$Q_{G',C}(\lambda) = \lambda^{(v'-t)} = \sum_{C \leq G'} P_{G',C}(\lambda)$$

This equation is saying that the number of ways to color (not necessarily properly) the vertices of G' using λ colors is equal to λ raised to the power of $(v' - t)$ which is the number of vertices in $G' - C$.

By Möbius inversion,

$$P_{G',C}(\lambda) = \sum_{C \leq G'} \mu(C, G')\lambda^{v'-t}$$

[14]

which shows that is is a polynomial.

In expanding this to Sudoku puzzles, let G be a Sudoku graph and C is given by the starting clues. Then the chromatic number of Sudoku would be nine because G cannot be colored with fewer than nine different colors. Thus, $P_{g,c}(x)$=0 if 1≤x≤8 because the number of ways of completing a partial coloring using eight or fewer colors is impossible; so the number of times it could be completed would be zero. A valid Sudoku puzzle has $P_{g,c}$(9)=1 because there is only one way to complete a partially colored Sudoku using nine colors and if the polynomial equals one, then this means the Sudoku is unique (i.e. only one valid solution). Now let s be the number of different digits included in the initial starting clues. Then:

$$P_{g,c}(x) = (x-s)[x-(s-1)][x-(s+2)]...(x-8)q(x) \tag{4}$$

[2, Chapter 7, Page 136]

Here $q(x)$ is a polynomial that has integer coefficients. For instance, if only five out of the nine digits were given in the initial starting clues, then $s = 5$. Then $P_{g,c}(x)$ would be zero because the Sudoku graph cannot be colored with five, six, seven, or even eight colors; it must be colored with nine. Hence, $(x-5), (x-6), (x-7)$, and $(x-8)$ would be factors of the polynomial. Dividing $P_{g,c}(x)$ by the previously stated factors would produce another polynomial, which is denoted by $q(x)$. The result is:

$$P_{g,c}(x) = (x-5)(x-6)(x-7)(x-8)q(x). \tag{5}$$

[2, Chapter 7, Page 136]

As previously stated, a Sudoku graph must be colored with nine colors. So, $P_{g,c}(9)$=1. Extending this to the polynomial equation gives:

$$P_{g,c}(9) = (9-s)[9-(s+1)][9-(s+2)]\cdots(9-8)q(9) = (9-s)!q(9). \tag{6}$$

[2, Chapter 7, Page 136]

If $s \leq 7$ then the polynomial would be equal to a value that is greater than one, which means it's not possible to have a unique Sudoku solution when fewer than eight digits are represented in the initial clues.

7 Extremes

The maximal number of starting clues a Sudoku can have is 81, although this is a boring case because the puzzle is completely filled in. So then we can ask: what is the maximal number of starting clues a Sudoku puzzle can have without having a unique solution? The answer is 77 and can be seen with the following example. [2, Chapter 9, Page 155]

7	3	5	6	1	2			4
6	4	9	3	8	5	1	2	7
1	2	8	4	7	9	3	5	6
2	5	1	9	6	3	7	4	8
4	9	6	8	2	7	5	3	1
8	7	3	1	5	4	2	6	9
9	8	4	2	3	1	6	7	5
5	1	2	7	4	6			3
3	6	7	5	9	8	4	1	2

Although this puzzle has a large number of initial clues, it does not have a unique solution. The two pairs in each Sudoku puzzle without a clue are missing an 8 and a 9. Switching the 8s and 9s in this example provides two possible solutions to the Sudoku.

In the above example, there are clearly two ways to finish the graph coloring using 9 colors. But how many ways would there be to complete the coloring using 10 colors? This number would grow to 18 different ways to finish the coloring. Using 11 colors, there are 84 different ways; for 12 colors there are 260 ways and for 13 colors there are 630 ways to finish the coloring.

Due to the fact that this is a degree 4 polynomial (total cells - filled in cells = 81-77= degree 4), there are 5 unknowns in the equation: $y = ax^4 + bx^3 + cx^2 + dx + e$. Plugging in the number of colors and their respective ways to finish the coloring, the following polynomial is obtained:

$$y = x^4 - 32x^3 + 384x^2 - 2047x + 4088$$

Here, x is the number of colors used in the coloring, and y is the number of different ways to complete the coloring given x.

Lemma 35. *For a Sudoku of size $n \times n$, the largest number of clues that doesn't produce a unique solution is given by $n^2 - 4$.*

The reason the number 4 is being subtracted from the total number of cells (n^2) is because 4 is the smallest ambiguity that can exist in a Sudoku. The switching of these 2 sets of 2 squares will lead to a non-unique solution. Known as an unavoidable set, not including these 4 cells is guaranteed to lead to a solution that isn't unique. Thus, the largest number of cells that doesn't produce a unique solution is given by $n^2 - 4$.

In the above Sudoku example, these two sets are known as unavoidable sets because they are crucial to the outcome of the Sudoku. Without them included in the initial starting clues, a Sudoku would have more than one solution.

Definition 36. *A puzzle is* irreducible *if each clue is essential and plays a part in getting to the unique Sudoku solution. These necessary clues are known as* independent *clues.*

Adding more (unnecessary) clues to an irreducible puzzle will create a reducible

Sudoku puzzle. In this case, 77 is the bound for the maximal number of independent starting clues. Looking at the extreme opposite case asks: what is the minimum number of starting clues possible in a Sudoku puzzle with a unique solution? Researchers have proved that a Sudoku square with 17 possible starting clues can produce a unique solution, however 16 has never been proven to lead to a unique solution. Researchers would have to check every 16-clue subset of the total 5,472,730,538 different Sudoku puzzles, totaling 183,851,407,423,359,414,572,057,730 different 16-clue candidates. [2, Chapter 9, Page 165]

8 Polynomials

In a 4×4 Shidoku, let w represent one of the sixteen cells. In order to satisfy a single row, column, or block, the following holds true: $(w-1)(w-2)(w-3)(w-4) = 0$. Now let w, x, y, and z be four cells in a region. Then $w + x + y + z = 10$ and $wxyz = 24$. Extending this to a 9×9 Sudoku, the following equations hold true:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 \tag{7}$$

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 = 362{,}880 \tag{8}$$

However, the sum and product equations can still work using different numbers:

$$1 + 2 + 4 + 4 + 4 + 5 + 7 + 9 + 9 = 45 \tag{9}$$

$$1 + 2 + 4 + 4 + 4 + 5 + 7 + 9 + 9 = 362{,}880 \tag{10}$$

These sum and product equations can no longer make certain that the nine cells in the same region are filled with distinct values.

9 Magic and Multi-Magic Squares

Each row and column in a 9×9 Sudoku puzzle add to the number 45.

Definition 37. *A* magic square *is an $n \times n$ array in which each row, column, and main diagonal adds to the same number. A* multimagic square *is a magic square which remains a magic square even when all the entries have been squared.*

Just like magic squares, one aspect of a valid Sudoku puzzle is that the rows and columns add to the same number. In the Sudoku case, this number is 45; in magic squares, this number varies.

Definition 38. *Let M be an $m \times m$ matrix that consists of the natural numbers from 1 to m^2. M is a magic square if the sum of each element in the row, column, and main diagonal is the same number; this number is called the* magic number.

Definition 39. *A magic square that uses the first m^2 numbers is called* normal.

Definition 40. *M^{*d} is the resulting matrix when every element in the matrix M is raised to the dth power. The M matrix is called an n-*multimagic square *if M^{*d} is a magic square for $d = 1,2,...,n$.*

[15]

For the previous definition, when $d \geq 1$, the magic square will no longer contain the first m^2 numbers.

Magic and multimagic squares can also be solved with graph coloring, however they are an extreme case because each cell requires a different graph color. Unlike a $n \times n$ Sudoku which uses n colors, a magic square of $m \times m$ size uses m^2 colors. In other words, the chromatic number for a magic square is m^2. Using graph coloring to solve a magic or multimagic square can be beneficial because it is imperative that no number (or color) is repeated in the process of creating a magic square.

The following is an example of a 3×3 magic square:

2	7	6
9	5	1
4	3	8

Adding the rows, columns, and the two main diagonals gives 15. Thus, the above is a valid magic square.

Magic squares date back to the 1700s when Euler created a magic square from a Greco-Latin square [3]. After superimposing two Latin squares, Euler came up with the following Greco-Latin square:

(1,1)	(2,5)	(3,4)	(4,3)	(5,2)
(2,2)	(3,1)	(4,5)	(5,4)	(1,3)
(3,3)	(4,2)	(5,1)	(1,5)	(2,4)
(4,4)	(5,3)	(1,2)	(2,1)	(3,5)
(5,5)	(1,4)	(2,3)	(3,2)	(4,1)

[3].

Given a Greco-latin square, there is an easy algorithm described below that produces a magic square.

Take this magic square and label each entry as ordered pairs (a,b) with the given order n. Here, the order n is 5. Then, take each entry and perform the following equation:

$$(a-1)n+b.$$

Doing so will create a magic square. In this example, the rows and columns sum to 65. More requirements are necessary if the diagonals must sum to 65 [3].

1	10	14	18	22
7	11	20	24	3
13	17	21	5	9
19	23	2	6	15
25	4	8	12	16

[3].

The earliest known multimagic squares was constructed by G. Pfeffermann in 1891, when he produced the following [15]. His differs from Euler's because Pfeffermann's includes the diagonal stipulation and it is also multimagic.

56	34	8	57	18	47	9	31
33	20	54	48	7	29	59	10
26	43	13	23	64	38	4	49
19	5	35	30	53	12	46	60
15	25	63	2	41	24	50	40
6	55	17	11	36	58	32	45
61	16	42	52	27	1	39	22
44	62	28	37	14	51	21	3

[15].

Adding each individual row, column, and diagonal will be 260. After squaring each entry, the rows, columns, and diagonals will all add up to 11180. Thus, it is a valid multimagic square.

There are several algorithms to make magic squares. Euler has one included in [3] which works for an odd $\times$ odd magic square.

Creating a magic square with large dimensions must be solved through an algorithm, and sometimes even requiring a computer in order to save time. There is a known algorithm for making multimagic squares.

A paper titled "Multimagic Squares" by Derksen, Eggermont, and van den Essen, explains an algorithm for constructing multimagic squares. [15]

This paper denotes R as a finite ring containing q elements and provides the following definition.

Definition 41. *For c that exists in R, a bijection* $N : R \rightarrow \{0,1,...,q-1\}$ *is of type c if* $N(a) + N(-a+c) = q-1$ *for all a existing in R [15].*

Bijections of type -1 always exist and will always be our choice. [15] Now that we have a bijection, the next step is to find generator matrices. First, choose some n. From the chosen n, choose q that is prime such that $q \geq 2n-1$.

Following a detailed proof in the paper [15], the matrix A must first be constructed. A is created with the following matrix given that $n \geq 2$. The following is the way to create matrix A. [15]

A=

$$\begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 4 & ... & 2^{n-1} \\ 1 & 3 & 9 & ... & 3^{n-1} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 1 & (2n-2) & (2n-2)^2 & ... & (2n-2)^{n-1} \\ & 0 & 0 & ... & 1 \end{bmatrix}$$

The dimensions of this matrix are $2n \times n$. Split up the matrix A into two $n \times n$ matrices, and call the top half $A1$ and the bottom half $A2$. The next step in creating our multimagic square is to make the matrix B. [15]

B=

$$\begin{bmatrix} 2 * A1 \\ -2 * A2 \end{bmatrix}$$

Thus, we now have our two generator matrices, A and B. The next step is to create the matrix X which is done by:

X=

$$\begin{bmatrix} A & B \end{bmatrix}$$

We now use matrix X to fill in our multi-magic square M as follows. We fill in the (i, j) entry of our magic square as follows: solve for $\vec{a}$ and $\vec{b}$ given i and j.

To find $\vec{a}$, whose dimensions are $n \times 1$, choose a number between 0 and $q - 1$ for each entry. Once you have done every possible vector with its entries between 0 and $q - 1$, the multimagic square will be complete. Thus, only doing one possible vector will only produce one entry value place. Once finished choosing the numbers, $\vec{a}$ is complete. Repeat this exact process for $\vec{b}$. Now $\vec{a}$ and $\vec{b}$ are complete.

For $\vec{a}$, label the entries as $\vec{a}_1$ through $\vec{a}_n$ and compute the following equation:

$$1 + \vec{a}_1 * q^0 + \vec{a}_2 * q^1 + ... + \vec{a}_n * q^{n-1} \tag{11}$$

The result of this equation is the row number of our entry in the eventual multimagic square.

Doing this for $\vec{b}$, label the entries as $\vec{b}_1$ through $\vec{b}_n$ and compute the following equation:

$$1 + \vec{b}_1 * q^0 + \vec{b}_2 * q^1 + ... + \vec{b}_n * q^{n-1} \tag{12}$$

The result of this will be the column number of the entry in the resulting multimagic square.

Now that we have determined exactly which entry we are going to compute, it is time to actually find the value of this entry place.

In order to do this, first stack $\vec{a}$ and $\vec{b}$ such that:

$$\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}$$

Take the matrix X and multiply it by this $2n \times 1$ matrix. The resulting is a $2n \times 1$ matrix, where each entry is considered modulo q. Call this matrix D where each entry is labeled D_1, D_2, ... , D_2n. In order to finally find our entry value for the previous calculated row and column, calculate the following equation:

$$1 + D_1 * q^0 + D_2 * q^1 + ... + D_2n * q^{2n-1} \tag{13}$$

This calculation will give the number for the entry value in our multimagic square. If doing a multimagic square completely by hand, each entry would be calculated using this algorithm.

The following uses the algorithm for $n = 2$ and $q = 5$.

Matrix A is $2n \times n = 4 * 2$ and looks like:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Matrix B is $2n \times n = 4 * 2$ and looks like:

$$\begin{bmatrix} 2 & 0 \\ 2 & 2 \\ 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Thus, matrix X is $2n \times 2n$ and looks like:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

To now find $\vec{a}$ and $\vec{b}$, the following uses numbers between 0 and $q - 1$, or between 0 and 4 in this case.

Say $\vec{a}$=

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then, using our equation:

$$1 + \vec{a}_1 * q^0 + \vec{a}_2 * q^1 + ... + \vec{a}_n * q^{n-1} = 1 + (0 * 5^0) + (1 * 5^1) = 6. \quad (14)$$

So, here our $\vec{a} = 6$ means our entry place is in the 6th row.

Additionally, say $\vec{b}$=

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then, using our equation:

$$1 + \vec{a}_1 * q^0 + \vec{a}_2 * q^1 + ... + \vec{a}_n * q^{n-1} = 1 + (1 * 5^0) + (1 * 5^1) = 7. \quad (15)$$

So, here our $\vec{b} = 7$ means our entry place is in the 7th column. Thus, when we compute the entry value, it will be in the 6th row, 7th column.

Then, stacking our vectors, the result is:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The next step is to multiply our matrix X by this stacked vector matrix.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

The result of this is the matrix D. So, here D_1 is 2, D_2 is 0, D_3 is 1, and finally D_{2n} is 4.

Then, using our equation:

$$1 + D_1 * q^0 + D_2 * q^1 + ... + D_2 n * q^{2n-1} \tag{16}$$

$$1 + (2 * 5^0) + (0 * 5^1) + (1 * 5^2) + (4 * 5^3) = 1 + 2 + 0 + 25 + 500 = 528 \tag{17}$$

Thus, this algorithm has shown that the entry value for the 6th row and 7th column is 528. Once this process has been completed for every possible $\vec{a}$ and $\vec{b}$, the multimagic square will be complete. This can be a very long process, so for the use of MATLAB on a computer, see 11 for how to use this algorithm in a computer program.

Our final multi-magic square for $n = 2$ and $q = 5$ can be seen on the next page.

1	88	50	107	69	411	498	435	392	454	196	133	220	152	239	581	543	605	562	524	366	303	265	347	284
32	119	51	13	100	442	379	461	423	485	202	164	246	183	145	612	574	506	593	530	272	334	291	353	315
63	25	82	44	101	473	410	492	429	386	233	195	127	214	171	518	580	537	624	556	278	365	322	259	341
94	26	113	75	7	479	436	398	460	417	139	221	158	245	177	549	606	568	505	587	309	266	328	290	372
125	57	19	76	38	385	467	404	486	448	170	227	189	146	208	555	512	599	531	618	340	297	359	316	253
181	143	205	162	249	591	528	615	572	509	351	313	275	332	294	11	98	35	117	54	421	483	445	377	464
212	174	231	193	130	622	559	516	578	540	257	344	276	363	325	42	104	61	23	85	427	389	471	408	495
243	180	137	224	156	503	590	547	609	566	288	375	307	269	326	73	10	92	29	111	458	420	477	439	396
149	206	168	230	187	534	616	553	515	597	319	251	338	300	357	79	36	123	60	17	489	446	383	470	402
155	237	199	131	218	565	522	584	541	603	350	282	369	301	263	110	67	4	86	48	395	452	414	496	433
361	323	260	342	279	21	83	45	102	64	406	493	430	387	474	191	128	215	172	234	576	538	625	557	519
267	329	286	373	310	27	114	71	8	95	437	399	456	418	480	222	159	241	178	140	607	569	501	588	550
298	360	317	254	336	58	20	77	39	121	468	405	487	449	381	228	190	147	209	166	513	600	532	619	551
304	261	348	285	367	89	46	108	70	2	499	431	393	455	412	134	216	153	240	197	544	601	563	525	582
335	292	354	311	273	120	52	14	96	33	380	462	424	481	443	165	247	184	141	203	575	507	594	526	613
416	478	440	397	459	176	138	225	157	244	586	548	610	567	504	371	308	270	327	289	6	93	30	112	74
447	384	466	403	490	207	169	226	188	150	617	554	511	598	535	252	339	296	358	320	37	124	56	18	80
453	415	497	434	391	238	200	132	219	151	523	585	542	604	561	283	370	302	264	346	68	5	87	49	106
484	441	378	465	422	144	201	163	250	182	529	611	573	510	592	314	271	333	295	352	99	31	118	55	12
390	472	409	491	428	175	232	194	126	213	560	517	579	536	623	345	277	364	321	258	105	62	24	81	43
596	533	620	552	514	356	318	255	337	299	16	78	40	122	59	401	488	450	382	469	186	148	210	167	229
602	564	521	583	545	262	349	281	368	305	47	109	66	3	90	432	394	451	413	500	217	154	236	198	135
508	595	527	614	571	293	355	312	274	331	53	15	97	34	116	463	425	482	444	376	248	185	142	204	161
539	621	558	520	577	324	256	343	280	362	84	41	103	65	22	494	426	388	475	407	129	211	173	235	192
570	502	589	546	608	330	287	374	306	268	115	72	9	91	28	400	457	419	476	438	160	242	179	136	223

Figure 1: Our 25 $\times$ 25 Multi Magic Square

10 The Next Step

What does the future hold for magic squares? As previously seen, this algorithm produces a multimagic square, but it only produces one unique multimagic square for a given n and q. Are there more multimagic squares that can be made with or without using this algorithm? In other words, does this algorithm produce every possible multimagic square? These questions are valid, but the research behind answering them would take a long time and a computer to answer.

11 Applications

What does a math puzzle have to do with the real world? In the opening line of his article titled "The secrets of solving mathematical puzzles", Matt Parker answers this question perfectly when he said, "Puzzles are the gateway drug of the world of mathematics." [16]. Additionally, math puzzles stimulate the brain to a large extent and are used with dementia and Alzheimer's patients. The problem-solving aspect involved in Sudoku is a cognitive exercise that has therapeutic value for people living with cognitive disorders. The stimulation provided by the puzzle improves brain function [17]. Puzzles such as Sudoku can teach our world more about not only math, but problem solving, strategy, and creativity while completing a logic puzzle.

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Appendix A 11

This appendix includes how to make a multimagic square in the computer program MATLAB.

For $\vec{b}$ and our q, this code outputs the number a that is the bijection of integers that is seen in Section 9

```
function [ a ] = addiebijection( b , q )
a = 1;
for i = 1:length(b)
    a = a + b(i)*q^(i-1);
end
```

This code returns another vector that gives the remainder of entries of $\vec{b}$ when divided by q.

```
function [ a ] = reductionvector( b , q )
for i = 1:length(b)
a(i) = mod(b(i),q);
end
a = transpose(a);
end
```

The following is the code for making matrices $A1$ and $A2$. Together $A1$ and $A2$ create the generator matrix A. In order to make $A1$ and $A2$, it requires inputs of n and q.

```
function [ A1,A2 ] = makeA1A2( n,q )
A1(1,1) = 1;
for i = 2:n
    A1(1,i) = 0;
end
for j = 2:n
    for i = 1:n
        A1(j,i) = (j-1)^(i-1);
    end
end
A2(n,n) = 1;
for i = 1:n-1
    A2(n,i) = 0;
end
for j = 1:n-1
    for i = 1:n
        A2(j,i) = (n+j-1)^(i-1);
    end
end
end
```

The code below is used for finding the matrix X, which is created from the matrices $A1$, $A2$, and B. This code requires inputs of $A1$ and $A2$. This is described in Section 9.

```
function [ X ] = makeX( A1, A2 )
A = [A1; A2];
B = [2*A1; -2*A2];
```

```
X = [ A B];
end
```

This code makes the matrix D which is the result of the stacked vector multiplied by the matrix X. D is used in Section 9.

```
function [ d ] = makeD( a, b, X, q )
c = [a;b];
d = X*c;
d = reductionvector(d,q);
end
```

This code uses previous matrices to go through the algorithm and produce the magic square M.

```
function [ M ] = makeM( X, q, n )
for i = 1:q^n
    a = makeA(i,q,n);
    for j = 1:q^n
        b = makeA(j,q,n);
        M(i,j) = addiebijection(makeD(a,b,X,q),q);
    end
end
```

This is the inverse of the code addiebijection. It requires inputs of i, q, and n.

```
function[ a ] = {\color{red}makeA}( i, q, n )
k = i-1;
for index = n:-1:1
a(index) = floor(k/q^(index-1));
k = rem(k,q^(index-1));
end
a=transpose(a);
```

```
end
```

This function is used to actually make the multi magic square for some given n and q. Included in our code is that if $q < 2 * n - 1$ then it will return the error message ('q too small'). Additionally, if q is not a prime number, then the program will return the message ('q not prime').

```
function [ M ] = makemultimagicsquare( n,q )
if logical(q < 2*n-1)
error('q too small')
else
if isprime(q)
[A1 A2] = makeA1A2(n,q);
X = makeX(A1,A2);
M = makeM(X,q,n);
else
error('q not prime')
end
end
```