2000

Non-Euclidean Geometry

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NON-EUCLIDEAN GEOMETRY

By

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B.S. University of Maine, 1990

A THESIS

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Master of Arts

(in Mathematics)

The Graduate School

University of Maine

May, 2000

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Acknowledgments

The Author would like to express his gratitude to the members of the thesis advisory committee for their time, effort and contributions, to Dr. Grattan Murphy who first introduced the author to non-Euclidean geometries, and to Jean-Marie Laborde for his permission to include the demonstration version of his software, Cabri II, with this thesis. Thanks also to Euclid, Henri Poincaré, Felix Klein, Janos Bolyai, and all other pioneers in the field of geometry. And thanks to those who wrote the texts studied by the author in preparation for this thesis.

A special debt of gratitude is due Dr. Eisso J. Atzema. His expert guidance and critique have made this thesis of much greater quality than the author could ever have produced on his own. His contribution is sincerely appreciated.

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The Birth of Geometry

We know that the study of geometry goes back at least four thousand years, as far back as the Babylonians (2000 to 1600 BC). Their geometry was empirical, and limited to those properties physically observable. Through their measurements they approximated the ratio of the circumference of a circle to its diameter to be 3, an error of less than five percent. They had knowledge of the Pythagorean Theorem, perhaps the most widely known of all geometric relationships, a full millennium prior to the birth of Pythagoras.

The Egyptians (about 1800 BC) had accurately determined the volume of the frustum of a square pyramid. It is not surprising that a formula relating to such an object should be discovered by their society.

Axiomatic geometry made its debut with the Greeks in the sixth century BC, who insisted that statements be derived by logic and reasoning rather than trial and error. We have the Greeks to thank for the axiomatic proof. (Though thanks would likely be slow in coming from most high school geometry students.)

This systematization manifested itself in the creation of several texts attempting to encompass the entire body of known geometry, culminating in the thirteen volume Elements by Euclid (300 BC). Though not the first geometry text, Euclid's Elements were sufficiently comprehensive to render superfluous all that came before it, earning Euclid the historical role of the father of all geometers. Today, the lay-person is familiar with only two, if any, names in geometry, Pythagoras, due the accessibility and utility of the theorem bearing his name, and Euclid, because the geometry studied by every high school student has been labeled "Euclidean Geometry".

The Elements is not a perfect text, but it succeeded in distilling the foundation of thirteen volumes worth of mathematics into a handful of common notions and five "obvious" truths, the so-called postulates.

The common notions are undefineable things, the nature of which we must agree on before any discussion of geometry is possible, such as what are points and lines, and what it means for a point to lie on a line. The ideas are accessible, even 'obvious' to children.

The five obvious truths from which all of Euclid's geometry is derived are:

The Euclidean Postulates

Postulate I: *To draw a straight line from any point to any point.* (That through any two distinct points there exists a unique line)

Postulate II: *To produce a finite line continuously in a straight line.* (That any segment may be extended without limit)

Postulate III: To *describe a circle with any center and distance.* (Meaning of course, radius)

Postulate IV: *All right angles are equal to one another.* (Where two angles that are congruent and supplementary are said to be right angles)

Postulate V: *If a straight line falling upon two straight lines makes the interior angles on the same side less than two right angles (in sum) then the two straight lines, if produced indefinitely, meet on that side on which are the two angles less than the two right angles.*

The first four of these postulates are, simply stated, basic assumptions. The fifth is something altogether different. It is not unlikely that Euclid himself thought so, as he put off using the fifth postulate until after he had proven the first twenty eight theorems of the Elements. It has been suggested that Euclid had tried in vain to prove the fifth postulate as a theorem following from the first four postulates, and reluctantly included it as a postulate when he was unable to do so. His attempts were followed by the attempts of scores, probably hundreds, of mathematicians who tried in vain to prove the fifth postulate redundant. So many, in fact, that in 1763, G.S.Klügel was able to submit his doctoral thesis finding the flaws in twenty eight “proofs” of the parallel postulate. We will discuss, here, a few of the ‘highlights’ from this two thousand year period.

The Search for a Proof of Euclid’s Fifth

Proclus (410-485 A.D.) said of the fifth postulate, “..ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties,....,The statement that since the two lines converge more and more as they are produced, they will sometime meet is plausible but not necessary.” John Wallis (1616-1703) replaced the wordy and cumbersome parallel postulate with the following. Given any triangle ABC and given any segment DE, there exists a triangle DEF that is similar to triangle ABC. He then proved Euclid’s parallel postulate from his new postulate. It turns out that his postulate and Euclid’s are logically equivalent.

The Italian Jesuit priest Saccheri (1667-1733) studied a particular quadrilateral, one with both base angles right, and both sides congruent. He knew that both summit angles were congruent, and that if he could, using only the first four postulates, prove them to be right angles, then he would have proven the fifth postulate. He was able to derive a contradiction if he assumed they were obtuse, but not in the case that they were acute. He argued instead that, “The hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line!” His sentiment was echoed much later in 1781 by Immanuel Kant. Kant’s position was that Euclidean space is, “inherent in the structure of our mind....(and) the concept of Euclidean space is...an inevitable necessity of thought.” The Swiss mathematician Lambert (1728-1777) also studied a particular quadrilateral that now bears his name, one having three right angles. The remaining angle must be acute, right or obtuse. Like Saccheri, Lambert was able to prove that the remaining angle can not be obtuse, but he also was unable to derive a contradiction in the case that it is acute. We will explore some of the characteristics of Saccheri and Lambert quadrilateral in Chapter II.

Adrien Legendre (French 1752-1833) continued the work of Saccheri and Lambert, but was still unable to derive a contradiction in the acute case. In 1823, just about the time that it was shown that no proof was possible, Legendre published the following "proof". (Figure 1.1)

Given P not on line l, drop perpendicular PQ from P to l at Q. Let m be the line through P perpendicular to PQ. Then m is parallel to l, since l and m have the common perpendicular PQ. Let n be any line through P distinct from m and PQ. We must show that n meets l. Let PR be a ray of n between PQ and a ray of m emanating from P. There is a point R' on the opposite side of PQ from R such that angles QPR' and QPR are congruent. Then Q lies in the interior of RPR'. Since line l passes through the point Q interior to angle RPR', l must intersect one of the sides of this angle. If l meets side PR, then certainly l meets n. Suppose l meets side PR' at a point A. Let B be the unique point on side PR such that segment PA is congruent to PB. Then triangles PQA and PQB are congruent by SAS, and PQB is a right angle so B lies on l and n. QED (Quite Erroneously Done?)

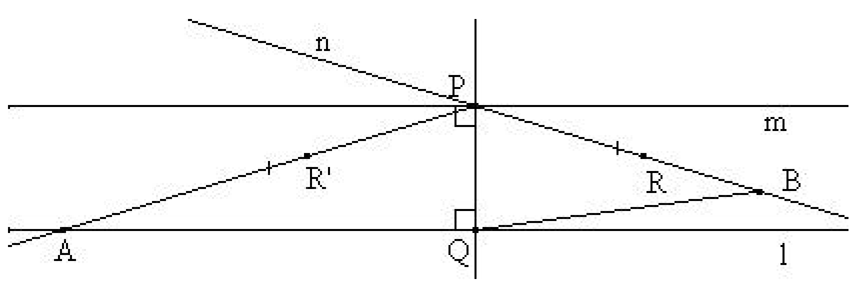

Figure 1.1 Legendre's 'proof' of the parallel postulate

The flaw is in the assumption that any line through a point interior to an angle must intersect one of the sides of the angle. We will show this to be false in Chapter II.

The Hungarian mathematician Wolfgang Bolyai also tried his hand at proving the parallel postulate. We include his “proof” here because it includes a false assumption of a different nature.

Given P not on l, PQ perpendicular to l at Q, and m perpendicular to PQ at P. Let n be any line through P distinct from m and PQ. We must show that n meets l. Let A be any point between P and Q, and B the unique point on line PQ such that Q is the midpoint of segment AB. (Figure 1.2) Let R be the foot of the perpendicular from A to n, and C be the unique point such that R is the midpoint of segment AC. Then A, B and C are not collinear, and there is a unique circle through A, B and C. Since l and n are the perpendicular bisectors of chords AB and AC of the circle, then l and n meet at the center of circle. QED (again, erroneously)

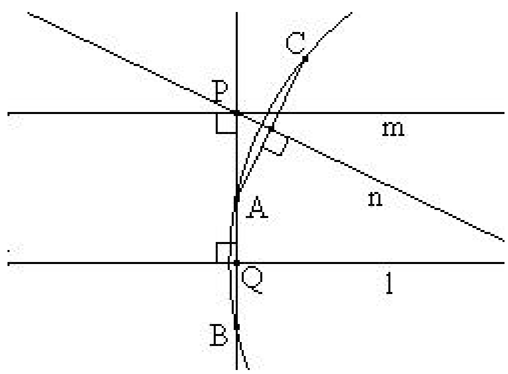

Figure 1.2 Bolyai’s ‘proof’ of the parallel postulate

The problem with this proof is that the existence of a circle through A, B and C may not exist, as we cannot show that lines l and n intersect. We will show, in Chapter II that this cannot be shown, and we will find a condition for the existence of the circle in Chapter VIII.

The End of the Search

Frustrated in his efforts to settle the issue of the parallel postulate, in 1823 Bolyai cautioned his son János to avoid the “science of parallels”, as he himself had gone further than others and felt that there would never be a satisfactory resolution to the situation, saying, “No man can reach the bottom of the night.”

Heedless of his fathers warning, János proceeded, that same year, to explore the “science of parallels”. He wrote to his father that, “Out of nothing I have created a strange new universe.” (hyperbolic geometry) The elder Bolyai agreed to include his son’s work at the end of his own book, and did so in 1832. Before publishing, however, he sent his son’s discoveries to his friend Carl Friedrich Gauss. Gauss replied that he had already done essentially the same work, but had not yet bothered to publish his findings. He declined to comment upon the younger Bolyai’s accomplishment, as praising his work would amount to praising himself. János was so disheartened by Gauss’s response that he never published in mathematics again.

Nicolai Ivanovitch Lobachevsky (1793-1856) had published his results in geometry without the parallel postulate in 1829-30, two or three years before the work of János Bolyai saw print, but Lobachevsky’s work had not reached Bolyai. Though he did not live to see his work acknowledged, hyperbolic geometry today is often referred to as Lobachevskian geometry.

Henri Poincaré and Felix Klein set about creating models within Euclidean geometry consistent with the first four postulates, but that allowed more than one parallel. They succeeded, proving that if there is an inconsistency in the Non-Euclidean geometry, then Euclidean geometry is also inconsistent, and that no proof of the parallel postulate was possible. We will explore their models in Chapter III.

In 1854 Riemann (1836-1866) developed a geometry based on the hypothesis that the non-right angles of the Saccheri quadrilateral are obtuse. To do so, he needed to modify some of the postulates, such as replacing the “infinitude” of the line with “unbounded ness”. The reader may be familiar with the popular model of geometry on the sphere. In this paper, we will deal only with the geometries derived from the first four postulates as stated by Euclid, and will not discuss the geometry of Riemann.

In 1871 Felix Klein gave the names Hyperbolic, Euclidean, and Elliptic to the geometries associated with acute, right, and obtuse angles in the Saccheri quadrilateral. The distinctions between these geometries may be illustrated as follows. Given any line l and any point P not on l, there exist(s)_____lines through P parallel to l. Parabolic (Euclidean) geometry guarantees a unique parallel, in Hyperbolic geometry there are an infinite number, and in Elliptic geometry there are none.

A More Complete Axiom System

Over the course of the two millennia following the work of Euclid, mathematicians determined that Euclid’s system of five postulates were not sufficient to serve as a foundation of Euclidean geometry. For example, the first postulate of Euclid guarantees that if we have two points, then we may draw a line, but none of the postulates guarantees the existence of any points, nor lines. Also, when we discuss the measure of a line segment or of an angle, we are assuming that measurement is possible and meaningful, but Euclid’s postulates are silent on this issue.

The following system of axioms is complete, (where Euclid’s postulate system is not) that is, it is a sufficient system from which to derive geometry. The geometry and its development are identical using both systems, but the problem in using Euclid’s system is that one must make many unstated assumptions, which is unacceptable.

Axiom I: *There exist at least two lines*

Axiom II: *Each line is a set of points having at least two elements* (This guarantees at least two points)

Axiom III: *To each pair of points P and Q, distinct or not, there corresponds a non-negative real number PQ which satisfies the following properties:*

(a) PQ = 0 iff P = Q and

(b) PQ = QP (This allows us to discuss measure)

Axiom IV: *Each pair of distinct points P and Q lie on at least one line, and if PQ < a, that line is unique* (If α is infinite we get Euclidean and/or hyperbolic geometry. If α is finite we get elliptic geometry)

Axiom V: *If l is any line and P and Q are any two points on l, there exists a one to one correspondence between the points of l and the real number system such that P corresponds to zero and Q corresponds to a positive number, and for any two points R and S on l, RS = | r - s | , where r and s are the real numbers corresponding to R and S respectively* (This allows us to impose a convenient coordinate system upon any line)

Axiom VI: *To each angle pq (the intersection of lines p and q), degenerate or not, there corresponds a non-negative real number pq which satisfies the following properties:*

(a) pq = 0 iff p = q and

(b) pq = qp

(This does for angles what Axiom III did for lines)

Axiom VII: *bis the measure of any straight angle* (We get the degree system by letting β be 180, π gives radians)

Axiom VIII: *If O is the common origin of a pencil of rays and p and q are any two rays in the pencil, then there exists a coordinate system g for pencil O whose coordinate set is the set* $\{ x : -\beta < x \leq \beta, x \in \mathbb{R} \}$ *and satisfying the properties:*

(a) $g(p) = 0$ *and* $g(q) > 0$ *and*

(b) For any two rays r and s in that pencil, if $g(r) = x$ *and* $g(x) = y$ *then* $rs = | x - y |$ *in the case* $| x - y | \leq \beta$, *and* $rs = 2\beta - | x - y |$ *in the case* $| x - y | > 2\beta$

(This does for angles what Axiom V did for lines)

Axiom IX (Plane separation principle): *There corresponds to each line l in the plane two regions* H_1 *and* H_2 *with the properties:*

(a) Each point in the plane belongs to exactly one of l, H_1 *and* H_2

(b) H_1 *and* H_2 *are each convex sets* *and*

(c) If $A \in H_1$ *and* $B \in H_2$ *and* $AB < \alpha$ *then l intersects line AB*

(This makes the discussion of the "sides" of a line possible)

Axiom X: *If the concurrent rays p, q, and r meet line l at respective points P, Q, and R and l does not pass through the origin of p, q and r, then Q is between P and R iff q is between p and r.* (This guarantees, essentially, that if a ray 'enters' a triangle at one vertex, then it must 'exit' somewhere on the opposite side. A slightly different wording of this is sometimes called the Crossbar Principle)

Axiom XI (SAS congruence criterion for triangles): *If in any two triangles there exists a correspondence in which two sides and the included angle of one are congruent, respectively, to the corresponding two sides and included angle of the other, the triangles are congruent.*

Axiom XII: *If a point and a line not passing through it be given, there exist(s)______ line(s) which pass through the given point parallel to the given line.* ("One" gives Euclidean geometry, "No" lines gives Elliptic, and "Two" gives Hyperbolic)

Note that axioms four and twelve are worded in such a way that different choices will lead to different geometries. Euclid's postulates lead to Euclidean geometry only, but this system gives us, with rather minor modifications, Euclidean, hyperbolic, and elliptic geometries.

We will begin our discussion of hyperbolic geometry by developing the geometry derived from the first four of Euclid's postulates, or more accurately, the first eleven axioms. During our discussion, we will refer to the postulates rather than the axioms because the geometry we will be discussing was originally developed using the postulates. In addition, the average reader is likely more familiar with the postulates than with the axioms.

Chapter II
Neutral and Hyperbolic Geometries

Neutral Geometry

Neutral geometry (sometimes called Absolute geometry) is the geometry derived from the first four postulates of Euclid, or the first eleven axioms (see Chapter I). As Euclid himself put off using his fifth postulate for the first twenty eight theorems in his Elements, these theorems might be viewed as the foundation of neutral geometry. We will see that Euclidean and hyperbolic geometries are contained within neutral geometry, that is the theorems of neutral geometry are valid in both.

We will develop neutral geometry to a degree sufficient to provide a foundation for hyperbolic geometry. It should not be surprising, since hyperbolic geometry was born as a result of the controversy over the fifth postulate, (the only postulate to address parallelism) that parallels will be the main focus of our discussion and the topic of our first few theorems of neutral geometry:

Theorem 2.1: *If two lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the lines are parallel. (Parallel at this point means nothing more than non-intersecting.)*

Proof: Suppose lines l and m are cut by transversal t with a pair of alternate interior angles congruent. Let t cut l and m in A and B respectively. Assume that l and m intersect at point C. (Figure 2.1) Let C' be the point on m such that B is between C and C' and AC≅BC', and let D be any point on l such that A is between D and C. Consider triangles ABC and BAC'. By SAS, they are congruent, so angles BAC' and ABC are congruent, which means that angles BAC' and BAC are supplementary, so CAC' is a straight angle and C' lies on l. But then we have l and m intersecting in two distinct points, which is a contradiction of Postulate I, so l and m do not intersect, and are parallel. QED

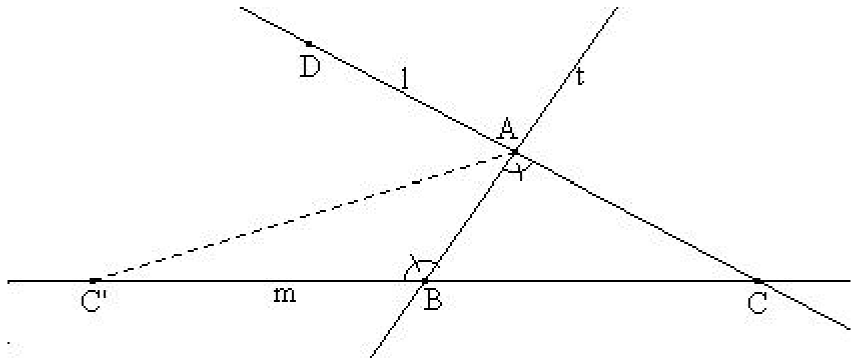

Figure 2.1 Congruent alternate interior angles implies parallelism

This theorem has two useful corollaries.

Corollary 2.2: *If two lines have a common perpendicular, they are parallel.*

Corollary 2.3: *Given line l and point P not on l, there exists at least one parallel to l through P.*

The parallel guaranteed here is simple to construct. Draw t, perpendicular to l through P, and m perpendicular to t through P. By Corollary 2.2, m and l are parallel.

Theorem 2.4: *The external angle of any triangle is greater than either remote interior angle.*

Proof: Given triangle ABC with D on ray AB such that B is between A and D, angle CBD is our external angle. (Figure 2.2) Assume that angle ACB is greater than angle CBD. Then there is a ray CE between rays CA and CB such that angles BCE and CBD are congruent. But these are the alternate interior angles formed by transversal CB cutting CE and BD, which tells us that CE and BD are parallel, by the preceding theorem. Since ray CE lies between rays CA and CB, it intersects segment AB and therefore line

BD, and we have a contradiction. The case for angle BAC is symmetric. QED

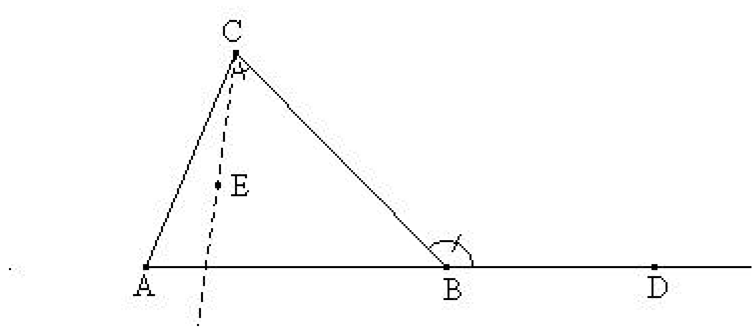

Figure 2.2 The external angle of a triangle is greater than either remote interior angle

This theorem is the key to proving the AAS condition for congruence. SAS and ASA criterion for triangle congruence are also valid in neutral geometry, but these are fairly obvious so we omit their proofs. AAS is not so intuitive.

Theorem 2.5 ***(AAS congruence):*** *Given two triangles ABC and A'B'C', if side AB@A'B', angle ABC@A'B'C', and angle BCA@B'C'A', then the two triangles are congruent.*

Proof: Suppose we have the triangles described. (Figure 2.3) If side BC≅B'C', the triangles are congruent by ASA, so assume that side B'C'>BC. If so, there is a unique point D on segment B'C' such that B'D is congruent to BC. Consider triangles ABC and A'B'D. By SAS, they are congruent, and angle A'DB'≅ACB≅A'C'B', which is a contradiction of Theorem 2.4, as angle A'DB' is the exterior angle and A'C'B' a remote interior angle of triangle A'C'D. QED

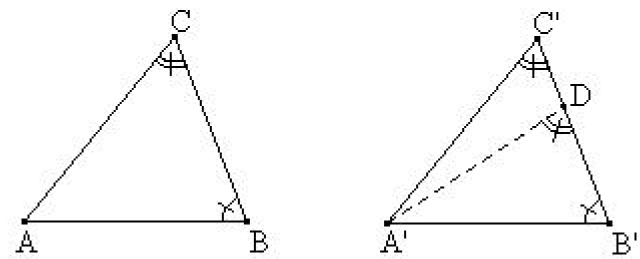

Figure 2.3 Angle-angle-side congruence of triangles

It happens that we have all of the congruence rules for triangles in hyperbolic geometry that we have in Euclidean; SAS, ASA, AAS, SSS, and HL (The proof of hypotenuse-leg congruence for right triangles is elementary, and we will not include it here.). Actually, we will see in Theorem 2.19 that we have another congruence criterion in hyperbolic geometry that is not valid in Euclidean.

Before we get to that, we must take look at several elementary properties of triangles in neutral geometry, starting with:

Theorem 2.6: *In any triangle, the greatest angle and the greatest side are opposite each other*.

Proof: Given any triangle ABC, assume that ABC is the greatest angle, and that AB is the greatest side. (Figure 2.4) There is a unique point D on segment AB such that AD≅AC. This means that triangle CAD is isosceles, and angle ACD≅ADC, but, by Theorem 2.4, angle ADC>ABC. So angle ACB>ABC, contradicting our assumption. QED

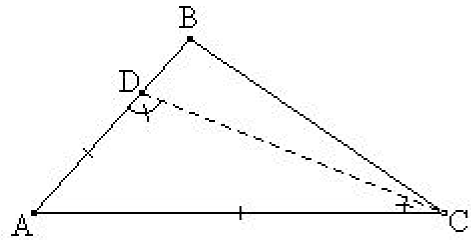

Figure 2.4 The greatest angle is opposite the greatest side

Theorem 2.7*: The sum of two angles of a triangle is less than 180°*

Proof: Given triangle ABC, assume that the sum of angles ABC and BAC is greater than 180°. (Figure 2.5) We can construct line AE interior to angle CAB such that angle BAE=180°–ABC. This gives us angle BAD=ABC, but this is a pair of alternate interior angles, so line AE is parallel to BC, an obvious contradiction. In the case where ABC+BAC=180°, point E lies on line AC, and we have AC parallel to BC, which is also absurd, so ABC+BAC<180°.

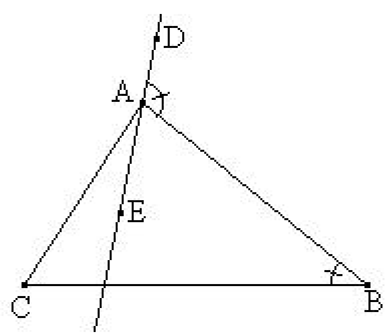

Figure 2.5 The sum of any two angles of a triangle is less than 180°

Up to this point, all of the theorems of neutral geometry are theorems that we recognize (in their exact form) from Euclidean geometry. Now we have come to a point where we will see a difference. Theorem 2.8 is slightly weaker than its Euclidean analogue.

Theorem 2.8 ***(Saccheri-Legendre):*** *The angle sum of a triangle is less than or equal to 180°.*

Proof (Max Dehn, 1900): Given triangle ABC, let D be the midpoint of segment BC, and let E be on ray AD such that D is between A and E, and AD≅DE. (Figure 2.6) By SAS, triangles ABD and ECD are congruent. Since angle BAC=BAD+EAC, and by substitution, BAC=AEC+EAC, either AEC or EAC must be less than or equal to ½BAC. Also, triangle AEC has the same angle sum as ABC. Assume now that the angle sum of any triangle ABC is greater than 180°, or =180°+p where p is positive. We see from above that we can create a triangle with the same angle sum as ABC, with one angle less than ½BAC. By repeated application of the construction, we can make one angle arbitrarily small, smaller than p. By this and the previous theorem, the angle sum of ABC must be less than 180°+p, a contradiction. So the angle sum of any triangle is ≤180°. QED

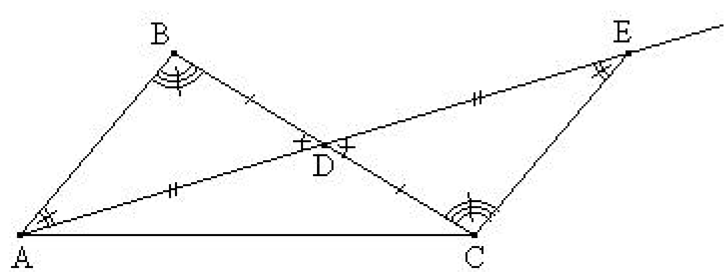

Figure 2.6 The angle sum of a triangle is less than or equal to 180°

In Euclidean geometry, the angle sum of a triangle is exactly 180°. To prove this we must use the Euclidean parallel postulate, or its logical equivalent. (The statement that the angle sum is 180° is actually equivalent to the parallel postulate) A common proof is given below.

Theorem 2.9: *In Euclidean Geometry, the angle sum of any triangle is 180°.*

Proof: Given triangle ABC, let l be the unique parallel to line BC through A. Let D be a point on l such that B and D are on the same side of AC, and E a point on l such that A is between D and E. (Figure 2.7) Because alternate interior angles formed by a transversal cutting two parallel lines are congruent, angle EAC≅ACB and angle DAB≅ABC. So the three angles add up to a straight angle, 180°. QED

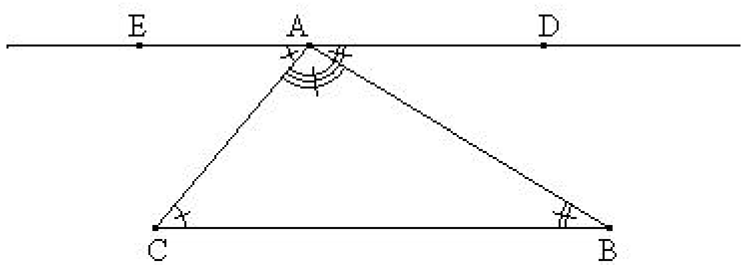

Figure 2.7 The angle sum of an Euclidean triangle is 180°

The reader is no doubt acquainted with this proof. It is included to illustrate how it uses the converse of Theorem 2.1, which is not valid in neutral geometry. A corollary of this theorem in Euclidean geometry is that the sum of any two angles of a triangle is equal to its remote exterior angle. In neutral geometry, the corollary to the Saccheri-Legendre theorem is as we might expect:

Corollary 2.10: *The sum of two angles of a triangle is less than or equal to the remote exterior angle.*

This is obvious: angle ABC+BCA+CAB≤180°, so angle ABC+BCA≤180°-CAB, which is the measure of the remote exterior angle at vertex A.

Corollary 2.11: *The angle sum of a quadrilateral is less than or equal to 360°.*

We can see this by noting that any quadrilateral can be dissected into two triangles by drawing one diagonal. The angle sum of the quadrilateral is the sum of the angle sums of the two triangles.

Let us look, again, at the parallel postulate of Euclid:

Parallel Postulate (Euclid): *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles (in sum), the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

or in language more palatable to modern readers:

Parallel Postulate (Euclid): *Given two lines l and m cut by a transversal t, if the sum of the interior angles on one side of t is less than 180°, then l intersects m on that side of t.*

The version we are more familiar with is that of John Playfair (1795):

Parallel Postulate (Playfair): *Given any line l and point P not on l, there exists a unique line m through P that is parallel to l.*

These two statements are logically equivalent.

Theorem 2.12: *Euclid's Parallel Postulate implies Playfair's Parallel Postulate, and vice versa.*

Proof: First suppose Playfair's is true. Let lines l and m be cut by a transversal t. Let t cut l in A, and m in B, and let C and D lie on l and m respectively on the same side of t. (Figure 2.8) Further, suppose that angle CAB+DBA<180°. Let n be the unique line through A such that the alternate interior angles cut by t crossing m and n are congruent. By Theorem 2.1, this line is parallel to m, and by Playfair, we know it is the only such line. By our conditions, n is distinct from m, and meets l in point E. Furthermore, E is on the same side of AB as C and D, else triangle ABE would have angle sum greater than 180°. So Playfair's implies Euclid's.

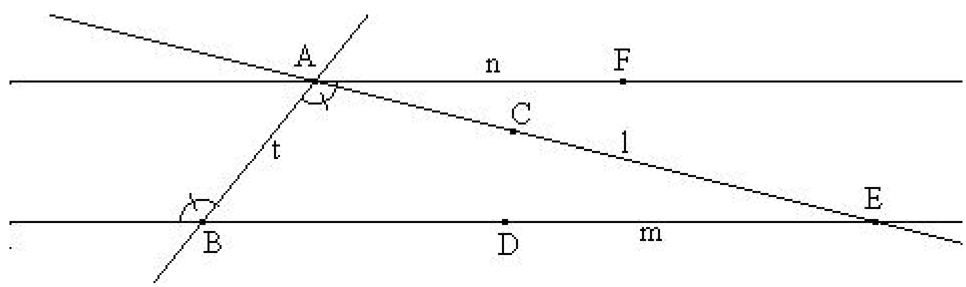

Figure 2.8 The postulates of Euclid and Playfair are equivalent

Now suppose Euclid's Parallel Postulate is true. Given line m and point A not on m, and any line t through A that cuts m in B. Let D be any point on m other than B. We know there is a unique ray AF such that angle BAF≅DBA, and that line n containing ray AF will be parallel to m. (Figure 2.8) Line m and any line l through A other than n, will not form congruent alternate interior angles when cut by t, so on one side of AB the sum of the interior angles will be less than 180°, and by Euclid, l and m will meet on that side, and l will not be parallel to m. So n is the unique parallel to m through A, proving Playfair and the postulates of Euclid and Playfair are equivalent. QED

In Euclidean geometry, the angle sum of a triangle is 180°, and we will show that in hyperbolic geometry it is less than 180°. Before we do so, we must define:

Definition: *The angle defect of a triangle is 180° minus the angle sum.*

In Euclidean geometry, the angle defect of every triangle is zero, which is why the term is never used. In hyperbolic geometry, the angle defect is always positive. We will explore the significance of the angle defect in Chapter V.

Theorem 2.13: *In any triangle ABC, with any point D on side AB, the angle defect of triangle ABC is equal to the sum of the angle defects of triangles ACD and BCD.*

The proof of this is trivial substitution and simplification, and we omit it.

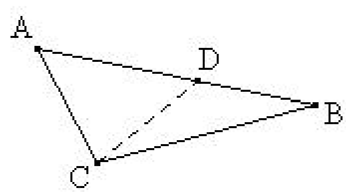

Figure 2.9 Angle defect is additive

Theorem 2.13 tells us that, like the area of triangles, angle defect (and angle sum) is additive, and gives us a useful corollary:

Corollary 2.14: *If the angle sum of any right triangle is 180°, than the angle sum of every triangle is 180°.*

Since any triangle can be divided into two right triangles,(this is shown in the proof of Theorem 2.15) its angle defect is the sum of the angle defects of the two right triangles, which are both zero.

The angle sum of the triangle is a striking difference between our two geometries. We have not yet proved that we can not have triangles with positive defect and zero defect residing within the same geometry. We show now that this is indeed the case.

Theorem 2.15: *If there exists a triangle with angle sum 180° then every triangle has angle sum 180°*

Proof: Suppose we have a triangle ABC with angle sum 180°. We know that any triangle has at least two acute angles. (If not, its angle sum would exceed 180°.) Let the angles at A and B be acute. Let D be the foot of the perpendicular from C to line AB. We claim that D lies between A and B. Suppose it does not, and assume that A lies between D and B. (Figure 2.10) By Theorem 2.4, angle BAC>BDC=90°. This contradicts our assumption that angle BAC is acute. By the same argument, B is not between A and D. It follows that D lies between A and B.

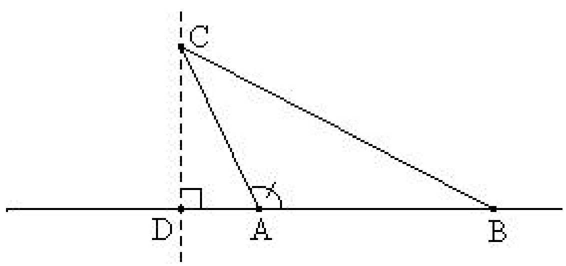

Figure 2.10 One altitude of a triangle must intersect the opposite side

So triangle ABC may be divided into two right triangles, both with angle defect of zero, since angle defect is additive and non-negative.

Consider now the right triangle ACD. From this we shall create a rectangle. (a quadrilateral with four right angles) There is a unique ray CE on the opposite side of AC from D such that angle ACE≅CAD, and there is a unique point F on ray CE such that segment CF≅AD. (Figure 2.11) By SAS, triangle ACF≅CAD, and by complementary angles, quadrilateral ADCF is a rectangle.

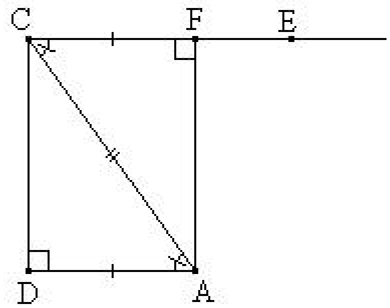

Figure 2.11 From any right triangle with angle sum 180° we can create a rectangle

Consider now any right triangle ABC with right angle at C. We can create a rectangle DEFG (by 'tiling' with the rectangle above) with EF>BC and FG>AC, and we can find the unique points H and K on sides EF and FG respectively such that FH≅BC and FK≅AC. Triangle KFH will be congruent to ABC by SAS. (Figure 2.12)

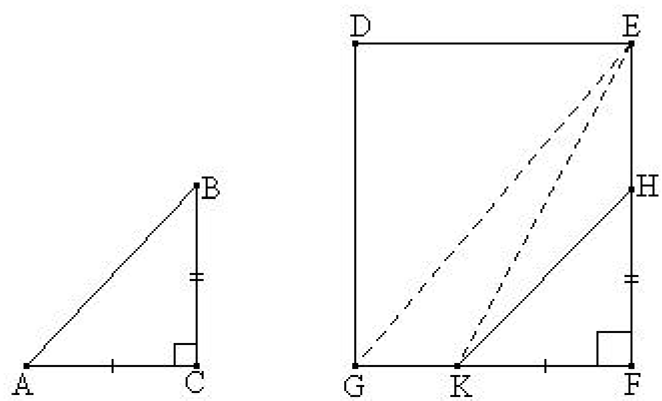

Figure 2.12 Fitting any right triangle into a rectangle

By drawing segments EG and EK, we divide the rectangle into triangles. By the additivity of angle defect, the angle sum of triangle KHF, and therefore ABC, is 180°. So the angle sum of any right triangle is 180°, and by Corollary 2.14 the angle sum of any triangle, is 180°. QED

Corollary 2.16: *If there exists a triangle with positive angle defect, then all triangles have positive angle defect.*

This neatly divides neutral geometry into two separate geometries, Euclidean where the angle sum is exactly 180°, and hyperbolic, where the angle sum is less than 180°. It is assumed that the reader is familiar with Euclidean geometry. We will now move on to:

Hyperbolic Geometry

Where the foundation of neutral geometry consists of the first four of Euclid's postulates, hyperbolic geometry is built upon the same four postulates with the addition of:

The Hyperbolic Parallel Postulate: *Given a line l and a point P not on l, then there are two distinct lines through P that are parallel to l.*

While the postulate states the existence of only two parallels, all of the lines through P between the two parallels will also be parallel to l. We can make this more precise. Let Q be the foot of the perpendicular from P to l, and A and B be points on m and n, the two parallels, respectively, such that A and B are on the same side of PQ. (Figure 2.13) Any line containing a ray PC between PA and PB must also be parallel to l.

In the Euclidean plane, given non-collinear rays PA and PB, and a point Q lying in the interior of angle APB, any line through Q must intersect either PA, PB or both. This is not the case in the hyperbolic plane. In Figure 2.13 line l through Q cuts neither line n nor m.

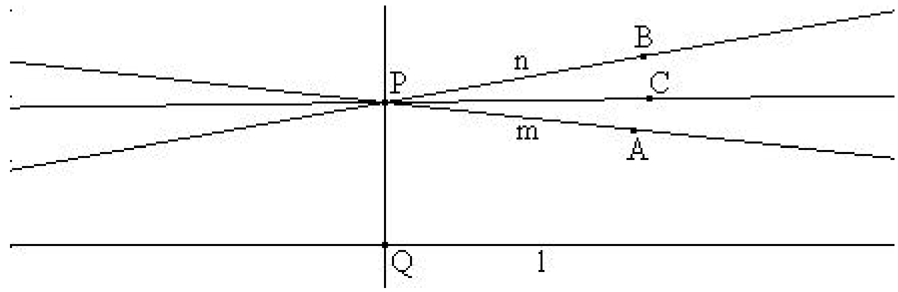

Figure 2.13 Two distinct parallels imply infinitely many parallels

Theorem 2.17 formalizes a couple of the ideas alluded to in Chapter I.

Theorem 2.17: *Every triangle has angle sum less than 180°.*

Proof: All we need to show is that there exists a triangle with angle sum less than 180°. It will follow by Corollary 2.16 that all triangles have angle sum less than 180°. Suppose we have line l and point P not on l. Let Q be the foot of the perpendicular from P to l, and line m perpendicular to PQ at P. Let n be any other parallel to l through P guaranteed by the hyperbolic parallel postulate, and suppose PA is a ray of n such that A is between m and l. Also let X be a point on m such that X and A are on the same side of PQ. (Figure 2.14)

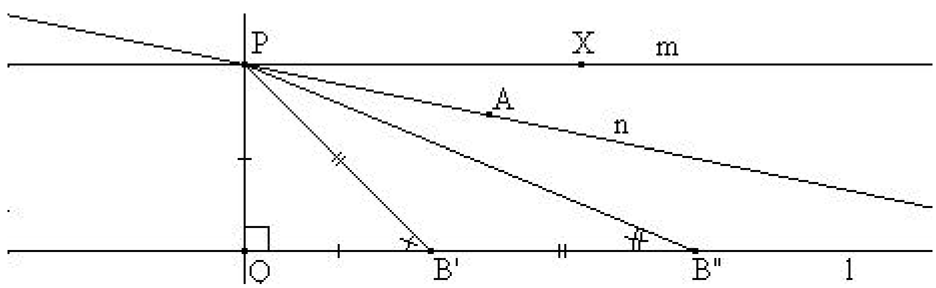

Figure 2.14 Finding a triangle with angle sum less than 180°

Angle XPA has positive measure p, and angle QPA has measure 90°-p. Then the angle QPB for any point B on l to the right of Q will be less than QPA. If we can find a point B on l such that the measure of angle QBP is less than p, then the angle sum of triangle QBP will be less than 90°+90°-p+p, or less than 180° which is what we want. To do this, we choose point B' on l to the right of Q such that QB'≅PQ. Triangle QPB' is an isosceles right triangle, so angle QB'P is at most 45°. If we then choose B" to the right of B' on l such that B'B"≅PB', then triangle PB'B" is an isosceles triangle with summit angle at least 135°, so angle PB"B' is at most 22½°. By continuing this process, eventually we will arrive at a point B such that angle PBQ is less than p, and we have our triangle PBQ with angle sum less than 180°. QED

So in the hyperbolic plane, all triangles have angle sum less than 180°.

Corollary 2.18: *All quadrilaterals have angle sum less than 360°.*

In Euclidean geometry triangles may be congruent or similar. (or neither), but in hyperbolic geometry:

Theorem 2.19: *Triangles that are similar are congruent.*

Proof: Given two similar triangles ABC and A'B'C', assume that they are not congruent, that is that corresponding angles are congruent, but corresponding sides are not. In fact, no corresponding pair of sides may be congruent, or by ASA, the triangles would be congruent. So one triangle must have two sides that are greater in length than their counterparts in the other triangle. Suppose that AB>A'B' and AC>A'C'. This means that we can find points D and E on sides AB and AC respectively such that AD≅A'B' and AE≅A'C'. (Figure 2.15) By SAS, triangle ADE≅A'B'C' and corresponding angles are congruent, in particular, angle ADE≅A'B'C'≅ABC and AED≅A'C'B'≅ACB. This tells us that quadrilateral DECB has angle sum 360°. This contradicts Corollary 2.18, and triangle ABC is congruent to triangle A'B'C'. QED

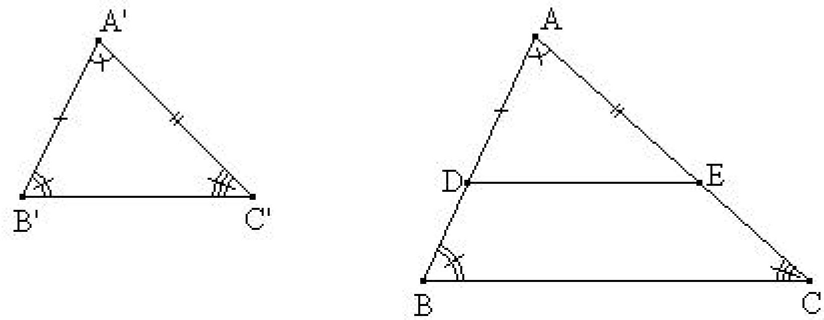

Figure 2.15 Similarity of triangles implies congruence

Note that this gives us another condition for congruence of triangles, AAA, which is not valid in Euclidean geometry.

We will explore several properties of triangles in Chapter V. We will now turn our attention to the nature of parallel lines in the hyperbolic plane. Before we look at parallel lines, we will need to learn a few things about some special quadrilaterals we mentioned in Chapter I.

Saccheri and Lambert quadrilaterals

Definition: *A quadrilateral with base angle right and sides congruent is called a* *Saccheri quadrilateral. The side opposite the base is the summit, and the angles formed by the sides and the summit are the summit angles*

In the Euclidean plane, this would of course be a rectangle, but by Corollary 2.18 there are no rectangles in the hyperbolic plane.

Note that the summit angles of a Saccheri quadrilateral are congruent and acute, and the segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both. These facts are easy to verify by considering the perpendicular bisector of the base. (MM' in Figure 2.16) By SAS, triangles MM'D and MM'C are congruent, and also by SAS, triangles AMD and BMC are congruent. This gives us that

M is the midpoint of, and perpendicular to, side AB, and also that angles DAM and CBM are congruent.

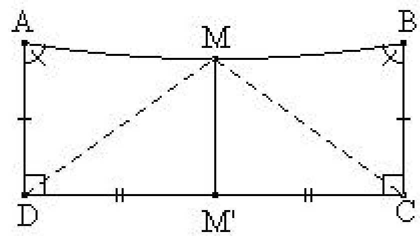

Figure 2.16 The Saccheri quadrilateral

There is one more fact we need to establish regarding the Saccheri quadrilateral. To do this we consider a more general quadrilateral.

Theorem 2.20: *Given quadrilateral ABCD with right angles at C and D, then side AD>BC iff angle ABC>BAD.*

Figure 2.17 should give the reader the idea of the proof.

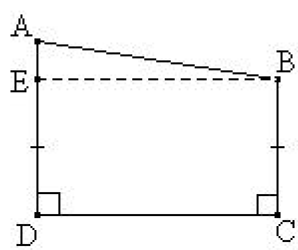

Figure 2.17 The longer side is opposite the larger angle

A direct consequence of this is that the segment connecting the midpoints of the summit and base of a Saccheri quadrilateral is shorter than its sides. We also know that this segment is the only segment perpendicular to the base and summit. (If there were another, then we would have a rectangle). We will state these facts together as:

Theorem 2.21: *The segment connecting the midpoints of the summit and base of a Saccheri quadrilateral is shorter than the sides, and is the unique segment perpendicular to both the summit and base.*

We now have what we need to examine and classify parallels in the hyperbolic plane.

Two kinds of hyperbolic parallels

In Euclidean geometry, parallel lines are often described as lines that are everywhere equidistant, like train tracks. This property is equivalent to the Euclidean parallel postulate, so as we would expect, this description is untrue in the hyperbolic plane.

Theorem 2.22: *If lines l and l' are distinct parallel lines, then the set of points on l that are equidistant from l' contains at most two points.*

Note that distance P is from l is defined in the usual way, as the length of segment PQ where Q is the foot of the perpendicular from P to l.

Proof: Given two parallel lines l and l', assume that distinct points A, B and C lie on l and are equidistant from l'. Let A', B' and C' be the feet of the perpendiculars from the corresponding points to l'. (Figure 2.18) ABB'A', ACC'A' and BCC'B' are all Saccheri quadrilaterals, and their summit angles are all congruent, so angles ABB' and CBB' are congruent supplementary angles, and therefore right. But we know they are acute, so we have a contradiction, and the set of points on l equidistant from l' contains fewer than three points. QED

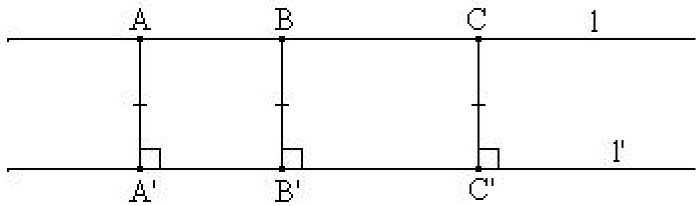

Figure 2.18 Three points on a line l equidistant from l' parallel to l

We have no guarantee that any set of points on l equidistant from l' has more than one element. If it does, there are some things we know about l and l'.

Theorem 2.23: *If l and l' are distinct parallel lines for which there are two points A and B on l equidistant from l', then l and l' have a common perpendicular segment that is the shortest segment from l to l'.*

Proof: Let A and B be on l equidistant from l', and let A' and B' be the feet of the perpendiculars from A and B to l'. (Figure 2.19) The existence of the common perpendicular is immediate by Theorem 2.21. To show that this common perpendicular is the shortest distance between l and l', choose any point C on l, and let C' be the foot of the perpendicular from C to l'. MM'C'C is a Lambert quadrilateral, and by Theorem 2.20, side CC' is greater than MM'. QED

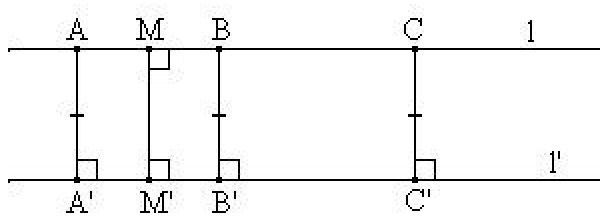

Figure 2.19 The mutual perpendicular is the shortest segment between two parallels

Theorem 2.24: *If lines l and l' have a common perpendicular segment MM' with M on l and M' on l', then l is parallel to l', MM' is the only segment perpendicular to both l and l', and if A and B lie on l such that M is the midpoint of segment AB, then A and B are equidistant from l'.*

Proof: We know that if l and l' have a common perpendicular MM', then l is parallel to l' by Theorem 2.1. We also know MM' is unique because if it were not, we would have a rectangle. It remains to be shown that A and B, so described above (Figure 2.20) are equidistant from l'. By SAS, triangles AMM' and BMM' are congruent, and by AAS, triangles AA'M' and BB'M' are congruent. So segments AA' and BB' are congruent. QED

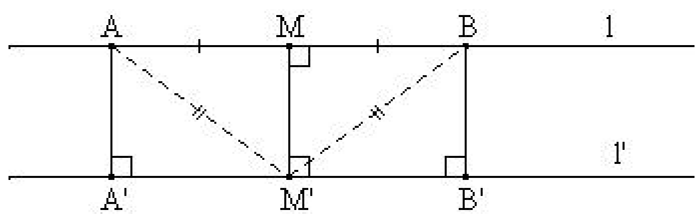

Figure 2.20 Points equidistant from the mutual perpendicular are equidistant from l'

We can add one more fact here about lines having a mutual perpendicular.

Theorem 2.25: *Given lines l and l' having common perpendicular MM', if points A and B are on l such that MB>MA, then A is closer to l' than B.*

Proof: Given the situation stated. If A is between M and B, let A' and B' be the feet of the perpendiculars from A and B to l', and consider the Sacchieri quadrilateral ABB'A' (Figure 2.21) We know that angles MAA' and ABB' are acute, so A'AB is obtuse, and therefore greater than ABB'. By Theorem 2.22 side BB'>AA', and B is farther from l' than is A. If M is between A and B, then there is a unique point C on segment MB such that M is the midpoint of segment AC. Let C' be the foot of the perpendicular from C to l'. Apply Theorem 2.22 to quadrilateral CBB'C', and the fact that CC'≅AA', and we have the theorem. QED

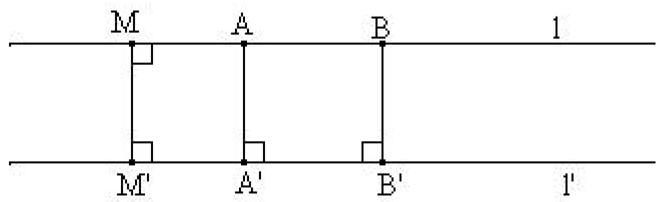

Figure 2.21 Points closer to the common perpendicular are closer to l'

So two lines having a mutual perpendicular diverge in both directions. We define such lines to be:

Definition: *Two lines having a common perpendicular are said to be* *divergently-parallel*.

It is also common for such lines to be called ultra-parallel or super-parallel. A more intuitive picture of ultra-parallel lines is shown in Figure 2.22.

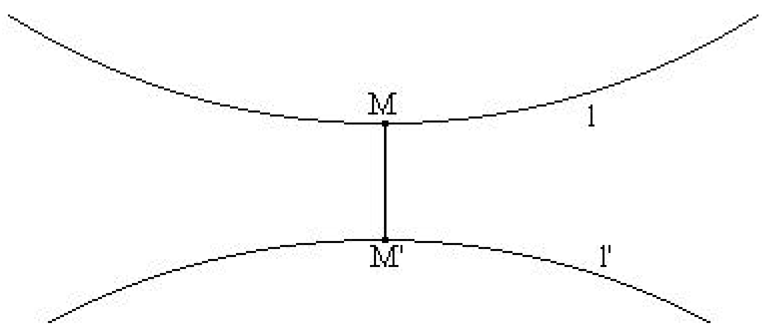

Figure 2.22 Divergently-parallel lines

We will state the following theorem, which is slightly different from Theorem 2.1, as we will be using it in later proofs.

Theorem 2.26: *If two lines are cut by a transversal such that alternate interior angles are congruent, then the lines are divergently-parallel.*

This differs from Theorem 2.1 because it guarantees not only that the lines do not intersect, but also that they diverge in both directions. There is another type of parallelism in hyperbolic geometry, those that diverge in one direction and converge in the other. We will look at this type now.

In Euclidean geometry, when two lines l and l' have a common perpendicular PQ, and you rotate l about P through even the smallest of angles, the lines will no longer parallel. In hyperbolic geometry, this is not the case, but how far can we rotate l about P? To answer this question, we first need to lay a little groundwork.

Theorem 2.27: *Given a line l and a point P not on l, with Q the foot of the perpendicular from P to l, then there exist two unique rays PX and PX' on opposite sides of PQ that do not meet l and have the property that any ray PY meets l iff PY is between PX and PX'. Also, the angles QPX and QPX' are congruent.*

Proof: Given line l and P not on l, with Q the foot of the perpendicular from P to l, let m be the line perpendicular to PQ at P. Line m is divergently parallel to l. Let S be a point on m to the left of P. Consider segment SQ. (Figure 2.23) Let Σ be the set of points T on segment SQ such that ray PT meets l, and Σ' the complement of Σ. We can see that if T on SQ is an element of Σ, than all of segment TQ is in Σ. Obviously, S is an element of Σ', so Σ' is non-empty. So there must be a unique point X on segment SQ such that all points on open segment XQ belong to Σ, and all points on open segment XS, to Σ'. PX is the ray with the property we are after.

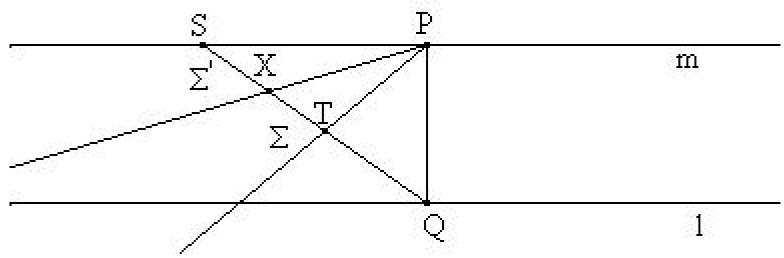

Figure 2.23 Rays from P parallel to, and intersecting l

It is easy to show that PX itself does not meet l. Suppose PX does meet l in A, then we can choose any point B on l such that A is between B and Q, and ray PB meets l, but cuts open segment XS, which contradicts what we know about X. (Figure 2.24) So PX can not meet l.

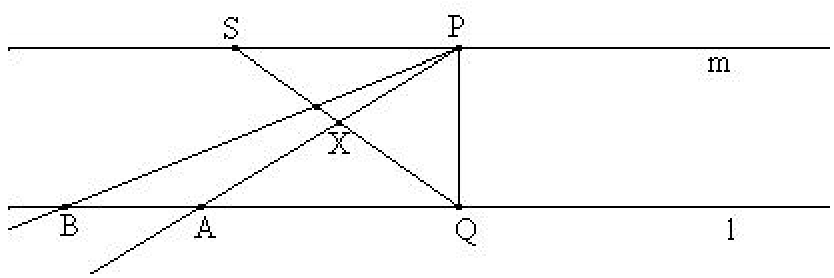

Figure 2.24 Rays from P intersecting l

We can find X' to the right of PQ in the same fashion, and all that remains to be shown is that angles QPX and QPX' are congruent. Assume that they are not, and that angle QPX>QPX'. Choose Y on the same side of PQ as X such that angle QPY≅QPX'. (Figure 2.25) PY will cut l in A. There is a unique point A' on l such that Q is the midpoint of segment AA'. By SAS, triangle PAQ≅PA'Q, and angle A'PQ≅APQ≅X'PX', and A' lies on PX', a contradiction, so angles QPX and QPX' are congruent. QED

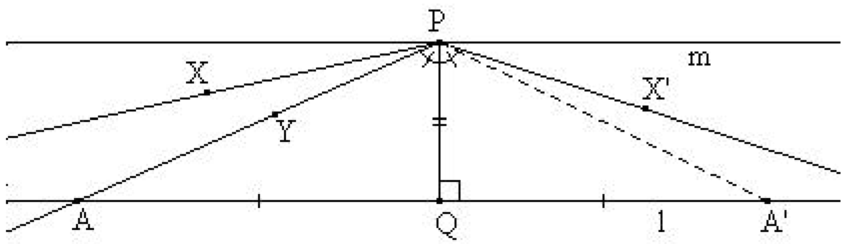

Figure 2.25 Limiting parallels form congruent angles with the perpendicular

Definition: *Given line l and point P not on l, the rays PX and PX' having the property that ray PY meets l iff PY is between PX and PX' are called the* *<u>limiting parallel rays</u>* *from P to l, and the lines containing rays PX and PX' are called the* *<u>limiting parallel lines</u>*, *or simply the* *<u>limiting parallels</u>*.

These lines are sometimes called asymptotically parallel. We will state a few fairly intuitive facts here about limiting parallels without proof, for sake of brevity.

First: Limiting parallelism is symmetric, that is if line l is limiting parallel from P to line m, and point Q is on m, then m is the limiting parallel from Q to l in the same direction.

Second: Limiting parallelism is transitive, if points P, Q and R lie on lines l, m and n respectively, and l is limiting parallel from P to m, and m is limiting parallel from Q to n in the same direction, then l is the limiting parallel from P to n in that direction.

Third: If line l is limiting parallel from P to m, and point Q is also on l, then the l is the limiting parallel from Q to m in the same direction.

Given these properties, it is reasonable to say that lines that are limiting parallels to one another in one direction intersect in a point at infinity. We call these points ideal points and denote them, for the moment, by capital Greek letters.

In Theorem 2.27, the angle QPX is not a constant, but changes with the distance of P from l. This angle will prove to be useful in our upcoming investigations and will require formal notation.

Definition: *Given line l, point P not on l, and Q the foot of the perpendicular from P to l, the measure of the angle formed by either limiting parallel ray from P to l and the segment PQ is called the angle of parallelism associated with the length d of segment PQ, and is denoted P(d). (Figure 2.26)*

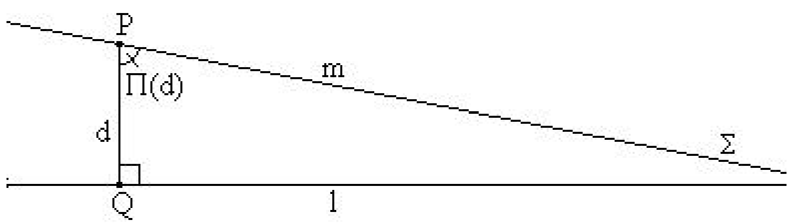

Figure 2.26 The angle of parallelism associated with a length

Note that Π(d) is a function of d only, so for any point at given distance d from any line, the angle of parallelism is the same. Also: Π(d) is acute for all d, approaches 90° as d approaches 0, and approaches 0° as d approaches ∞. These are not obvious facts, and we will prove them in Chapter V when we derive a formula for Π(d).

It is intuitive (and true) that as a point on l moves along l in the direction of parallelism, its distance from m becomes smaller, and as it moves in the other direction, its distance grows. So limiting parallels approach each other in one direction and diverge in the other. This distinguishes them from divergent parallels. We can show that they approach each other asymptotically and diverge to infinity.

Suppose, then, that we have lines l and m limiting parallel to each other, to the right. Select any point A on l, and let Q be the foot of the perpendicular from A to m. (Figure 2.27) We can choose any point R on segment AQ such that segment QR has any length less than AQ. Let line n be the limiting parallel from R to m, to the left. Since n can not meet m, and can not be limiting parallel to m to the right, (or n=m) n will meet l in point S. Let T be the foot of the perpendicular from S to m, and choose Q' on m such that T is the midpoint of segment QQ'. By SAS, triangles STQ and STQ' are congruent, and SQ≅SQ'. The perpendicular to m at Q' will cut l in R'. By subtraction of angles and congruent triangles, we see that Q'R'≅QR, which was arbitrarily small.

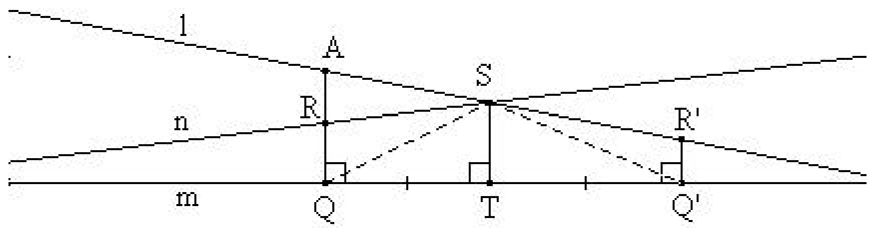

Figure 2.27 Limiting parallels are asymptotic and divergent in opposite directions

A symmetric argument, choosing R on line AQ such that A is between Q and R, will give us Q'R' arbitrarily large. So Limiting parallels are asymptotic in the direction of parallelism, and diverge without bound in the other. Also, since R was chosen at an arbitrary distance from m, there exists a point P on either line such that the distance from P to the other line is d. So:

Theorem 2.28: *Limiting parallels approach one another asymptotically in the direction of parallelism, diverge without limit in the other, and the distance from one to the other takes on all positive values.*

We now need one more theorem pertaining to a special kind of triangle

Definition: *A triangle having one or more of its vertices at infinity (an ideal point) is an* *<u>asymptotic triangle</u>. <u>Singly, doubly and trebly asymptotic triangles</u> have one, two and three vertices at infinity, respectively.*

An example of each type of asymptotic triangle is shown in Figure 2.28. A singly asymptotic triangle has only one finite side and two non-zero angles. A doubly asymptotic triangle has one non-zero angle and no finite sides, and is therefore defined entirely by the one non-zero angle. A trebly asymptotic triangle has no finite sides and no non-zero angles, (the measure of the asymptotic angle is taken to be zero), so all trebly asymptotic triangles are congruent. Note that the angle sum of any asymptotic triangle is less than 180°.

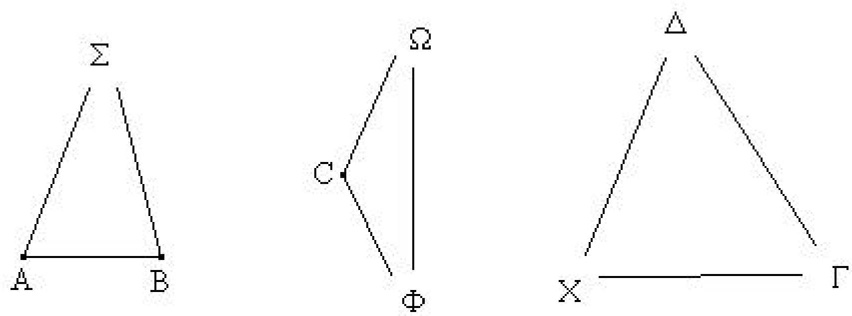

Figure 2.28 Singly, doubly and trebly asymptotic triangles

The following theorem establishes that the AAA criterion for congruence of singly asymptotic triangles.

Theorem 2.29: *Let two asymptotic triangles be given such that their non-zero angles are pairwise congruent. Then their finite sides are congruent.*

Proof: Suppose we are given ABΣ and PQΩ, both singly asymptotic triangles such that pairs of angles ABΣ and PQΩ, and BAΣ and QPΩ are congruent. (Figure 2.29) Let A' and P' be the feet of the perpendiculars from A and P to BΣ and QΩ respectively. Assume that segment AB>PQ, then AA'>PP'. We show this by Letting C be on segment AB such that BC is congruent to PQ, and letting C' be the foot of the perpendicular from C to BΣ. AAS congruence tells us that CC' is congruent to PP', and it is obviously less than AA'.

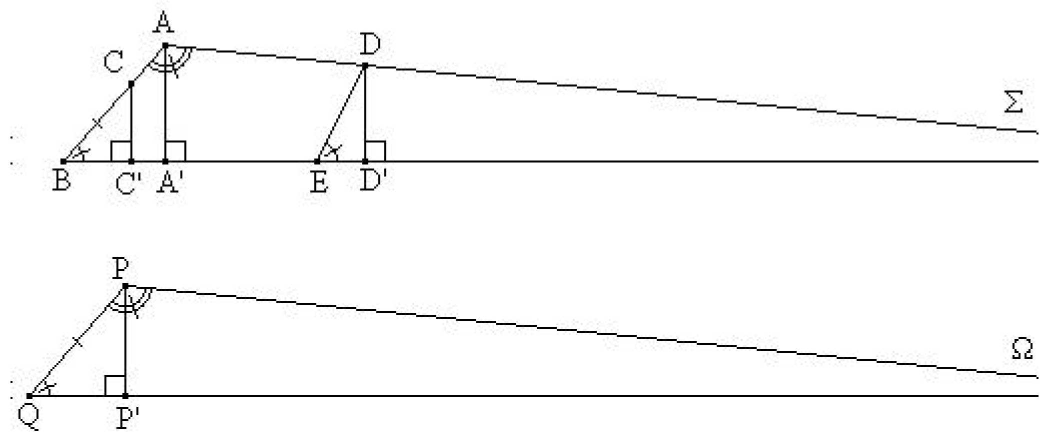

Figure 2.29 AAS condition for congruence of singly asymptotic triangles

Since AA'>PP', and since AΣ is asymptotic to BΣ, we can find the unique point D on AΣ such that PP' is congruent to DD', where D' is the foot of the perpendicular from D to BΣ. (Figure 2.29) The angle of parallelism D'DΣ is congruent to P'PΩ. By choosing point E on ray DB such that D'E is congruent to P'Q, we get triangle DD'E≅PP'Q, and angle DED'≅PQP'≅ABA'. AB is parallel to DE, by Theorem 2.1, and ADEB is a quadrilateral with angle sum 360°, a contradiction of Corollary 2.18, so AB≅PQ. QED

Recall from Chapter I the proof of the parallel postulate given by Legendre. The assumption was made that any line through a point in the interior of an angle must intersect at least one side of the angle. The following theorem shows that this is not the case.

Theorem 2.30 (The Line of Enclosure): *Given any two intersecting lines, there exists a third line that is the limiting parallel to each of the given lines, in opposite directions.*

Proof: Given lines l and m intersecting in point O, consider any one of the four angles formed by them. Let the ideal points at the 'ends' of l and m be Σ and Ω respectively Choose points A and B on OΣ and OΩ respectively such that OA≅OB. Draw segment AB, and the limiting parallels from A to m (AΩ), and from B to l (BΣ). These lines will intersect in point C. Next, draw the angle bisectors n and p of angles

ΣAΩ and ΣBΩ. These will cut BΣ and AΩ in F and G respectively. Also, let D be a point on ray AF such that F is between A and D. (Figure 2.30) We can see that angles OAC and OBC are congruent, and therefore angle ΣAC≅ΩBC, and we have ΣAF≅FAC≅CBG≅GBΩ. We will show that n and p are ultra-parallel, and therefore have a common perpendicular, and we will see that this common perpendicular is parallel to both l and m.

First, assume that rays AF and BG intersect in H. If so, then angles BAH and ABG are congruent, by angle subtraction, and AH≅BH. By a fairly trivial congruence argument, H is equidistant from AΩ and BΩ, so if we draw ray HΩ, then angle AHΩ≅BHΩ, which cannot be. So rays AF and BG do not intersect. Since angle AFΣ+FAΣ<180°, by substitution, GBF+BFD<180°, so rays FA and GB can not intersect, and the lines n and p do not intersect.

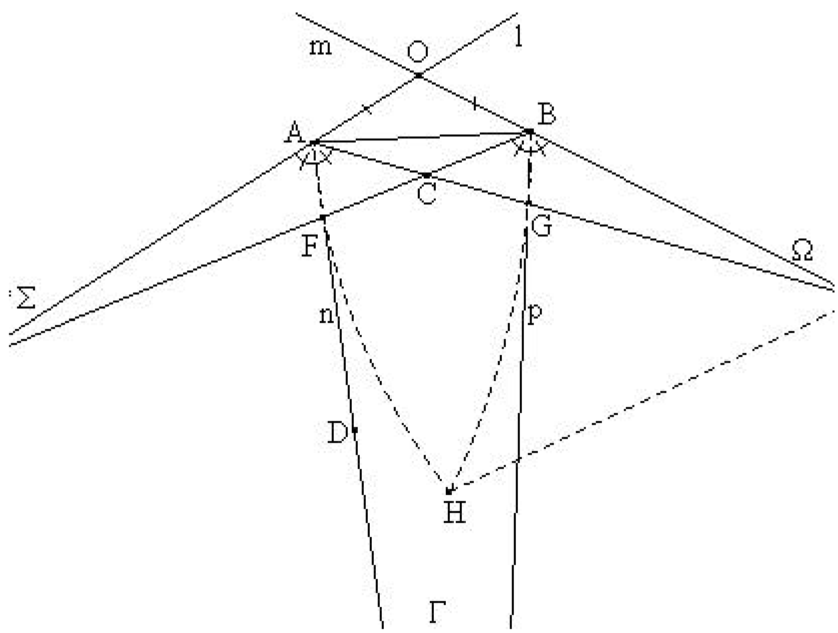

Figure 2.30 The line of enclosure of two intersecting lines I

Now assume that n and p are limiting parallels. Again, since angle DFB+FBG<180°, we know that n and p must be limiting parallels in the direction of ray AF, and 'intersect' in ideal point Γ. By applying Theorem 2.31 to the singly asymptotic triangles FAΣ and FBΓ, we see that FA≅FB, and therefore angle BAF≅ABF which is impossible. So n and p are not limiting parallels, and the only case remaining is that they are ultra-parallel and have a common perpendicular.

Let this perpendicular cut n in N and p in P. (Figure 2.31) ABPN is a Saccheri quadrilateral, so AN≅BP. Assume that NP is not limiting parallel to m, and draw NΩ and PΩ. Considering that N and P are equidistant from AΩ and BΩ respectively (by dropping the perpendiculars and using AAS) angles ANΩ and BPΩ are congruent, but this tells us that triangle NPΩ has one exterior angle congruent to the alternate interior angle, a contradiction of Theorem 2.4. So ray NP is limiting parallel to m, and by the symmetric argument, also to l, and line NP is limiting parallel to both intersecting lines l and m. There are, of course, three other such lines, one for each angle formed by l and m. QED

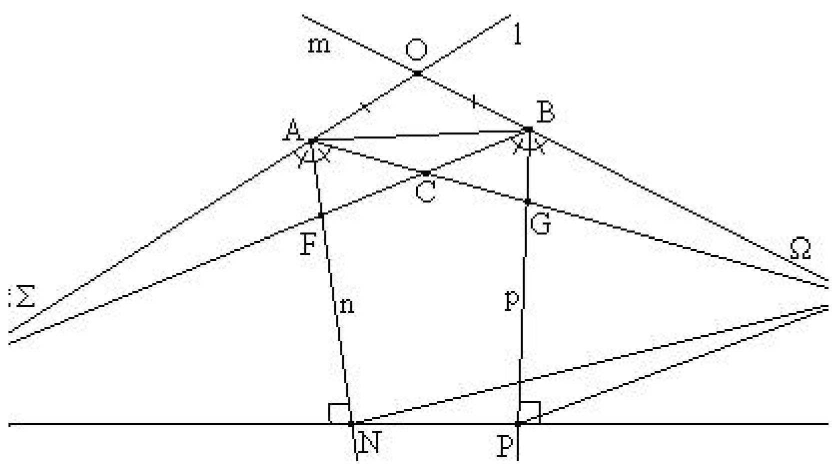

Figure 2.31 The line of enclosure of two intersecting lines II

Definition: *Given angle ABC, the line lying interior to the angle, and limiting parallel to both rays BA and BC is the* <u>*line of enclosure*</u> *of angle ABC.*

This theorem also shows that our angle of parallelism may be as small as we like, because no matter how small we choose the angle AOB, there is a line of enclosure l such that the angle of parallelism associated with distance from O to l is one half of AOB.

There is one more topic we will cover before we move on to the next chapter.

The in-circle and circum-circle of a triangle

In Euclidean geometry, every triangle has an inscribed circle, and the center of this circle is the intersection of the angle bisectors of the triangle. To prove this, we show that the three angle bisectors coincide, and that their mutual intersection point is equidistant from all three sides. The reader is no doubt acquainted with the Euclidean proof. This proof is also valid in hyperbolic geometry.

Theorem 2.31: *Inside any given triangle can be inscribed a circle tangent to all three sides.*

Every triangle in Euclidean geometry also has a circumscribed circle, whose center is the intersection point of the perpendicular bisectors of the three sides. In contrast to the angle bisectors, the perpendicular bisectors of the three sides of a triangle in hyperbolic geometry will not always intersect.

`**Theorem 2.32**: *Give any triangle, the perpendicular bisectors of the three sides either; intersect in the same point, are limiting parallels to each other, or are divergently parallel and share a common perpendicular.*

The circumscribed circle exists only for the case where the three bisectors intersect. We will examine this condition more closely in Chapter VIII.

Proof: Suppose we have triangle ABC with l and m the perpendicular bisectors of segments AB and BC.

Case I: Suppose l meets m in O. (Figure 2.32) We need to show that the perpendicular bisector of AC passes through O. By SAS congruence of the appropriate triangles, we can see that AO, BO and CO are all congruent, so triangle AOC is isosceles, so the perpendicular from O to AC will bisect AC, by HL congruence, and the fact that the perpendicular bisector of AC is unique, it passes through O, and we are done.

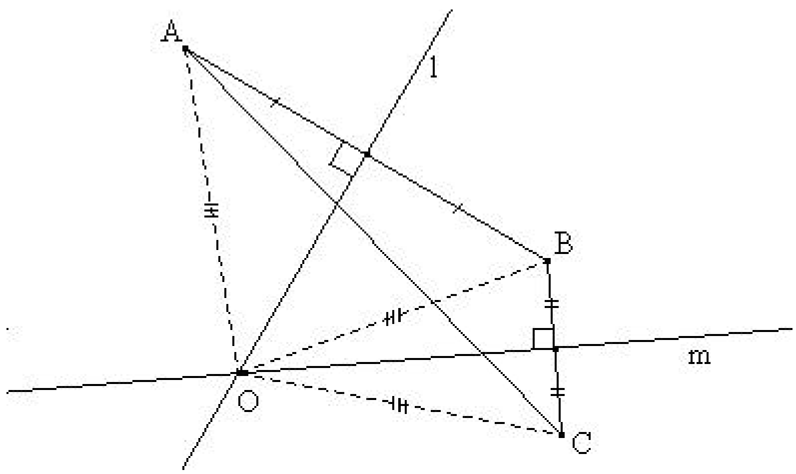

Figure 2.32 The circum-center of a triangle

Case II: Suppose that l and m are divergently parallel with common perpendicular p. (Figure 2.33) We need to show that the perpendicular bisector of AC is perpendicular to p. Drop perpendiculars AA', BB' and CC' from A, B and C to p, and let l meet AB and p in L and L', and m meet BC and p in M and M' respectively. Now, by SAS, triangles AL'L and BL'L are congruent, so segment AL'≅BL', and angle

AL'L≅BL'L. By angle subtraction, we have angle AL'A'≅BL'B', and by AAS, triangle AL'A'≅BL'B'. This gives us AA'≅BB', and by the same argument, BB'≅CC'. ACC'A' is a Saccheri quadrilateral, and the segment connecting the midpoints of A'C' and AC are perpendicular to both, and is therefore the perpendicular bisector of side AC, and perpendicular to p, and we are done.

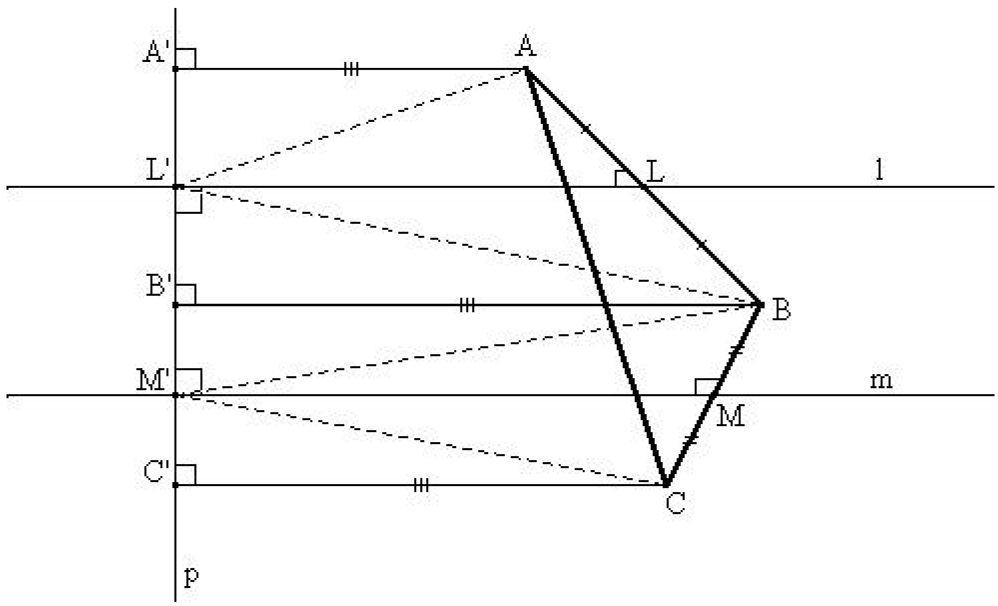

Figure 2.33 The pairwise parallel perpendicular bisectors of the sides of a triangle

Case III: This case trivial since, if l and m are limiting parallels, the perpendicular bisector of AC being anything other than limiting parallel to both would be contradictory to one of the first two cases, and we have proven the theorem. QED

We will look more at the properties of triangles and circles in hyperbolic geometry. Before we do so, however, we will introduce some models of the hyperbolic geometry that we have studied abstractly so far. These models will allow us to visualize the properties of non-Euclidean geometry much more clearly.

Chapter III
The Models

So far, we have developed hyperbolic geometry axiomatically, that is independent of the interpretation of the words ‘point’ and ‘line’. To help visualize objects within the geometry, and to make calculations more convenient we use a model. We define points and lines as certain ‘idealized’ physical objects that are consistent with the axioms. This system of lines and points is the model of the geometry. Though the pictures drawn in the model are consistent with the axiomatic development of the geometry it represents, they are not the geometry, merely a way of picturing objects and operations within the geometry. Probably the best known model of a geometry is:

The Euclidean Model

This model is derived by defining a ***point*** to be an ordered pair of real numbers (x,y), a ***line*** to be the sets of ordered pairs (points) that solve an equation having the form ax + by = c where a, b and c are given real numbers, and the ***plane*** to be the collection of all points. Two lines ax + by = c and dx + ey = f are said to ***intersect*** if there exists a point (x,y) that satisfies both equations.

The distance between two points A(x,y) and B(z,w) in the plane is given by:

$$d(A,B) = \sqrt{(z-x)^2 + (w-x)^2}$$

And the angle between two lines ax + by = c and dx + ey = f by:

$$angle = \left| \tan^{-1}\left(\frac{-d}{e}\right) - \tan^{-1}\left(\frac{-a}{b}\right) \right|$$

(or by π minus this value.)

This model is consistent with the five postulates of Euclidean geometry, and is usually referred to as the Euclidean Plane, the Real Plane, or R^2. It is assumed that the reader is familiar with the Euclidean Plane Model, and we will move on to the hyperbolic models. There are three models that dominate the discussion of elementary hyperbolic geometry; the Klein Disk, the Poincaré Disk and The Poincaré Upper Half-Plane models. All three are realized with the Euclidean Plane, but all three have entirely different flavors, especially when constructing objects within them. (These will be explored in the Appendix) All three also have their advantages and disadvantages. The Upper Half-Plane is the most convenient for employing the Calculus and analytic geometry to derive formulae and prove relationships, and we shall use this model for most of our development of hyperbolic geometry. Before we do, we will look at the two other models.

The Klein Disk Model

For the actual definition and construction of the most basic objects such as points and lines, the Klein Disk model is the easiest of the three. For this reason we introduce it first. For anything more complicated, such as calculating angle measures, it is considerably less convenient.

When introducing parallel lines to middle school or high school students, teachers often say something along the lines of, "Parallel lines never meet no matter how far you extend them. Lines that are not parallel will eventually meet if you extend them far enough." The Klein Disk Model, (or KDM) removes this distinction by eliminating the infinitude (in the Euclidean sense) of the line.

The model consists of the interior of the unit circle. The *points* are Euclidean points within the unit circle $\{(x,y) : x^2 + y^2 < 1\}$, *ideal points* lie on the circle $\{(x,y) : x^2 + y^2 = 1\}$, and *ultra-ideal points* lie without$\{(x,y) : x^2 + y^2 > 1\}$. The *lines* are the portions of Euclidean lines lying within the unit circle, or the chords of the circle, and two lines *intersect* if they intersect in the Euclidean sense and the point of intersection lies inside the unit circle.

Figure 3.1 illustrates the model. A, B, C, D and O are points; P, Q, R and S are ideal points; and AB, CD, OC, and CP are lines. Notice that line AB may also be referred to as AP, BQ, PQ or any combination of two distinct points or ideal points lying on it.

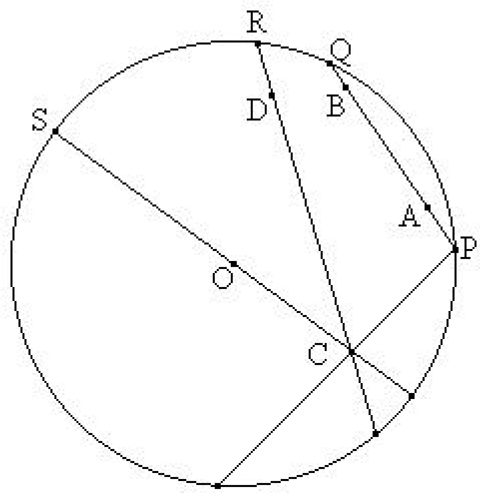

Figure 3.1 Points and lines in KDM

Note that line AB is limiting parallel to line CP, and divergently parallel to CD and CO.

A tool that we will be using in the discussions of metric in all three models is the cross ratio. For that reason we will introduce it here. Given four points in the plane, A, B, P and Q, we define the cross ratio (AB,PQ) by:

$$(AB, PQ) = \frac{(AP)(BQ)}{(AQ)(BP)}$$

where, e.g., AP is the length of the Euclidean segment AP.

The metric of KDM

The distance between any two points A to B in KDM is defined as follows:

$$h(A,B)=\frac{1}{2}\left|\ln\left(\frac{AP\cdot BQ}{AQ\cdot BP}\right)\right|=\frac{1}{2}\left|\ln(AB,PQ)\right|$$

where P and Q are the ideal points associated with line AB.

If A and B coincide, then (AB,PQ)=1, and h(A,B)=0, so h(A,A)=0.

The cross ratios (AB,PQ) and (BA,PQ) are merely reciprocals of each other, so the absolute values of the logs of these expressions will be equal, and h(A,B)=h(B,A).

We will not show the triangle inequality for the metric, but we can confirm easily that h(A,B) + h(B,C) = h(A,C) if A, B and C are collinear:

$$h(A,B)+h(B,C)=\frac{1}{2}\left|\ln(AB,PQ)\right|+\frac{1}{2}\left|\ln(BC,PQ)\right|=\frac{1}{2}\left|\ln\left(\frac{AP\cdot BQ}{AQ\cdot BP}\right)\right|+\frac{1}{2}\left|\ln\left(\frac{BP\cdot CQ}{BQ\cdot CP}\right)\right|$$

$$=\frac{1}{2}\left|\ln\left(\frac{AP\cdot BQ}{AQ\cdot BP}\right)\cdot\left(\frac{BP\cdot CQ}{BQ\cdot CP}\right)\right|=\frac{1}{2}\left|\ln\left(\frac{AP\cdot CQ}{AQ\cdot CP}\right)\right|=h(A,C)$$

Notice that as A and B become very close to each other (AB,PQ) approaches 1 and the metric approaches zero. Notice also that as A (or B) approaches P (or Q) the cross ratio (AB,PQ) approaches either zero or infinity, and h(A,B) approaches infinity. So, with this metric, our lines are indeed infinite.

Angle measure in KDM

A disadvantage of KDM is that it does not represent angles ‘accurately’, in fact the definition of angle measure is rather inconvenient. For lines l and m intersecting in point A, we define the measure of the angle formed by l and m at A as the angle formed by l' and m' at A' where l' and m' are the arcs of circles orthogonal to the unit circle at the endpoints of l and m, and A' is the intersection of l' and m'.

This is illustrated in Figure 3.2.

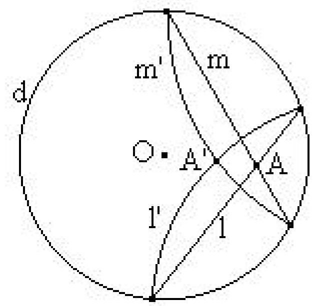

Figure 3.2 Angle measure in KDM

This angle measure gives us a curious definition for perpendicularity in KDM. In KDM, each line has associated with it an ultra-ideal point exterior to d (the unit circle) called the *polar point* of the line. It is defined for line l in KDM as the intersection L of the e-lines tangent to d at the endpoints of l. (Figure 3.3) A line through O will have no polar point. (We can think of it as having its polar point at infinity)

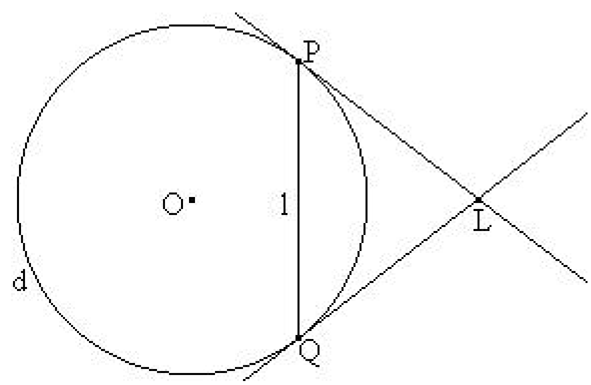

Figure 3.3 The polar point L of line l in KDM

We define a line m as *perpendicular* to line l if the extension of line m contains the polar point L of line l, (l will contain M) (Figure 3.4)

This definition is easier to understand when we consider the definition of angle measure. Lines l and m are perpendicular if their related Euclidean circles l' and m' are. But if l' and m' are perpendicular, then l', m' and d are pairwise orthogonal, and the center of each lies on the radical axis of the other two. In Euclidean Geometry, given two circles intersecting in two points P and Q, any circle centered on line PQ that is orthogonal to one of the circles will be orthogonal to the other, and in fact, to any other circle containing P and Q. The set of circles containing both P and Q form a *pencil* of circles, and the line PQ is the *radical axis* of the pencil. (A development of pencils and radical axes can be found in Greenberg pp232-3) The radical axis of l' and d is the extended line l and the center of m' is M, so the extended line l must contain M, as line m must contain L. (Figure 3.4)

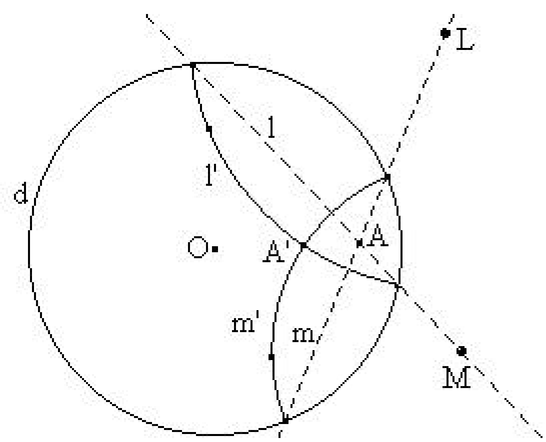

Figure 3.4 Perpendicular lines in KDM

One nice thing about KDM is that it has rotational symmetry, so regular polygons and tessellations have a pleasing and complete appearance that reminds one of, and may well have inspired, some of the works of M.C. Escher. Figure 3.5 depicts a partial tessellations of KDM by equilateral triangles.

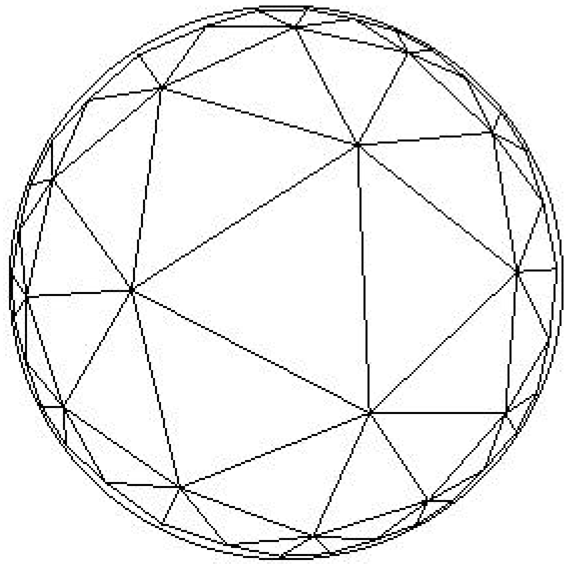

Figure 3.5 A partial tessellation of KDM

The Poincaré Disk Model

The second model we will consider is the Poincaré Disk Model, or PDM. It is somewhat similar to KDM in appearance. The slightly more complicated definition of lines in PDM gives it an important advantage over KDM. It is conformal.

PDM also resides in the interior of the unit circle d in the Euclidean plane. As in KDM, the points of PDM are the points lying interior to d, ideal points lie on d, and ultra ideal points lie exterior to d. The lines of PDM are general Euclidean circles (Euclidean lines and circles) orthogonal to d. These will either be arcs of Euclidean circles orthogonal to d (line AB in Figure 3.6), or diameters of d (line OC in Figure 3.6). Note that Figure 3.6 shows the same situation for PDM as was shown for KDM in Figure 3.1.

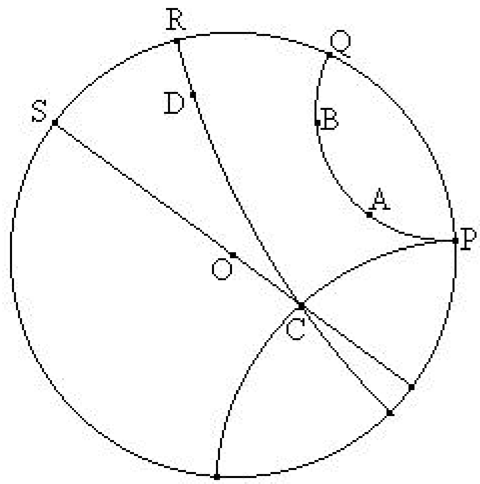

Figure 3.6 Points and lines in PDM

The metric of PDM

The metric in PDM is the same as in KDM:

$$h(AB) = \frac{1}{2}\left|\ln\left(\frac{AP \cdot BQ}{AQ \cdot BP}\right)\right| = \left|\frac{1}{2}\ln(AB, PQ)\right|$$

where P and Q are the ideal points at the 'ends' of the line AB. All the same properties of the metric hold.

Angle measure in PDM

The measure of the angle formed at point A by lines l and m is defined as the measure of the angle formed by lines l' and m' at A where l' and m' are the Euclidean lines tangent to l and m, respectively, at A.

The polar points of our lines in PDM (defined the same way as in KDM) make calculating angle measure simple. The angle formed by lines l and m at point A is equal to the measure of angle LAM, or its complement, where L and M are the polar points of l and m respectively. (Figure 3.7) It is evident that rotation through a right angle about A sends the tangents to the Euclidean circles l and m at A to the lines LA and MA, which are the radii of the Euclidean circles l and m.

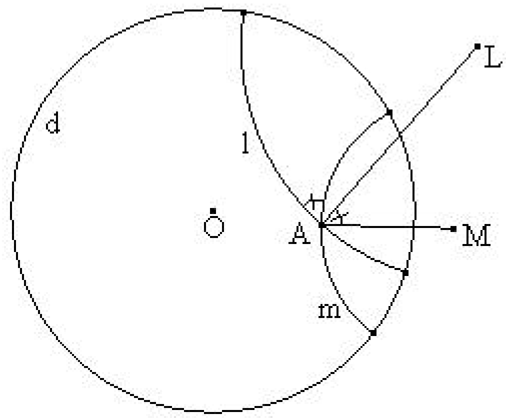

Figure 3.7 Measuring angles in PDM

While constructions in PDM tend to be more complicated than in KDM, the fact that PDM is conformal makes the pictures of objects look more like they ‘should’. For example, Figure 3.8 shows a right triangle in both KDM and PDM. The right angle at C looks right in PDM, but not in KDM.

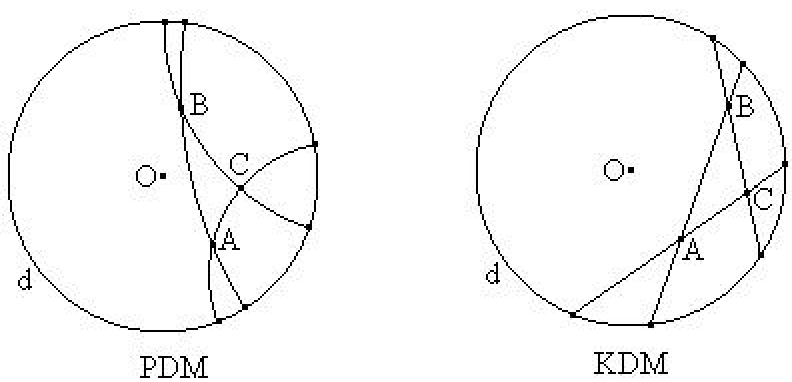

Figure 3.8 Right triangles in KDM and PDM

Tessellations are also symmetric and nice in PDM. Figure 3.9 shows a partial tiling of the plane by equilateral triangles.

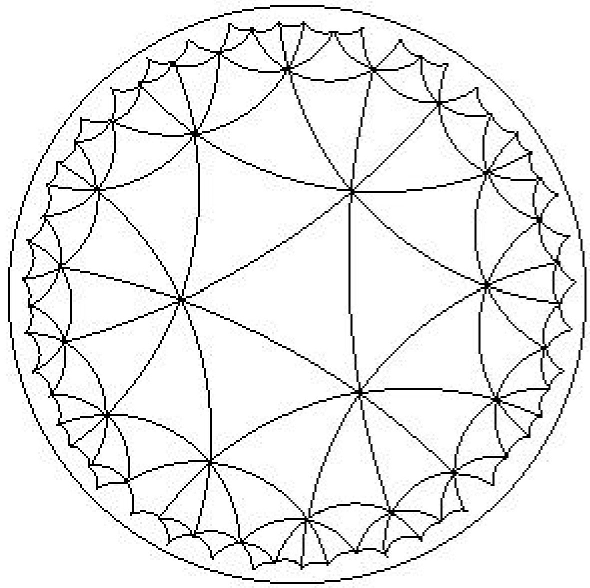

Figure 3.9 A partial tessellation of PDM

While both KDM and PDM allow for easy visualization, a major disadvantage of both is that any calculations are tedious and messy. Our next model, The Upper Half-Plane Model (UHP) is much more convenient for calculations and we will use it to investigate many theorems and formulae of hyperbolic geometry.

The Upper Half-Plane Model

We will now introduce the third of our three models of hyperbolic geometry. The *Upper Half-Plane Model*, (or *UHP*) is defined as follows.

UHP resides within R^2. The ***points*** of UHP are:

$$\{(x, y) : x, y \in R, y > 0\}$$

Which is the half-plane lying above the x-axis (or x) in R^2. We will refer to these points by capital letters from the beginning of the alphabet. (A, B, C, ...) In addition to ordinary points, it will be useful for us to define the set of ***ideal-points*** (or i-points) in UHP as:

$$\{(x,0) : x \in R\} \cup \{\infty\}$$

Which are the Euclidean points on the x-axis with the addition of the Euclidean point at infinity. We will denote i-points by capital letters beginning with P (P, Q, R, ...) and we will reserve the label 'Z' for the point at infinity. It is not entirely incorrect to think of the set of ideal points of UHP as surrounding the model with the point at infinity 'tying' the ends of the x-axis together 'above' the plane, much like the i-points of KDM and PDM surround the ordinary points.

A ***line*** in UHP is defined as the set of points satisfying the conditions $x = b$ and $y > 0$, or $(x - c)^2 + y^2 = r^2$ and $y > 0$, where b, c, and r are real numbers and r is positive. These are obviously of two types. The first type is an open vertical ray emanating from the x-axis, and the second is the upper half of a circle centered on the x-axis. (Note that we can consider the vertical ray as a circle of infinite radius). Both types of lines are orthogonal to the x axis. ($y = 0$) We will denote lines by lower case letters from the second half of the alphabet. (l, m, n, ...) Note also that each line 'contains' two i-points, one at each 'end'. Lines of the Euclidean circle type 'contain' two i-points on x while lines of the vertical Euclidean ray type 'contain' one i-point on x and Z at the other 'end'. (Figure 3.10) Notice that this means that all lines of this type are limiting parallel to each other, as they all contain the same i-point.

Two lines are said to ***intersect*** if there is a point of UHP that satisfies the equations of both lines (if they intersect in the Euclidean sense)

Figure 3.10 shows three lines l, m, and n. Lines l and m intersect in point A, m and n are limiting parallels, as they share i-point S, and lines l and n are divergently parallel. Curves p and q are not lines, as they are not orthogonal to x, but they do have a significance we will discuss in Chapter VI.

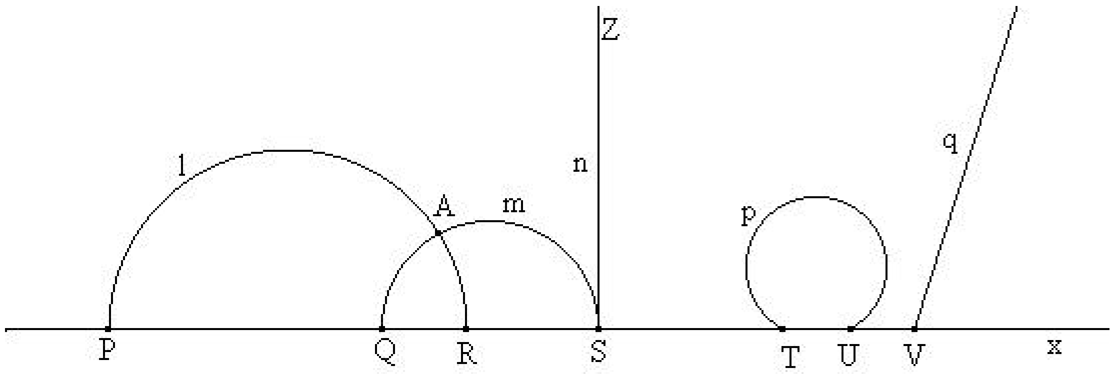

Figure 3.10 Lines and Non-lines in UHP

Figure 3.11 shows some triangles in UHP. Triangle ABC is an ordinary triangle, while DER is singly-asymptotic, both PEQ and QER are doubly-asymptotic, and PQR is trebly-asymptotic.

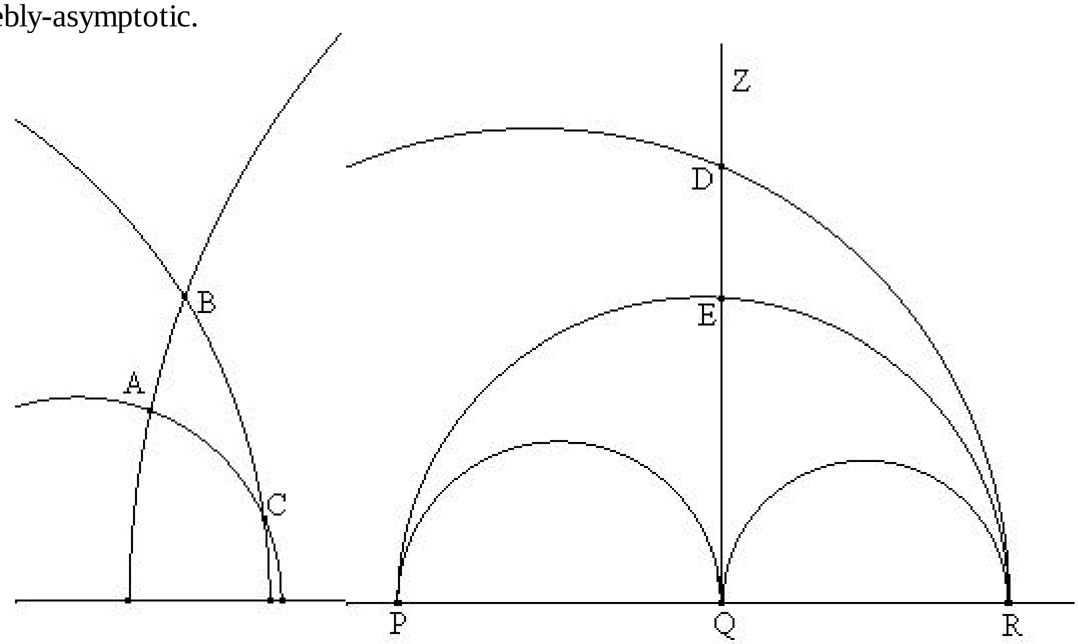

Figure 3.11 Triangles in UHP

We will sometimes need to refer to an object in UHP by its role in the Euclidean Plane. For example, in Figure 3.10, the object labeled l is a line in UHP, but is a half circle in R^2. To avoid confusion, when we are referring to the role an object plays in R^2, we will prefix an e- to the front of the name. So instead of saying, "the radius of the

Euclidean circle associated with line l", we will say, "the radius of e-circle l". (Since it does not make sense to refer to the radius of a line.) Similarly, line n might also be called e-ray n.

Angle measure in UHP

The measurement of angles in UHP is straightforward. The measure of angle ABC is defined as the measure of the angle formed by the e-rays tangent to BA and BC at B in the same direction as rays BA and BC. (Figure 3.12) In other words, the UHP measure for the angle between lines is the same as the Euclidean measure of the angle between the half circles. We say that UHP is conformal, (angles are as they appear).

The measure may also be thought of as the measure of e-angle OBP (or its complement, according as BA and BC are in the same or opposite clockwise directions, respectively) where O and P are the centers of the e-circles AB and BC respectively. In Figure 3.12, angle ABC is the complement of e-angle OBP.

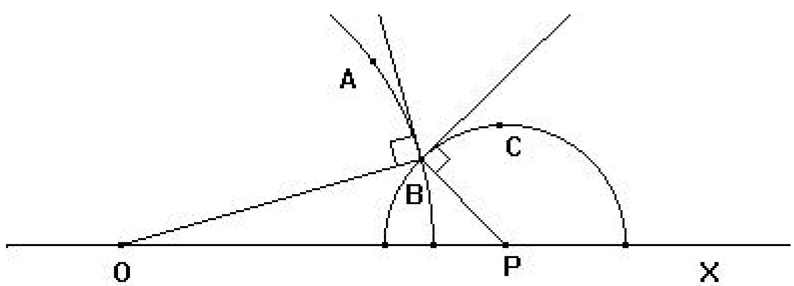

Figure 3.12 Measurement of angles in UHP I

Angles formed by lines of the vertical e-line and e-circle type are measured similarly, but somewhat more simply. In figure 3.13, the measure of angle ABC is equal to the measure of e-angle OPB (or its complement, should C and P lie on opposite sides of AB), where O and P are the intersection of AB with x and the center of e-circle BC respectively. This can be shown by a simple counterclockwise rotation through a right angle. This simpler method of measurement will be invaluable in our development of trigonometry in Chapter V.

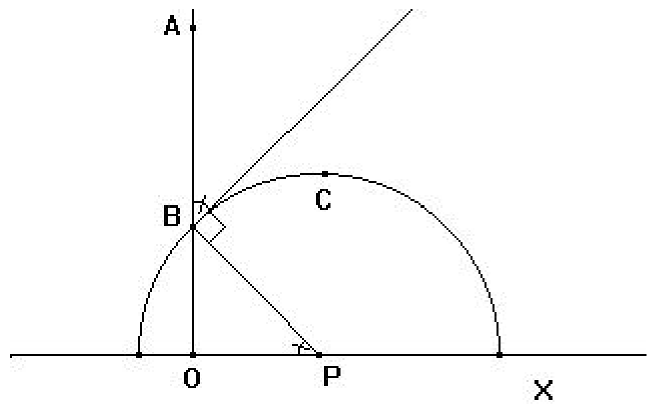

Figure 3.13 Measurement of angles in UHP II

The metric of UHP

Using the Euclidean metric in UHP would be problematic, because our model would fail to adhere to Euclid's second postulate, essentially that lines are infinite in both directions. So, we need to adjust our metric. Since, as P approaches the 'end' of line l, its y-coordinate approaches zero, it seems that division by the y-coordinate might be in order.

We define a metric as:

$$h(A,B) = \inf\left\{\int_s F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right)dt\right\}$$

where s is any path from A to B, and F is a function. For ease of notation, we will shorten dx/dt and dy/dt to x-dot and y-dot. We call the path that yields the minimum distance (if it exists) the geodesic. In R^2, the geodesic is the line segment AB and the function used to define the metric is:

$$F_E(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

If we alter this slightly, by division by the y-coordinate, we get:

$$F_{UHP}(x, y, \dot{x}, \dot{y}) = \frac{1}{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$

To find the extremal curve for this function, in this case the curve that yields the minimal distance, or geodesic, we must satisfy the two differential equations:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = 0$$

The four partial derivatives of F are:

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = \frac{-\sqrt{\dot{x}^2 + \dot{y}^2}}{y^2}, \quad \frac{\partial F}{\partial \dot{x}} = \frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}, \text{ and } \frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}$$

We can reparameterize by letting x = t , giving us:

$$F_x = 0, \quad F_y = \frac{-\sqrt{1 + \dot{y}^2}}{y^2}, \quad F_{\dot{x}} = \frac{1}{y\sqrt{1 + \dot{y}^2}}, \text{ and } F_{\dot{y}} = \frac{\dot{y}}{y\sqrt{1 + \dot{y}^2}}$$

Substituting these into the first of our differential equations we get:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 - \frac{d}{dt}\frac{1}{y\sqrt{1 + \dot{y}^2}} = 0$$

or:

$$\frac{1}{y\sqrt{1 + \dot{y}^2}} = c$$

$$y\sqrt{1 + \dot{y}^2} = R$$

$$\sqrt{y^2 + (y\dot{y})^2} = R$$

$$y\dot{y} = \sqrt{R^2 - y^2}$$

If we let z = y^2 and substitute:

$$\frac{1}{2}\dot{z} = \sqrt{R^2 - z} \qquad \text{or} \qquad 1 = \frac{\dot{z}}{2\sqrt{R^2 - z}}$$

and integrate both sides over x and re-substitute:

$$\int_x 1 = \int_x \frac{\dot{z}}{2\sqrt{R^2 - z}}$$

$$x = \sqrt{R^2 - z} + k = \sqrt{R^2 - y^2} + k$$

which is exactly the equation of a circle of radius R centered at k on the x-axis, and we have that the lines of UHP are the geodesics. It turns out that the solution to the second differential equation is the same.

Note that this solution only confirms that lines of the e-circle type are geodesics and says nothing about lines of the vertical e-ray type. In Chapter IV we will see that lines of e-circle type and vertical e-ray type in UHP may be sent to each other by isometries. Since isometries preserve metric, our vertical lines are also geodesics. Note also that since the equation is valid for all values of R and k, any line of the e-circle type is a geodesic, regardless of its position on x, or its radius.

To find a useful expression for our metric, we impose upon UHP the polar coordinate system with the center of e-circle AB (which has e-radius R) at the origin. (Figure 3.14)

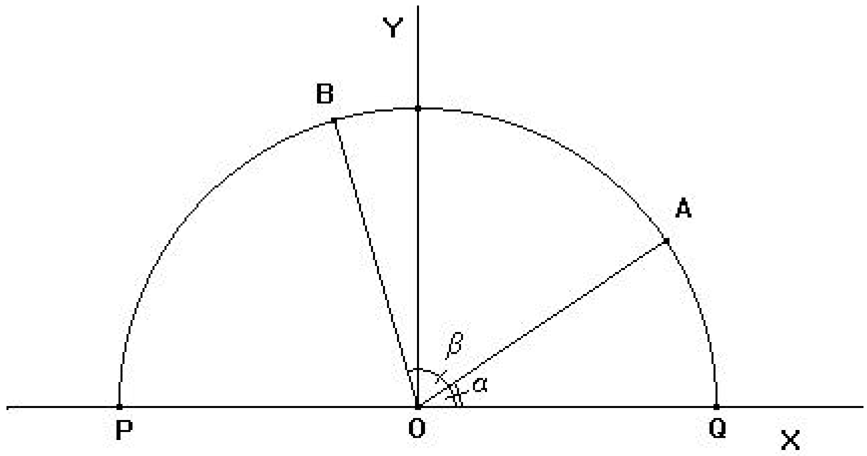

Figure 3.14 Line AB in UHP with center at O

This gives us the following parametric representation:

$$A = R \cdot \cos(a) \text{ and } B = R \cdot \sin(b)$$

and if we let (x,y) on the line, our geodesic, be written in polar coordinates:

$$x = R \cdot \cos(t) \text{ and } y = R \cdot \sin(t)$$

we get:

$$\dot{x} = -R\sin(t) \text{ and } \dot{y} = R\cos(t)$$

Plugging these into our formula for F gives us:

$$\begin{aligned} F(x, y, \dot{x}, \dot{y}) &= \frac{1}{y}\sqrt{\dot{x}^2 + \dot{y}^2} \\ &= \frac{1}{R \cdot \sin(t)}\sqrt{(-R \cdot \sin(t))^2 + (R \cdot \cos(t))^2} \\ &= \frac{\sqrt{R^2}}{R \cdot \sin(t)} \\ &= \frac{1}{\sin(t)} \end{aligned}$$

And integrating along our geodesic we get:

$$h(A,B)=\int_a^b \frac{dt}{\sin(t)}$$

$$=\ln\left(\frac{|\sin(t)|}{|1+\cos(t)|}\right)\Bigg|_a^b=\ln\left(\frac{\sin(b)\cdot(1+\cos(a))}{\sin(a)\cdot(1+\cos(b))}\right)$$

$$=\ln\left(\frac{\csc(b)-\cot(b)}{\csc(a)-\cot(a)}\right)=\ln\left(\frac{\tan\left(\frac{b}{2}\right)}{\tan\left(\frac{a}{2}\right)}\right)$$

Note that the radius of our e-circle has been eliminated from the expression. This tells us that the length of segment AB depends only upon the position of A and B on the e-circle relative to the positive x-axis, that is the angles formed by e-rays OA OB with x.

If we consider lines of the vertical e-ray type to be e-circles with their centers at infinity we find an even simpler expression for the metric along these types of lines. (Figure 3.15) As the center O of the e-circle containing segment AB moves to infinity (Z, not an i-point on x) both angles α and β go to zero. The ratio of the tangents of these angles approaches the ratio of the y-coordinates, a and b, of points A and B, and the interior of our metric approaches:

$$\frac{\tan\left(b/2\right)}{\tan\left(a/2\right)}\rightarrow\frac{b/2}{a/2}=\frac{b}{a}$$

and

$$h(a,b)=\ln\left(\frac{b}{a}\right)$$

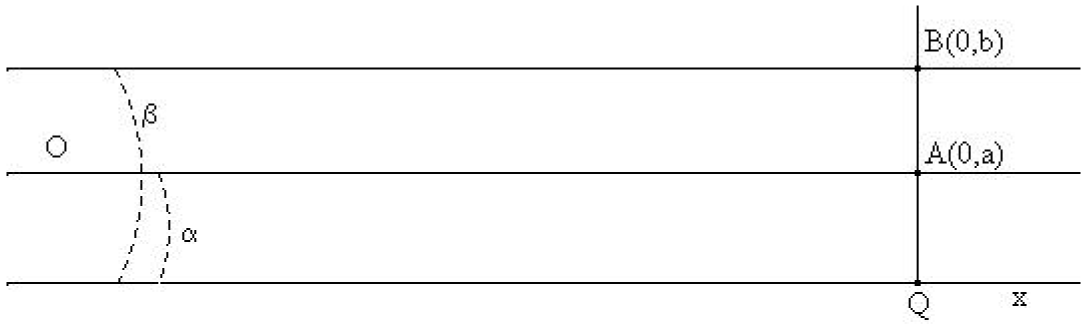

Figure 3.15 Metric for segments of vertical e-lines in UHP

Clearly both expressions for distance will have negative value when α > β or when a > b. Since we want h(A,B) to be non-negative we take the absolute value and get:

$$h(A,B) = \left| \ln\left(\frac{\tan \frac{b}{2}}{\tan \frac{a}{2}} \right) \right| \qquad \text{or} \qquad h(A,B) = \left| \ln\left(\frac{b}{a} \right) \right|$$

according as line AB is of the e-circle or vertical e-ray type.

Though this is the most common, and useful for our purposes, form of the metric, there is another form that will be important to us when we look at isometries in UHP. It turns out that the metric in UHP is equivalent to the metrics of PDM and KDM. To show this we consider points A and B on a line of the e-circle type centered at point O on x, and let P and Q be the i-points at the 'ends' of the line. We say angle QOA=α, and angle QOB=β. (Figure 3.16) Using the Law of Cosines to express the cross ratio (AB;PQ) in terms of α and β give us:

$$(AB,PQ) = \frac{BQ \cdot AP}{BP \cdot AQ} = \frac{\sqrt{2r^2 - 2r^2 \cdot \cos(b)} \cdot \sqrt{2r^2 - 2r^2 \cdot \cos(p-a)}}{\sqrt{2r^2 - 2r^2 \cdot \cos(p-b)} \cdot \sqrt{2r^2 - 2r^2 \cdot \cos(a)}}$$

$$= \sqrt{\frac{(1-\cos(b))(\cdot 1+\cos(a))}{(1+\cos(b))(\cdot 1-\cos(a))}} \cdot \sqrt{\frac{(1-\cos(b)) \cdot (1-\cos(a))}{(1-\cos(b)) \cdot (1-\cos(a))}}$$

$$= \sqrt{\frac{(1-\cos(b))^2 \cdot \sin^2(a)}{\sin^2(b)(\cdot 1-\cos(a))^2}} = \frac{(1-\cos(b)) \cdot \sin(a)}{\sin(b)(\cdot 1-\cos(a))}$$

$$(AB;PQ) = \frac{\tan\left(b/2\right)}{\tan\left(a/2\right)}$$

which is exactly the interior portion of our expression for the hyperbolic metric. This gives us an alternative expression for h(A,B):

$$h(A,B) = \left|\ln(AB,PQ)\right|$$

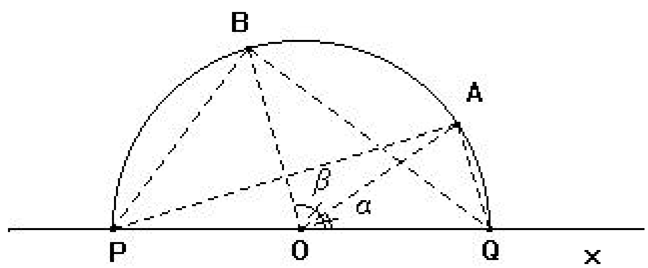

Figure 3.16 Metric of UHP as cross-ratio

We note here some basic properties of the metric that follow immediately from the properties of logs.

h(A,B) ≥ 0 with equality iff B = A and h(A,B)=h(B,A).

The fact that we are measuring along geodesics gives us the triangle inequality;

h(A,C) + h(C,B) µ h(A,B).

We will now verify that UHP satisfies the first four Euclidean postulates, as well as the hyperbolic parallel postulate.

The hyperbolic postulates in UHP

Postulate I: Through any two distinct points there exists a unique line.

This is obvious by the definition of the lines of UHP. If two points A and B are vertically related, there is a unique vertical e-ray through them. If not, then the e-perpendicular bisector of e-segment AB will intersect x in a unique i-point. The e-circle containing A and B that is centered at this i-point gives us the line.

Postulate II: To produce a finite line continuously in a straight line. (lines are infinite)

This can be shown by an examination of our metric. For lines of e-circle type, as A (or B) approaches either 'end' of the line, α (or β) approaches 0 or π, and the interior expression in our metric formula approaches either 0 or infinity. Taking the log and the absolute value, the distance goes to infinity. The same is true of lines of the vertical e-ray type. As A (or B) approaches the x-axis, or the point Z at infinity, the interior of the metric formula goes to 0 or infinity, and the distance approaches infinity. Since the distance formula is continuous for both types of lines, and $h(A,B) = 0$, we can extend a segment to any length.

Postulate III: To describe a circle with any center and distance.

This follows almost directly from the metric. If we consider all of the lines through a given point C, and all the points on these lines at a given distance r from C, we get a circle. We will see what this circle looks like in Chapter 6, and examine its properties in Chapter 7.

Postulate IV: All right angles are equal to one another.

This follows immediately from the fact that our model is conformal.

Postulate V: (the Hyperbolic Parallel Postulate) Given a line l and a point P not on l, then there are two distinct lines through P that are parallel to l.

This is evident by the definition of line in UHP. We can see in Figure 3.17 that there are two distinct lines through A (indeed an infinite number) that are parallel to l.

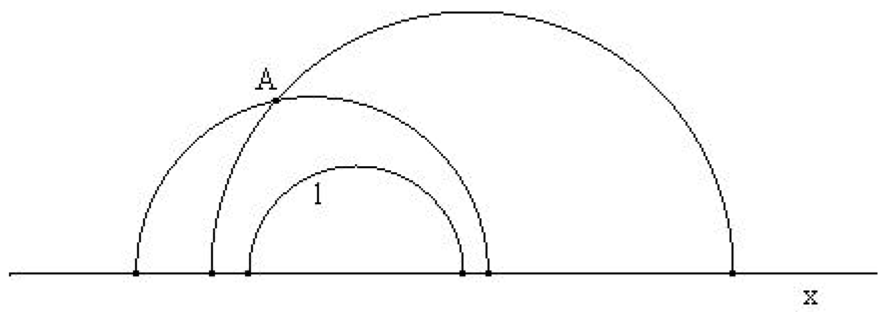

Figure 3.17 Illustration of the hyperbolic parallel postulate in UHP

Since all five postulates of hyperbolic geometry hold in UHP, it is a valid model of hyperbolic geometry. We will use this model to explore many formulae and theorems relating to triangles and circles in hyperbolic geometry. And we will discuss a couple objects in hyperbolic geometry that do not exist in Euclidean geometry. Before we do so, it will be helpful to examine the isometries in UHP.

Chapter IV
Isometries on UHP

Before we explore triangles in Chapter V, we must introduce isometries on UHP, and before we do that we will discuss:

Isometries on the Euclidean Plane

Plane isometries are functions from the plane onto itself that preserve the metric and angles. In the Euclidean plane there are 4 different isometries; reflection, rotation, translation, and glide-reflection. We will discuss the isometries of the Euclidean plane here as a basis for comparison to the isometries on the hyperbolic plane, specifically in the Upper Half-plane Model of hyperbolic geometry.

Reflection

The reflection in a given line l (called the mirror of the reflection) is defined as follows:

$$r_l : R^2 \to R^2 \qquad\qquad r_l(P) = P'$$

where l is the perpendicular bisector of every segment PP'. (Figure 4.1) The points of the mirror l are fixed under the reflection.

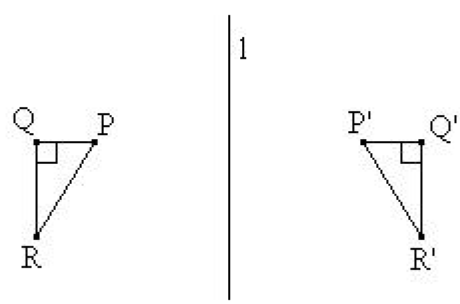

Figure 4.1 Reflection in the Euclidean Plane

Note that reflection reverses the 'sense' of an object. In Figure 4.1 triangle PQR is 'counter-clockwise', but its image, triangle P'Q'R' is 'clockwise'. Also notice that reflection in line l is self- inversive. That is:

$$r_l(r_l(P)) = P$$

Finding the mirror of a reflection given any point P and its image P' under the reflection is simple, merely construct the perpendicular bisector of segment PP'. Since any segment of positive length has a unique perpendicular bisector, any point P can be sent to any point Q, distinct from P, by reflection in exactly one mirror.

Translation

Translation through a given vector AB is defined as follows:

$$t_{AB} : R^2 \rightarrow R^2 \qquad\qquad r_{AB}(P) = P'$$

where vector PP' is of the same length and parallel to, or collinear with, vector AB. Equivalently for all P not on line AB, quadrilateral ABP'P is a parallelogram. (Figure 4.2) A translation in a non-zero vector has no fixed points.

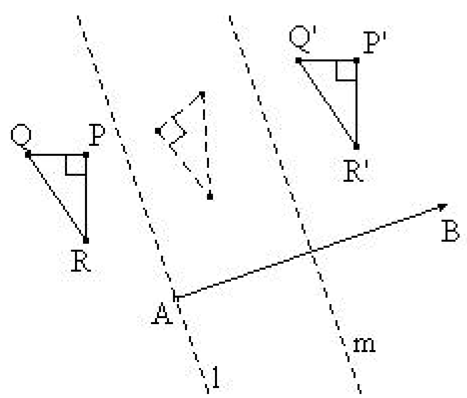

Figure 4.2 Translation in the Euclidean Plane

Note that translation retains the sense of an object. In Figure 4.2 both triangles

PQR and P'Q'R' are counter-clockwise. Also notice that the inverse of the translation in vector AB is the translation in vector BA, or:

$$t_{AB}\left(t_{BA}(P)\right) = P$$

The vector of translation that sends P to P' is merely the vector PP', or any vector having the same length and direction.

We can describe the translation in vector AB as the composition of two successive reflections. The first in line l, the line through A perpendicular to vector AB, and then in line m, the perpendicular bisector of segment AB. (Figure 4.2). Note that the distance between l and m is half the length of vector AB.

Even though a given vector yields a unique translation, each translation is defined by infinitely many vectors, all congruent and in the same direction as each other.

Rotation

The rotation about a point C (called the center of the rotation) through oriented angle α (called the angle of the rotation) is defined as follows:

$$R_{C,a} : R^2 \rightarrow R^2 \qquad R_{C,a}(P) = P'$$

where segments CP and CP' are congruent, and angle PCP' has directed measure α. (Figure 4.3) Only the center of the rotation is fixed.

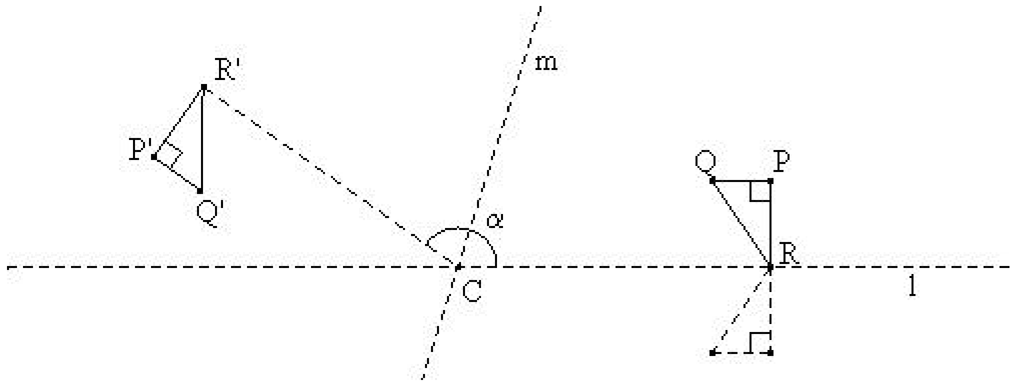

Figure 4.3 Rotation in the Euclidean Plane

Note that rotation, like translation, preserves the sense of the object, and that the inverse of rotation about center C by angle α is the rotation about C by -α, or:

$$R_{C,-a}\left(R_{C,a}(P)\right)=P$$

Given any two points P and Q and their images P' and Q' under the reflection, the center and angle of reflection can be found as follows. Construct the perpendicular bisectors l and m of segments PP' and QQ'. Since these are not parallel, they will intersect in point C, the center of the rotation. Directed angle PCP' gives us α.

We can describe any rotation with center C and directed angle α, as the composition of two successive reflections. The first in l, the line through C and P, and the second in line m, the angle bisector of angle PCP', where P is any point other than C and P' is its image under the rotation. (Figure 4.3) Note that the angle formed by l and m at C is one-half α.

Glide-reflection

Glide-reflection in a vector AB is defined as the composition of translation by vector AB with reflection in line AB. (Figure 4.4)

$$G_{AB} : R^2 \rightarrow R^2 \qquad G_{AB}(P) = r_{AB} \circ t_{AB}(P)$$

The order of the translation and reflection is unimportant. No points are fixed under glide-reflection in a vector of positive length.

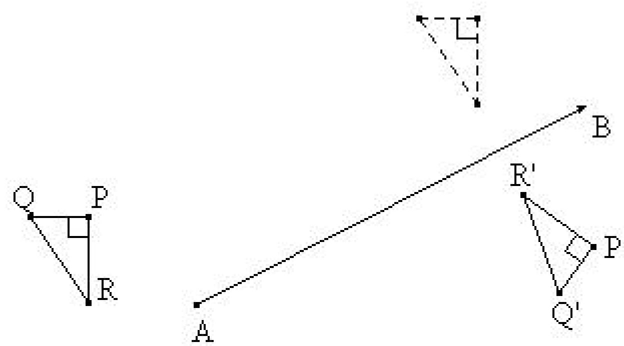

Figure 4.4 Glide-reflection in the Euclidean Plane

Note that glide-reflection reverses the sense of an object.

Finding the vector AB of a glide-reflection given two points P and Q and their images P' and Q' under the glide-reflection takes a little bit of work. First, find the midpoints M and N of segments PP' and QQ', then drop perpendiculars from each of P and P' to the line MN. The feet of these perpendiculars are A and B respectively. (Figure 4.5) Remember that the vector of a translation is not unique. This is also true of the glide-reflection. Any vector contained within line AB that is congruent to vector AB and in the same direction will define the same glide-reflection as vector AB.

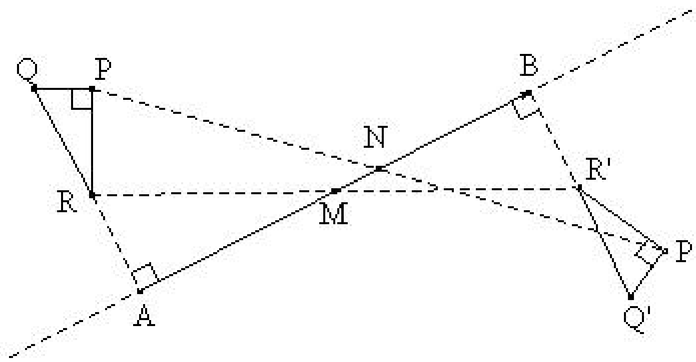

Figure 4.5 Finding the vector of a Glide-reflection

The glide-reflection, being the composition of a translation and a reflection, may also be described as the composition of three successive reflections in two parallel mirrors and a third which is mutually perpendicular to them. Specifically, these are line l, perpendicular to line AB at A, line m, the perpendicular bisector of segment AB, and line AB itself.

All isometries on the Euclidean plane are of one of these four types, and all are completely defined by three non-collinear points and their images. This means that given any two congruent triangles, we can find a unique isometry that will send one to the other. Furthermore, if the two triangles have the same sense, they are related to each other by either a translation or a rotation, each of which are the composition of two reflections. If the triangles have opposite sense they are related to each other by a reflection or a glide-reflection, either a single reflection or the product of three reflections. A much more complete treatment of Euclidean plane isometries may be found in Dodge [1].

Before we discuss our hyperbolic isometries, we need to look at one more Euclidean transformation:

Euclidean Inversion

Before we define inversion, we must first extend R^2 by attaching a point we call the point at infinity, giving us the Extended Euclidean Plane. Or:

$$R^2_\infty = R^2 \cup \{\infty\}$$

We need this infinite point to define the effect of inversion on the center of inversion. While we defined the isometries as mapping R^2 to itself, we can easily define them on the extended plane by merely stating that the point at infinity is fixed under all of them. It is not fixed under inversion.

Given any circle γ with center O and radius r, we define inversion in this circle as:

$$I_\gamma : R^2_\infty \to R^2_\infty \quad I_\gamma(P) = P'$$

where O, P and P' are collinear, OP · OP' = r^2, and:

$$I_\gamma(O) = \infty \qquad \text{and} \qquad I_\gamma(\infty) = O$$

In the extended Euclidean plane, inversion preserves neither lines nor the metric, but we will see that it does preserve angles. Also, inversion reverses the sense of an object. (Figure 4.6) The only fixed points under inversion are the points lying on the circle.

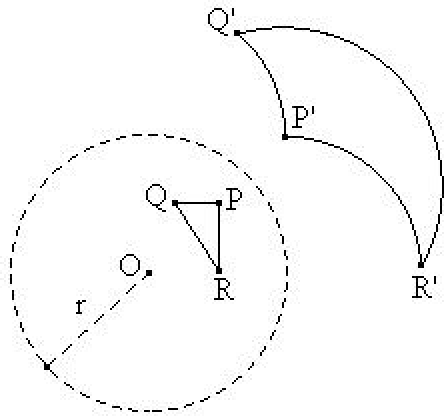

Figure 4.6 Inversion in the Extended Euclidean Plane

We note a few fairly obvious facts about inversion: first, Iγ is self-inversive, that is Iγ(Iγ(P)) = P, second, Iγ maps the interior of γ to the exterior, and vice versa. So the points of γ are the only fixed points.

The following two theorems will show that inversion in a circle preserves angles.

Theorem 4.1: *Given circle g with center O and points P and Q such that P, Q and O are not collinear. Assume P' and Q' are the images of P and Q under inversion in g Then triangle OPQ is similar to triangle OQ'P'. (Figure 4.7)*

Proof: We know from the definition of inversion that:

$$OP \cdot OP' = r^2 \quad \text{and} \quad OQ \cdot OQ' = r^2$$

Where r is the radius of γ, So:

$$OP \cdot OP' = OQ \cdot OQ'$$

$$\frac{OP}{OQ} = \frac{OQ'}{OP'}$$

And since angle POQ is equal to angle Q'OP',by SAS triangles OPQ and OQ'P' are similar. QED

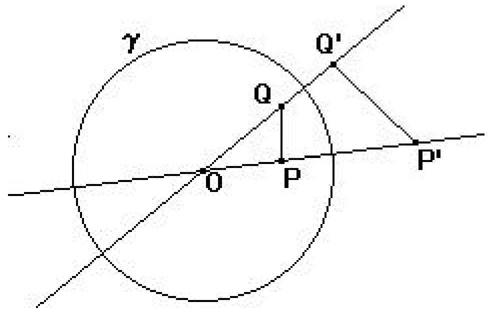

Figure 4.7 Similar triangles under inversion

Theorem 4.2: *Angles formed by curves are invariant under inversion. We say that inversion is conformal.*

Proof: Let **a** and **b** be curves intersecting in R, and let OP, (where O is the center of γ, the circle of inversion) be a ray different from OR that intersects **a** and **b** in P and Q respectively. Let **a** ', **b** ', R', P', and Q' be the images of **a** , **b** , R, P, and Q under Iγ. (Figure 4.8) We need to show that angle QRP = angle Q'R'P'. We know that the exterior angle of a triangle is equal to the sum of the two remote angles. So by simple angle subtraction:

$$PRQ \cong OPR - PQR \quad \text{and} \quad P'R'Q' \cong OR'P' - OR'Q'$$

And from Theorem 4.1:

$$PQR \cong OR'Q' \quad \text{and} \quad OPR \cong OR'P''$$

Simple substitution gives us:

$$PRQ \cong P'R'Q'$$

Now, as ray OP approaches OR, the lines RP, RQ, R'P' and R'Q' approach continuously the tangent lines to **a** , **b** , **a** ' and **b** '. It follows that the angles formed by **a** and **b** , and **a** ' and **b** ' are equal. QED

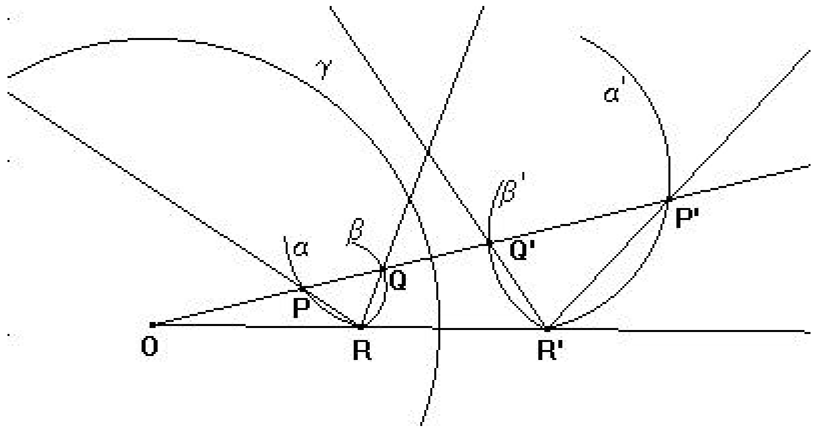

Figure 4.8 Preservation of angles under inversion

We can see that inversion does not preserve lines, but if we consider lines to be circles centered at infinity, and generalized circles to be the set of all circles and all lines, inversion does preserve generalized circles. We will look at how inversion affects these generalized circles.

Theorem 4.3: *The image of a circle not containing the center of inversion is another such circle. (Figure 4.9)*

Proof: Let α be a circle not containing O, the center of inversion. Let the ray OP through the center of α cut α in P and Q. Let ray OR cut α in any point R not on OP. Let P', Q' and R' be the images of P,Q and R under the inversion with center O and any radius. Since the angle PQ is a diameter of α, PRQ is a right angle. By the same argument we used in proving the preceding theorem, P'R'Q' is also a right angle, and therefore R' lies on the circle β having diameter P'Q'. So the image of any point R on α is the point R' on β, and the image of any circle α not through O is another circle which does not contain O. QED

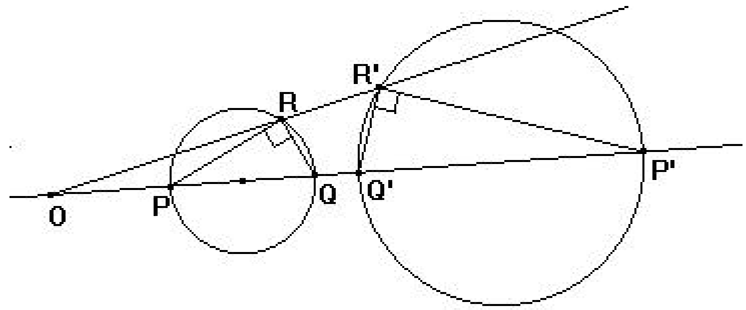

Figure 4.9 Circle mapping to circle under inversion

The condition in the preceding theorem prompts us to ask, What happens to a circle that does contain the center of inversion.

Theorem 4.4: *The image under inversion of a circle a containing O, the center of inversion, is a line orthogonal to the line containing O and the center of circle a. Also, the image of a line l not through O is a circle containing O and centered on the line through O orthogonal to l. (Figure 4.10)*

Proof: Let α be any circle containing O, the center of inversion. Let OP be a diameter of α, Q be any point of α, save O and P, and P' and Q' be the images of P and Q under the inversion . We know from Theorem 4.1 that triangles OPQ and OQ'P' are similar and therefore, angles OQP and QP'Q' are both right angles, so the image of Q lies on the line orthogonal to OP at P'. Also, inversion is bijective, so the image under inversion with center O of a circle containing O is the line orthogonal to the diameter OP of the circle at the image of the point P. The converse is an immediate consequence of the fact that an inversion is its own inverse. QED

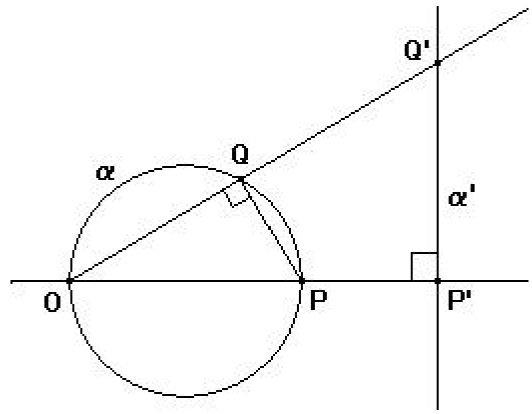

Figure 4.10 Circle mapping to line under inversion

So we know that the image under inversion of a circle is either a line or a circle, according as it does or does not contain the center of inversion. We also know that the image under inversion of a line not containing the center O of inversion is a circle containing O. These theorems tell us that inversion preserves generalized circles. There is one rather special situation left to consider: When a circle maps to itself.

Suppose that circle α maps to itself under inversion in circle γ, (with center O) (Figure 4.11). Since each point outside γ maps to a point inside γ, α must contain a point outside, and a point inside γ, and by the nature of circles, must intersect γ in two points which will be fixed under inversion in γ. Choose either of these points and call it Q and suppose that line OQ intersects α in another point P. If P is not on γ (Figure 4.11), since α maps to itself, OQ must also intersect α in P'. This means that line OQ intersects a circle in three points, which cannot be. If line OQ intersects α in a point P on γ (Figure 4.11), then O lies on chord PQ and is in the interior of α. We may choose any ray OR through any point R on α. Obviously R is not on γ, or else α and γ would intersect in three points, so ray OR also contains R', and intersects α in two points, something a ray emanating from the interior of a circle cannot do.

So OQ intersects α in just one point, Q, and is therefore tangent to α, but OQ is a radius of γ, so α must be orthogonal to γ. This tells us that if a circle maps to itself under inversion, it is orthogonal to the circle of inversion.

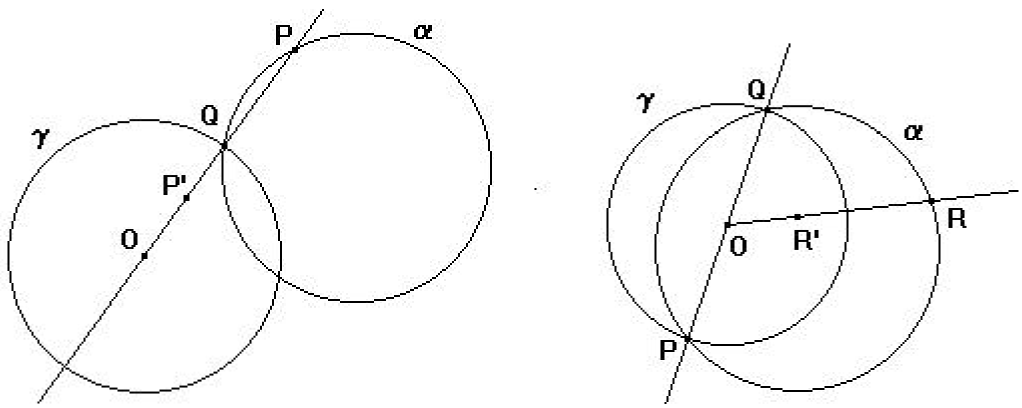

Figure 4.11 Inversion of orthogonal circle I

We demonstrate the converse of this statement as follows.

Suppose circle α is orthogonal to γ, the circle of inversion at P and Q. Rays OP and OQ , where O is the center of inversion, are tangent to α. Both rays and points P and Q are fixed by the inversion. Since inversion is conformal, tangency is maintained and the image of α must also be tangent to OP and OQ at P and Q respectively. But only one circle fits that condition, and that is α, so α maps to itself under inversion, and we have the following:

Theorem 4.5: *Circles and lines map to themselves under inversion iff they are orthogonal to the circle of inversion.*

It is evident that Theorem 4.5 is true for lines when one considers that a line orthogonal to the circle of inversion must contain the center of inversion.

We will now show how inversion affects the cross-ratio, (from our discussion of the metrics of the models in Chapter III).

Theorem 4.6: *Given four points A, B, P and Q such that none of the pairs AP, AQ, BP or BQ are collinear with point O, then the cross ratio of A, B, P and Q is preserved by inversion centered at O.*

Proof: Given a pair of points A and P, and their images A' and P' in an inversion about O, we have that triangle OAP is similar to triangle OP'A'. Applying this to our four relevant pairs gives us:

$$\frac{AP}{AO} = \frac{A'P'}{P'O} \quad , \quad \frac{BP}{BO} = \frac{B'P'}{P'O} \quad , \quad \frac{AQ}{AO} = \frac{A'Q'}{Q'O} \quad \text{and} \quad \frac{BQ}{BO} = \frac{B'Q'}{Q'O}$$

Now simple substitution gives us:

$$(A'B', P'Q') = \left(\frac{A'Q' \cdot B'P'}{A'P' \cdot B'Q'}\right) = \left(\frac{AQ \cdot Q'O \cdot AO \cdot BP \cdot P'O \cdot BO}{AO \cdot AP \cdot P'O \cdot BO \cdot BQ \cdot Q'O}\right) = \left(\frac{AQ \cdot BP}{AP \cdot BQ}\right) = (AB, PQ)$$

and we have the theorem. QED

We now have the tools we need to begin our discussion of:

Isometries on UHP

We will approach our hyperbolic isometries in a slightly different way. Since an isometry preserves metric and angle, it is completely defined by a triangle and its image under the isometry. We will look at how, given two congruent triangles, we may find the isometry that will send one to the other. We will begin by looking at:

Reflection

In Euclidean geometry, a line is sometimes viewed as a circle with its center at infinity, and reflection in the line as inversion in the infinite circle. Since the lines of UHP are e-circles, it seems natural that reflection in a line of UHP is the Euclidean inversion in the associated e-circle. This turns out to be the case.

Consider the congruent triangles ABC and A'B'C' with different orientations. (Figure 4.13) Suppose the perpendicular bisectors of segments AA' and BB' coincide and call this line m. We know that the e-circles associated with lines AA' and BB' are orthogonal to the e-circle associated with line m, and will remain fixed under inversion in e-circle m. Since our metric is preserved, this inversion will send A and B to A' and B' respectively. Also by preservation of angle and metric, and by the fact that inversion is orientation reversing, C will be sent to C', and we have that the Euclidean inversion in e-circle m acts as the hyperbolic reflection in line m.

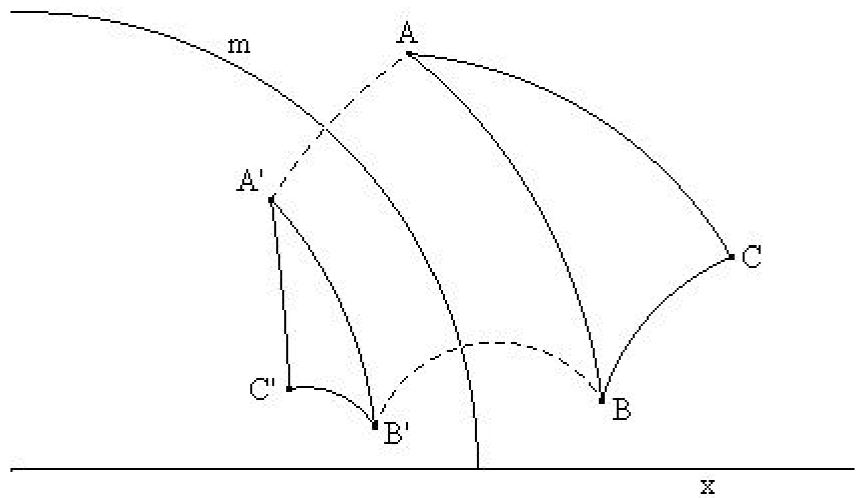

Figure 4.12 Reflection in UHP

Recall that under Euclidean inversion circles orthogonal to the circle of inversion, as well as lines through the center of inversion, remain fixed. This means that in UHP, lines perpendicular to the line of reflection remain fixed, and our hyperbolic reflection is defined entirely by the line (mirror) and is a direct analog of Euclidean reflection. Reflection in a line of vertical e-ray type is simply the Euclidean reflection in the associated e-line.

We will deal shortly with the case of an orientation reversing isometry where the perpendicular bisectors of AA' and BB' do not coincide, but before we do, we will examine the orientation preserving isometries.

We saw that in Euclidean geometry that the product of two reflections is either a rotation or a translation, according as the mirrors of reflection intersect or are parallel. In hyperbolic geometry two lines either intersect, are limiting parallels or are divergent parallels, so the product of two reflections in UHP will give us three distinct isometries. We will begin with the case where the mirrors intersect:

Rotation

Suppose we are given congruent triangles ABC and A'B'C' having the same orientation, and that the perpendicular bisectors l and m of segments AA' and BB' intersect in point O. (Figure 4.13)

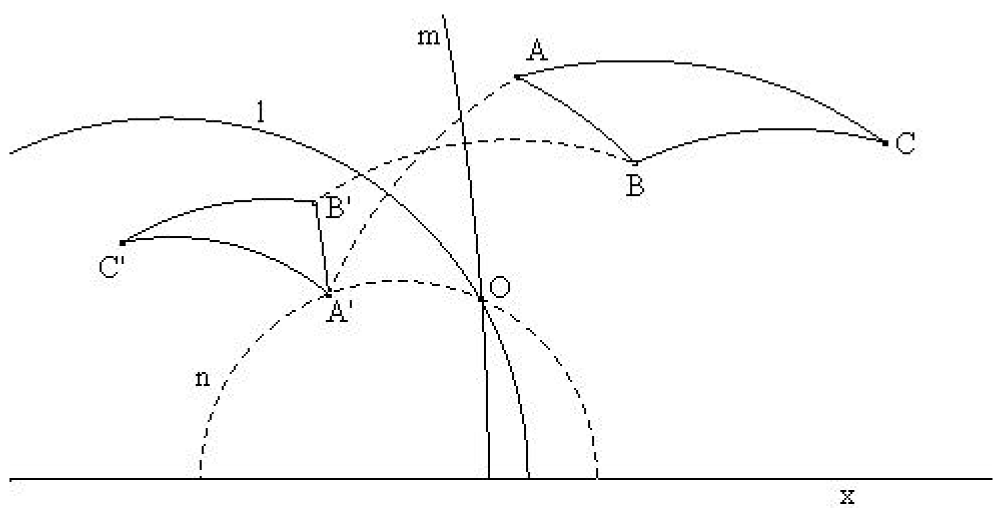

Figure 4.13 Rotation in UHP

The point O is the center of rotation. We can use line l as one of the mirrors and the line n through O and A' as the other. It is evident that reflection in line l will send A to A', and that reflection in line n will leave A' fixed. By the preservation of angle and the fact that point O is equidistant from B and B', that the rotation will send B to B', and therefore C to C'.

Since O is equidistant from A and A', and since line l is the perpendicular bisector of segment AA', line l is an altitude of isosceles triangle AOA' and therefore bisects angle AOA'. So the successive reflections in the mirrors l and n gives us a rotation about point O through the twice the angle between the mirrors l and n. Thus the rotation is completely defined by a point (center) and an angle, and is a direct analog to Euclidean rotation. As with Euclidean rotation, the only fixed objects are the e-circles mutually orthogonal to the mirrors l and n. We will discuss the role of these objects in UHP in Chapter VII.

We will now consider the case where the mirrors are limiting parallels to each other:

°-Rotation

Suppose we have the same situation as in Figure 4.13, except that the perpendicular bisectors l and m are limiting parallels sharing the point O at infinity. (Figure 4.14) As with rotation, this isometry is achieved by taking successive reflections in lines l and n (through O and A'). This isometry is different from a rotation because angle AOA' has measure zero. It also differs from Euclidean translation because corresponding line segments of the triangles are not always parallel to each other. (Note that AB and A'B' in Figure 4.14 will probably intersect if extended.)

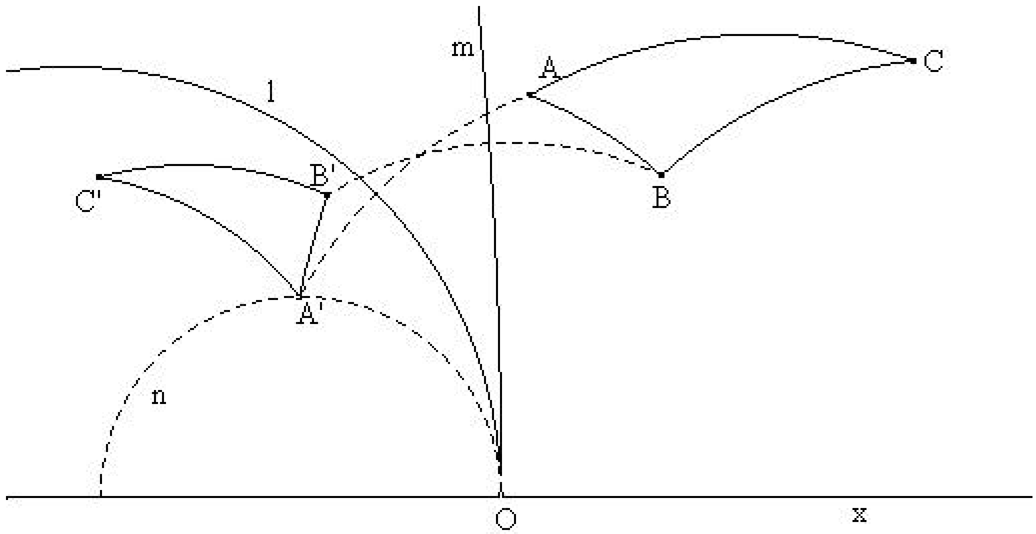

Figure 4.14 ≡-Rotation in UHP

Because the 'angle' of rotation has measure zero, we cannot define this rotation by a center and an angle measure. We must instead define it by the specific angle AOA' where A and A' are any point and its image under the ≡-rotation, and O is the point at infinity at one end of the perpendicular bisector of segment AA'. (The i-point at the other end of the perpendicular bisector will yield a different ≡-rotation.)

The fixed objects under ≡-rotation are e-circles that are tangent to x at O. We will discuss these objects in Chapter VII. This brings us to our last case of the orientation preserving isometries:

Translation

Suppose, again, that we have the situation described in Figure 4.13, except that the perpendicular bisectors l and m are divergently parallel to each other. (Figure 4.15)

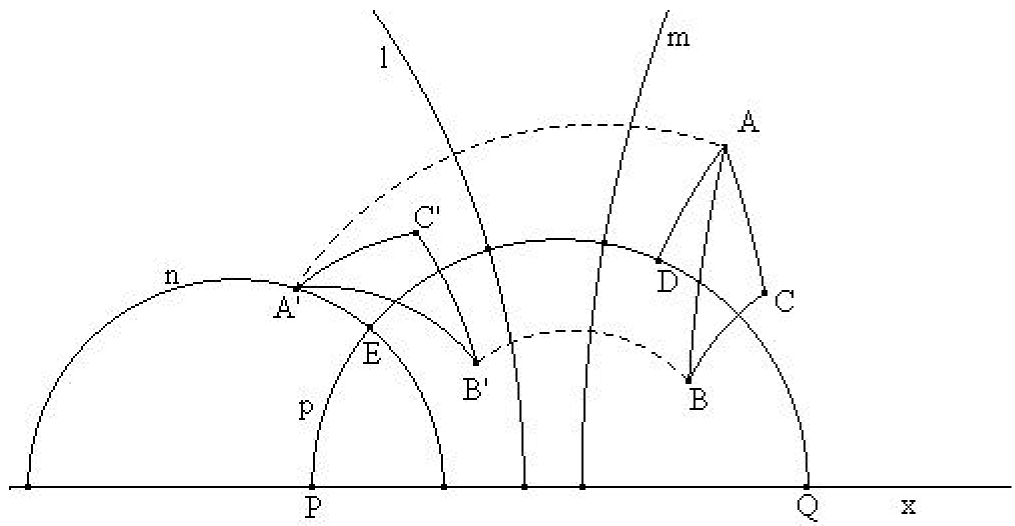

Figure 4.15 Translation in UHP

Since l and m are divergently parallel, they have a unique mutual perpendicular p. Consider the perpendicular from point A' to line p, and call this line n. Successive reflection in l and n will map A to A' and will leave p fixed, as l and n are orthogonal to p. We can see by the preservation of angles (specifically the angle formed by lines AB and p) and metric, that B will map to B', and C to C'.

This translation is defined entirely by the vector DE, where D and E are the feet of the perpendiculars from A and A' to line p. (line l is the perpendicular bisector of this vector). This makes this isometry most closely related to the translation of the Euclidean plane, but it is not a direct analog. In Euclidean geometry, each point is 'moved' by the same distance. This is not the case in hyperbolic geometry. Segments AA' and BB' are not necessarily the same length.

The objects that remain fixed under translation in UHP are e-circles orthogonal to both mirrors. These turn out to be the e-circles containing the i-points P and Q at the 'ends' of line p. We will study these objects in Chapter VII.

This takes care of the three types of orientation preserving isometries, or products of two reflections. We have only one case remaining to consider, that of the orientation reversing isometry that is not a simple reflection. This is the hyperbolic analog of Euclidean glide-reflection:

Glide-reflection

Suppose we have congruent triangles ABC and A'B'C' with opposite orientations, and that the perpendicular bisectors l and m of segments AA' and BB' do not coincide. Consider the midpoints M and N of the segments AA' and BB', the line l through M and N, and points D and E, the feet of the perpendiculars from A and A' to line l. (Figure 4.16)

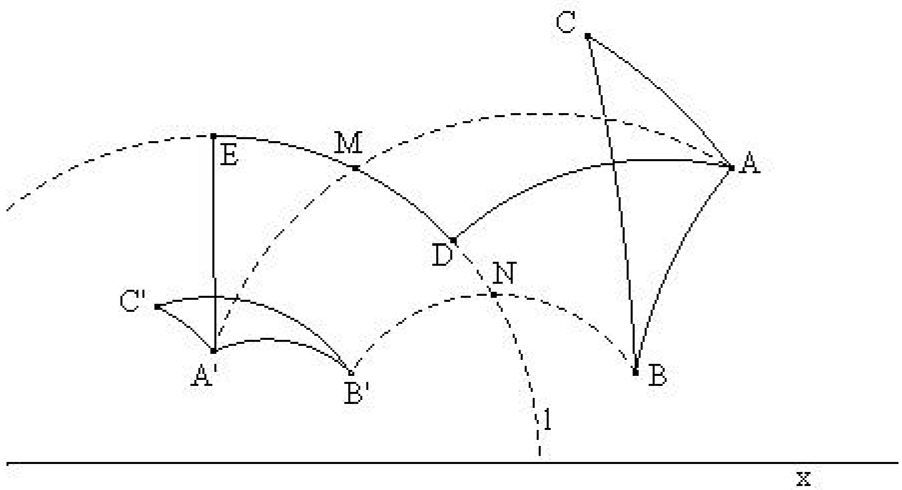

Figure 4.16 Glide-reflection in UHP

The glide-reflection that sends triangle ABC to A'B'C' is the product of the translation through vector DE and the reflection in line l (through D and E), or the product of three successive reflections. The only fixed object is line l itself.

So in the hyperbolic plane, as in the Euclidean plane, any triangle can be sent to any congruent triangle using three or fewer reflections. This allows us to place any object in UHP in a 'standard' position, and will greatly facilitate our discussion of triangles in the next chapter, and of circles in Chapter VII.

Chapter V
Triangles in UHP

To facilitate our study of triangles and trigonometry it will be necessary to place them in a standard position, or more accurately to examine a triangle in standard position congruent to the triangle in which we are interested. We showed in Chapter IV that this is possible.

Definition: *A triangle in UHP is in* <u>*standard position*</u> *if it has vertices A(0,k), B(s,t) and C(0,1) where k>1, and both s and t positive.*

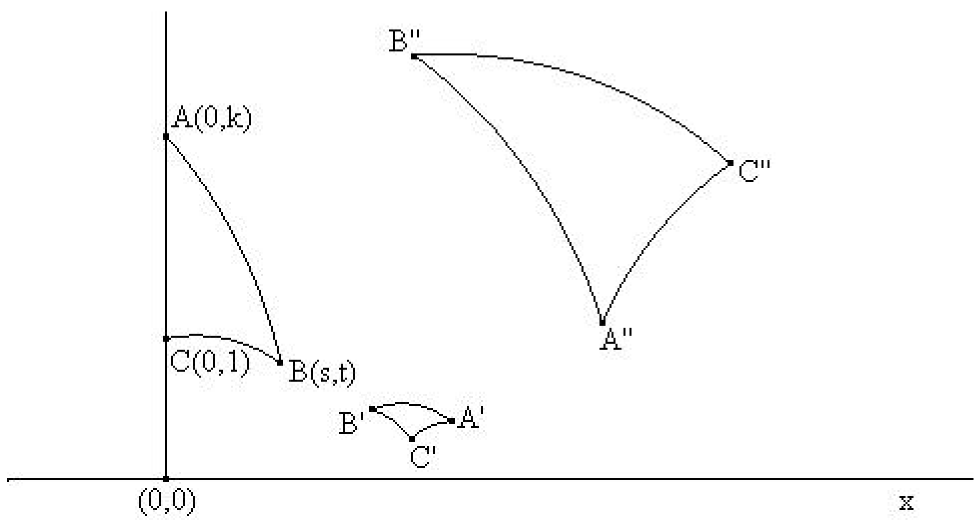

Figure 5.1 Triangle in standard position

Triangles ABC, A'B'C', and A"B"C" in Figure 5.1 are congruent to each other and triangle ABC is in standard position. For our discussion of triangles, we will assume that all of our triangles are in standard position.

Angle Sum and Area

We know from Theorem 2.17 that if the hyperbolic parallel postulate holds, which it does in UHP, then every triangle has a positive defect. So:

Theorem 5.1: *Every triangle has positive angle defect.*

We will now examine the relationship between the angle sum of a triangle and its area. Recall that the angle defect is π minus the angle sum.

In Chapter II we discussed asymptotic triangles, those having one or more vertices at ideal points. We will begin our investigation of the area of triangles by looking at singly asymptotic triangle ABZ, where A and B are ordinary points and Z is ideal. (Figure 5.2) If we let the 'center' of line AB lie at the origin, and r be the 'radius' of line AB, then line AB has the equation:

$$y = \sqrt{r^2 - x^2}$$

and points A and B have x-coordinates:

$$a = r \cdot \cos(p - a) = -r \cdot \cos(a) \quad \text{and} \quad b = r \cdot \cos(b)$$

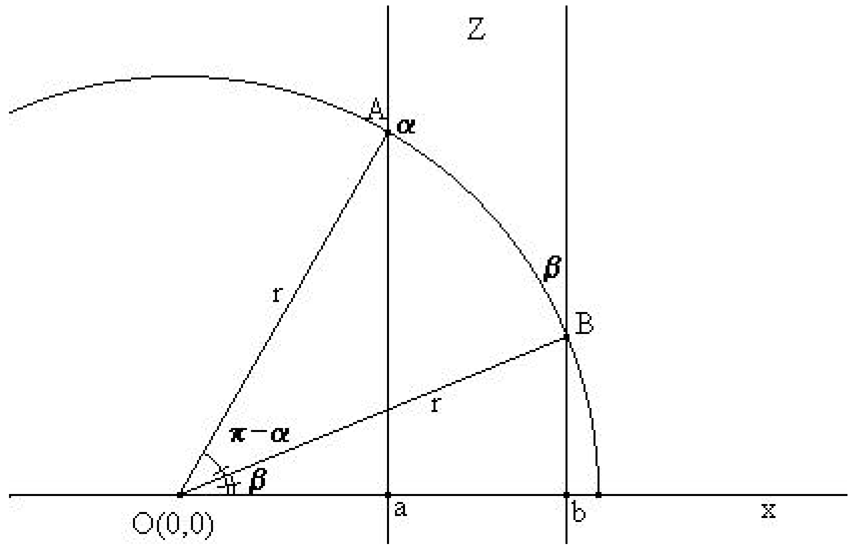

Figure 5.2 Singly asymptotic triangle ABZ

Double integration over x and y, using our UHP metric, gives us:

$$Area = \int_{-r\cdot\cos(a)}^{r\cdot\cos(b)} \int_{\sqrt{r^2-x^2}}^{\infty} \frac{dy}{y}\cdot\frac{dx}{y} = \int_{-r\cdot\cos(a)}^{r\cdot\cos(b)} \left.\frac{-1}{y}\right|_{\sqrt{r^2-x^2}}^{\infty} \cdot dx$$

$$= \int_{-r\cdot\cos(a)}^{r\cdot\cos(b)} \frac{dx}{\sqrt{r^2-x^2}} = \sin^{-1}\left(\frac{x}{r}\right)\Bigg|_{-r\cdot\cos(a)}^{r\cdot\cos(b)}$$

$$= \sin^{-1}(\cos(b)) - \sin^{-1}(-\cos(a))$$

$$= \left(\frac{p}{2} - b\right) + \left(\frac{p}{2} - a\right)$$

$$= p - a - b$$

It is a short jump from here to our formula. Refer to Figure 5.3, the picture of a general triangle in standard position with angles α, β and γ (where β=β'-β").

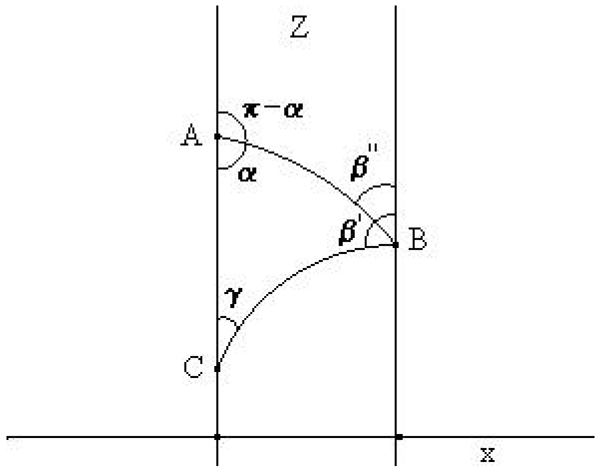

Figure 5.3 The triangle as the difference of two singly asymptotic triangles

By subtracting the area of ABZ from that of CBZ, we find the area ABC to be:

$$
\begin{aligned}
(p - g - b') - (p - (p - a) - b'') &= p - g - b' - p + p - a + b'' \\
&= p - a - (b' - b'') - g \\
&= p - a - b - g
\end{aligned}
$$

which is exactly the angle defect of triangle ABC, so:

Theorem 5.2: *The area of a triangle is equal to its angle defect.*

We will move on to triangle trigonometry next, beginning with the trigonometry of the singly asymptotic right triangle. This has a significance to which we have previously alluded:

Trigonometry of the Singly Asymptotic Right Triangle

Recall that given a line l and a point A at a distance of d from l, that the angle of parallelism of d is the angle CAP where AC is the perpendicular from A to l, and AP is asymptotically parallel to l. Consider Figure 5.4, the singly asymptotic right triangle ACP in standard position with right angle at C. Let the length of segment AC be d, and angle CAP is α, the angle of parallelism associated with d. We can make the relationship between α and d precise.

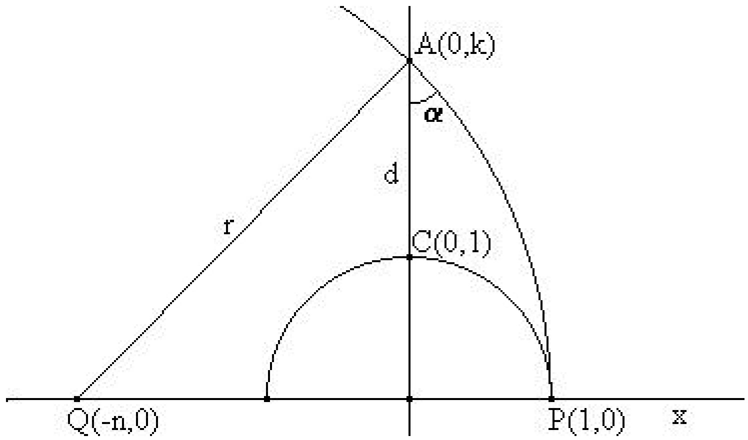

Figure 5.4 Singly asymptotic right triangle in standard position

First of all, we know from our metric that d=ln(k), so $k=e^{d}$. Also we have the following relationships:

$$n^2 + k^2 = r^2$$

$$(r-1)^2 + k^2 = r^2$$

$$r^2 - 2r + 1 + k^2 = r^2$$

$$r = \frac{k^2+1}{2}$$

and

$$(1) \qquad \sin(a) = \frac{k}{r} = \frac{2k}{k^2+1} = \frac{2 \cdot e^d}{e^{2d}+1} = \frac{1}{\cosh(d)}$$

Similarly we get:

$$(2) \qquad \tan(a) = \frac{1}{\sinh(d)} \qquad \text{and} \qquad (3) \qquad \cos(a) = \tanh(d)$$

These three equations relate distance and its related angle of parallelism. We will use the third to express the angle of parallelism as a function of distance:

$$p(d) = \cos^{-1}(\tanh(d))$$

We assumed in Chapter II that this relationship was a function, independent of our choice of line l and point P. Now that we have transformational geometry on UHP, we know it is. Given any line l and P not on l, we may place l on the unit circle, P on the y-axis, and consider them to be one infinite side and the opposite vertex of a singly asymptotic right triangle in standard position. We also claimed in Chapter II that as d approaches 0, $\pi(d)$ approaches $\pi/2$, and as d approaches ≡, $\pi(d)$ approaches 0. These claims are now evident by the formula.

These relationships will form the basis for our development of the trigonometry of the hyperbolic plane.

As in Euclidean geometry, it is helpful to begin the investigation of trigonometry with the study of the simplest (right) triangles first and then apply the results to general triangles. We just looked at the relationship between the one finite side and the one non-zero non-right angle of the singly asymptotic right triangle. We will apply those results to the general singly asymptotic triangle, then the right triangle, and finally the general triangle. From the angle of parallelism results, the relationships of the singly asymptotic triangle are almost immediate.

Trigonometry of the General Singly Asymptotic Triangle

Consider singly asymptotic triangle ABZ with finite side AB having length d. If we place side AB on the unit circle, the y-axis will be perpendicular to side AB at E(0,1). (Think of this 'segment' EZ as an 'altitude' of the triangle) Suppose for the moment that E is between A and B, and segments AE and BE have lengths d_1 and d_2, and let angles

EAZ and EBZ have measures α and β respectively. (Figure 5.5) Note that $\alpha = \pi(d_1)$ and $\beta = \pi(d_2)$.

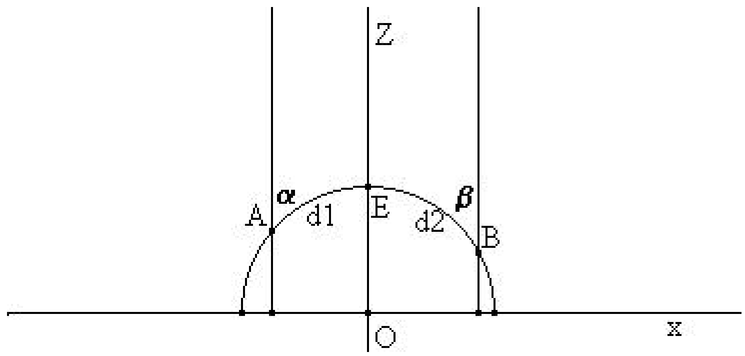

Figure 5.5 Singly asymptotic triangle as sum of two singly asymptotic right triangles

So using relationships (1), (2) and (3) for substitution, we have the following:

$$\cosh(d) = \cosh(d_1 + d_2) = \cosh(d_1)\cdot\cosh(d_2) + \sinh(d_1)\cdot\sinh(d_2)$$
$$= \csc(a)\cdot\csc(b) + \cot(a)\cdot\cot(b)$$

so

$$\cosh(d) = \frac{1+\cos(a)\cdot\cos(b)}{\sin(a)\cdot\sin(b)} \qquad (4)$$

And similarly:

$$\sinh(d) = \sinh(d_1 + d_2) = \sin(a)\cdot\cosh(b) + \sinh(b)\cdot\cosh(a)$$
$$= \cot(a)\cdot\csc(b) + \cot(b)\cdot\csc(a)$$

$$\sinh(d) = \frac{\cos(a)+\cos(b)}{\sin(a)\cdot\sin(b)} \qquad (5)$$

And combining these:

$$\tanh(d) = \frac{\cos(a)+\cos(b)}{1+\cos(a)\cdot\cos(b)} \qquad (6)$$

It is not correct to assume that E lies between A and B, but the calculations work out the same if it does not, as we might expect form our experience with the same type of calculations in Euclidean space. We now have what we need to begin our investigation of the trigonometry of the right triangle.

Trigonometry of the Right Triangle

Let ABC be a right triangle in standard position, with right angle at C, sides a, b and c opposite A, B and C, and angles α and β at vertices A and B, respectively. Let AE and BD be vertical rays forming angles ABD=β_1 and BAE=π-α. (Figure 5.6)

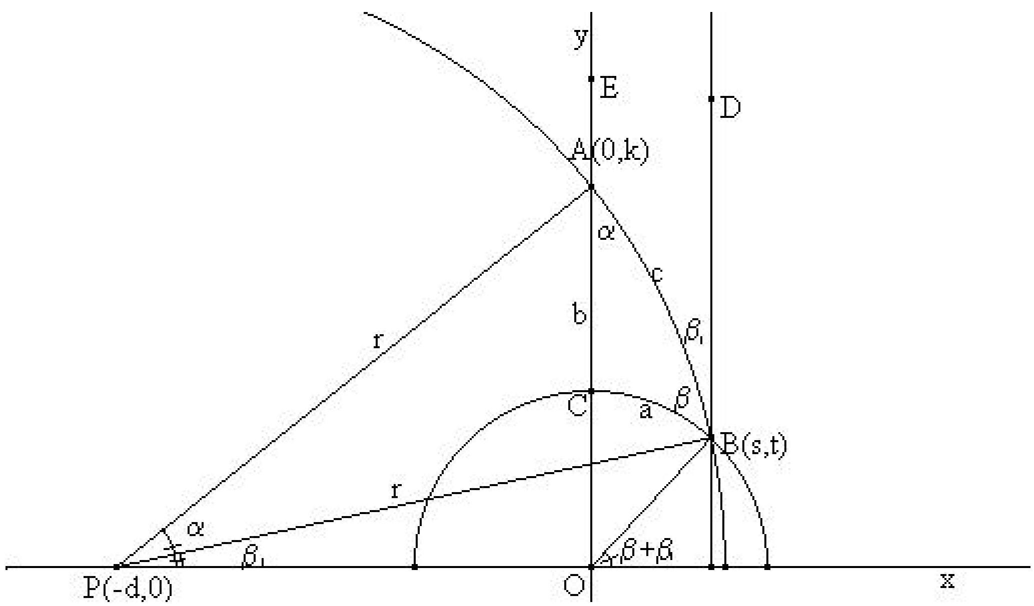

Figure 5.6 The right triangle in standard position

Before we get the relationships we are after, we need several preliminary results from simple Euclidean trigonometry:

$$\sin(a) = \frac{k}{r} \quad , \quad \cos(a) = \frac{d}{r} \quad \text{and} \quad \tan(a) = \frac{k}{d}$$

$$\sin(\beta_1) = \frac{t}{r} \quad , \quad \cos(\beta_1) = \frac{d+s}{r} \quad \text{and} \quad \tan(\beta_1) = \frac{t}{d+s}$$

$$\sin(\beta + \beta_1) = t \quad , \quad \cos(\beta + \beta_1) = s \quad \text{and} \quad \tan(\beta + \beta_1) = \frac{t}{s}$$

The angle difference formulae, and $\beta=(\beta+\beta_1)-\beta_1$ give us:

$$\sin(\beta) = \frac{d \cdot t}{r} \quad , \quad \cos(\beta) = \frac{k^2+1}{2 \cdot r} \quad \text{and} \quad \tan(\beta) = \frac{2 \cdot d \cdot t}{k^2+1} = \frac{d \cdot t}{s \cdot d + 1}$$

Applying our metric to b we get $b=e^k$, so:

$$\sinh(b) = \frac{k - \frac{1}{k}}{2} = \frac{k^2-1}{2 \cdot k} \quad , \quad \cosh(b) = \frac{k^2+1}{2 \cdot k} \quad \text{and} \quad \tanh(b) = \frac{k^2-1}{k^2+1}$$

Using formulae (1), (2) and (3) with the ordinary trig ratios of α, β and γ, we have:

$$\sinh(a) = \frac{1}{\tan(\beta + \beta_1)} = \frac{s}{t} \quad , \quad \cosh(a) = \frac{1}{\sin(\beta + \beta_1)} = \frac{1}{t} \quad \text{and} \quad \tanh(a) = s$$

Combining formulae (4), (5) and (6) with the ordinary trigonometry and the following relationships:

$$d^2 + k^2 = r^2 \quad \text{and} \quad (d+s)^2 + t^2 = r^2 \quad \text{which combine to give us} \quad d \cdot s = \frac{k^2-1}{2}$$

we get:

$$\sinh(c) = \frac{\cos(p-\alpha) + \cos(\beta_1)}{\sin(p-\alpha) \cdot \sin(\beta_1)} = \frac{\cos(\beta_1) - \cos(\alpha)}{\sin(\alpha) \cdot \sin(\beta_1)} = \frac{\frac{d+s}{r} - \frac{d}{r}}{\frac{k}{r} \cdot \frac{t}{r}} = \frac{s \cdot r}{k \cdot t}$$

$$\cosh(c) = \frac{1+\cos(p-a)\cdot\cos(b_1)}{\sin(p-a)\cdot\sin(b_1)} = \frac{1-\cos(a)\cdot\cos(b_1)}{\sin(a)\cdot\sin(b_1)}$$

$$= \frac{1-\frac{d}{r}\cdot\frac{d+s}{r}}{\frac{k}{r}\cdot\frac{t}{r}} = \frac{r^2-d^2-d\cdot s}{k\cdot t} = \frac{k^2-\frac{k^2-1}{2}}{k\cdot t} = \frac{k^2+1}{2\cdot k\cdot t}$$

and

$$\tanh(c) = \frac{2\cdot s\cdot r}{k^2+1}$$

Now we get to the more significant and meaningful relationships. We can combine these numerous expressions for our regular and hyperbolic trig functions to get the following:

(7) $$\sinh(c)\cdot\sin(a) = \frac{s\cdot r}{k\cdot t}\cdot\frac{k}{r} = \frac{s}{t} = \sinh(a)$$

(8) $$\sinh(b)\cdot\tan(a) = \frac{k^2-1}{2\cdot k}\cdot\frac{k}{d} = \frac{k^2-1}{2\cdot d} = s = \tanh(a)$$, remember $s\cdot d = \frac{k^2-1}{2}$

(9) $$\tanh(c)\cdot\cos(a) = \frac{2\cdot s\cdot r}{k^2+1}\cdot\frac{d}{r} = \frac{2\cdot s\cdot d}{k^2+1} = \frac{k^2-1}{k^2+1} = \tanh(b)$$

(10) $$\cosh(b)\cdot\sin(a) = \frac{k^2-1}{2\cdot k}\cdot\frac{k}{r} = \frac{k^2-1}{2\cdot r} = \cos(b)$$

(11) $$\cot(a)\cdot\cot(b) = \frac{d}{k}\cdot\frac{s\cdot d+1}{d\cdot t} = \frac{s\cdot d+1}{k\cdot t} = \frac{k^2+1}{2\cdot k\cdot t} = \cosh(c)$$

and, of course, their counterparts:

(12)	$\sinh(a)\cdot\tan(b)=\tanh(b)$
(13)	$\sinh(c)\cdot\sin(b)=\sinh(b)$
(14)	$\tanh(c)\cdot\cos(b)=\tanh(a)$
(15)	$\cosh(a)\cdot\sin(b)=\cos(a)$

These relationships may be seen as similar to the trig ratios of the right triangle in the Euclidean plane. Of special note are (13) and (9) which, written differently, look familiar:

$$\sin(a)=\frac{\sinh(a)}{\sinh(c)}=\frac{\sinh(opp)}{\sinh(hyp)} \quad \text{and} \quad \cos(a)=\frac{\tanh(b)}{\tanh(c)}=\frac{\tanh(adj)}{\tanh(hyp)}$$

and are almost direct analogues to their Euclidean counterparts.

The Euclidean Pythagorean Theorem also has its hyperbolic counterpart, a simple and elegant relationship between the three sides of the right triangle.

Theorem 5.3 ***(The Hyperbolic Pythagorean Theorem):*** *In any right triangle ABC, with right angle at C, then the lengths of the three sides are related by:*

cosh(c)=cosh(a)cosh(b)

Proof: $$\cosh(c)=\frac{k^2+1}{2\cdot k\cdot t}=\frac{k^2+1}{2\cdot k}\cdot\frac{1}{t}=\cosh(a)\cdot\cosh(b)$$ QED

Trigonometry of the General Triangle

We are now ready to consider the general triangle ABC in the hyperbolic plane. Assume that the altitude from A intersects the opposite side (we know from the proof of Theorem 2.15 that this must be true of at least one altitude), and let E be the foot of this altitude, such that E divides side a into a_1 and a_2 and ray AE divides angle α into α_1 and α_2. (Figure 5.7) This decomposes our triangle into two right triangles AEC and AEB. We may use the information from the preceding section to derive three interesting relationships within the hyperbolic triangle.

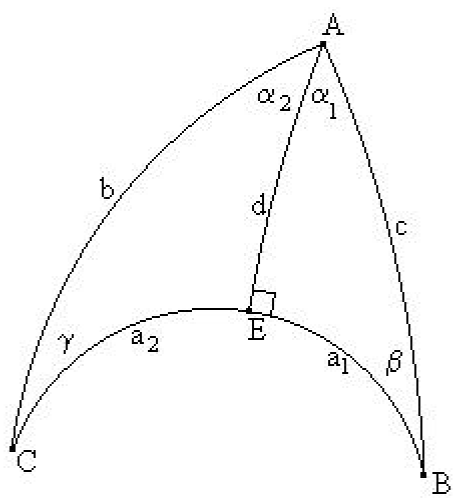

Figure 5.7 The general triangle decomposed into two right triangles

By using the angle sum and difference formulae on $\cos(\alpha)$, and making substitutions using the right triangle relationships from the preceding section, (equations 7-15) we get the following:

$$\cos(a) = \cos(a_1 + a_2) = \cos(a_1)\cdot\cos(a_2) - \sin(a_1)\cdot\sin(a_2)$$

$$= \frac{\tanh(d)}{\tanh(c)}\cdot\frac{\tanh(d)}{\tanh(b)} - \frac{\sinh(a_1)}{\sinh(b)}\cdot\frac{\sinh(a_2)}{\sinh(c)}$$

$$= \frac{\cosh(c)\cdot\cosh(b)\cdot\tanh^2(d) - \sinh(a_1)\cdot\sinh(a_2)}{\sinh(c)\cdot\sinh(b)}$$

$$= \frac{\cosh(c)\cdot\cosh(b)\cdot\left(1-\sec h^2(d)\right) - \sinh(a_1)\cdot\sinh(a_2)}{\sinh(c)\cdot\sinh(b)}$$

$$= \frac{\cosh(c)\cdot\cosh(b) - \frac{\cosh(b)}{\cosh(d)}\cdot\frac{\cosh(c)}{\cosh(d)} - \sinh(a_1)\cdot\sinh(a_2)}{\sinh(c)\cdot\sinh(b)}$$

$$= \frac{\cosh(c)\cdot\cosh(b) - \left(\cosh(a_1)\cdot\cosh(a_2) + \sinh(a_1)\cdot\sinh(a_2)\right)}{\sinh(c)\cdot\sinh(b)}$$

$$= \frac{\cosh(c)\cdot\cosh(b) - \cosh(a_1 \pm a_2)}{\sinh(c)\cdot\sinh(b)}$$

$$\cos(a) = \frac{\cosh(c)\cdot\cosh(b) - \cosh(a)}{\sinh(c)\cdot\sinh(b)}$$

This formula relates one angle and the three sides of a general triangle in the hyperbolic plane, as does the Law of Cosines in the Euclidean plane.

We now apply the same technique to cosh(a):

$$\cosh(a) = \cosh(a_1 + a_2) = \cosh(a_1)\cdot\cosh(a_2) + \sinh(a_1)\cdot\sinh(a_2)$$

$$= \frac{\cos(a_1)}{\sin(b)}\cdot\frac{\cos(a_2)}{\sin(g)} + \frac{\tanh(d)}{\tan(b)}\cdot\frac{\tanh(d)}{\tan(g)}$$

$$= \frac{\cos(a_1)\cdot\cos(a_2) + \left(1 - \sec h^2(d)\right)\cdot\cos(b)\cdot\cos(g)}{\sin(b)\sin(g)}$$

$$= \frac{\cos(a_1)\cdot\cos(a_2) + \cos(b)\cdot\cos(g) - \dfrac{\cos(b)}{\cosh(d)}\cdot\dfrac{\cos(g)}{\cosh(d)}}{\sin(b)\sin(g)}$$

$$= \frac{\cos(b)\cdot\cos(g) + \cos(a_1)\cdot\cos(a_2) - \sin(a_1)\cdot\sin(a_2)}{\sin(b)\sin(g)}$$

$$\cosh(a) = \frac{\cos(b)\cdot\cos(g) + \cos(a)}{\sin(b)\sin(g)}$$

This formula relates one side and the three angles of a triangle, something that is not possible in the Euclidean plane. This formula is a consequence of the absence of similar triangles in hyperbolic geometry.

Finally, we turn our attention to the sine. Using the first cosine relationship for substitution, we get:

$$\frac{\sin^2(a)}{\sinh^2(a)} = \frac{1 - \cos^2(a)}{\sinh^2(a)} = \frac{\sinh^2(b)\cdot\sinh^2(c) - \left(\cosh(b)\cdot\cosh(c) - \cosh(a)\right)^2}{\sinh^2(a)\cdot\sinh^2(b)\cdot\sinh^2(c)}$$

$$= \frac{\left(1 - \cosh^2(b)\right)\cdot\left(1 - \cosh^2(c)\right) - \left(\cosh(b)\cdot\cosh(c) - \cosh(a)\right)^2}{\sinh^2(a)\cdot\sinh^2(b)\cdot\sinh^2(c)}$$

$$= \frac{1 - \cosh^2(b) - \cosh^2(b) - \cosh^2(c) + 2\cdot\cosh(a)\cdot\cosh(b)\cdot\cosh(c)}{\sinh^2(a)\cdot\sinh^2(b)\cdot\sinh^2(c)}$$

Which is symmetric in terms of a, b and c, so the ratio is the same for all three pairs of α and a, β and b, and γ and c. Also, since all three angles are between 0 and π, all terms are positive, so we can take the square root and get:

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}$$

We state these results formally as:

Theorem 5.4 (Hyperbolic Laws of Cosines): *Given triangle ABC, labeled in the usual manner:*

$$\cos(\alpha) = \frac{\cosh(b)\cdot\cosh(c) - \cosh(a)}{\sinh(b)\cdot\sinh(c)} \quad \text{and} \quad \cosh(a) = \frac{\cos(\beta)\cdot\cos(\gamma) + \cos(\alpha)}{\sin(\beta)\sin(\gamma)}$$

and:

Theorem 5.5 (Hyperbolic Law of Sines): *Given triangle ABC, labeled in the usual manner:*

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}$$

The similarity between the two laws of cosines leads us to believe that they are closely related. Since we get one from the other by simply exchanging corresponding angles and sides, and regular and hyperbolic trig functions, we might consider them to be duals. We have yet to find any simple direct link between them.

This concludes our discussion of trigonometry, and of triangles for the moment. We will look at the inscribed and circumscribed circles of a triangle in Chapter VIII, but first we must investigate the roles played by Euclidean circles in UHP.

Chapter VI
Euclidean Circles in UHP

A Euclidean circle in UHP can play several roles, depending on its position relative to the x axis (x for sake of brevity). We have four cases to consider, the first of which we have already seen. An e-circle centered on the x, or more accurately the portion lying above x of an e-circle centered on x, is a line in UHP. We will look now at the other three cases: intersecting x in two points, tangent to x, and lying entirely above x. (The case of an e-circle lying entirely below x is irrelevant, as it contains no points in UHP)

The first case, that of a Euclidean circle intersecting x in two points, plays a curious role in UHP, one that is played by a line in Euclidean space:

Hypercycles

Given a line l in UHP with i-points P and Q, let A be any point at distance d from l. Consider the e-circle a through A, P and Q. (Figure 6.1) This obviously intersects x in the two points P and Q, and is not a line, else it would coincide with l, and A is not on l. The x-axis is the radical axis of a and l, and any line (e-circle centered on x) orthogonal to l is also orthogonal to a. So the line perpendicular to a at A is perpendicular to l at A' and h(A,A') is the distance d from A to l.

Now choose any point B on e-circle a distinct from A and let the mutual perpendicular to l and a through B meet l in B'. h(B,B') is the distance from B to l. We can show that h(B,B')=h(A,A')=d by considering the perpendicular bisector m of segment AB. Since m is perpendicular to both a and l, reflection in m sends A to B. Since angles are preserved, this reflection also sends line AA' to BB'. Since l is fixed under the reflection, A' is mapped to B'. This tells us that A and B are equidistant from l, and since B was chosen at random, every point on a is at distance d from l.

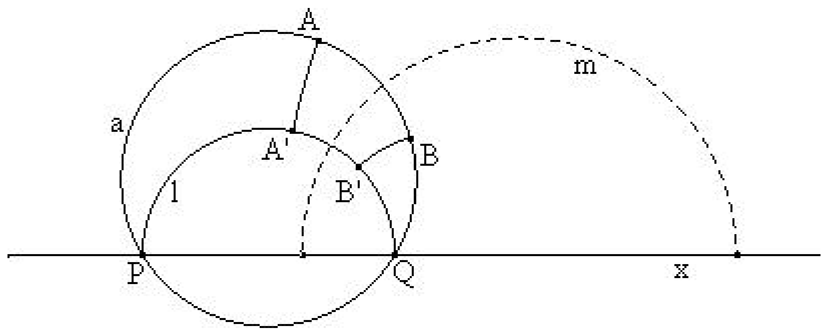

Figure 6.1 An e-circle intersecting x in two points

This is not the entire set of points at distance d from l, but the rest are easy to find. All we need to do is to reflect a in l, and by preservation of distance, we get the other 'half' of the curve. (Figure 6.2) This portion of the curve also happens to be the reflection of the lower portion of e-circle a in the x-axis.

Definition 6.1: *The* *<u>hypercycle</u>* *of distance d from l is {A : h(A,l)=d}. This is also sometimes called the* *<u>curve of constant distance</u>*.

Though not Euclidean circles, we will at this point discuss the hypercycle of distance d from a line of vertical e-ray type. The illustration is simple. All we need to do is reflect the objects l and a as defined above in any line centered at P (or Q). This will send Q to Q', P to Z (the ideal point 'above'), l to l' of vertical e-ray type, and a to two straight e-rays (not vertical) a_1 and a_2 forming the same angles with l' at Q' that a formed with l at Q. (Figure 6.2)

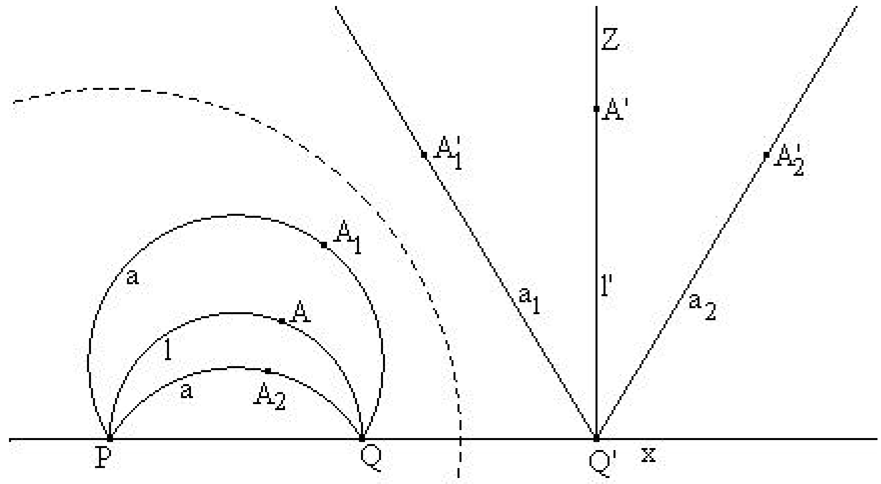

Figure 6.2 Curves of constant distance to lines of both types

We state these results as:

Theorem 6.1: *Given line l having i-points P and Q, and point A_1 at distance d from l, both in UHP. The hypercycle at distance d from l is the portion lying above x of the e-circles through P, Q and A_1, and P, Q and A_2, the reflection of A_1 in l. If line l is of vertical e-ray type having i-points P and Z, the horocycle consists of the e-rays PA_1 and PA_2*

As mentioned in Chapter II, most teachers of elementary geometry describe parallel lines as a set of train tracks that are everywhere equidistant (an excellent description). In Hyperbolic space, however, a pair of train tracks would not be a pair of lines, but rather a pair of hypercycles or one hypercycle and one line.

We look now at our second case, that of e-circles lying entirely above x in UHP. These turn out to be hyperbolic:

Circles

Consider the e-circle c lying entirely above x in UHP, and its reflection -c in x, lying entirely below x. (Figure 6.3) The x axis is the radical axis of c and -c, and any e-circle centered on x orthogonal to c is also orthogonal to -c. Furthermore, each e-circle in the intersecting orthogonal pencil defined by c and -c passes through two points we will call C and -C lying within c and -c respectively. This tells us that any line orthogonal to c must pass through a point C in c's interior, (also that any line through C is orthogonal to c) We shall, without justification for the moment, call C the center of c.

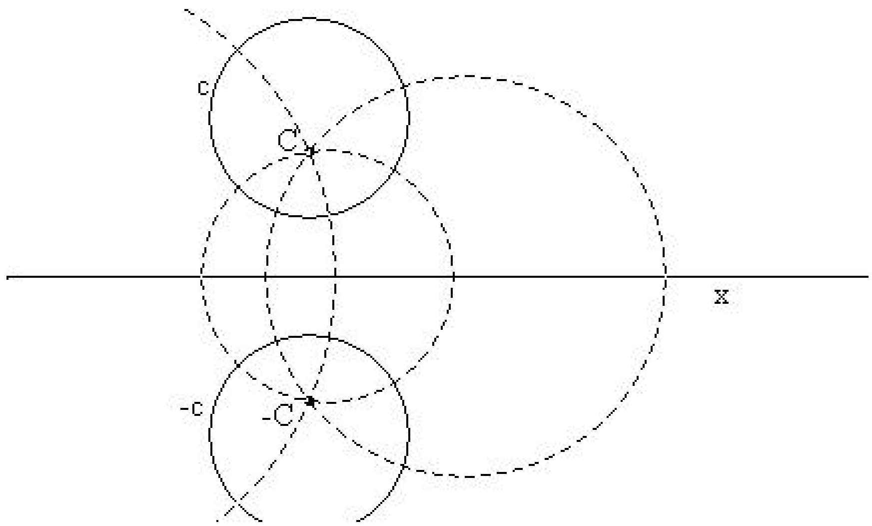

Figure 6.3 The center of a circle

Let A be any point on c, and call r = h(C,A) the radius of c. Choose any point B on c distinct from A and consider lines CA and CB. (Figure 6.4) Take m to be the angle bisector of ACB. Since m passes through C, it is orthogonal to c, and c is fixed under reflection in m. This reflection sends segment CA to CB, and therefore h(C,A) = r = h(C,B). Since B on c was chosen at random, this shows that every point on the e-circle c is at distance r from C, so c is the hyperbolic circle centered at C with radius r. Note that the hyperbolic center of circle c does not coincide with the Euclidean center of e-circle c.

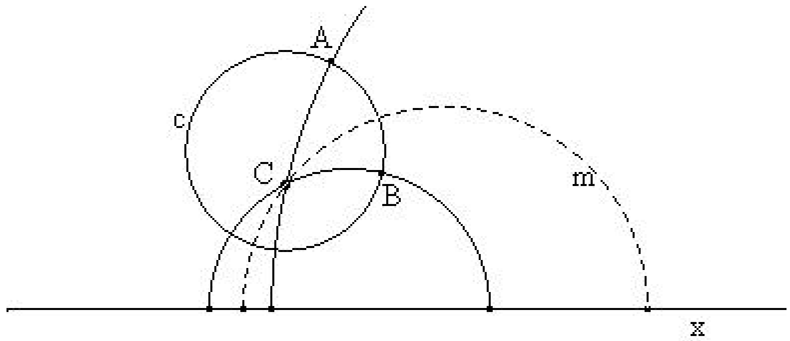

Figure 6.4 The hyperbolic circle

Showing that hyperbolic circles are Euclidean circles is simple. Given center C and point A (radius r = h(C,A)), consider the intersecting pencil of e-circles defined by C and -C. This gives us the pencil of lines through C. The image of A under reflection in all of these lines will give us the circle c, but this is the unique e-circle through A that is orthogonal to the pencil of e-circles. And we have:

Theorem 6.2: *The set of circles in UHP is exactly the set of e-circles lying entirely above x.*

We now reach our third and final case, that of e-circles in UHP tangent to x. These also play a role played by lines in Euclidean space:

Horocycles

Often when discussing circles and lines in Euclidean geometry, we include a line at infinity, and define all circles and lines as generalized circles, with lines being circles centered on the line at infinity. This is not possible in hyperbolic space, because as the center of a circle approaches infinity the circle does not approach a line. We can see this by considering a circle c containing A centered at C as C approaches i-point P. (Figure 6.5) As A remains fixed and C approaches P on x, the radius of c approaches infinity. Since P is a point at infinity, the distance from C to P is always infinite. Circle c intersects ray CP in a point B such that h(C,B) = h(C,A), and as C approaches P on x, B approaches C and P. (in the Euclidean sense) Since c always lies entirely above x, the limit of circle c as C approaches P is the e-circle through A and tangent to x at P.

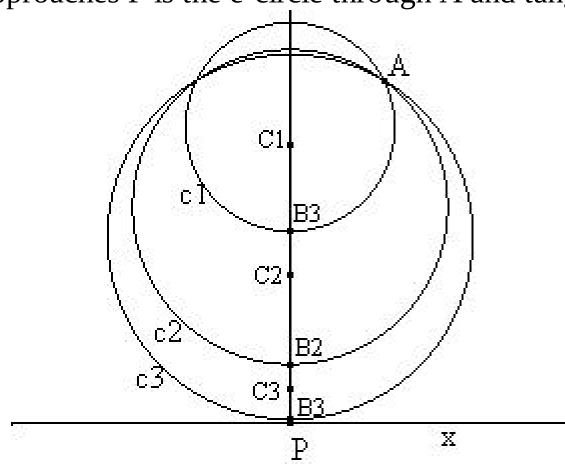

Figure 6.5 The limit of a circle as its center approaches P on x

If we let C approach Z, the point at infinity "above", c approaches the horizontal e-line through A. (Figure 6.6) Though they look like circles, horocycles are not closed curves and therefore do not define, in the strict sense, an interior and exterior region, as both regions defined are unbounded. We will consider the interior of the horocycle to be the interior of the associated e-circle, or the portion above the associated horizontal e-line

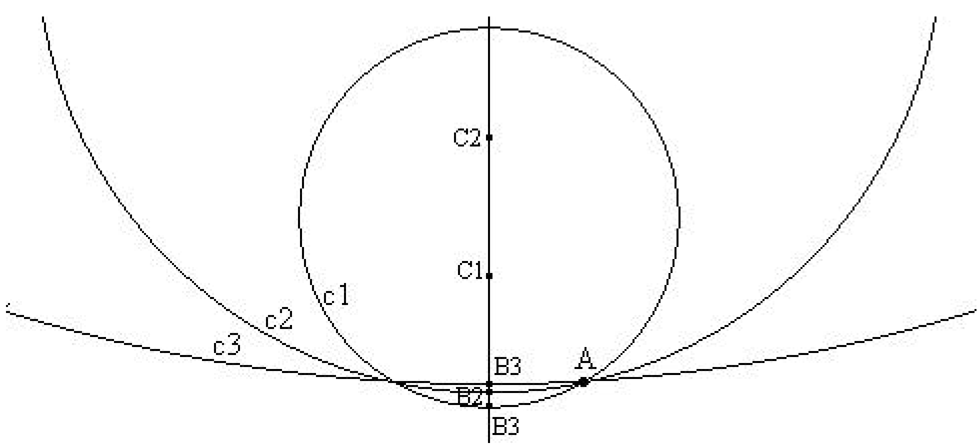

Figure 6.6 The limit of a circle as C approaches i-point Z "above"

The horocycle is an interesting and useful object in the hyperbolic plane, and is the key to proving, without the use of a model, many of the relationships and theorems we have discussed here. We will look now at a few facts about horocycles that will prove useful to us in Chapter VIII.

First, each pair of points is contained in two distinct horocycles. This is evident from Figure 6.7. If the points are horizontally related, one of the horocycles is the horizontal e-line through the two points. If they are vertically related, then the horocycles are congruent (in the Euclidean sense). All horocycles are congruent in the hyperbolic sense.

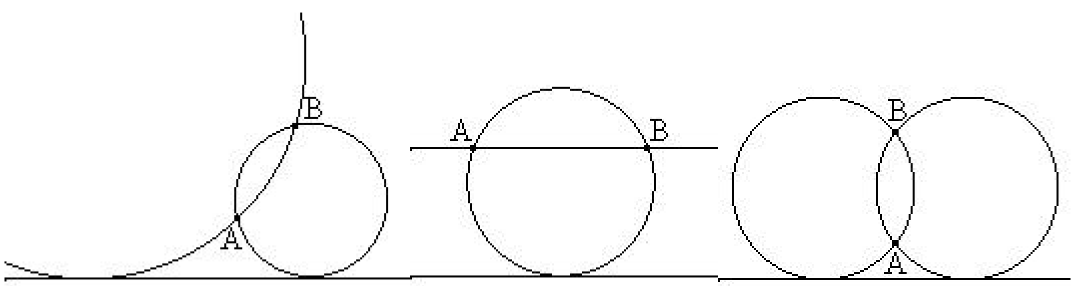

Figure 6.7 Horocycles defined by two points

Second, any 'radius' r (line through center C) of a horocycle h is orthogonal to h. This is obvious by both a simple continuous limit argument and also by noting that any line r through C is orthogonal to e-circle h at C, and therefore also at the other point of intersection. Figure 6.8

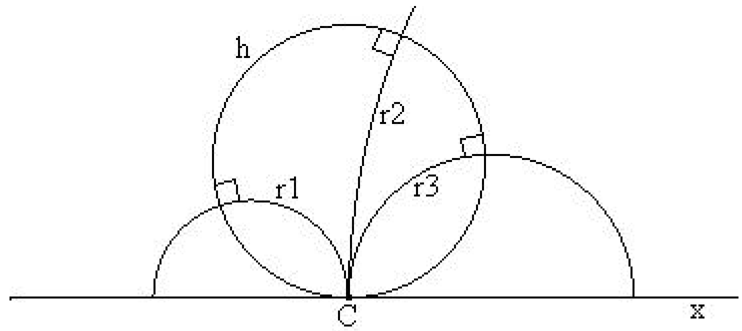

Figure 6.8 Any radius of a horocycle is orthogonal to the horocycle

Third, Given C, the 'center' of a horocycle defined by points A and B, then angles CAB and CBA are congruent. To see this, consider the perpendicular bisector of segment AB. This will contain C, the 'center' of the horocycle. Since reflection in this line sends A and B to each other, leaves C fixed, and preserves angles, angles CAB and CBA are congruent. This is illustrated in Figure 6.9.

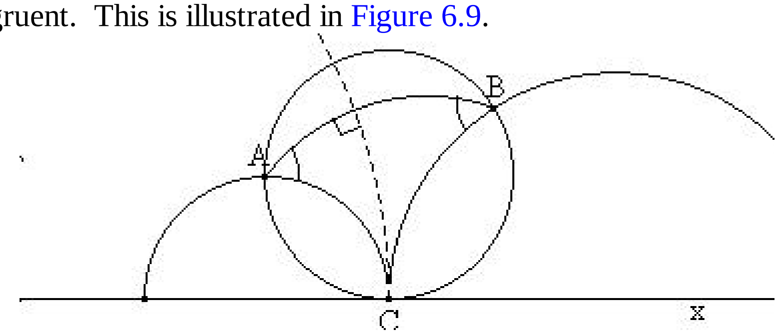

Figure 6.9 Non-zero angles of a singly asymptotic triangle inscribed in a horocycle

Also evident from Figure 6.9 is that angles CAB and CBA, are each the angle of parallelism associated with half the length of AB, or:

$$CAB = CBA = \cos^{-1}\left(\tanh\left(\frac{h(A,B)}{2}\right)\right) = p\left(\frac{h(A,B)}{2}\right)$$

We will use these facts when we discuss circum-circles in Chapter VIII.

This concludes our discussion of the basics of hyperbolic geometry and UHP. We will now examine a few topics in more depth.

Chapter VII
The Hyperbolic Circle

Since, in UHP, Euclidean and hyperbolic circles coincide, we should ask how the coordinates of their centers, and their radii, relate to each other.

The Hyperbolic and Euclidean Center and Radius

For this discussion we let C and R denote Euclidean center (y-coordinate), and radius, and c and r denote its hyperbolic center and radius.

Suppose we have the circle with hyperbolic center P(*,c) and radius r. Let the vertical e-line through the center cut the circle at A(*,a) and B(*,b) with a>b. (Figure 7.1)

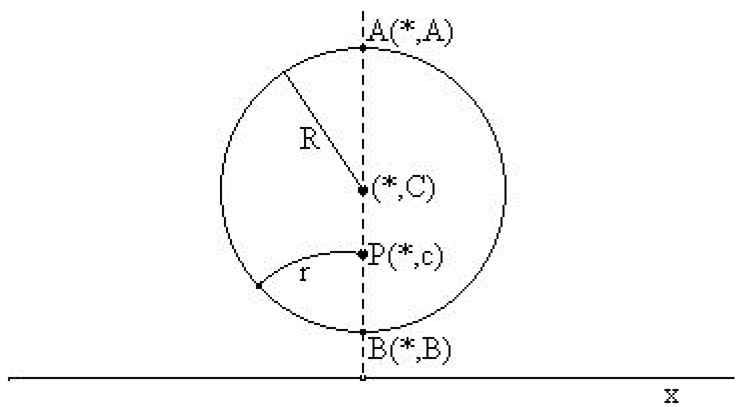

Figure 7.1 The Euclidean and hyperbolic center and radius of the circle

Since:

$$h(A,P) = \ln\left(\frac{a}{c}\right) = r \quad \text{and} \quad h(P,B) = \ln\left(\frac{c}{b}\right) = r$$

We know:

$$a = c \cdot e^{r} \quad \text{and} \quad b = c \cdot e^{-r}$$

and the Euclidean center C is the e-midpoint of segment AB:

$$C = \frac{a+b}{2} = \frac{c \cdot e^{r} + c \cdot e^{-r}}{2} = c \cdot \cosh(r)$$

and the Euclidean radius R is half the length of AB:

$$R = \frac{a-b}{2} = \frac{c \cdot e^{r} - c \cdot e^{-r}}{2} = c \cdot \sinh(r)$$

We can use these to find the inverse relationships.

$$\frac{R}{C} = \frac{c \cdot \sinh(r)}{c \cdot \cosh(r)} = \tanh(r) \quad \text{and} \quad C^2 - R^2 = c^2 \cdot \cosh^2(r) - c^2 \cdot \sinh^2(r) = c^2$$

And we have:

$$C = c \cdot \cosh(r), \quad R = c \cdot \sinh(r), \quad r = \tanh^{-1}\left(R/C\right), \text{ and } \quad c = \sqrt{C^2 - R^2}$$

These relationships are interesting and elegant in themselves, and they also will be invaluable as we develop the formulae for the circumference and area of the circle.

Circumference

To find the formula for circumference, we will use the parametric form of the equation for our circle, and the corresponding modified integration for arc length.

$$x = R \cdot \cos(t) \qquad y = C + R \cdot \sin(t)$$

$$\frac{dx}{dt} = -R \cdot \sin(t) \quad \text{and} \quad \frac{dy}{dt} = R \cdot \cos(t)$$

So:

$$Circumference = 2 \cdot \int_{-\frac{p}{2}}^{\frac{p}{2}} \sqrt{\left(\frac{dx/dt}{y}\right)^2 + \left(\frac{dy/dt}{y}\right)^2}\, dt = 2 \cdot \int_{-\frac{p}{2}}^{\frac{p}{2}} \sqrt{\left(\frac{-R \cdot \sin(t)}{C + R \cdot \sin(t)}\right)^2 + \left(\frac{R \cdot \cos(t)}{C + R \cdot \sin(t)}\right)^2}\, dt$$

$$= 2 \cdot \int_{-\frac{p}{2}}^{\frac{p}{2}} \sqrt{\frac{R^2}{(C + R \cdot \sin(t))^2}}\, dt = 2 \cdot \int_{-\frac{p}{2}}^{\frac{p}{2}} \frac{R}{C + R \cdot \sin(t)} dt$$

$$= 2 \cdot R \cdot \left[\frac{-2}{\sqrt{C^2 - R^2}} \cdot \tan^{-1}\left[\sqrt{\frac{C - R}{R - C}} \cdot \tan\left(\frac{p}{4} - \frac{t}{2}\right)\right]\right]_{-\frac{p}{2}}^{\frac{p}{2}}$$

$$= \frac{-4 \cdot R}{\sqrt{C^2 - R^2}} \left[\tan^{-1}\left(\sqrt{\frac{C - R}{R - C}} \cdot \tan(0)\right) - \tan^{-1}\left(\sqrt{\frac{C - R}{R - C}} \cdot \tan\left(\frac{p}{2}\right)\right)\right]$$

$$= \frac{-4 \cdot R}{\sqrt{C^2 - R^2}} \left[0 - \frac{p}{2} \right] = \frac{2 \cdot p \cdot R}{\sqrt{C^2 - R^2}}$$

which, by using the relationships between Euclidean and hyperbolic centers and radii:

$$C = \frac{2 \cdot p \cdot R}{c} = 2 \cdot p \cdot \sinh(r)$$

Notice that, though the formula initially contained the y-coordinate of the center, the final formula does not. This is as we would hope, and the circumference of the circle depends only upon its radius, and not its position in the plane.

Area

The analogous direct integration we could use to find the area of the circle is much more difficult than for the circumference, and we will avoid it.

Consider the regular n-gon with circum-radius r, and its decomposition into 2n congruent right triangles. We can do this by connecting the center to each vertex and to the midpoint of each side. (Figure 7.2)

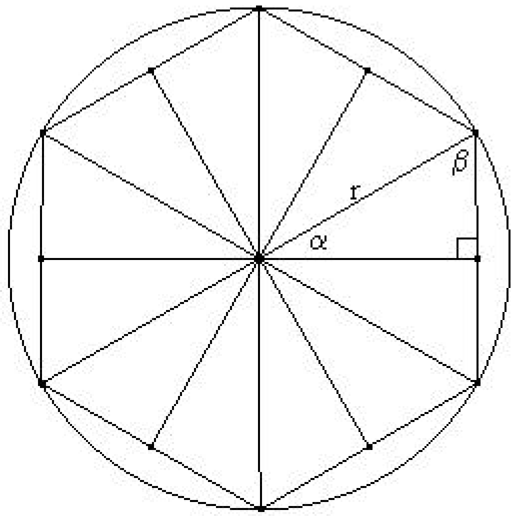

Figure 7.2 The regular n-gon divided into 2n right triangles

As the number of sides approaches infinity, the area of the n-gon, the sum of the areas of the 2n triangles, approaches the area of the circle.

We know that the measure of central angle α is 2π/2n or π/n, the hypotenuse is r, and we can find the measure of angle β by using the formula cot(α)cot(β) = cosh(c) from our investigation of the trigonometry of right triangles. This gives us:

$$b = \cot^{-1}(\cosh(r)\cdot\tan(a)) = \cot^{-1}\left(\cosh(r)\cdot\tan\left(\frac{p}{n}\right)\right)$$

and remembering that the area of the triangle is equal to its angle defect we get that the area of the circle of radius r is:

$$A(r) = \lim_{n\to\infty}\left[2n\cdot\left(p - \frac{p}{2} - a - b\right)\right] = \lim_{n\to\infty}\left[2n\cdot\left(\frac{p}{2} - \frac{p}{n} - \cot^{-1}\left(\cosh(r)\cdot\tan\left(\frac{p}{n}\right)\right)\right)\right]$$

using the fact that arccot(α) = π/2 - arctan(α) we get:

$$\lim_{n\to\infty}\left[2n\cdot\left(\frac{p}{2} - \frac{p}{n} - \frac{p}{2} + \tan^{-1}\left(\cosh(r)\cdot\tan\left(\frac{p}{n}\right)\right)\right)\right] = \lim_{n\to\infty}\left[-2p + 2n\cdot\tan^{-1}\left(\cosh(r)\cdot\tan\left(\frac{p}{n}\right)\right)\right]$$

As u approaches zero and c is a constant, arctan(c·u) approaches c·arctan(u). This gives us:

$$A(r) = -2p + 2n \cdot \cosh(r) \cdot \tan^{-1}\left(\tan\left(\frac{p}{n}\right)\right) = -2p + 2n \cdot \cosh(r) \cdot \frac{p}{n}$$
$$= 2p \cdot \cosh(r) - 2p$$

or alternately:

$$A(r) = 4p \sinh^2\left(r/2\right)$$

We state these together as:

Theorem 7.1: *The circumference and area of a circle are given by: $C = 2p\sinh(r)$, and $A = 4p\sinh^2(r/2)$, where r is the radius of the circle.*

Note that in hyperbolic geometry, as in Euclidean, the circumference formula is the derivative of the area formula with respect to r.

The Limiting Case

We know that, for triangles, as their area approaches 0, their properties (e.g. angle sum) approach those of triangles in Euclidean space. We can easily confirm that this is also true of circles. (It is true of all objects in the hyperbolic plane.)

To confirm the limiting case of the circle as its radius approaches zero, we consider the ratio of the hyperbolic and Euclidean formulae for circle area and take the limit, using L'Hopital's rule:

$$\lim_{r\to 0} \frac{2 \cdot p \cdot (\cosh(r) - 1)}{p \cdot r^2} = \lim_{r\to 0} \frac{2 \cdot (\cosh(r) - 1)}{\cdot r^2} = \lim_{r\to 0} \frac{2 \cdot \sinh(r)}{2 \cdot r} = \lim_{r\to 0} \frac{2 \cdot \cosh(r)}{2} = 1$$

so, as $r \to 0$, our hyperbolic formula for area approaches the Euclidean formula. The same is true of our hyperbolic formula for circumference:

$$\lim_{r\to 0}\frac{2\cdot p\cdot \sinh(r)}{2\cdot p\cdot r}=\lim_{r\to 0}\frac{\sinh(r)}{r}=\lim_{r\to 0}\frac{\cosh(r)}{1}=\frac{1}{1}=1$$

Unlike the triangle, though, the appearance of the circle in UHP remains the same as its area approaches zero.

Hyperbolic Π

In Euclidean space, π is defined as the ratio of the circumference of any circle to its diameter, and this is a constant. In hyperbolic geometry, as we can see from the formulae above, the hyperbolic π ratio for any given circle is equal to:

$$\frac{p\cdot \sinh(r)}{r}$$

where r is the radius of the circle and π is Euclidean pi. Obviously, this is not constant, but as r approaches zero, sinh(r)/r approaches 1, and hyperbolic π approaches Euclidean π.

The Angle Inscribed in a Semicircle

It is a well known fact in Euclidean geometry that any angle inscribed in a semicircle is a right angle. A common proof of this uses the fact that the angle sum of a triangle is π. A similar proof in hyperbolic geometry will show:

Theorem 7.2: *The measure of an angle inscribed in a semicircle is less than a right angle.*

Proof: Let angle ACB be inscribed in a circle. Consider triangles ACD and BCD where D is the center of the circle. (Figure 7.3) Triangles ADC and BDC are isosceles, so the measure of angle ACB is α + β, and the angle sum of triangle ABC is 2α + 2β which must be less than π, so angle the measure of ACB is less than π/2. QED

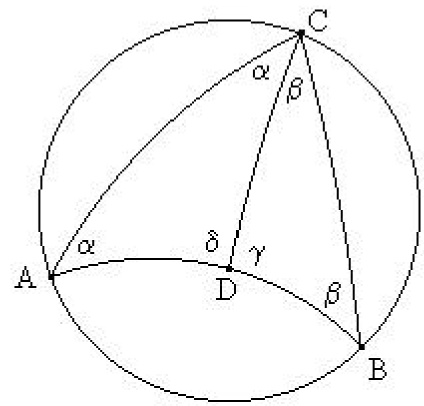

Figure 7.3 An angle inscribed in a semi-circle

Since the angle sum of a triangle goes to π as the area of the triangle goes to zero, as C approaches B or A, angle ACB will approach $\pi/2$.

We saw in Chapter I that the assumption of the existence of a circum-circle to every triangle led Wolfgang Bolyai to a false ‘proof’ of the Euclidean parallel postulate, and we saw in Chapter II that not every triangle has a circum-circle. The natural question to ask is which triangles do have circum-circles, and which do not. We answer this question in the next chapter.

Chapter VIII
In-Circles and Circum-Circles

We have looked at circles and triangles in the hyperbolic plane, and how they look in UHP. We will now look at two examples of the interaction of these objects, the inscribed and circumscribed circles.

In-Circles

Remember, from Chapter II that there are four kinds of triangles in hyperbolic geometry: ordinary, singly asymptotic, doubly asymptotic, and trebly asymptotic. We can find the in-circle of each of these kinds of triangles. We will consider the ordinary triangle first.

The in-circle of the ordinary triangle

We showed in Chapter II that any ordinary triangle has an inscribed circle. This is easy to find. All we need is the center and any point on the circle. These are found in exactly the same manner as in Euclidean geometry. An example is shown in figure 8.1, where l and m bisect angles CAB and ACB respectively and intersect in D, and E is the foot of the perpendicular n from D to side AC. The circle is centered at D with radius DE.

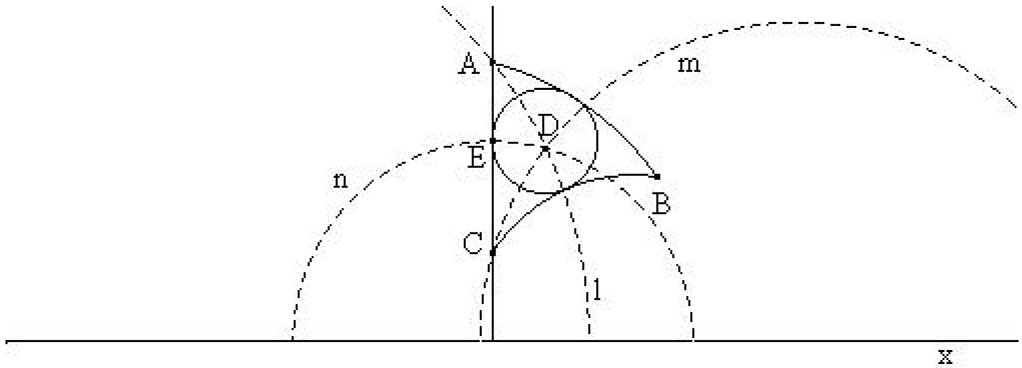

Figure 8.1 The In-circle of a triangle in standard position

The in-circle of the asymptotic triangle

It is no more difficult to find the in-circle of a singly asymptotic triangle. Since we have two non-zero angles, we can construct the angle bisectors of these. The intersection of these angle bisectors will be equidistant from all three sides. Essentially, the construction is the same as that for the ordinary triangle and is illustrated in Figure 8.2. (Recall that one vertex of the ordinary triangle was never used in the construction above.)

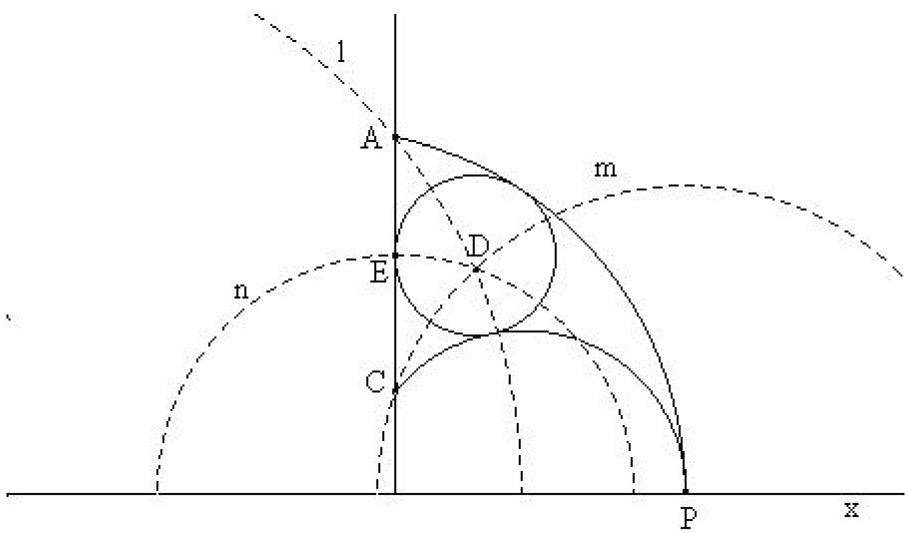

Figure 8.2 The In-Circle of the Singly Asymptotic Triangle

The doubly and trebly asymptotic triangles pose a little more of a problem since neither of them has a pair of non-zero angles and the angle bisector does not exist for an angle of measure zero with its vertex at infinity. There is, however, for any two limiting parallels, a mirror of reflection that will send one to the other. Since our metric is preserved by reflection, this mirror will act as an angle bisector in the sense that it is the set of points equidistant from both lines. The intersection of these mirrors of reflection with the angle bisector of the non-zero angle will be equidistant from all three sides, and therefore the center of our in-circle. Figure 8.3 illustrates this for the doubly asymptotic triangle. Line l is the angle bisector of PAQ, lines m and n are the mirrors of reflection from side PQ to AQ and AP respectively, and line p is the perpendicular from the in-center to side AQ.

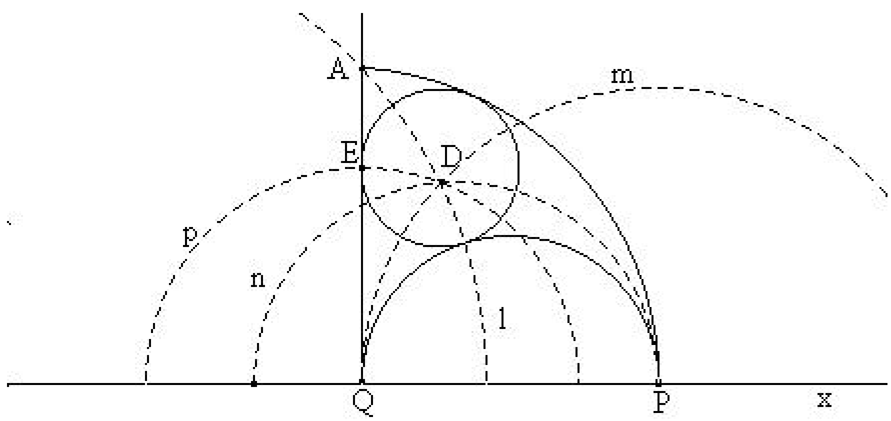

Figure 8.3 The In-Circle of the Doubly Asymptotic Triangle I

Since the doubly asymptotic triangle is defined entirely by the one non-zero angle, it seems natural that there ought to be a relationship between this angle and the radius of the in-circle. Consider the situation pictured in Figure 8.4. Since each of the four angles marked β are the angle of parallelism associated with the radius of the circle, $\Pi(r)$, we know that they are all congruent, and each of the angles marked δ has measure $\pi - 2\Pi(r)$.

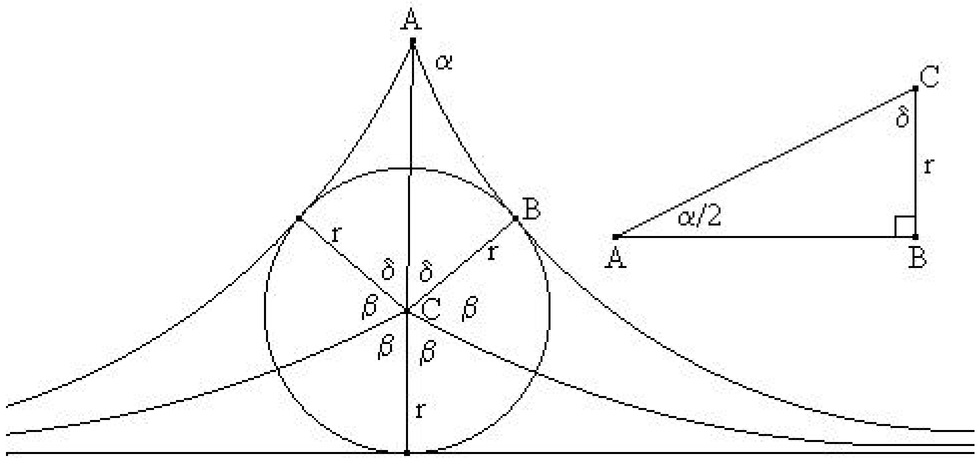

Figure 8.4 The In-Circle of the Doubly Asymptotic Triangle II

Applying equation 10 from our discussion of trigonometry in Chapter V to triangle ABC, we get:

$$\cos\left(\frac{\alpha}{2}\right) = \cosh(r)\cdot\sin(\delta) = \cosh(r)\cdot\sin(\pi - 2\Pi(r))$$
$$= \cosh(r)\cdot\sin(2\Pi(r)) = \cosh(r)\cdot 2\cdot\sin(\Pi(r))\cos(\Pi(r))$$

Using equations 1 and 3 from Chapter V for substitution gives us:

$$\cos\left(\frac{\alpha}{2}\right) = \cosh(r)\cdot 2\cdot\sin(\Pi(r))\cos(\Pi(r)) = 2\cdot\cosh(r)\cdot\frac{1}{\cosh(r)}\cdot\tanh(r) = 2\cdot\tanh(r)$$

and we have:

Theorem 8.1: *The measure of the non-zero angle α of a doubly asymptotic triangle and the radius r of its in-circle satisfy: cos (α/2) = 2 tanh(r).*

The in-circle for the trebly asymptotic triangle is constructed in much the same manner, by constructing the three mirrors (only two are needed) that reflect the sides to each other pairwise. (Figure 8.5) Note that we do not need to construct the perpendicular from the center to any side of the triangle, as each of the mirrors is perpendicular to the side that it does not reflect.

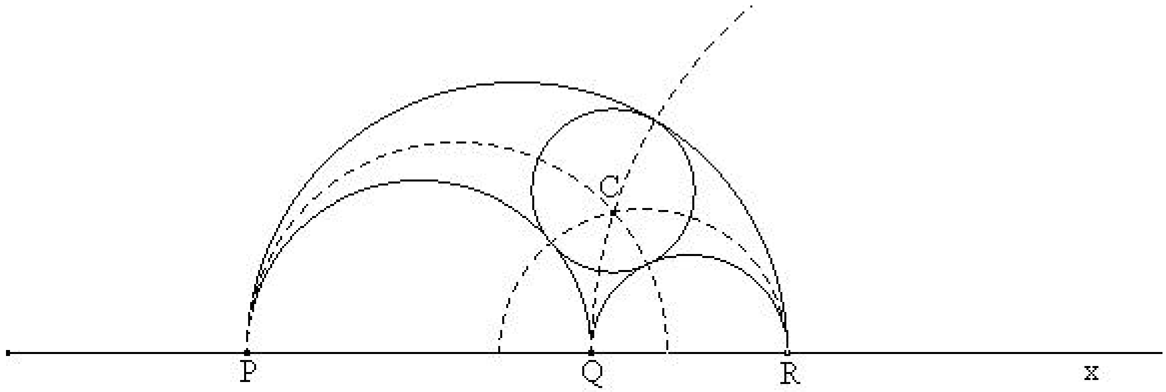

Figure 8.5 The In-Circle of the Trebly Asymptotic Triangle I

Since a trebly asymptotic triangle has three infinite sides, and three angles of measure zero, all trebly asymptotic triangles are congruent. We would expect that the radius of the in-circle is a constant. Figure 8.6 shows that the three mirrors of reflection intersect at the center of the circle forming six congruent angles. We know they are congruent because they are each the angle of parallelism associated with the radius of the circle.

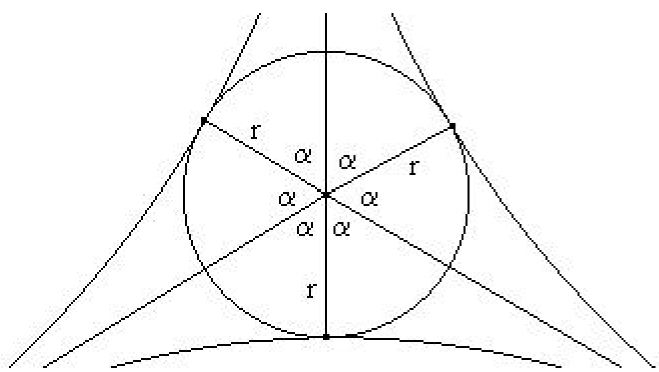

Figure 8.6 The In-Circle of the Trebly Asymptotic Triangle II

We can apply equation 3 from Chapter V to the angle α, which we know to be π/3, and get:

$$\tanh(r) = \cos\left(\frac{p}{3}\right) = \frac{1}{2}$$

$$r = \tanh^{-1}\left(\frac{1}{2}\right) = \frac{\ln(3)}{2}$$

so:

Theorem 8.2: *The radius of the in-circle of any trebly asymptotic triangle is ln(3)/2*

If we consider the equilateral triangle inscribed within the circle of radius ln(3)/2, we discover something curious about the vertex angle of this triangle. Figure 8.7 shows this triangle divided into six congruent right triangles, each having the radius of the circle as its hypotenuse.

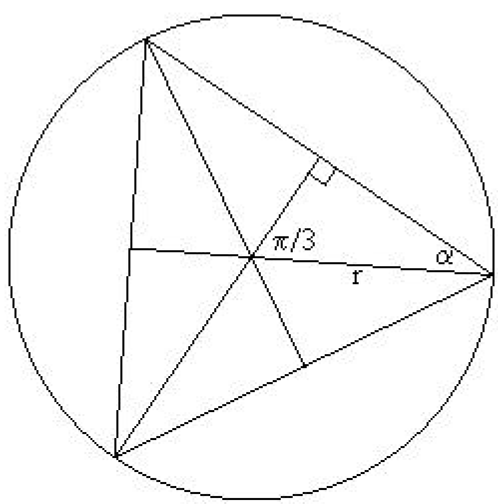

Figure 8.7 The equilateral triangle inscribed in a circle of radius ln(3)/2

Applying equation 11 from Chapter V to any one of the six right triangles, we get:

$$\cot\left(\frac{p}{3}\right) \cdot \cot(a) = \cosh\left(\frac{\ln(3)}{2}\right) \qquad \text{or} \qquad \cot(a) = 2$$

Which tells us that:

$$\sin(a) = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos(a) = \frac{2}{\sqrt{5}}$$

Since the vertex angle of the triangle is 2α, the double angle formula for sine gives us:

$$\sin(2a) = 2 \cdot \sin(a) \cdot \cos(a) = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}$$

This tells us that the measure of the vertex angle of this equilateral triangle is the same as the larger of the two non-right angles of the ubiquitous 3-4-5 triangle. (approximately 53.13°) This is not particularly significant, merely curious.

This concludes our discussion of in-circles, and we move on to:

Circum-Circles

We saw in Theorem 2.34 that the perpendicular bisectors of the sides of a triangle are either: concurrent, parallel in the same direction, or ultra-parallel. Figure 8.8 illustrates this for UHP. Since the intersection of the perpendicular bisectors of the sides of a triangle is the center of the circum-circle, this circle will exist only if the point of intersection exists. We can see that this is not always the case in UHP.

We know that any circle in UHP is an e-circle, which means that the hyperbolic circum-circle of triangle ABC is also its Euclidean circum-circle. The problem occurs when the Euclidean circum-circle of triangle ABC does not lie entirely above x. Figure 8.8 also illustrates this.

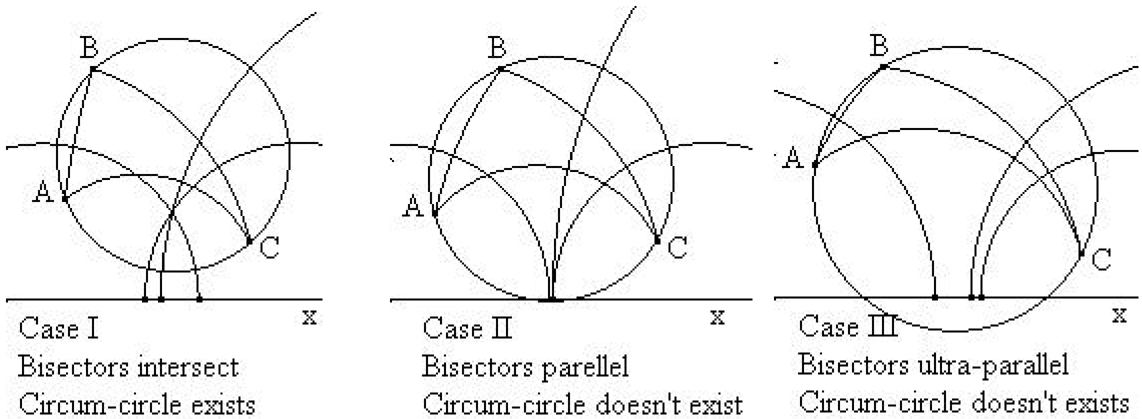

Figure 8.8 The three cases of the Euclidean circum-circle of triangle ABC

Note that if the Euclidean circum-circle lies entirely above x (Case I), then it is the hyperbolic circum-circle. If it is tangent to x (Case II), it is a horocycle, and if it intersects x (Case III) it is a hypercycle.

We know that triangle ABC will not have a circum-circle if A, B and C are collinear in the Euclidean sense, because its Euclidean circum-circle will not exist. If A, B and C are collinear in the hyperbolic sense, then the Euclidean circum-circle is the line through the three points, and is not a hyperbolic circle because half of it lies below x. We will restrict our discussion to triples of points that are non-collinear in both the Euclidean and the hyperbolic sense.

In the case where the circum-circle of triangle ABC does exist, its construction is simple, and procedurally identical to its construction in Euclidean Geometry. The perpendicular bisectors of any two sides will intersect in the circum-center. (Figure 8.9)

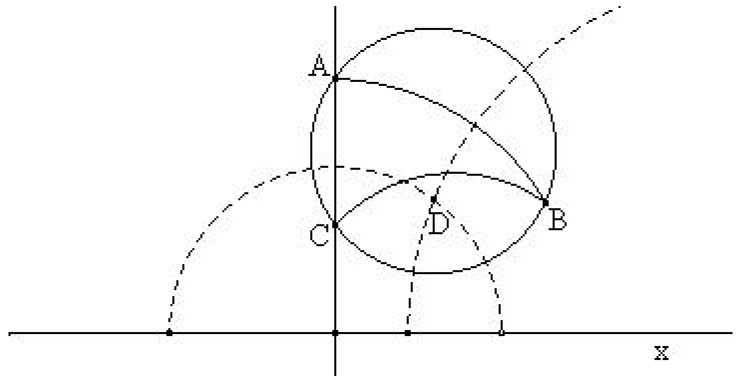

Figure 8.9 The circum-circle of triangle ABC

The question is, when does the circum-circle of a triangle exist and when does it not? We saw in Figure 8.8 and in Chapter VI that the horocycle acts as the 'limit' of the circle in UHP, in the sense that if the e-circle grows any 'larger' downward, it ceases to represent a hyperbolic circle. This is the key to finding a condition for the existence of the circum-circle

Remember the relationship shown in Figure 6.9, that given a singly asymptotic triangle ABC with all three points lying on horocycle h and C being the i-point where h is tangent to x, we have:

$$CAB = CBA = \cos^{-1}\left(\tanh\left(\frac{h(A,B)}{2}\right)\right) = p\left(\frac{h(A,B)}{2}\right)$$

Consider any triangle ABC in standard position in UHP, with A(0,1), B(0,k) with k>1 and C to the right of y. (Figure 8.10) Let h and k be the two horocycles containing A and B. They will be symmetric about y. We can see by inspection that if C lies on h or k, outside both h and k, or within the intersection of the interiors of h and k, that the circum-circle fails to exist. If C lies within the interior of h or k, but not both, the circum-circle exists. (In Figure 8.10, the horocycles are solid curves and the 'circles' are dashed.)

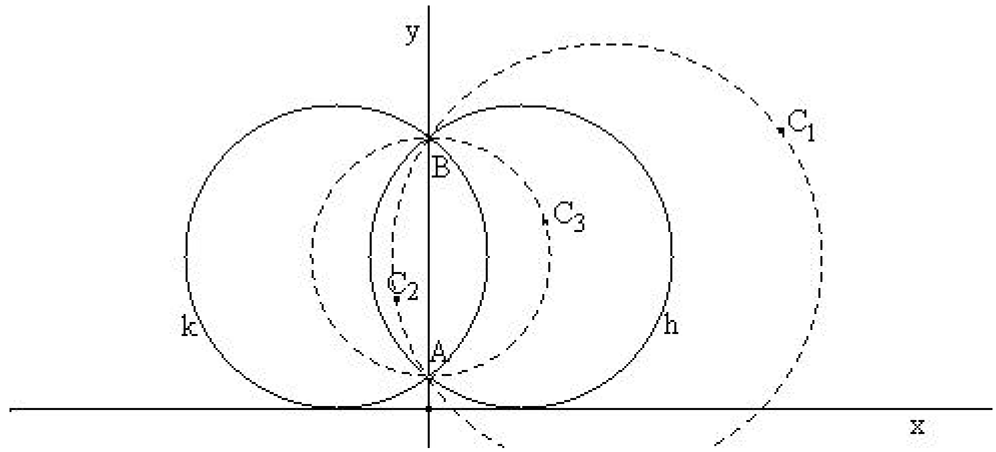

Figure 8.10 The relationship between horocycles and circum-circles

We can show this more concisely by using a few facts about circles. We know two intersecting circles intersect in exactly two points, and we know that one arc of each circle lies entirely inside the other circle, and one arc lies outside. We also know that, in the situation pictured, any continuous path from A to B external to both circles h and k must intersect x.

This tells us:

Case I: If D lies on the exterior of h and k,(C1 in Figure 8.10) then arc ACB, and therefore circle ACB, will intersect x, and triangle ACB has no circum-circle.

Case II: If C lies on h or k, then the e-circle through A, B and C is the horocycle h or k, and the circumcircle does not exist.

Case III: If C lies inside both h and k,(C2) then the arc AB not containing C lies outside both h and k, and intersects x, and the circumcircle fails to exist.

Case IV: If C lies inside h and outside k,(C3) (or inside k and outside h) then the arc AB not containing C lies inside k and outside h, (or outside k and inside h) and does not intersect x, and the circum circle exists.

So the condition for the existence of the circum circle is that any vertex of the triangle must lie in the interior of one, but not both, of the horocycles containing the other two. This is rather wordy and difficult to check. We can do much better.

Consider the construction illustrated in Figure 8.10 and reflect in the mirror that will send the perpendicular bisector of side AB to the y-axis. The images of A and B will be horizontally related, the image of horocycle h will be the horizontal e-line AB, and the image of k will be the e-circle through A, B and O (the origin). If the image of C lies below AB, we reflect in line AB. This will place C above line AB and will map the horocycles to each other. This arrangement is illustrated in Figure 8.11

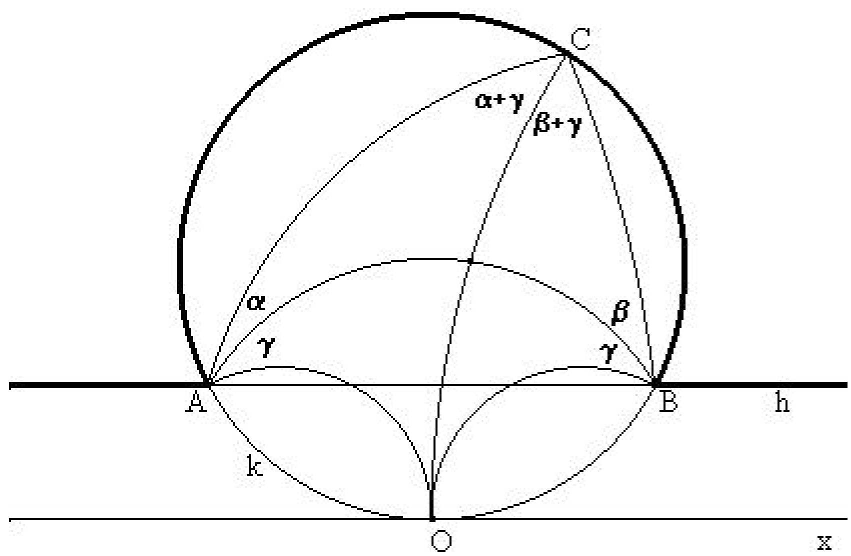

Figure 8.11 The relationship between horocycles and circum-circles II

The interior of the horocycle h is the portion above it, so the region interior to h and exterior to k is the region lying above the darkened line. This is where C must lie for the circum-circle to exist. (It cannot lie inside k and outside h because that region lies below line AB and C lies above.) We know that if C lies on h or k, then the circum-circle does not exist, but it will be helpful to examine this situation.

Assume that C lies on k, and remember the following facts from our discussion of horocycles in Chapter VI. (Figure 8.11) First, that the non-zero angles of a singly asymptotic triangle inscribed in a horocycle are congruent. This tells us that angles ACO≅CAO, BCO≅CBO and ABO≅BAO. Second, that each of these non-zero angles is the angle of parallelism associated with one half the length of the finite side of the triangle, or ABO=Π(AB/2). Taken together, these give us:

$$ACB = CAB + BAO + ABO + ABC$$

$$ACB = CAB + ABC + 2p\left(\frac{AB}{2}\right)$$

Suppose, now, that C lies above the darkened line, (inside h but outside k). This is the case in which the circum-circle exists. (Figure 8.12)

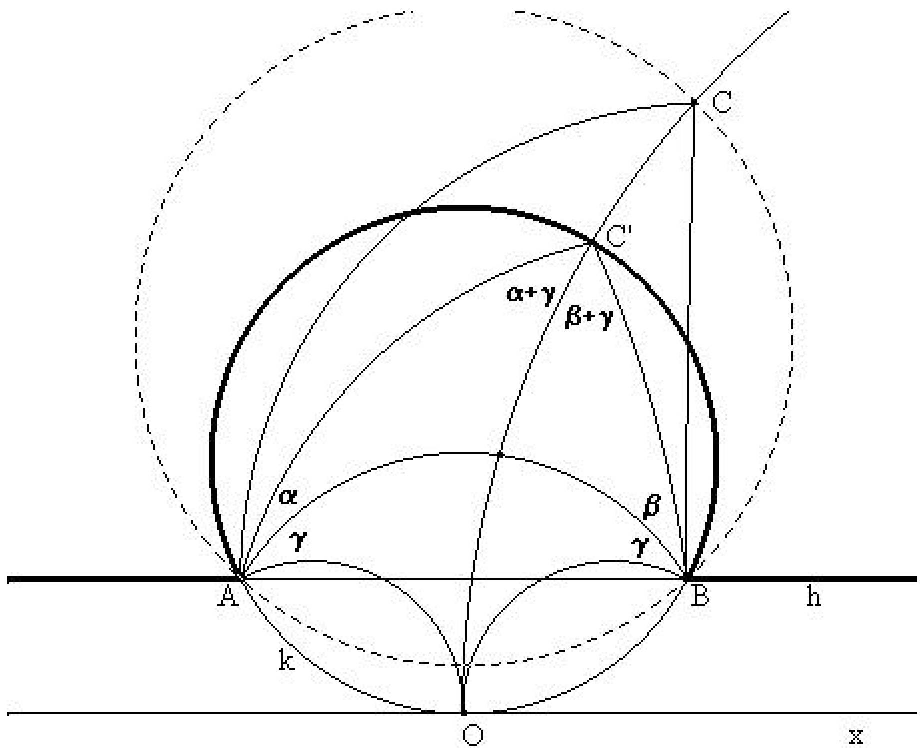

Figure 8.12 The relationship between horocycles and circum-circles III

Let C' be the intersection of line CO and horocycle k. We can see that:

$$ACB < AC'B = C'AB + C'BA + 2p\left(\frac{AB}{2}\right) < CAB + CBA + 2p\left(\frac{AB}{2}\right)$$

This is the case where the circum-circle exists. By a similar argument, if C lies inside both h and k, (beneath the darkened line and above segment AB) and C' the intersection of line CO and horocycle k, we get:

$$ACB > CAB + CBA + 2p\left(\frac{AB}{2}\right)$$

The case where C lies in the exterior of both horocycles can be handled by adding the assumption that angle ACB is the largest angle. Figure 8.13 shows that if C lies outside (below) h, then angle ACB is not the largest angle.

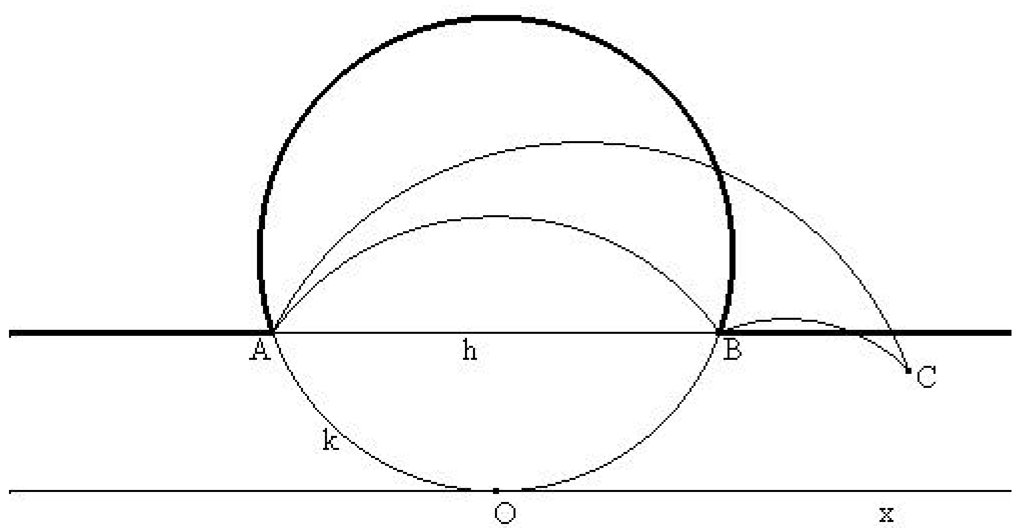

Figure 8.13 The relationship between horocycles and circum-circles IV

This gives us our condition for the existence of the circumcircle for triangle ABD:

Theorem 8.3: *The circumcircle exists for a given triangle iff the measure of any one of its angles is less than the sum of the measures of the other two angles plus twice the angle of parallelism associated with half the length of the longest side.*

$$ACB < CAB + CBA + 2p\left(\frac{AB}{2}\right)$$

Note that two angles of every triangle fit this condition by virtue of their not being the largest, so we only need to check the largest angle.

We may think of this condition in the following way: As the largest angle grows such that it exceeds the sum of the other two angles, the angle of parallelism associated with the length of the opposite side must grow larger, meaning that the opposite side, the longest side, must grow smaller. In other words, the circum-circle exists for 'very' obtuse triangles, provided they are 'very' small.

References

[1] Dodge, Clayton W., Euclidean Geometry and Transformations, Addison-Wesley Publishing Company, 1972

[2] Eves, Howard, An Introduction to the History of Mathematics, 6th Edition, Saunders College Publishing, 1953

[3] Greenburg, Marvin Jay, Euclidean and Non-Euclidean Geometries, Development and History, W. H. Freeman and Company, 1972

[4] Guggenheimer, Heinrich W., Differential Geometry, Dover Publications, Inc., 1963

[5] Stahl, Saul, The Poincaré Half-Plane, A Gateway to Modern Geometry, Jones and Bartlet Publishers, 1993

[6] Wolfe, Harold E., Non-Euclidean Geometry, Holt, Rinehart and Winston, Inc., 1945

[7] Kay, David C., College Geometry, Holt, Rinehart and Winston, Inc., 1969

Appendix
Constructions

These are some of the basic constructions for the three models of hyperbolic geometry discussed in Chapter III. The instructions are appropriate for both pencil-and-paper constructions as well as for dynamic geometry software such as Cabri II. Macros for most of these constructions in Cabri II are included on a CDRom with this thesis, as well as a demonstration version of Cabri II.

In the figures, original and final objects are drawn solid, while intermediate objects are dashed.

Constructions in Euclidean Space

Construction E.1 (Orthogonal Circles): *Given a circle c with center C, and a point P outside c, construct the circle p with center P that is orthogonal to circle c. (Figure A.1)*

1) Draw circle a on diameter PC
2) Circles a and c intersect in point Q
3) Draw circle p centered at P through Q

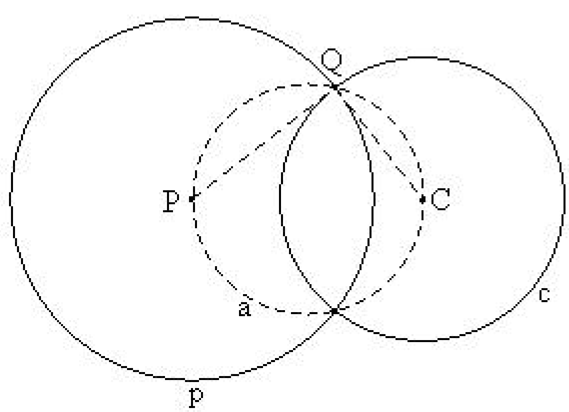

Figure A.1 Constructing a circle orthogonal to a given circle

Segments PQ and CQ are radii of circles p and c respectively, and are orthogonal because angle PQC is inscribed in a semi-circle.

Construction E.2 (Inversion): *Given a circle c with center C and a point P, construct the image P' of P under inversion in circle c.*

Case I: P lies inside c (Figure A.2)

1) Draw the line l through C and P
2) Draw the line m perpendicular to l at P
3) Line m and circle c intersect in Q
4) Draw line n tangent to c at Q
5) Lines n and l intersect in P'

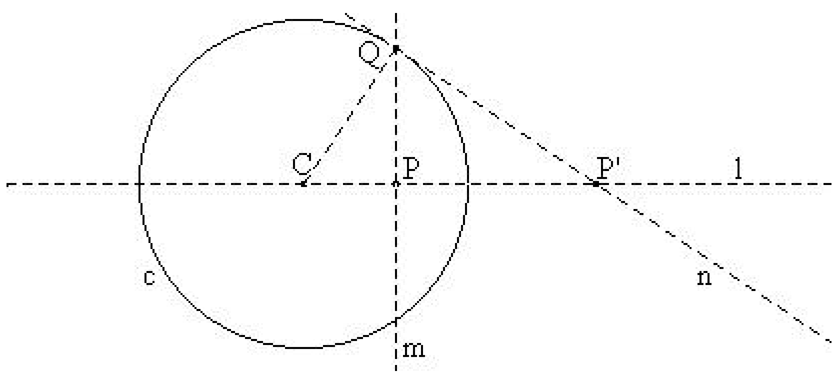

Figure A.2 Constructing the image of a point under inversion I

Case II: P lies outside c (Figure A.3)

1) Draw line l through P and C
2) Draw circle a on diameter PC
3) Circles a and c intersect in Q
4) Draw line m through Q perpendicular to l
5) Lines l and m intersect in P'

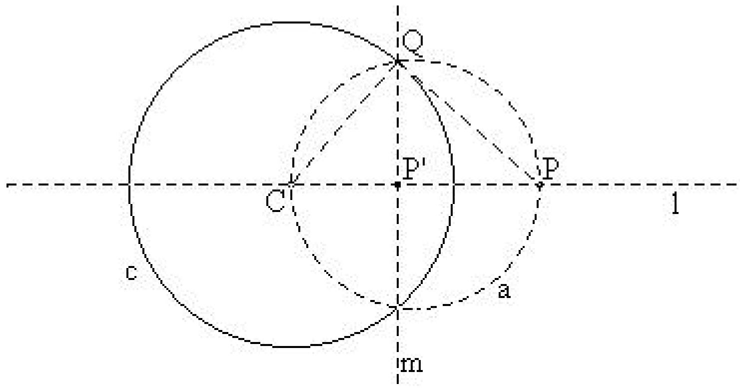

Figure A.3 Constructing the image of a point under inversion II

In both cases, triangles CPQ and CQP' are similar by AAA, so $CP \cdot CP' = CQ^2$.

Constructions in KDM

Construction K.1 (Line/Segment): *Given two points A and B, construct the line/segment through them.*

1) Draw the Euclidean line/segment l through A and B

Construction K.2 (Polar Point): *Given line l, construct the polar point L of l. (Figure A.4)*

1) Line l has ideal points P and Q
2) Draw the tangents m and n to d at P and Q respectively
3) Lines m and n meet at L

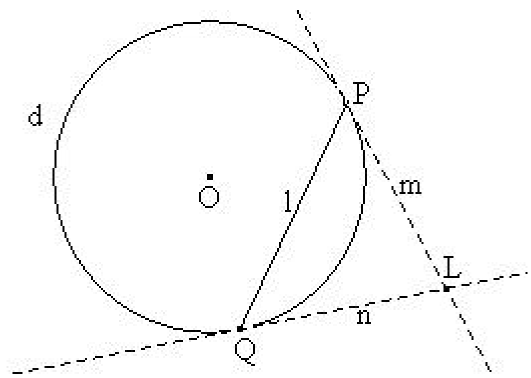

Figure A.4 Constructing the polar point of a line in KDM

Point L is the polar point of line l by definition of the polar point in KDM.

Construction K.3 (Perpendicular): *Given a line l and a point A, construct the line p through A perpendicular to l. (Figure A.5)*

1) Draw point L, the polar point of L
2) Draw line p through points L and A

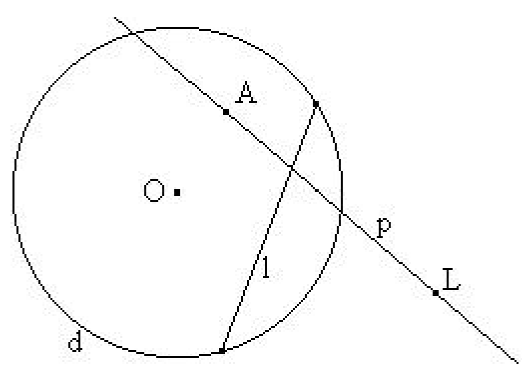

Figure A.5 Constructing perpendiculars in KDM

Line p is perpendicular to l by the definition of parallel in KDM.

Construction K.4 (Perpendicular Bisector/Midpoint): *Given two points A and B, construct C and p, the midpoint and perpendicular bisector of segment AB. (Figure A.6)*

1) Draw line m through A and B
2) Draw the polar point M of line m
3) Draw e-line AM
4) Line AM cuts d in P such that P is between A and M
5) Draw e-line BM
6) Line BM cuts d in Q such that B is between Q and M
7) Draw e-line PQ
8) Line PW cuts m in C
9) Draw line p through C perpendicular to m

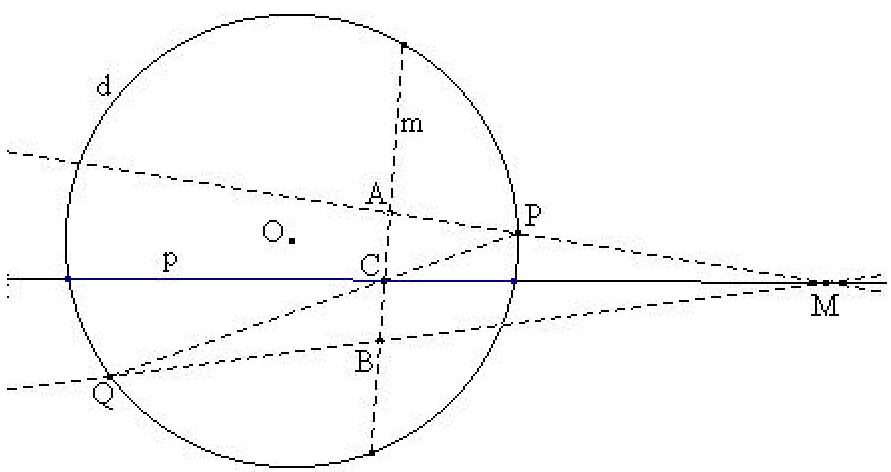

Figure A.6 Constructing the perpendicular bisector/midpoint in KDM

Angles ACP and BCQ are congruent (vertical) so segments AC and BC are congruent since they have the same angle of parallelism. (Angles CBQ and CAP are right angles)

Construction K.5 (Angle Bisector): *Given three points, A, B and C in KDM, construct line n, the angle bisector of angle ABC. (Figure A.7)*

1) Let ray BA cut d in P
4) Let ray BC cut d in Q
5) Draw line PQ
6) Draw line n through B perpendicular to line PQ

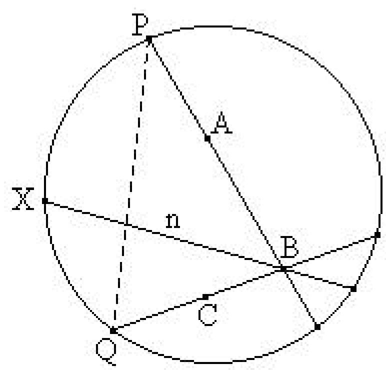

Figure A.7 Constructing the angle bisector in KDM

Since line n is perpendicular to line PQ, angles ABX and CBX are both the angles of parallelism associated with the distance of B from PQ, and therefore congruent.

Construction K.6 (Mutual Perpendicular): *Given two lines l and m in KDM, construct the line p perpendicular to both l and m. (Figure A.8)*

1) Draw the pole L of l
2) Draw the pole M of m
3) Draw the e-line p through L and M

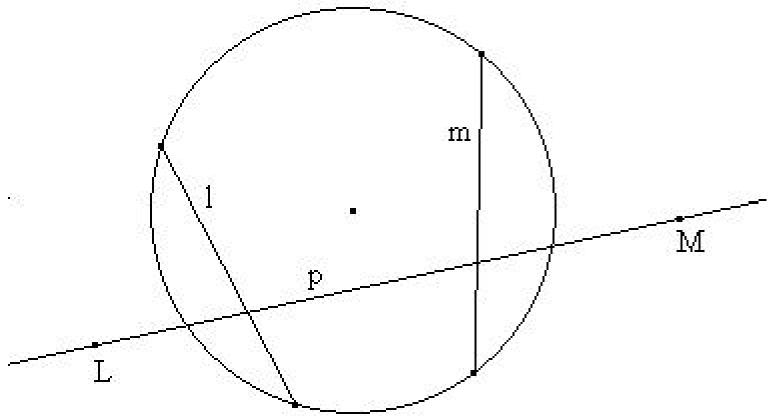

Figure A.8 Constructing the mutual perpendicular to two lines in KDM

Because line P passes through the polar points of both l and m, it is perpendicular to both.

Construction K.7 (Reflection of a Point in a Line): *Given a line l and a point A, construct the reflection A' of A in l. (Figure A.9)*

1) Draw line m through A perpendicular to l
2) Line m meets line l in C
3) Draw line n through A perpendicular to m
4) Line n meets d in P and Q
5) Draw the e-line through P and C, cutting d in R
6) Draw the e-line through M and R, cutting m in A'

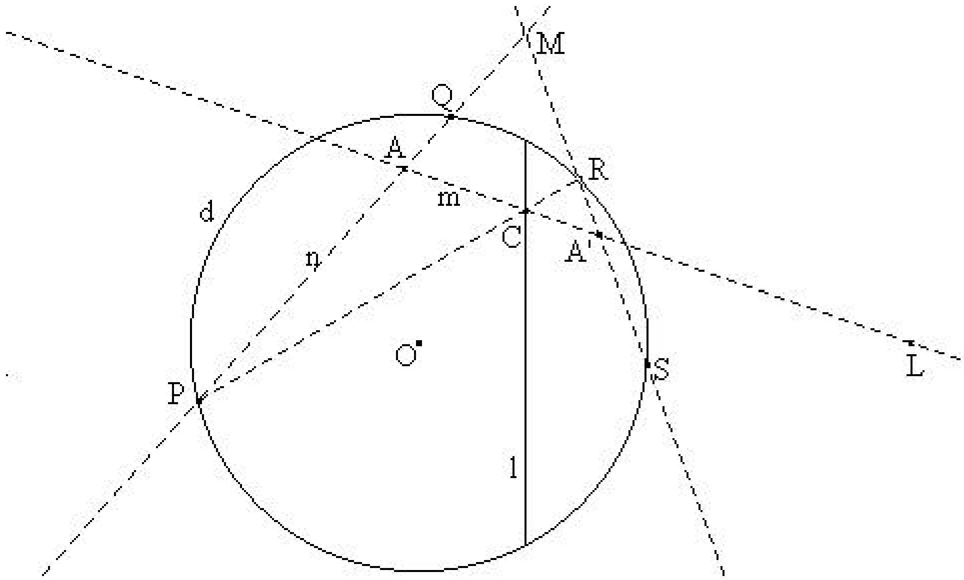

Figure A.9 Constructing the reflection of a point in a line in KDM

The justification for this construction is the same as for Construction K.5

Construction K.8 (Circle): Given two points C and A, construct the circle centered at C with radius CA. (Figure A.10)

1) Draw line AC
2) Draw line OC
3) Draw A', the reflection of A in OC
4) Draw A", the reflection of A' in AC
5) Draw the Euclidean circle through A, A' and A"

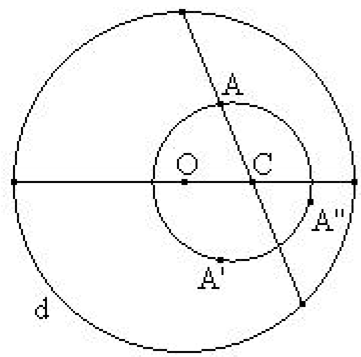

Figure A.10 Constructing a circle in KDM

The three points A, A' and A'', are all equidistant from C, and, therefore, lie on the circle centered at C. The Euclidean circle through the three points is the hyperbolic circle.

Constructions in PDM

Construction P.1 (Polar Point/Line/Segment): *Given two points A and B, construct P and l, the polar point and line/segment through them. (Figure A.11)*

Case I: A and B collinear with the center O of d.

1) Line l is the e-line through A and B (and O)

Case II: A and B are not collinear with the center O of d.

1) Draw A', the Euclidean inverse of A in d
2) Draw the e-perpendicular bisector m of segment AB
3) Draw the e-perpendicular bisector n of segment AA'
4) Lines m and n intersect in ultra-ideal point L, the polar point of line l
5) Draw the e-circle l with center L and radius LA

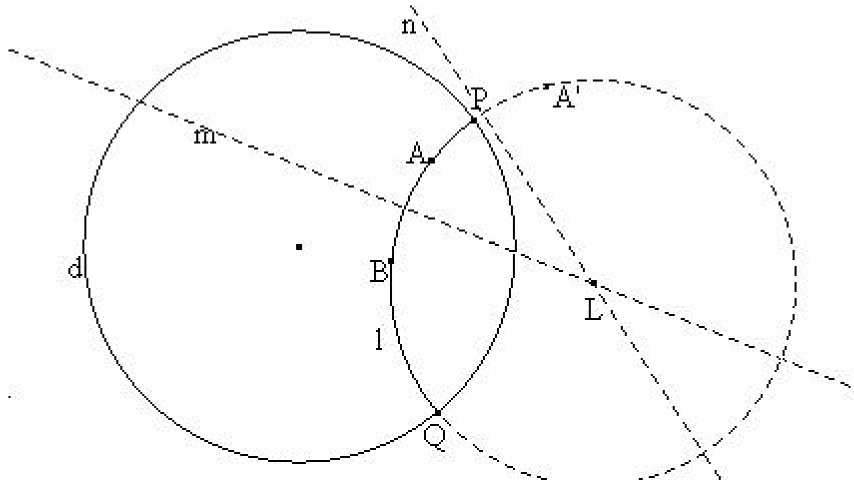

Figure A.11 Constructing the line/segment in PDM

Since l contains a pair of inverse points under inversion in d, l and d are orthogonal.

Construction P.2 (Perpendicular): *Given a line l and a point A, construct the line p through A perpendicular to l.*

Case I: Line l is through O (Figure A.12)

1) Draw A', the inverse of A in d
2) Draw the e-perpendicular bisector m of segment AA'
3) Line m intersects l in P (l will need to be extended)
4) Draw e-circle p with center P and radius PA

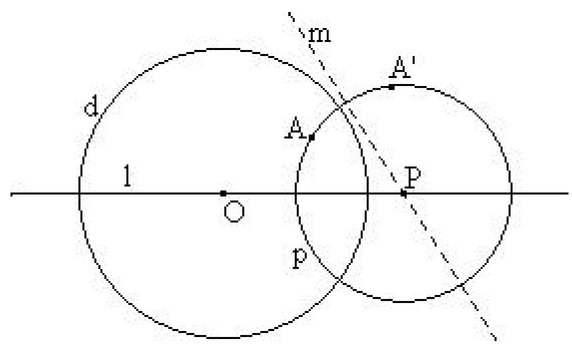

Figure A.12 Constructing a perpendicular in PDM I

Case II: Line l is not through O (Figure A.13)

1) Draw the Euclidean inverse A' of A in d
2) Draw the e-line m through Q and R, the ideal points of l
3) Draw the e-perpendicular bisector n of segment AA'
4) Lines m and n intersect in P
5) Draw e-circle p with center P and radius PA

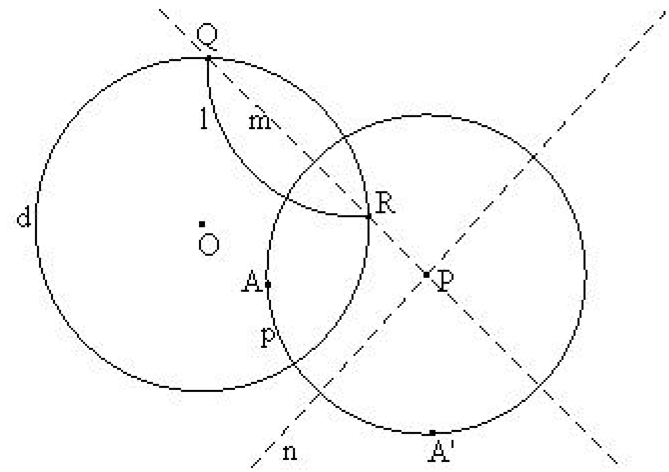

Figure A.13 Constructing a perpendicular in PDM II

In both cases, p contains a pair of inverse points, A and A' under reflection in d, so p is orthogonal to d. In the first case, since P is on l, p is orthogonal to l, and in the second, P is on the radical axis of l and d, so p is orthogonal to l. Both constructions also works if A is on l. In the second case, if e-lines l and m are parallel, then line OA is perpendicular to line l.

Construction P.3 (Perpendicular Bisector/Midpoint): *Given two points A and B, construct C and p, the midpoint and perpendicular bisector of segment AB. (Figure A.14)*

1) Draw e-line l through A and B
2) Draw e-circle c on diameter AB
3) Draw the radical axis r of c and d

4) line r intersects l in P

5) Draw e-circle p with center P orthogonal to d

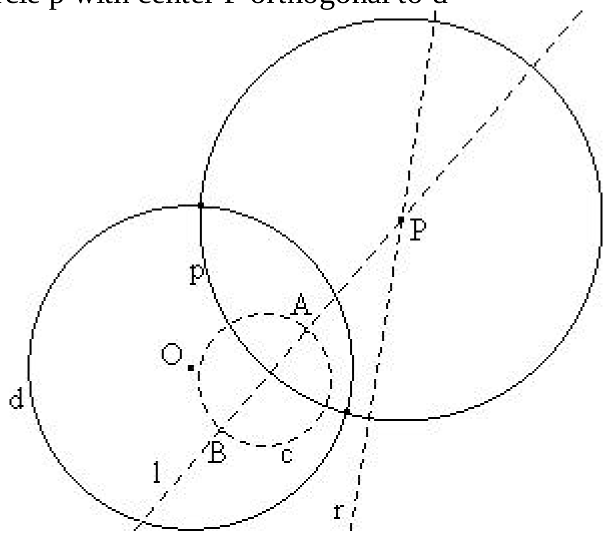

Figure A.14 Constructing the perpendicular bisector/midpoint in PDM

Since P is on the radical axis of the circles p and d, it is orthogonal to c, and since it is on the line AB, inversion in p will map A to B, so it is the perpendicular bisector of segment AB. Line p intersects segment AB at its midpoint.

Construction P.4 (Angle Bisector): *Given three points, A, B and C in KDM, construct line l, the angle bisector of angle ABC. (Figure A.15)*

1) Draw the inverse B' of B in d

2) Draw n the e-perpendicular bisector of segment BB'

3) Ray BA meets d in P

4) Ray BC meets d in Q

5) Draw line e-line t through P and Q

6) Lines n and t intersect in L

7) Draw e-circle l with center L and radius LB

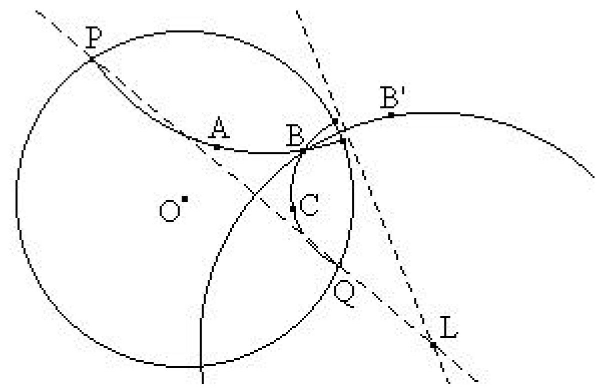

Figure A.15 Constructing the angle bisector in PDM

Since l is orthogonal to d, inversion in l maps P and Q to each other, and leaves B fixed.

Construction P.5 (Mutual Perpendicular): *Given two lines l and m in KDM, construct the line p perpendicular to both l and m. (Figure A.16*)

1) Draw r, the radical axis of e-circles d and l
2) Draw q, the radical axis of e-circles d and m
3) Lines r and q intersect in ultra-ideal point P
4) Draw p, the e-circle centered at P and orthogonal to d

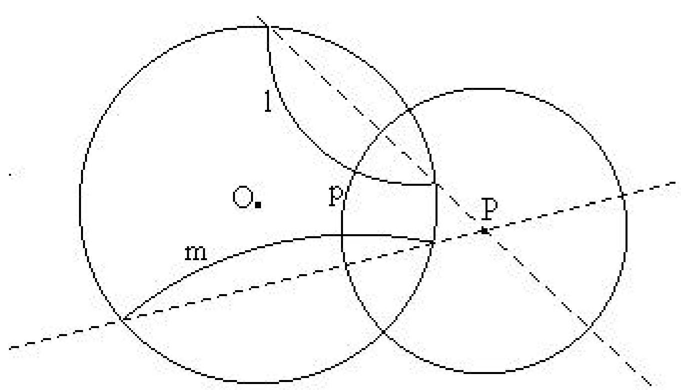

Figure A.16 Constructing the mutual perpendicular in PDM

Since P is on the radical axis of both d and l, and d and m, p is perpendicular to l and m. Should r and q be perpendicular, then the line through O perpendicular to l is also perpendicular to m.

Construction P.6 (Reflection of a Point in a Line): *Given a line l and a point A, construct the reflection A' of A in l.*

Case I: Line l contains O

1) Draw point A', the image of A under reflection in the e-line l

Case II: Line l does not contain O

1) Draw point A', the image of A under inversion in the e-circle l

Construction P.7 (Circle): *Given two points C and A, construct the circle centered at C with radius CA. (Figure A.17)*

1) Draw line AC
2) Draw line OC
3) Draw point A', the reflection of A in OC
4) Draw point A", the reflection of A' in AC
5) Draw the e-circle c through A, A' and A"

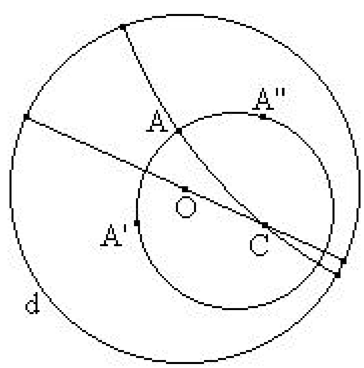

Figure A.17 Constructing the circle in PDM

Segments CA, CA', and CA" are all congruent, so circle c will contain all three. The e-circle c is the hyperbolic circle.

Constructions in UHP

Construction U.1 (Line/Segment): *Given two points A and B, construct l, the line/segment through A and B. (Figure A.18)*

Case I: Points A and B are vertically related

1) Draw the vertical e-line through A and B

Case II: Points A and B are not vertically related

1) Draw p, the Euclidean perpendicular bisector of e-segment AB
2) E-line p meets x in L
3) Draw e-circle l with center L and radius LA

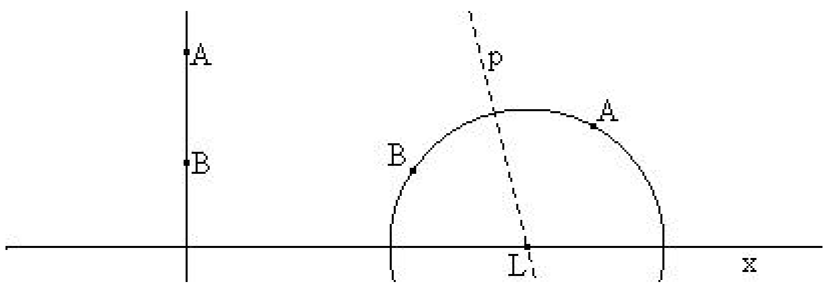

Figure A.18 Constructing the line/segment in UHP

Construction U.2 (Perpendicular): *Given line l and point A, construct line m through A perpendicular to l. (Figure A.19)*

Case I: Line l is of vertical e-line type

1) Line l meets x in M
2) Draw e-circle m with center M and radius MA

Case II: Line l is of e-circle type and A is on l

1) Draw e-line n through A tangent to l
2) Line n meets x in P
3) Draw e-circle m with center P and radius PA

Case III: Line l is of e-circle type and A is not on l

1) Draw A', the Euclidean inverse of A in l

2) Draw e-line n, the perpendicular bisector of e-segment AA'

3) Line n meets x in P

4) Draw e-circle m with center P and radius PA

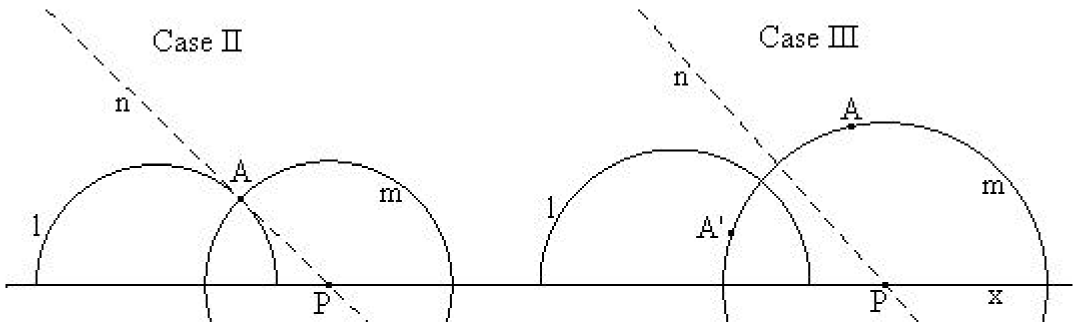

Figure A.19 Constructing perpendiculars in UHP

The first and second cases are obvious. In the third, A and A' are inverses in l and are both on m, so m maps to itself under inversion in l, and is therefore orthogonal to l.

Construction U.3 (Perpendicular Bisector/Midpoint): *Given two points A and B, construct C and p, the midpoint and perpendicular bisector of segment AB.*

Case I: Points A and B are horizontally related

1) Draw e-line m, the Euclidean perpendicular bisector of e-segment AB

2) Line m meets segment AB in C

Case II: Points A and B are not horizontally related (Figure A.20)

1) Draw line l through A and B

2) Draw e-line m through A and B

3) Line m meets x in P

4) Draw e-circle p centered at P orthogonal to l

5) Line p meets line l in C

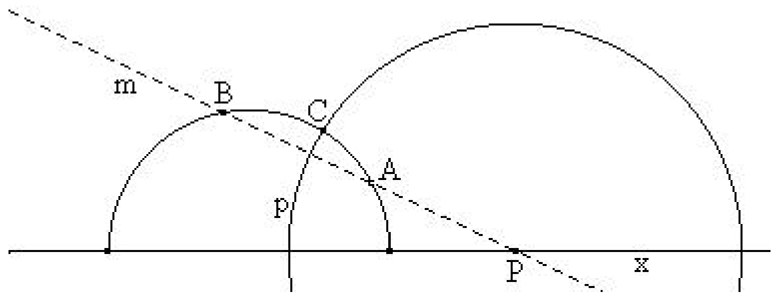

Figure A.20 Constructing the perpendicular bisector/midpoint in UHP

Inversion in p maps l to itself. Since A and B are on l and collinear with P, they map to each other, and m is the perpendicular bisector of segment AB.

Construction U.4 (Angle Bisector): Given three points A, B and C, construct line l, the angle bisector of angle ABC. (Figure A.21)

Case I: Points A and B (or B and C) are vertically related

1) Ray BC meets x in L (if B is above A, then ray CB meets x in L)
2) Draw e-circle l with center L and radius LB

Case II: Neither A and B, nor B and C are vertically related

1) Choose point D on vertical e-line through B
2) Draw e-circle d with center D orthogonal to e-circle BC
3) Circle d intersects rays BA and BC in E and F respectively
4) Draw e-line e through E and F
5) Line e meets x in L
6) Draw e-circle l with center L orthogonal to d

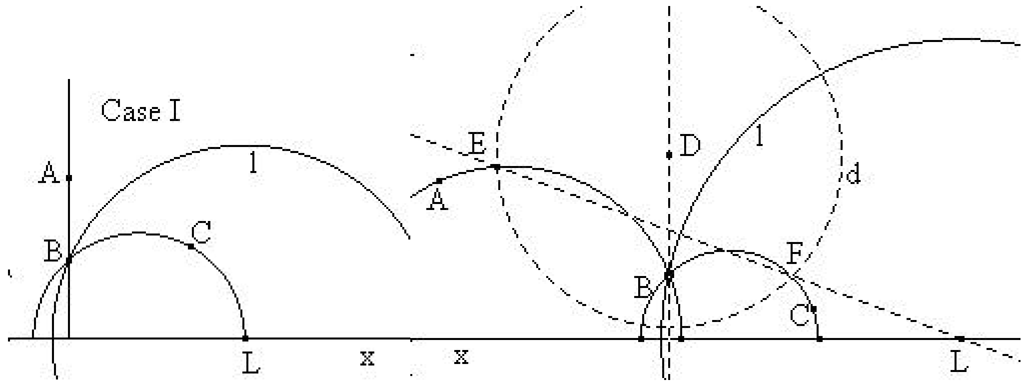

Figure A.21 Constructing the angle bisector in UHP

In Case I, inversion in e-circle l will send L to Z and leave B fixed, therefore rays BA and BC are sent to each other, and l is the angle bisector of ABC.

In Case II, D is on the radical axis of e-circles AB and BC, so e-circle d is orthogonal to both AB and BC. Since l is orthogonal to d and L is collinear with E and F, inversion in l will send E and F to each other. By preservation of angles, this sends rays BA and BC to each other, so B is fixed (on l) and l is the angle bisector of angle ABC.

Construction U.5 (Mutual Perpendicular): *Given two lines l and m, construct the line p perpendicular to both l and m.*

Case I: Line l is of vertical e-line type (Figure A.22)

1) Line l meets x in P
2) Draw e-circle with center P orthogonal to m

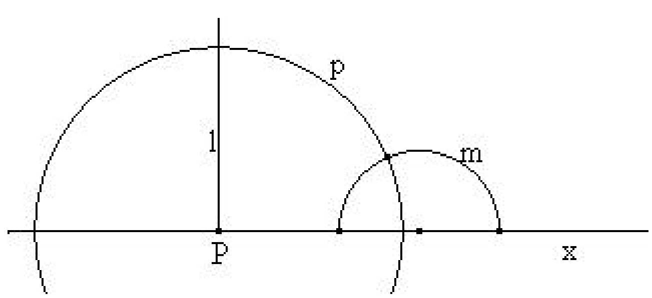

Figure A.22 Construction of the mutual perpendicular in UHP I

Case II: Both lines are of e-circle type (Figure A.23)

1) Draw any e-circle a centered above x that intersects both l and m
2) Circle a intersects l in A and B, and m in C and D
3) Draw e-line n through A and B and e-line q through C and D
5) Line n intersects q in E
6) Draw e-line r through E perpendicular to x
7) Line r intersects x in P
8) Draw circle p with center P orthogonal to l

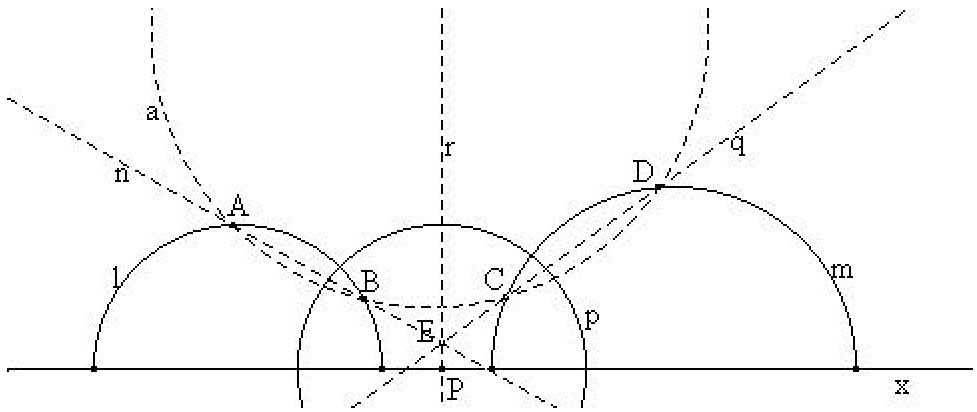

Figure A.23 Construction of the mutual perpendicular in UHP II

In Case I, line p is obviously perpendicular to l.

In Case II, P lies on r, the radical axis of l and m, and since p is perpendicular to l, it is also perpendicular to m.

Construction U.6 (Reflection of a Point in a Line): *Given a line l and a point A, construct the reflection A' of A in l.*

Case I: Line l is of the vertical e-ray type

1) Draw the Euclidean reflection A' of A in e-line l

Case II: Line l is of the e-circle type

1) Draw the Euclidean inverse A' of A in e-circle l

Construction U.7 (Circle): *Given two points C and A, construct the circle centered at C with radius CA. (Figure A.24)*

1) Draw the vertical e-line l through C
2) Draw line m through C and A
3) Draw point A', the reflection of A in l
4) Draw point A", the reflection of A' in m
5) Draw the e-circle through A, A' and A"

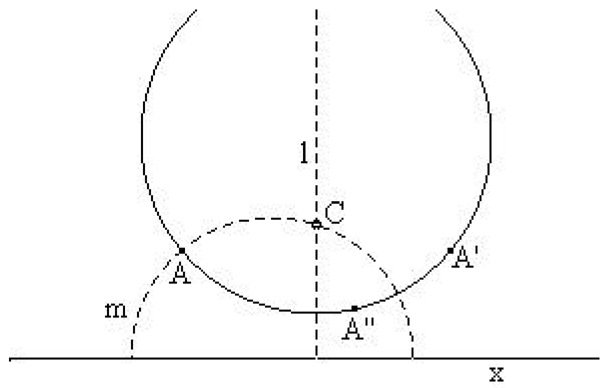

Figure A.24 Constructing the circle in UHP

Segments CA, CA', and CA" are all congruent, so circle c will contain all three. The e-circle c is the hyperbolic circle.

Biography of the Author

Skyler William Ross was born in Framingham, Massachusetts in January of 1966. His family moved to the Greater Bangor Area in the early Seventies. Skyler graduated from Orono High School in 1984, and earned his B.S. in Education from the University of Maine in 1990. After teaching high school Mathematics in Maine and Texas, he returned to the University of Maine to pursue his Master's degree.

He is a candidate for the Master of Arts degree in Mathematics from the University of Maine in May, 2000