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Ramanujan's Formula for the Riemann Zeta Function Extended to L-Functions

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RAMANUJAN'S FORMULA FOR THE RIEMANN ZETA FUNCTION EXTENDED TO L-FUNCTIONS

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Ramanujan's formula for the Riemann-zeta function is one of his most celebrated. Beginning with M. Lerch in 1900, there have been many mathematicians who have worked with this formula. Many proofs of this formula have been given over the last 100 years utilizing many techniques and extending the formula.

This thesis provides a proof of this formula by the Mittag-Leffler partial fraction expansion technique. In comparison to the most recent proof by utilizing contour integration, the proof in this thesis is based on a more natural growth hypothesis. In addition to a less artificial approach, the partial fraction expansion technique used in this thesis yields a stronger convergence result.

In addition to providing a new proof of this formula, the work in this thesis extends this formula to a series acceleration formula for Dirichlet L-series with periodic coefficients. The result is a generalized character analog, which can be reduced to the original formula.

To Edwin C. Gibson, first a mathematician, foremost a friend.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

HISTORY AND BACKGROUND

This chapter will introduce Ramanujan by giving an outline of his life and work, and will elaborate on the work by other mathematicians regarding Ramanujan's formula for the Riemann zeta function.

1.1 S. Ramanujan, His Life and Work

Srinivasa Ramanujan Iyengar was born on December 22, 1887 in Erode, India to Komalatammal, his mother, and Srinivasa Aiyangar, his father. Although he was born a Brahmin, the highest caste, it was during a time when the Brahmin class was in economic decline. Patrons were not providing financial support as they had in the past. Since his father was a clerk in a silk shop earning only 20 rupees per month, Ramanujan grew up poor. Yet, he was still predisposed to the life of a Brahmin that would have permitted and encouraged meditation on the higher order of the world, including mathematics.

In 1898, Ramanujan entered his first form at Town High School in Kumbakonam, where he studied for six years. It was during his studies at Town High School around 1903, that he worked with G. S. Carr's *A Synopsis of Elementary Results in Pure and*

Applied Mathematics and began the *Notebooks* [23]. Carr was a tutor for the Tripos examination in Cambridge and Carr's text was his study guide for his students. It contains some 6000 theorems with virtually no proofs. Carr's text influenced Ramanujan's writing style. Ramanujan's Notebooks contain over 3000 results without any elaboration of proofs. In 1904 upon graduation from Town High School, Ramanujan received the *K. Ranganatha Rao* prize for Mathematics, a local prize given by the high school.

Ramanujan entered Government College in Kumbakonam in 1904 with a scholarship. The scholarship covered the tuition, which was 32 rupees per term. However, Ramanujan had started writing the *Notebooks* and became totally absorbed in his Mathematics to the exclusion of his other studies. The consequence was the loss of his scholarship. His parents were not in a position to help him, since a term's cost was one and a half months salary for his father. Faced with this failure, Ramanujan ran away into seclusion for a few months.

In 1906, Ramanujan entered Pachaiyappa's College in Madras, where he failed again. For the next three years, Ramanujan was out of school with no degree, no qualifications for a job, and no contact with other mathematicians. However, this freed him to work on his *Notebooks* with no other commitments.

Ramanujan continued in this way, until his mother arranged a marriage to Srimathi Janaki for him. The wedding was on July 14, 1909. Janaki was not to join him for three more years. At this time, with a sense of responsibility to his family and to his new wife, Ramanujan went looking for employment in Madras with letters of introduction. One letter of introduction was for Ranachandra Rao, educated at Madras's Presidency College, a high ranking official in the provincial civil service, and secretary of the Indian Mathematical Society. Ranachandra Rao received Ramanujan reluctantly. At first, Ranachandra Rao would not help him, when Ramanujan had asked for a stipend to just live on while continuing his work in mathematics. At the third visit with Ranachandra Rao, Ramanujan was finally able to communicate to Rao at Rao's level of mathematical expertise and convince Rao of his mathematical ability. After this visit in 1911, Rao gave Ramanujan 25 rupees a month. Ramanujan also published his first paper, "Some Properties of Bernoulli's Numbers," in the *Journal of the Indian Mathematical Society.*

On March 1, 1912, Ramanujan finally got a job as a Class III, Grade IV clerk with the Madras Port Trust at 30 rupees per month. Shortly thereafter, his mother and Janaki joined him in Georgetown. Feeling confident in himself, he started contacting well known English mathematicians for recognition. He was ignored by M. J. M. Hill, H. F. Baker and E. W. Hobson. It was a letter dated "Madras, 16th January 1913" to G. H. Hardy at Trinity College in Cambridge that was pivotal to the rise of Ramanujan and subsequently to his demise.

Hardy read the letter and glanced at the attached nine pages of formulas and theorems. In a calculating fashion that was typically Hardy, he analyzed the scenario and set the letter aside to continue on with the day. However, his thoughts kept coming back to the letter. "Genius or fraud?" [17] By evening, he had decided that J. E. Littlewood would have to see the letter also. They met after Hall, and "before midnight, Hardy and Littlewood began to appreciate that for the past three hours they had been rummaging through the papers of a mathematical genius" [17].

On February 3rd, Hardy advised the India Office of London of his interest in Ramanujan. On February 8th, 1913, Hardy wrote back to Ramanujan. A whirlwind of communication between many people in England and India, led to Ramanujan being awarded a two year research scholarship of 75 rupees per month from Madras University within a month. However, Ramanujan was unwilling to fulfill Hardy's request to go to Cambridge since Brahmins were not permitted to cross the seas. Then Ramanujan's mother had a dream of the goddess Namagiri, the family patron, urging her not to stand between her son and his life's work. On March 17, 1914, Ramanujan set sail for England and arrived on April 14th.

Upon his arrival, he lived with E. H. Neville and his wife for a short time. He then moved into Whewell's Court at Trinity. As the war began, Hardy and Ramanujan began their work together. Hardy then saw the *Notebooks* in their entirety with what he estimated between three to four thousand entries. Ramanujan took on the formal rigour of verifying his theorems and started publishing results in the *Journal of the London Mathematical Society.* At this point, he had stopped working in his *Notebooks.*

By the winter of 1916, Ramanujan was both physically and emotionally fatigued. He was having night fevers, weight loss, and generalized pain. He had also stopped receiving letters from home. He then entered a series of nursing homes in Cambridge and London. There were a number of different diagnoses, ranging from gastric ulcer to cancer. The physicians finally decided to treat him for Tuberculosis. R. Rankin [7, p. 74] states that Ramanujan more than likely had Hepatic Amoebiasis from dysentery that he contracted in 1906. Nonetheless, his health deteriorated.

Hardy and Littlewood worked quickly to establish Ramanujan as a great mathematician and honour him as such. They compared him with Euler and Jacobi. In December of 1917, he was elected to the London Mathematical Society. In May of

1918, he was elected a Fellow of the Royal Society. Then finally, he was elected a Fellow of Trinity College. The war ended and Ramanujan's health stabilized. He was able to return to India on March 27, 1919. He died on April 26th, 1920 at the age of 32.

With the legacy of his *Notebooks*, Hardy wrote in 1921 that there is a mass of unpublished material that still awaited analysis. Hardy published Ramanujan's *Collected Papers* in 1927. In 1929, G. N. Watson and B. W. Wilson started going through Ramanujan's Notebooks to verify the results. The effort was thwarted when Wilson died unexpectedly in 1935. By 1940, there were 105 papers devoted to Ramanujan's work. In 1957, the Tata Institute of Fundamental Research in Bombay printed fascimiles of the *Notebooks* in two volumes. In 1965, J. M. Whittaker found what Whittaker and Rankin later called *The Lost Notebook* in Watson's home. This was given to Trinity College in 1968. In 1976, G. Andrews rediscovered the *Lost Notebook* at Trinity. And in 1977, B. C. Berndt started working on the results where Watson and Wilson left off. Berndt has successfully verified the 3,254 results.

Ramanujan's mathematical work was primarily in the areas of number theory and classical analysis. In particular, he worked extensively with infinite series, integrals, continued fractions, modular forms, q-series, theta functions, elliptic functions, the Riemann Zeta-Function, and other special functions.

1.2 Entries from Ramanujan's *Notebooks*

Hardy wrote in Ramanujan's obituary [14]:

> *There is always more in one of Ramanujan's formulae than meets the eye, as anyone who sets to work to verify those which look the easiest will soon*

discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining.

The Riemann zeta-function makes its first appearance in Ramanujan's *Notebooks* as Euler's famous formula, showing that for n a positive integer $\zeta(2n)$ is a rational multiple of π^{2n}:

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}, \tag{1.2.1}$$

where B_{2n} denotes the $2n$th Bernoulli number defined by

$$\frac{z}{e^z-1} = \sum_{j=0}^{\infty}\frac{B_j}{j!}z^j, \qquad |z| < 2\pi.$$

The formulas relating to the Riemann zeta-function are in Chapter 14 of the second *Notebook*, where Ramanujan works extensively with sums of powers, Bernoulli numbers and the Riemann zeta-function. It is here where Ramanujan discovers the functional equation of the zeta-function:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right) \tag{1.2.2}$$

and demonstrates phenomenal numerical calculations.

The progression of formulas relating to the work in this thesis begins with a question that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [24] in 1912:

QUESTION 387 *Show that:*

$$\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}. \tag{1.2.3}$$

Berndt, Choi, and Kang [6] state that although in 1914 Ramanujan published a proof of this in his paper [22], it was first established in 1877 by O. Schlömilch [27]. Also, A. Hurwitz established this result in his 1881 thesis [16], but did not state it explicitly. Ramanujan proved this result with his work on q-series in [22].

Question 387 (1.2.3) is a special case of Ramanujan's Corollary(i), [4, p. 255], when $\alpha = \beta = \pi$:

Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then

$$\alpha \sum_{k=1}^{\infty} \frac{k}{e^{2\alpha k} - 1} + \beta \sum_{k=1}^{\infty} \frac{k}{e^{2\beta k} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \tag{1.2.4}$$

Corollary(i) can be derived from (1.2.6), [4, Entry 21(i), pp. 275-276], where $n = -1$ and using $\zeta(-1) = -\frac{1}{12}$.

Let α, $\beta > 0$ with $\alpha\beta = \pi^2$, and let n be an integer greater than 1. Then

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k} - 1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k} - 1} = \{\alpha^n - (-\beta)^n\} \frac{B_{2n}}{4n}. \tag{1.2.5}$$

Formula (1.2.5) is derived from Entry 21(i), Ramanujan's formula for the Riemann zeta function for odd valued integers [4, pp. 275-276], when n is replaced by $-n$ and using $\zeta(1-2n) = -\frac{B_{2n}}{2n}$ for $n > 0$.

Entry 21(i) *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$, and let n be any nonzero integer. Then*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}$$

$$=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$

$$-\ 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k, \qquad (1.2.6)$$

where B_j denotes the jth *Bernoulli number.*

Lerch [20] proved (1.2.6) for $\alpha = \beta = \pi$, n odd and positive. This formula can also be recovered from the more general formula in Entry 20 [5, p. 430] when we choose $\varphi \equiv 1$, and replacing α by $\sqrt{\alpha}$ and β by $\sqrt{\beta}$:

Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi$ and $\varphi(z)$ be an entire function. Then :

$$\frac{(-1)^n}{(2\beta)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ki)+\varphi(-2\beta ki)}{k^{2n+1}(e^{2k\beta^2}-1)}+\frac{(-1)^n}{(2\beta)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ki)}{k^{2n+1}}$$

$$-\frac{1}{(2\alpha)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\alpha k)+\varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2}-1)}-\frac{1}{(2\alpha)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\alpha k)}{k^{2n+1}}$$

$$=-\frac{\pi i}{2}\frac{\varphi^{(2n)}(0)}{(2n)!}+\alpha\sum_{k=0}^{n}(-1)^{n+k}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\beta^{2n-2k}$$

$$-\,i\beta\sum_{k=0}^{n}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\alpha^{2n-2k}$$

$$+\,2\sum_{k=0}^{n+1}\frac{\varphi^{(2k)}(0)}{(2k)!}\sum_{j=0}^{n+1-k}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!}\alpha^{2n+2-2k-2j}\beta^{2j}. \qquad (1.2.7)$$

Ramanujan stated (1.2.7) without any proof and without the necessary hypotheses on the growth of φ to make the formula universally true. Berndt gives an implicit restriction on φ in his proof [5, pp. 429-432], summarized in Chapter 2 herein.

1.3 Work Inspired by these Formulas

The formulas from "Question 387" (1.2.3) to $\zeta(2n+1)$ (1.2.6) have been known for over a century. Here, we will give an outline and a lineage of the Mathematics associated with the formulas.

1.3.1 Question 387

Formula (1.2.3) was first established by O. Schlömilch in 1877 [27] while investigating finite series of integrals with sine and cosine, combined with the exponential function utilizing Fourier transforms.

Krishnamachari [19] was working directly to solve questions submitted to the *Journal of the Indian Mathematical Society.* For Question 387, his proof utilized residue calculus and contour integration.

As part of the original effort to verify the results in Ramanujan's *Notebooks*, G. N. Watson derived (1.2.3) directly from a formula due to Jacobi on elliptic functions [30].

In 1954, H. F. Sandham proved (1.2.3) by partial fraction expansion of the hyperbolic functions [26].

Finally, C. Ling has established a general method to put certain summations in closed form, including (1.2.3) [21].

Other mathematicians have worked on the formulas (1.2.4) and (1.2.5) utilizing many different techniques, such as the Mellin transform, modular transformations of the elliptic function, and Lambert series.

1.3.2 Formula for $\zeta(2n+1)$

Following M. Lerch [20] in 1900, who showed (1.2.6) for the case when $\alpha = \beta = \pi$, Malurkar used the Mellin transform, and Grosswald and Guinand employed transformations formulae. Guinand's result is given in the work of H. Glaeske [12], and K. Chandrasekharan and R. Narasimhan [10].

There are several mathematicians who have been working with Epstein zeta functions and have produced the formula for $\zeta(2n+1)$, including J. R. Smart [28], A. Terras [29], and N. Zhang [31]. Similar to Grosswald's work using the Mellin transform of the Dedekind eta function [13], Fourier expansions of the Epstein zeta function yield formulas for $\zeta(2n+1)$.

Berndt [2] has perhaps the most extensive work in generalizing modular transformation formulas that yields Ramanujan's formula from the function

$$f(z,-m) = \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1}, \tag{1.3.1}$$

which comes from the theory of elliptic functions. Berndt [2] shows that from the same modular transformation of (1.3.1) we have:

THEOREM 2.2. For $z \in \mathcal{H} = \{z : \mathrm{Im}(z) > 0\}$, m integral, and $Vz = -1/z$ or $Vz = -(z+1)/z$, we have

$$\begin{aligned} z^m(1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k V z} - 1} \\ = (1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1} + g(z, -m) \\ + (2\pi i)^{m+1} \sum_{k=0}^{m+2} \frac{B_k(1)}{k!} \frac{B_{m+2-k}}{(m+2-k)!} (-z)^{k-1}, \end{aligned} \tag{1.3.2}$$

where $g(z,-m) = \{1 - (-cz - d)^m\}\zeta(m+1)$ if $m \neq 0$ and $g(z,0) = \pi i - \log(cz + d)$, where $c = 1$, $d = 0$, $B_k(x)$ is a Bernoulli polynomial, and B_k is a Bernoulli number.

We get Euler's formula for $\zeta(2n)$ (1.2.1) (when specifying $m = 2N - 1$) and Ramanujan's formula for $\zeta(2n+1)$ (when specifying $m = 2N$, $N \neq 0$), let $Vz = -1/z$ and $z = i\pi/\alpha$. Through this modular transformation, it is shown that Ramanujan's formula is one of an infinite class of such formulas [2].

Ramanujan stated his formula twice in his *Notebooks.* The first time was Entry number 15, [23, Volume I, p. 259], and the second time was Entry number 21(i), [23, Volume II, p. 177]. With both of these entries, Ramanujan's hypothesis states "If $\alpha\beta = \pi^2$ and n any integer" and distinguishes "according as n even or odd", [23], but it is not universally true. Additional hypotheses have been added for both proofs in this thesis.

1.3.3 Character Analogs

Berndt developed transformation formulae of Eisenstein series with characters [3]. Then by utilizing the Lipschitz summation formula, the theorems could be converted to theorems giving transformation formulae for certain Lambert series with characters or certain character generalizations of the classical Dedekind eta-function (involving characters and generalized Bernoulli numbers). The transformation formulae yield immediately formulae for L-functions, including Ramanujan's formula for $\zeta(2n+1)$.

Katayama's work [18] follows Grosswald's function theoretic methods and are considered to be special cases of an infinite class of similar formulae.

Bradley [8] has developed series acceleration formulas for Dirichlet series with periodic coefficients by the classical Mittag-Leffler partial fraction expansion method. In [8], the formulas cover the different cases of negative exponents with odd and even characters, satisfying the reciprocity relation $\alpha\beta = \pi^2$. Of the many recognizable corollaries, Ramanujan's formula for $\zeta(2n+1)$ is also recovered.

Chapter 5 of this thesis establishes a character analog of "Entry 20" in Berndt's *Ramanujan's Notebooks, Part IV* [5], where again Ramanujan's formula for $\zeta(2n+1)$ is recovered.

CHAPTER 2

RAMANUJAN'S NOTEBOOK, PART IV, ENTRY 20

Chapter 2 presents Entry 20 [5, pp. 429-432]. Entry 20 has a function that is shown to have a power series expansion that converges for φ specified. This formula (1.2.7), reduces to Ramanujan's $\zeta(2n+1)$ formula under certain specifications. Details of this, and an outline of the contour integration proof will be given.

2.1 Entry 20

Entry 20. *Let α, β, $t > 0$ with $\alpha\beta = \pi$ and $t = \alpha/\beta$. Let C denote the positively oriented parallelogram with vertices $\pm\, i$ and $\pm\, t$. Let $\varphi(z)$ be an entire function. Let m be a positive integer and define M by $M = m + \frac{1}{2}$. Define for each positive integer n,*

$$f_m(z) = \frac{\varphi(2\beta M z)}{z^{2n+1}(e^{-2\pi M z} - 1)(e^{2\pi i M z/t} - 1)} \tag{2.1.1}$$

and assume that $f_m(z)/M^{2n}$ tends to 0 boundedly on $C \setminus \{\pm i, \pm t\}$ as m tends to ∞, i.e., the functions are uniformly bounded and tend to zero on the parallelogram minus the vertices. This type of convergence on a set is called convergence boundedly. Let B_j, $0 \le j < \infty$, denote the jth Bernoulli number. Then

$$
\begin{aligned}
&\frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta ki)+\varphi(-2\beta ki)}{k^{2n+1}(e^{2k\beta^2}-1)} + \frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta ki)}{k^{2n+1}} \\
&\qquad - \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k)+\varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2}-1)} - \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k)}{k^{2n+1}} \\
&= -\frac{\pi i}{2} \frac{\varphi^{(2n)}(0)}{(2n)!} + \alpha \sum_{k=0}^{n} (-1)^{n+k} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \beta^{2n-2k} \\
&\quad - i\beta \sum_{k=0}^{n} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \alpha^{2n-2k} \\
&\quad + 2 \sum_{k=0}^{n+1} \frac{\varphi^{(2k)}(0)}{(2k)!} \sum_{j=0}^{n+1-k} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!} \alpha^{2n+2-2k-2j} \beta^{2j}.
\end{aligned}
\tag{2.1.2}
$$

2.2 Proof by Contour Integration

In addition to the given hypotheses, we must add the hypothesis that the entire function φ does not vanish at $z = \{\pm 2\beta ik, \pm 2\beta kt\}$. The function (2.1.1) is meromorphic with simple poles at $z = \pm ik/M$ and $\pm kt/M$ for all positive integer k. There is also a pole of order $2n+3$ at the origin. From complex analysis we have the following Cauchy theorem [11]:

> **Theorem.** *If C is a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points z_k $(k = 1, 2, ..., n)$ interior to C, then*
>
> $$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

In Entry 20, let C be the positively oriented parallelogram with vertices $\pm\, i$ and $\pm\, t$. The function (2.1.1) is analytic within and on C except at a finite number of points,

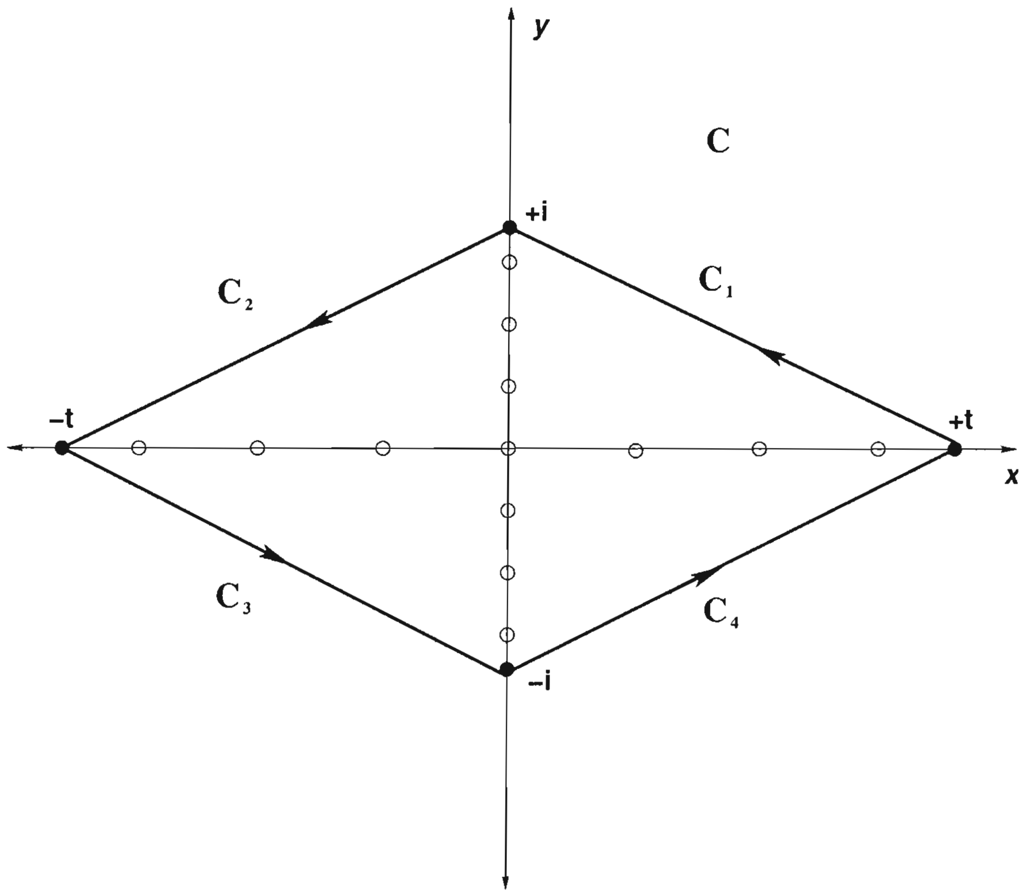

Figure 2.1: Contour Integration Parallelogram, C

namely, $z = \pm ik/M$ and $\pm kt/M$ for $1 \leq k \leq m$. This contour is represented in Figure 2.1. As an example, Figure 2.1 indicates the simple poles inside C and the pole at the origin when $M = \frac{7}{2}$, and $m = 3$.

Let R_{z_k} be the residue at $z = z_k$ for $1 \leq k \leq m$. Then:

$$\begin{aligned}
R_{ik/M} &= \frac{(-1)^n M^{2n} \varphi(2\beta ki)}{2\pi i k^{2n+1}} \left(\frac{1}{e^{2\pi k/t} - 1} + 1 \right), \\
R_{-ik/M} &= \frac{(-1)^n M^{2n} \varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t} - 1)}, \\
R_{kt/M} &= -\frac{M^{2n} \varphi(2\beta kt)}{2\pi i k^{2n+1} t^{2n}} \left(\frac{1}{e^{2\pi kt} - 1} + 1 \right), \\
R_{-kt/M} &= -\frac{M^{2n} \varphi(-2\beta kt)}{2\pi i k^{2n+1} t^{2n} (e^{2\pi kt} - 1)}.
\end{aligned}$$

For the pole of order $2n+3$ at the origin, we use the generating function of the Bernoulli numbers:

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j, \qquad |z| < 2\pi \tag{2.2.1}$$

for the exponential functions in the denominator of (2.1.1). Since φ is entire, it can be expanded in a Taylor series expansion about the origin. We obtain

$$\begin{aligned}
f_m(z) &= \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (2\beta M z)^k \sum_{r=0}^{\infty} \frac{B_r}{r!} (-2\pi M z)^r \sum_{j=0}^{\infty} \frac{B_j}{j!} (2\pi i M z/t)^j \\
&= \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (2\beta M z)^k \\
&\quad \times \left(-\frac{\pi^2 i M^2 z^2}{t} + \pi M z \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!} (2\pi i M z/t)^{2j} \right. \\
&\quad \left. - \frac{\pi i M z}{t} \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} (2\pi M z)^{2r} + \sum_{s=0}^{\infty} \sum_{j=0}^{s} \frac{B_{2j}}{(2j)!} \frac{B_{2s-2j}}{(2s-2j)!} \left(\frac{i}{t} \right)^{2j} (2\pi M z)^{2s} \right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
R_0 = {} & \frac{1}{4}(2\beta M)^{2n}\frac{\varphi^{(2n)}(0)}{(2n)!} \\
& + \frac{1}{2}i(2M)^{2n}\sum_{k=0}^{n}(-1)^{n+k}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\beta^{4n-2k-1} \\
& + \frac{(2M)^{2n}}{2\pi}\beta^{2n+1}\sum_{k=0}^{n}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\alpha^{2n-2k} \\
& + \frac{i(2M)^{2n}}{\pi}\sum_{k=0}^{n+1}\frac{\varphi^{(2k)}(0)}{(2k)!} \\
& \times \sum_{j=0}^{n+1-k}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!}\alpha^{2n+2-2k-2j}\beta^{2n+2j}.
\end{aligned}
$$

By the residue theorem, we have

$$
\begin{aligned}
& \frac{1}{2\pi i(2M)^{2n}}\int_C f_m(z)dz \\
& = \frac{(-1)^n}{2^{2n}2\pi i}\sum_{k=1}^{m}\frac{\varphi(2\beta ki)+\varphi(-2\beta ki)}{k^{2n+1}(e^{2k\beta^2}-1)} + \frac{(-1)^n}{2^{2n}2\pi i}\sum_{k=1}^{m}\frac{\varphi(2\beta ki)}{k^{2n+1}} \\
& \quad - \frac{1}{2\pi i}\left(\frac{\beta}{2\alpha}\right)^{2n}\sum_{k=1}^{m}\frac{\varphi(2\alpha k)+\varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2}-1)} - \frac{1}{2\pi i}\left(\frac{\beta}{2\alpha}\right)^{2n}\sum_{k=1}^{m}\frac{\varphi(2\alpha k)}{k^{2n+1}} \\
& \quad + \frac{1}{4}\beta^{2n}\frac{\varphi^{(2n)}(0)}{(2n)!} + \frac{1}{2}i\sum_{k=0}^{n}(-1)^{n+k}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\beta^{4n-2k-1} \\
& \quad + \frac{1}{2\pi}\beta^{2n+1}\sum_{k=0}^{n}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\alpha^{2n-2k} \\
& + \frac{i}{\pi}\sum_{k=0}^{n+1}\frac{\varphi^{(2k)}(0)}{(2k)!}\sum_{j=0}^{n+1-k}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!}\alpha^{2n+2-2k-2j}\beta^{2n+2j}. \qquad (2.2.2)
\end{aligned}
$$

Royden states the bounded convergence theorem as [25]:

> **6. Proposition (Bounded Convergence Theorem):** *Let $\{f_m\}$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number B such that $|f_m| \le B$ for all m and*

all x. If

$$f(x) = \lim_{m\to\infty} f_m(x)$$

for each x in E, then

$$\int_E f = \lim_{m\to\infty} \int_E f_m.$$

Now, letting m tend to ∞ and using the bounded convergence theorem, we conclude that the limit on the lefthand side of (2.2.2) equals 0. Then multiplying both sides by $2\pi i/\beta^{2n}$, we obtain (2.1.2).

2.3 Ramanujan's $\zeta(2n+1)$ Formula

Entry 20 concludes by recovering Ramanujan's $\zeta(2n+1)$ formula (1.2.6) by letting $\varphi(z) \equiv 1$ and replacing α and β by $\sqrt{\alpha}$ and $\sqrt{\beta}$, respectively.

2.4 Bounded Convergence Verification

Verification of the bounded convergence hypothesis, "*assume that $f_m(z)/M^{2n}$ tends to 0 boundedly on $C\backslash\{\pm i, \pm t\}$ as m tends to ∞*" [5] will be given as follows.

From *Entry 20*, we have

$$\begin{aligned}\left|\frac{f_m(z)}{M^{2n}}\right| &= \left|\frac{\varphi(2\beta Mz)}{M^{2n}z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}\right| \\ &= |\varphi(2\beta Mz)|\left|\frac{1}{M^{2n}z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}\right| \to 0.\end{aligned}$$

Since φ is entire, it is bounded on any compact set. Therefore, it suffices to show that every factor in the denominator on the righthand side of (2.1.1) is bounded away from 0 on each side of the parallelogram.

Lemma 2.4.1. $\inf\{|z|^{2n+1} : z \in C\} = (t^2/(t^2+1))^{(2n+1)/2} > 0.$

Proof: Let $z \in C_1$ (see Figure 2.2).

$$|z| = \sqrt{x^2 + \left(\frac{-1}{t}x + 1\right)^2}$$

$$|z|^2 = x^2 + \left(\frac{-1}{t}x + 1\right)^2 \tag{2.4.1}$$

Since equation (2.4.1) is a quadratic, we can obtain the minimum by using the formula for the axis of symmetry:

$$x = \frac{-\left(\frac{-2}{t}\right)}{2\left(1+\frac{1}{t^2}\right)} = \left(\frac{t}{t^2+1}\right)$$

Therefore,

$$|z|^2 = \left(1+\frac{1}{t^2}\right)\left(\frac{t}{t^2+1}\right)^2 - \frac{2}{t}\left(\frac{t}{t^2+1}\right) + 1 = \frac{t^2}{t^2+1}$$

and

$$|z|^{2n+1} = \left(t^2/(t^2+1)\right)^{(2n+1)/2} > 0.$$

Similarly for the other three sides of the parallelogram. This completes the proof of Lemma 2.4.1.

Lemma 2.4.2. $\inf\{|e^{-2\pi Mz} - 1| \ : \ z \in \ C, \ m \in \mathbb{Z}^+, \ M = m + \frac{1}{2}\} > 0.$

Proof: We will begin by parameterizing the parallelogram as illustrated in Figure 2.2. On each side of the contour, we will show that $|e^{-2\pi Mz} - 1|$ is bounded away from zero. We need to determine the values of the $\cos(2\pi M\xi)$ close to the vertices of the parallelogram for $0 \leq \xi \leq 1$. For each side of the contour, we will have two cases, breaking up the interval $\xi \in [0,1]$ into two sections. For sides C_1 and C_3, the interval will be split into $0 \leq \xi < 1 - \frac{1}{4M}$ and $1 - \frac{1}{4M} \leq \xi \leq 1$. For sides C_2 and C_4, the interval will be split into $0 \leq \xi \leq \frac{1}{4M}$ and $\frac{1}{4M} < \xi \leq 1$.

Let $z \in C_1$. We have $z = (1-\xi)t + \xi i$, $0 \leq \xi \leq 1$. Suppose that $0 \leq \xi < 1 - 1/(4M)$. Then, since $|z_1 - z_2| \geq |z_1| - |z_2|$ for complex numbers z_1 and z_2, we have

$$\left|e^{-2\pi Mz} - 1\right| = \left|1 - e^{-2\pi M(1-\xi)t}e^{-2\pi M\xi i}\right| \geq 1 - e^{-2\pi M(1-\xi)t} \geq 1 - e^{-\pi t/2},$$

which is bounded away from 0 because $t > 0$ is fixed.

Now suppose $1 - 1/(4M) \leq \xi \leq 1$. Then $\cos(2\pi M\xi) \leq 0$. Hence

$$\begin{aligned}
\left|e^{-2\pi Mz} - 1\right| &= \left|e^{-2\pi M(1-\xi)t}e^{-2\pi M\xi i} - 1\right| \\
&= \left|e^{-2\pi M(1-\xi)t}(\cos(2\pi M\xi) - i\sin(2\pi M\xi)) - 1\right| \\
&= \left|(e^{-2\pi M(1-\xi)t}\cos(2\pi M\xi) - 1) + e^{-2\pi M(1-\xi)t} i \sin(2\pi M\xi)\right| \\
&= \sqrt{(e^{-2\pi M(1-\xi)t}\cos(2\pi M\xi) - 1)^2 + e^{-4\pi M(1-\xi)t}\sin^2(2\pi M\xi)} \\
&= \sqrt{1 + e^{-4\pi M(1-\xi)t} - 2e^{-2\pi M(1-\xi)t}\cos(2\pi M\xi)} \\
&\geq \sqrt{1 + e^{-4\pi M(1-\xi)t}} \\
&\geq 1.
\end{aligned}$$

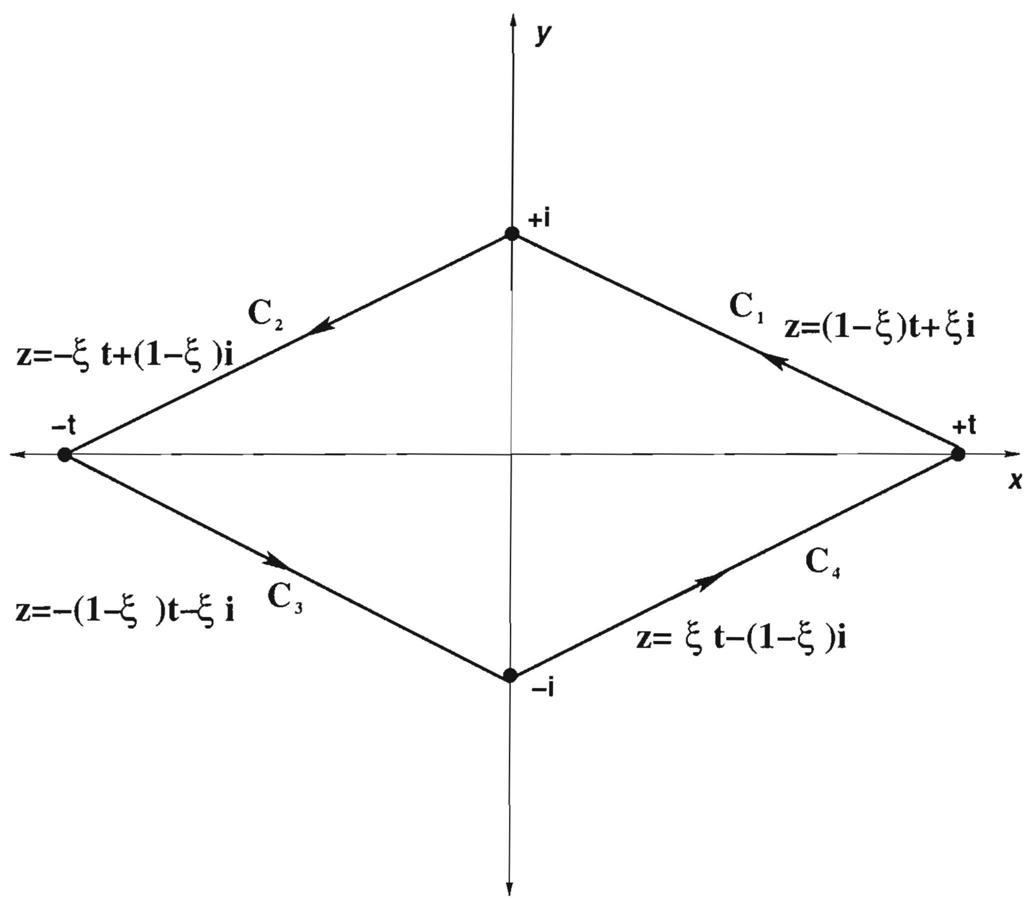

Figure 2.2: Parameterized Contour, $0 \leq \xi \leq 1$

Let $z \in C_2$. We have $z = -\xi t + (1-\xi)i,\ 0 \leq \xi \leq 1$. Suppose that $\xi > 1/(4M)$. Then

$$\begin{aligned} \left|e^{-2\pi M z} - 1\right| &= \left|e^{2\pi M\xi t} e^{-2\pi M(1-\xi)i} - 1\right| \\ &\geq e^{2\pi M \xi t} - 1 \\ &\geq e^{\pi t/2} - 1 \\ &> 0. \end{aligned}$$

Now suppose $0 \leq \xi \leq 1/(4M)$. Then $\cos(2\pi M\xi) \geq 0$, hence

$$\begin{aligned} \left|e^{-2\pi M z} - 1\right| &= \left|e^{2\pi M\xi t} e^{-2\pi M(1-\xi)i} - 1\right| \\ &= \left|e^{2\pi M\xi t} e^{-2\pi M i} e^{2\pi M \xi i} - 1\right| \\ &= \left| - e^{2\pi M\xi t} e^{2\pi M \xi i} - 1\right| \\ &= \left|1 + e^{2\pi M\xi t}(\cos(2\pi M\xi) + i\sin(2\pi M\xi))\right| \\ &= \sqrt{(1 + e^{2\pi M\xi t}\cos(2\pi M\xi))^2 + e^{4\pi M\xi t}\sin^2(2\pi M\xi)} \\ &= \sqrt{1 + e^{4\pi M\xi t} + 2e^{2\pi M\xi t}\cos(2\pi M\xi)} \\ &\geq \sqrt{1 + e^{4\pi M\xi t}} \\ &\geq 1. \end{aligned}$$

Similar arguments can be constructed for sides C_3 and C_4. Therefore, $\inf\{|e^{-2\pi M z} - 1| \ :\ z \in \mathrm{C},\ m \in \mathbb{Z}^+,\ M = m + \frac{1}{2}\} > 0$. This completes the proof of Lemma 2.4.2.

Lemma 2.4.3. $\inf\{|e^{2\pi i M z/t} - 1| \ :\ z \in \mathrm{C},\ m \in \mathbb{Z}^+,\ M = m + \frac{1}{2}\} > 0.$

Proof: We will use the same parameterized parallelogram as illustrated in Figure 2.2. For each side of the contour, we will have two cases, breaking up the interval $\xi \in [0,1]$ into two sections. For sides C_1 and C_3, the interval will be split into $0 \leq \xi < 1/(4M)$ and $1/(4M) \leq \xi \leq 1$. For sides C_2 and C_4, the interval will be split into $0 \leq \xi \leq 1 - 1/(4M)$ and $1 - 1/(4M) < \xi \leq 1$.

Let $z \in C_1$. We have $z = (1-\xi)t + \xi i$, $0 \leq \xi \leq 1$. Suppose that $\xi > 1/(4M)$. Then

$$\begin{aligned}
\left|e^{2\pi i M z/t} - 1\right| &= \left|1 - e^{2\pi i M(1-\xi)} e^{-2\pi M \xi/t}\right| \\
&\geq 1 - e^{-2\pi M\xi/t} \\
&\geq 1 - e^{-\pi/2t} \\
&> 0.
\end{aligned}$$

Now suppose $0 \leq \xi < 1/(4M)$. Then $\cos(2\pi M\xi) \geq 0$, hence

$$\begin{aligned}
\left|e^{2\pi i M z/t} - 1\right| &= \left|e^{2\pi i M(1-\xi)} e^{-2\pi M\xi/t} - 1\right| \\
&= \left|e^{2\pi i M} e^{-2\pi M i \xi} e^{-2\pi M\xi/t} - 1\right| \\
&= \left|-e^{-2\pi i M\xi} e^{-2\pi M\xi/t} - 1\right| \\
&= \left|1 + e^{-2\pi M\xi/t}(\cos(2\pi M\xi) - i\sin(2\pi M\xi))\right| \\
&= \sqrt{(1 + e^{-2\pi M\xi/t}\cos(2\pi M\xi))^2 + e^{-4\pi M\xi/t}\sin^2(2\pi M\xi)} \\
&= \sqrt{1 + e^{-4\pi M\xi/t} + 2e^{-2\pi M\xi/t}\cos(2\pi M\xi)} \\
&\geq \sqrt{1 + e^{-4\pi M\xi/t}} \\
&\geq 1.
\end{aligned}$$

Let $z \in C_2$. We have $z = -\xi t + (1-\xi)i$, $0 \le \xi \le 1$. Suppose that $0 \le \xi \le 1 - 1/(4M)$. Then

$$\begin{aligned}
\left|e^{2\pi i M z/t} - 1\right| &= \left|1 - e^{-2\pi i M\xi} e^{-2\pi M(1-\xi)/t}\right| \\
&\ge 1 - e^{-2\pi M(1-\xi)/t} \\
&\ge 1 - e^{-\pi/2t} \\
&> 0.
\end{aligned}$$

Now suppose $\xi > 1 - 1/(4M)$. Then $\cos(2\pi M\xi) \le 0$, hence

$$\begin{aligned}
\left|e^{2\pi i M z/t} - 1\right| &= \left|e^{-2\pi i M\xi} e^{-2\pi M(1-\xi)/t} - 1\right| \\
&= \left|e^{-2\pi M(1-\xi)/t}(\cos(2\pi M\xi) - i\sin(2\pi M\xi)) - 1\right| \\
&= \left|(e^{-2\pi M(1-\xi)/t}\cos(2\pi M\xi) - 1) + i\sin(2\pi M\xi)\right| \\
&= \sqrt{(1 - e^{2\pi M\xi/t}\cos(2\pi M\xi))^2 + e^{-4\pi M(1-\xi)/t}\sin^2(2\pi M\xi)} \\
&= \sqrt{1 + e^{-4\pi M(1-\xi)/t} - 2e^{-2\pi M(1-\xi)/t}\cos(2\pi M\xi)} \\
&\ge \sqrt{1 + e^{-4\pi M(1-\xi)/t}} \\
&\ge 1.
\end{aligned}$$

Similar arguments can be constructed for sides C_3 and C_4. This completes the proof of Lemma (2.4.3). Therefore, the bounded convergence hypothesis holds.

CHAPTER 3

MITTAG-LEFFLER'S PARTIAL FRACTION METHOD

In this chapter, a new proof of Entry 20 will be given by Mittag-Leffler's partial fraction expansion method. The hypotheses will be less restrictive. Residue calculus will be employed to calculate the principal parts of the function. The function will be expanded into powers of z. From this expansion, the coefficient of z^p will be set equal to the corresponding coefficient of the Laurent expansion, proving the result (2.1.2). It will be shown that under suitable conditions, Ramanujan's formula for $\zeta(2n+1)$ and Euler's formula for $\zeta(2n)$ are recovered.

3.1 Proof by Partial Fraction Expansion

In this and subsequent chapters, let $D_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$, where $r > 0$ and $a \in \mathbb{C}$.

Theorem 3.1.1. *Let α, β, $t > 0$ with $\alpha\beta = \pi$ and let $t = \alpha/\beta$. For $\varepsilon > 0$, let $D_\varepsilon = \bigcup_{k \in \mathbb{Z}} \{D_\varepsilon\left(\frac{kt}{M}\right) \cup D_\varepsilon\left(\frac{ik}{M}\right)\}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be entire and does not vanish at $z = \{\pm 2\beta ik, \pm 2\beta kt\}$. Let m be a positive integer and let $M = m + \frac{1}{2}$. Define for each positive integer n,*

$$f_m(z) = \frac{\varphi(2\beta M z)}{z^{2n+1}(e^{-2\pi M z} - 1)(e^{2\pi i M z/t} - 1)}$$

and assume that

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(z) = 0 \tag{3.1.1}$$

for some $0 < \varepsilon < \min\left(\frac{t}{4M}, \frac{1}{4M}\right)$. *Also, assume that*

$$\sum_{0\neq k\in\mathbb{Z}} \left| \frac{\varphi(2\beta i k)}{k^{2n+1}} \right| < \infty$$

and

$$\sum_{0\neq k\in\mathbb{Z}} \left| \frac{\varphi(2\beta k t)}{k^{2n+1}} \right| < \infty.$$

Then

$$\begin{aligned}
&\frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta k i) + \varphi(-2\beta k i)}{k^{2n+1}(e^{2k\beta^2} - 1)} + \frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta k i)}{k^{2n+1}} \\
&\qquad - \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k) + \varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2} - 1)} - \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k)}{k^{2n+1}} \\
&= -\frac{\pi i}{2} \frac{\varphi^{(2n)}(0)}{(2n)!} + \alpha \sum_{k=0}^{n} (-1)^{n+k} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \beta^{2n-2k} \\
&- i\beta \sum_{k=0}^{n} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \alpha^{2n-2k} \\
&\qquad + 2 \sum_{k=0}^{n+1} \frac{\varphi^{(2k)}(0)}{(2k)!} \sum_{j=0}^{n+1-k} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!} \alpha^{2n+2-2k-2j} \beta^{2j}.
\end{aligned} \tag{3.1.2}$$

The proof of Theorem 3.1.1 depends on the following partial fraction expansion:

Theorem 3.1.2. *The partial fraction expansion of $f_m(z)$ is given by*

$$\begin{aligned}
f_m(z) = &\frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ki)}{k^{2n+1}}\left(\frac{1}{e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-\frac{ik}{M}}\right)\\
&+\frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(-2\beta ki)}{k^{2n+1}(e^{2\pi k/t}-1)}\left(\frac{1}{z+\frac{ik}{M}}\right)\\
&-\frac{M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(2\beta kt)}{k^{2n+1}t^{2n}}\left(\frac{1}{e^{2\pi kt}-1}+1\right)\left(\frac{1}{z-\frac{kt}{M}}\right)\\
&-\frac{M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(-2\beta kt)}{k^{2n+1}t^{2n}(e^{2\pi kt}-1)}\left(\frac{1}{z+\frac{kt}{M}}\right)\\
&+\frac{1}{4}\sum_{k=0}^{2n}\frac{\varphi^{(k)}(0)}{k!}\left(\frac{(2\beta M)^k}{z^{2n+1-k}}\right)\\
&+\frac{it}{4\pi M}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{B_{2k-2j}}{(2k-2j)!}\left(\frac{\pi i}{t}\right)^{2k-2j}\\
&\quad\times\left(\frac{(2M)^{2k+1}}{z^{2n+1-2k}}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1}+\frac{(2M)^{2k}}{z^{2n+2-2k}}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\right)\\
&+\frac{1}{4\pi M}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{B_{2k-2j}}{(2k-2j)!}\pi^{2k-2j}\\
&\quad\times\left(\frac{(2M)^{2k+1}}{z^{2n+1-2k}}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1}+\frac{(2M)^{2k}}{z^{2n+2-2k}}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\right)\\
&+\frac{it}{(2\pi M)^2}\sum_{k=0}^{n+1}\sum_{j=0}^{k}\frac{(2M)^{2k}}{z^{2n+3-2k}}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\pi^{2k-2j}\\
&\quad\times\sum_{l=0}^{k-j}\frac{B_{2k-2j-2l}}{(2k-2j-2l)!}\frac{B_{2l}}{(2l)!}\left(\frac{i}{t}\right)^{2l}\\
&+\frac{it}{(2\pi M)^2}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(2M)^{2k+1}}{z^{2n+2-2k}}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1}\pi^{2k-2j}\\
&\quad\times\sum_{l=0}^{k-j}\frac{B_{2k-2j-2l}}{(2k-2j-2l)!}\frac{B_{2l}}{(2l)!}\left(\frac{i}{t}\right)^{2l}.
\end{aligned}\qquad(3.1.3)$$

Proof. E. Hille [15] states the Mittag-Leffler theorem as:

> **Theorem 8.5.2.** *Given a sequence of distinct complex numbers $z_0,\ z_1,\ z_2,\ \ldots$ having no limit point in the finite plane, and given a sequence of polynomials $Q_{z_0}, Q_{z_1}, Q_{z_2}, \ldots$ which may be distinct or equal. Given $Q_{z_p}(0) = 0$, for each $p = 0, 1, 2, \ldots$. Then there exists a meromorphic function f having the principal parts*
>
> $$Q_{z_p}\left(\frac{1}{z - z_p}\right) \text{ at } z = z_p,\ p = 0, 1, 2, \ldots.$$
>
> *Assuming $z_0 = 0$, such a function may be given the form*
>
> $$f(z) = Q_0\left(\frac{1}{z}\right) + \sum_{p=1}^{\infty}\left\{Q_{z_p}\left(\frac{1}{z - z_p}\right) - \sum_{j=0}^{j_p} A_{p,j} z^j\right\}, \tag{3.1.4}$$
>
> *where*
>
> $$Q_{z_p}\left(\frac{1}{z - z_p}\right) \equiv \sum_{j=0}^{\infty} A_{p,j} z^j \text{ for } |z| < |z_p|.$$
>
> *It is possible to choose the sequence of integers $\{j_p\}$ in such a manner that the series (3.1.4) converges absolutely and uniformly on compact sets not containing any of the poles. The most general meromorphic function with these poles and principal parts is obtained by adding an arbitrary entire function to f.*

To find the partial fraction expansion of $f_m(z)$ it is necessary to calculate the principal part at each pole. At simple poles, we have the formula:

$$\text{If } A(z_0) \neq 0 = B(z_0) \neq B'(z_0), \text{ then } \operatorname*{Res}_{z=z_0} \frac{A(z)}{B(z)} = \frac{A(z_0)}{B'(z_0)}.$$

Let R_a denote the residue of $f_m(z)$ at $z = a$. For $k \in \mathbb{Z}^+$ the residues at the simple poles are:

$$\begin{aligned}
R_{ik/M} &= \frac{(-1)^n M^{2n}\varphi(2\beta ki)}{2\pi i k^{2n+1}}\left(\frac{1}{e^{2\pi k/t}-1}+1\right),\\
R_{-ik/M} &= \frac{(-1)^n M^{2n}\varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t}-1)},\\
R_{kt/M} &= -\frac{M^{2n}\varphi(2\beta kt)}{2\pi i k^{2n+1}t^{2n}}\left(\frac{1}{e^{2\pi kt}-1}+1\right),\\
R_{-kt/M} &= -\frac{M^{2n}\varphi(-2\beta kt)}{2\pi i k^{2n+1}t^{2n}(e^{2\pi kt}-1)}.
\end{aligned}$$

The corresponding principal parts are:

$$\begin{aligned}
Q_{ik/M} &= \frac{(-1)^n M^{2n}\varphi(2\beta ki)}{2\pi i k^{2n+1}}\left(\frac{1}{e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-\frac{ik}{M}}\right),\\
Q_{-ik/M} &= \frac{(-1)^n M^{2n}\varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t}-1)}\left(\frac{1}{z+\frac{ik}{M}}\right),\\
Q_{kt/M} &= -\frac{M^{2n}\varphi(2\beta kt)}{2\pi i k^{2n+1}t^{2n}}\left(\frac{1}{e^{2\pi kt}-1}+1\right)\left(\frac{1}{z-\frac{kt}{M}}\right),\\
Q_{-kt/M} &= -\frac{M^{2n}\varphi(-2\beta kt)}{2\pi i k^{2n+1}t^{2n}(e^{2\pi kt}-1)}\left(\frac{1}{z+\frac{kt}{M}}\right).
\end{aligned}$$

To find the principal part for the pole at the origin of order $2n+3$, we use the generating function for the Bernoulli numbers (2.2.1) and find that $f_m(z)$ has the

Laurent expansion about $z = 0$ given by

$$\frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \sum_{r=0}^{\infty} \frac{B_r}{r!}(-2\pi Mz)^r \sum_{j=0}^{\infty} \frac{B_j}{j!}(2\pi iMz/t)^j$$

$$= \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \left(-\frac{\pi^2 iM^2 z^2}{t} + \pi Mz \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!}(2\pi iMz/t)^{2j} \right.$$

$$\left. -\frac{\pi iMz}{t} \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!}(2\pi Mz)^{2r} + \sum_{s=0}^{\infty}(2\pi Mz)^{2s} \sum_{j=0}^{s} \frac{B_{2j}}{(2j)!}\frac{B_{2s-2j}}{(2s-2j)!}\left(\frac{i}{t}\right)^{2j} \right)$$

$$= \frac{1}{4z^{2n+1}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k$$

$$+ \frac{it}{4\pi Mz^{2n+2}} \left(\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \right) \left(\sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!}\left(\frac{2\pi iMz}{t}\right)^{2j} \right)$$

$$+ \frac{1}{4\pi Mz^{2n+2}} \left(\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \right) \left(\sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!}(2\pi Mz)^{2r} \right)$$

$$+ \frac{it}{(2\pi M)^2 z^{2n+3}} \left(\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \right)$$

$$\times \left(\sum_{s=0}^{\infty}(2\pi Mz)^{2s} \sum_{j=0}^{s} \frac{B_{2j}}{(2j)!}\frac{B_{2s-2j}}{(2s-2j)!}\left(\frac{i}{t}\right)^{2j} \right).$$

The principal part of $f_m(z)$ at $z = 0$ is:

$$
\begin{aligned}
Q_0 = &\sum_{k=0}^{2n} \frac{1}{4}(2\beta M)^k \frac{\varphi^{(k)}(0)}{k!}\left(\frac{1}{z^{2n+1-k}}\right) \\
&+ \frac{it}{4\pi M}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{B_{2k-2j}}{(2k-2j)!}\left(\frac{\pi i}{t}\right)^{2k-2j} \\
&\quad \times \left(\frac{1}{z^{2n+1-2k}}(2M)^{2k+1}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1} + \frac{1}{z^{2n+2-2k}}(2M)^{2k}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\right) \\
&+ \frac{1}{4\pi M}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{B_{2k-2j}}{(2k-2j)!}\pi^{2k-2j} \\
&\quad \times \left(\frac{1}{z^{2n+1-2k}}(2M)^{2k+1}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1} + \frac{1}{z^{2n+2-2k}}(2M)^{2k}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\right) \\
&+ \frac{it}{(2\pi M)^2}\sum_{k=0}^{n+1}\sum_{j=0}^{k}\frac{(2M)^{2k}}{z^{2n+3-2k}}\frac{\varphi^{(2j)}(0)}{(2j)!}\beta^{2j}\pi^{2k-2j} \\
&\quad \times \sum_{l=0}^{k-j}\frac{B_{2k-2j-2l}}{(2k-2j-2l)!}\frac{B_{2l}}{(2l)!}\left(\frac{i}{t}\right)^{2l} \\
&+ \frac{it}{(2\pi M)^2}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(2M)^{2k+1}}{z^{2n+2-2k}}\frac{\varphi^{(2j+1)}(0)}{(2j+1)!}\beta^{2j+1}\pi^{2k-2j} \\
&\quad \times \sum_{l=0}^{k-j}\frac{B_{2k-2j-2l}}{(2k-2j-2l)!}\frac{B_{2l}}{(2l)!}\left(\frac{i}{t}\right)^{2l}.
\end{aligned}
\tag{3.1.5}
$$

It will be shown that the sum of the principal parts, $\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)$, converges uniformly on compact subsets by the Weierstrass M-test.

First,

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(Q_{\frac{ik}{M}} + Q_{\frac{-ik}{M}}\right) \\
&= \frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\varphi(2\beta ik)\left(\frac{1}{e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-\frac{ik}{M}}\right)\right. \\
&\qquad \left. +\varphi(-2\beta ik)\left(\frac{1}{e^{2\pi k/t}-1}\right)\left(\frac{1}{z+\frac{ik}{M}}\right)\right) \\
&= \frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{\varphi(2\beta ik)}{e^{2\pi k/t}-1}\left(\frac{1}{z-\frac{ik}{M}}\right)+\frac{\varphi(-2\beta ik)}{e^{2\pi k/t}-1}\left(\frac{1}{z+\frac{ik}{M}}\right)\right) \\
&\qquad + \frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ik)}{k^{2n+1}}\left(\frac{1}{z-\frac{ik}{M}}\right) \\
&= \frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{\left(\varphi(2\beta ik)+\varphi(-2\beta ik)\right)z+\left(\varphi(2\beta ik)-\varphi(-2\beta ik)\right)\frac{ik}{M}}{(e^{2\pi k/t}-1)(z^2+\left(\frac{ik}{M}\right)^2)}\right) \\
&\qquad + \frac{(-1)^n M^{2n}}{2\pi i}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ik)}{k^{2n+1}}\left(\frac{1}{z-\frac{ik}{M}}\right).
\end{aligned}
$$

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4M}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$ and $|z| \leq N$,

$$\left| \sum_{k=2NM}^{\infty} \left(Q_{\frac{ik}{M}} + Q_{\frac{-ik}{M}} \right) \right|$$

$$\leq \left| \frac{(-1)^n M^{2n}}{2\pi i} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \left(\frac{(\varphi(2\beta ik) + \varphi(-2\beta ik))\, z + (\varphi(2\beta ik) - \varphi(-2\beta ik))\, \frac{ik}{M}}{(e^{2\pi k/t} - 1)(z^2 + \left(\frac{ik}{M}\right)^2)} \right) \right|$$

$$+ \left| \frac{(-1)^n M^{2n}}{2\pi i} \sum_{k=2NM}^{\infty} \frac{\varphi(2\beta ik)}{k^{2n+1}} \right| \left| \left(\frac{1}{z - \frac{ik}{M}} \right) \right|$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \frac{|(\varphi(2\beta ik) + \varphi(-2\beta ik))z| + (|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|)|\frac{ik}{M}|}{(e^{2\pi k/t} - 1)|(z^2 + \left(\frac{ik}{M}\right)^2)|}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \left| \left(\frac{1}{z - \frac{ik}{M}} \right) \right|$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \frac{(|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|)}{(e^{2\pi k/t} - 1)} \frac{(|z| + |\frac{ik}{M}|)}{|(z^2 + \left(\frac{ik}{M}\right)^2)|}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \left| \left(\frac{1}{z - \frac{ik}{M}} \right) \right|$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \frac{|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|}{(e^{2\pi k/t} - 1)} \frac{N + \frac{k}{M}}{-N^2 + \left(\frac{k}{M}\right)^2}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \left| \frac{1}{N - \frac{k}{M}} \right|$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \frac{|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|}{(e^{2\pi k/t} - 1)} \frac{1}{\frac{k}{M} - N}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \left| \frac{1}{\frac{k}{M} - N} \right|$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \frac{|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|}{(e^{2\pi k/t} - 1)} \frac{1}{\frac{k}{M}\left(1 - N/\frac{k}{M}\right)}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \frac{1}{\frac{k}{M}\left(1 - N/\frac{k}{M}\right)}$$

$$\leq \frac{M^{2n+1}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+2}} \frac{|\varphi(2\beta ik)| + |\varphi(-2\beta ik)|}{(e^{2\pi k/t} - 1)} \frac{2M}{k}$$

$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{|\varphi(2\beta ik)|}{k^{2n+1}} \frac{2M}{k} < \infty, \text{ since } \frac{k}{M} \geq 2N.$$

Therefore, $\sum_{k=1}^{\infty}\left(Q_{\frac{ik}{M}} + Q_{\frac{-ik}{M}}\right)$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Next,

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(Q_{\frac{kt}{M}} + Q_{\frac{-kt}{M}}\right) \\
&= \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \left(\varphi(2\beta kt)\left(\frac{1}{e^{2\pi kt}-1}+1\right)\left(\frac{1}{z-\frac{kt}{M}}\right)\right. \\
&\qquad \left. + \varphi(-2\beta kt)\left(\frac{1}{e^{2\pi kt}-1}\right)\left(\frac{1}{z+\frac{kt}{M}}\right)\right) \\
&= \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \left(\frac{\varphi(2\beta tk)}{e^{2\pi kt}-1}\left(\frac{1}{z-\frac{kt}{M}}\right) + \frac{\varphi(-2\beta kt)}{e^{2\pi kt}-1}\left(\frac{1}{z+\frac{kt}{M}}\right)\right) \\
&\qquad + \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta kt)}{k^{2n+1}}\left(\frac{1}{z-\frac{kt}{M}}\right) \\
&= \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \\
&\qquad \times \left(\frac{(\varphi(2\beta kt)+\varphi(-2\beta kt))\, z + (\varphi(2\beta kt)-\varphi(-2\beta kt))\, \frac{kt}{M}}{(e^{2\pi kt}-1)(z^2+\left(\frac{kt}{M}\right)^2)}\right) \\
&\qquad + \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta kt)}{k^{2n+1}}\left(\frac{1}{z-\frac{kt}{M}}\right).
\end{aligned}
$$

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4M}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$ and $|z| \leq N$,

$$\left| \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \left(Q_{\frac{kt}{M}} + Q_{\frac{-kt}{M}} \right) \right|$$

$$\leq \left| \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+1}} \left(\frac{(\varphi(2\beta kt) + \varphi(-2\beta kt))\, z + (\varphi(2\beta kt) - \varphi(-2\beta kt))\, \frac{kt}{M}}{(e^{2\pi kt} - 1)(z^2 + \left(\frac{kt}{M}\right)^2)} \right) \right|$$

$$+ \left| \frac{-M^{2n}}{2\pi i t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{\varphi(2\beta kt)}{k^{2n+1}} \right| \left| \frac{1}{z - \frac{kt}{M}} \right|$$

$$\leq \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+1}} \frac{|(\varphi(2\beta kt) + \varphi(-2\beta kt))z| + (|\varphi(2\beta kt)| + |\varphi(-2\beta kt)|)|\frac{kt}{M}|}{(e^{2\pi kt} - 1)|(z^2 + \left(\frac{kt}{M}\right)^2)|}$$

$$+ \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{|\varphi(2\beta kt)|}{k^{2n+1}} \left| \frac{1}{z - \frac{kt}{M}} \right|$$

$$\leq \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+1}} \frac{(|\varphi(2\beta kt)| + |\varphi(-2\beta kt)|)}{(e^{2\pi kt} - 1)} \frac{(|z| + |\frac{kt}{M}|)}{|(z^2 + \left(\frac{kt}{M}\right)^2)|}$$

$$+ \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{|\varphi(2\beta kt)|}{k^{2n+1}} \left| \frac{1}{z - \frac{kt}{M}} \right|$$

$$\leq \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+1}} \frac{|\varphi(2\beta kt)| + |\varphi(-2\beta kt)|}{(e^{2\pi kt} - 1)} \frac{N + \frac{kt}{M}}{\left(\frac{kt}{M}\right)^2 - N^2}$$

$$+ \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{|\varphi(2\beta kt)|}{k^{2n+1}} \left| \frac{1}{N - \frac{kt}{M}} \right|$$

$$\leq \frac{M^{2n}}{2\pi t^{2n+1}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+1}} \frac{|\varphi(2\beta kt)| + |\varphi(-2\beta kt)|}{(e^{2\pi kt} - 1)} \frac{1}{\frac{kt}{M} - N}$$

$$+ \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{|\varphi(2\beta kt)|}{k^{2n+1}} \left| \frac{1}{\frac{kt}{M} - N} \right|$$

$$\leq \frac{M^{2n+1}}{2\pi t^{2n+1}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{1}{k^{2n+2}} \frac{|\varphi(2\beta kt)| + |\varphi(-2\beta kt)|}{(e^{2\pi kt} - 1)} \frac{2M}{kt}$$

$$+ \frac{M^{2n}}{2\pi t^{2n}} \sum_{k=\lfloor 2NM/t \rfloor+1}^{\infty} \frac{|\varphi(2\beta kt)|}{k^{2n+1}} \frac{2M}{kt} < \infty.$$

Therefore, $\sum_{k=1}^{\infty}\left(Q_{\frac{kt}{M}} + Q_{\frac{-kt}{M}}\right)$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Clearly,

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) = 0 \tag{3.1.6}$$

because uniform convergence allows us to interchange the limit and the summation.

By Mittag-Leffler (Theorem 8.5.2), there exists an entire function g such that

$$f_m(z) = \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) + g(z).$$

It will be shown that $g(z) = 0$, by the assumption on $f_m(z)$ and the fact that $\sum_p Q_p\left(\frac{1}{z-z_p}\right)$ converges uniformly on compact sets. First, we will show that g is bounded. Hence, g is constant by Liouville's theorem. Then taking the limit as $|z|$ approaches infinity, it will be shown that $g = 0$.

Now, it will be shown that g is bounded on $\mathbb{C} \setminus D_\varepsilon$. We first look at the two strips $-\frac{t}{2M} \leq \operatorname{Re} z \leq t/(2M)$ and $-1/(2M) \leq \operatorname{Im} z \leq 1/(2M)$. We will show that g is bounded in each strip. Since g is entire, g is analytic and continuous in these strips. Also since g is entire, g assumes it maximum on the boundary of any connected compact set, by the Maximum Modulus Principle.

For the vertical strip in Figure 3.1, we have $-t/(2M) \leq \operatorname{Re} z \leq t/(2M)$. For $k \in \mathbb{Z}^+$, we examine E_k, the compact region with the boundary $-t/(2M) \leq \operatorname{Re} z \leq t/(2M)$ and $(k-1/2)/M \leq y \leq (k+1/2)/M$. Since the boundary of E_k avoids the poles, $|f_m(z)|$ is uniformly bounded by (3.1.1). Also, $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ is bounded on

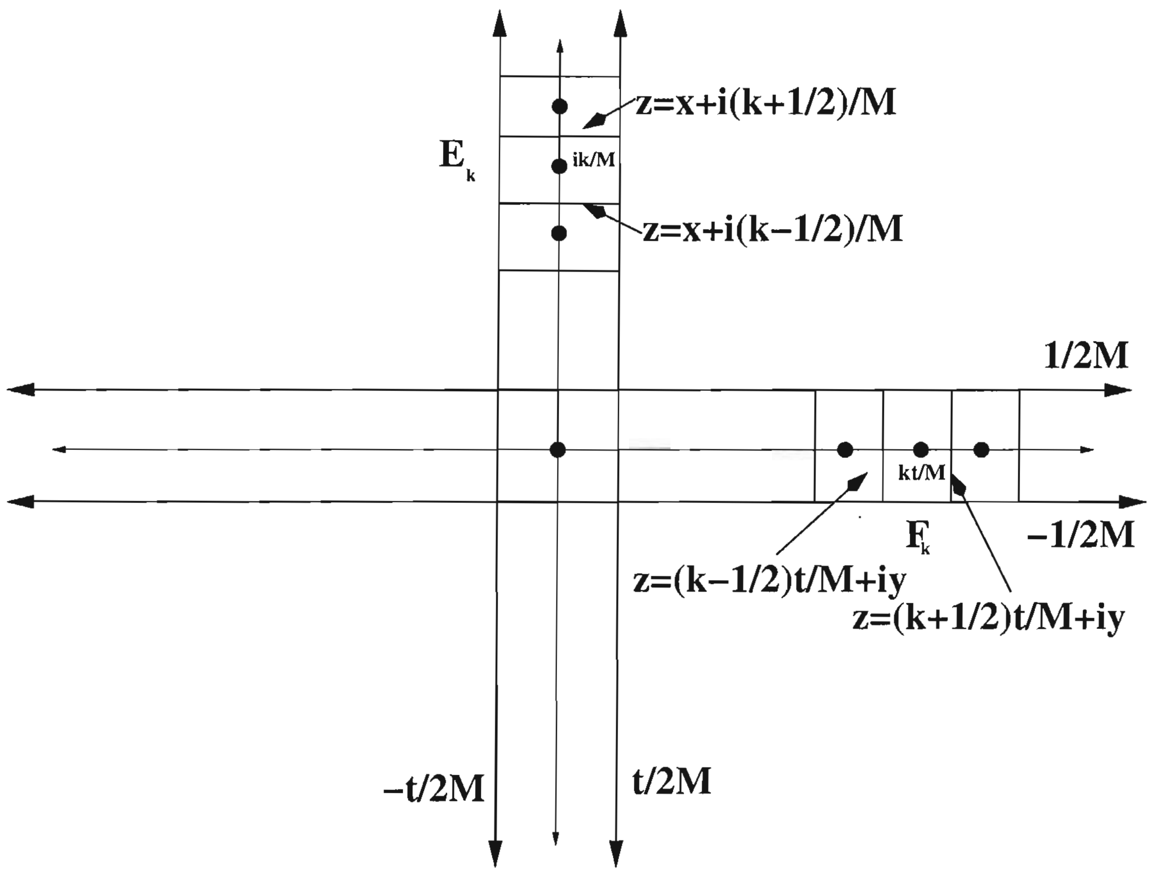

Figure 3.1: Contour Strips: $-\frac{t}{2M} \leq \operatorname{Re} z \leq \frac{t}{2M}$ and $-\frac{1}{2M} \leq \operatorname{Im} z \leq \frac{1}{2M}$

the boundary of E_k by (3.1.6). Since k is arbitrary, $|f_m(z)|$ and $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ are bounded on the entire vertical strip. Therefore g, their difference, is likewise bounded on the entire vertical strip.

For the horizontal strip in Figure 3.1, we have $-1/(2M) \leq \operatorname{Im} z \leq 1/(2M)$. For $k \in \mathbb{Z}^+$, we examine F_k, the compact region with the boundary $(k-1/2)t/M \leq \operatorname{Re} z \leq (k+1/2)t/M$ and $-1/(2M) \leq y \leq 1/(2M)$. Since the boundary of F_k avoids the poles, $|f_m(z)|$ is uniformly bounded by (3.1.1). Also, $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ is bounded on the boundary of F_k by (3.1.6). Since k is arbitrary, $|f_m(z)|$ and $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ are bounded on the entire horizontal strip. Therefore g, their difference, is likewise bounded on the entire horizontal strip.

Now inside any disc of radius R, minus the strips, g is obviously bounded. Since g is bounded on the strips, g is bounded inside the disc and strips. Outside the disc and strips, g is bounded by (3.1.1) and (3.1.6). Therefore, g is bounded on $\mathbb{C}$. Hence, by Liouville's theorem, g is constant.

Now take the limit as $|z|$ approaches infinity,

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(z) = \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) + \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} g(z).$$

By the hypothesis (3.1.1), $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(z) = 0$. By uniform convergence and (3.1.6), $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) = 0$. Therefore, $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} g(z) = 0$. Since g is constant, it follows that $g \equiv 0$. Hence, $f_m(z)$ has the partial fraction expansion (3.1.3). This completes the proof of Theorem 3.1.2.

Proof of Theorem 3.1.1. To obtain (3.1.2), the partial fraction expansion will be expanded into powers of z. Each of the principal parts are expanded in powers of z as follows:

$$\begin{aligned}
Q_{ik/M} &= \frac{(-1)^n M^{2n}\varphi(2\beta ki)}{2\pi i k^{2n+1}} \left(\frac{1}{e^{2\pi k/t}-1}+1\right)\left((-1)\sum_{j=0}^{\infty}\left(\frac{M}{ik}\right)^{j+1} z^j\right), \\
Q_{-ik/M} &= \frac{(-1)^n M^{2n}\varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t}-1)} \left(\sum_{j=0}^{\infty}(-1)^j\left(\frac{M}{ik}\right)^{j+1} z^j\right), \\
Q_{kt/M} &= -\frac{M^{2n}\varphi(2\beta kt)}{2\pi i k^{2n+1}t^{2n}} \left(\frac{1}{e^{2\pi kt}-1}+1\right)\left((-1)\sum_{j=0}^{\infty}\left(\frac{M}{kt}\right)^{j+1} z^j\right), \\
Q_{-kt/M} &= -\frac{M^{2n}\varphi(-2\beta kt)}{2\pi i k^{2n+1}t^{2n}(e^{2\pi kt}-1)} \left(\sum_{j=0}^{\infty}(-1)^j\left(\frac{M}{kt}\right)^{j+1} z^j\right).
\end{aligned}$$

We now expand the partial fraction decomposition (3.1.3) in powers of z:

$$
\begin{aligned}
f_m(z) = {} & \sum_{k=1}^{\infty} \frac{(-1)^n M^{2n} \varphi(2\beta k i)}{2\pi i k^{2n+1}} \left(\frac{1}{e^{2\pi k/t} - 1} + 1 \right) \left((-1) \sum_{j=0}^{\infty} \left(\frac{M}{ik} \right)^{j+1} z^j \right) \\
& + \sum_{k=1}^{\infty} \frac{(-1)^n M^{2n} \varphi(-2\beta k i)}{2\pi i k^{2n+1} (e^{2\pi k/t} - 1)} \left(\sum_{j=0}^{\infty} (-1)^j \left(\frac{M}{ik} \right)^{j+1} z^j \right) \\
& + \sum_{k=1}^{\infty} \frac{(-1) M^{2n} \varphi(2\beta k t)}{2\pi i k^{2n+1} t^{2n}} \left(\frac{1}{e^{2\pi k t} - 1} + 1 \right) \left((-1) \sum_{j=0}^{\infty} \left(\frac{M}{kt} \right)^{j+1} z^j \right) \\
& + \sum_{k=1}^{\infty} \frac{(-1) M^{2n} \varphi(-2\beta k t)}{2\pi i k^{2n+1} t^{2n} (e^{2\pi k t} - 1)} \left(\sum_{j=0}^{\infty} (-1)^j \left(\frac{M}{kt} \right)^{j+1} z^j \right) \\
& + \sum_{k=0}^{2n} \frac{1}{4} (2\beta M)^k \frac{\varphi^{(k)}(0)}{k!} \left(\frac{1}{z^{2n+1-k}} \right) \\
& + \frac{it}{4\pi M} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{B_{2k-2j}}{(2k-2j)!} \left(\frac{\pi i}{t} \right)^{2k-2j} \\
& \quad \times \left(\frac{1}{z^{2n+1-2k}} (2M)^{2k+1} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} + \frac{1}{z^{2n+2-2k}} (2M)^{2k} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \right) \\
& + \frac{1}{4\pi M} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{B_{2k-2j}}{(2k-2j)!} \pi^{2k-2j} \\
& \quad \times \left(\frac{1}{z^{2n+1-2k}} (2M)^{2k+1} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} + \frac{1}{z^{2n+2-2k}} (2M)^{2k} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \right) \\
& + \frac{it}{(2\pi M)^2} \sum_{k=0}^{n+1} \sum_{j=0}^{k} \frac{(2M)^{2k}}{z^{2n+3-2k}} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \pi^{2k-2j} \\
& \quad \times \sum_{l=0}^{k-j} \frac{B_{2k-2j-2l}}{(2k-2j-2l)!} \frac{B_{2l}}{(2l)!} \left(\frac{i}{t} \right)^{2l} \\
& + \frac{it}{(2\pi M)^2} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(2M)^{2k+1}}{z^{2n+2-2k}} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} \pi^{2k-2j} \\
& \quad \times \sum_{l=0}^{k-j} \frac{B_{2k-2j-2l}}{(2k-2j-2l)!} \frac{B_{2l}}{(2l)!} \left(\frac{i}{t} \right)^{2l} .
\end{aligned}
\tag{3.1.7}
$$

On the other hand, we can also write $f_m(z)$ as a Laurent series directly from its definition using (2.2.1). We have

$$f_m(z) = \frac{\varphi(2\beta Mz)}{z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}.$$

Since φ is entire, it can be rewritten as a Taylor series about the origin:

$$\varphi(2\beta Mz) = \sum_{k=0}^{\infty}(2\beta Mz)^k \frac{\varphi^{(k)}(0)}{k!}.$$

So we rewrite $f_m(z)$ as:

$$\begin{aligned} f_m(z) &= \frac{\varphi(2\beta Mz)}{(-2\pi Mz)(2\pi iMz/t)z^{2n+1}} \frac{(-2\pi Mz)}{(e^{-2\pi Mz}-1)} \frac{(2\pi iMz/t)}{(e^{2\pi iMz/t}-1)} \\ &= \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \sum_{j=0}^{\infty} \frac{(-2\pi Mz)^j}{j!} B_j \sum_{l=0}^{\infty} \frac{(2\pi iMz/t)^l}{l!} B_l. \end{aligned} \tag{3.1.8}$$

To obtain the formula stated as the result of the partial fraction expansion method, compare the coefficients of z^p in (3.1.7) and (3.1.8) .

From (3.1.8):

$$\begin{aligned} [z^p]f_m(z) &= [z^{p+2n+3}]\frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(2\beta Mz)^k \sum_{j=0}^{\infty} \frac{(-2\pi Mz)^j}{j!} B_j \sum_{l=0}^{\infty} \frac{(2\pi iMz/t)^l}{l!} B_l \\ &= \frac{it(2M)^{p+2n+1}}{\pi^2} \sum_{k+j+l=p+2n+3} \beta^k \frac{\varphi^{(k)}(0)}{k!} \frac{(-\pi)^j}{j!} B_j \frac{(\pi i/t)^l}{l!} B_l. \end{aligned} \tag{3.1.9}$$

From (3.1.7):

$$
\begin{aligned}
[z^p]f_m(z) = [z^p] &\left(\sum_{k=1}^{\infty} \frac{(-1)^n M^{2n} \varphi(2\beta k i)}{2\pi i k^{2n+1}} \left(\frac{1}{e^{2\pi k/t} - 1} + 1 \right) \left((-1) \sum_{j=0}^{\infty} \left(\frac{M}{ik} \right)^{j+1} z^j \right) \right. \\
&+ \sum_{k=1}^{\infty} \frac{(-1)^n M^{2n} \varphi(-2\beta k i)}{2\pi i k^{2n+1}(e^{2\pi k/t} - 1)} \left(\sum_{j=0}^{\infty} (-1)^j \left(\frac{M}{ik} \right)^{j+1} z^j \right) \\
&+ \sum_{k=1}^{\infty} \frac{(-1) M^{2n} \varphi(2\beta k t)}{2\pi i k^{2n+1} t^{2n}} \left(\frac{1}{e^{2\pi k t} - 1} + 1 \right) \left((-1) \sum_{j=0}^{\infty} \left(\frac{M}{kt} \right)^{j+1} z^j \right) \\
&+ \sum_{k=1}^{\infty} \frac{(-1) M^{2n} \varphi(-2\beta k t)}{2\pi i k^{2n+1} t^{2n} (e^{2\pi k t} - 1)} \left(\sum_{j=0}^{\infty} (-1)^j \left(\frac{M}{kt} \right)^{j+1} z^j \right) \\
&+ \sum_{k=0}^{2n} \frac{1}{4} (2\beta M)^k \frac{\varphi^{(k)}(0)}{k!} \left(\frac{1}{z^{2n+1-k}} \right) \\
&+ \frac{it}{4\pi M} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{B_{2k-2j}}{(2k-2j)!} \left(\frac{\pi i}{t} \right)^{2k-2j} \\
&\quad \times \left(\frac{1}{z^{2n+1-2k}} (2M)^{2k+1} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} + \frac{1}{z^{2n+2-2k}} (2M)^{2k} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \right) \\
&+ \frac{1}{4\pi M} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{B_{2k-2j}}{(2k-2j)!} \pi^{2k-2j} \\
&\quad \times \left(\frac{1}{z^{2n+1-2k}} (2M)^{2k+1} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} + \frac{1}{z^{2n+2-2k}} (2M)^{2k} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \right) \\
&+ \frac{it}{(2\pi M)^2} \sum_{k=0}^{n+1} \sum_{j=0}^{k} \frac{(2M)^{2k}}{z^{2n+3-2k}} \frac{\varphi^{(2j)}(0)}{(2j)!} \beta^{2j} \pi^{2k-2j} \\
&\quad \times \sum_{l=0}^{k-j} \frac{B_{2k-2j-2l}}{(2k-2j-2l)!} \frac{B_{2l}}{(2l)!} \left(\frac{i}{t} \right)^{2l} \\
&+ \frac{it}{(2\pi M)^2} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(2M)^{2k+1}}{z^{2n+2-2k}} \frac{\varphi^{(2j+1)}(0)}{(2j+1)!} \beta^{2j+1} \pi^{2k-2j} \\
&\quad \times \sum_{l=0}^{k-j} \frac{B_{2k-2j-2l}}{(2k-2j-2l)!} \frac{B_{2l}}{(2l)!} \left(\frac{i}{t} \right)^{2l}
\end{aligned}
$$

Since the principal part at $z=0$ does not have $[z^p]$, we have

$$
\begin{aligned}
&= [z^p]\left(\sum_{k=1}^{\infty}\frac{(-1)^n M^{2n}\varphi(2\beta ki)}{2\pi i k^{2n+1}}\left(\frac{1}{e^{2\pi k/t}-1}+1\right)\left((-1)\sum_{j=0}^{\infty}\left(\frac{M}{ik}\right)^{j+1} z^j\right)\right.\\
&\quad+\sum_{k=1}^{\infty}\frac{(-1)^n M^{2n}\varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t}-1)}\left(\sum_{j=0}^{\infty}(-1)^j\left(\frac{M}{ik}\right)^{j+1} z^j\right)\\
&\quad+\sum_{k=1}^{\infty}\frac{(-1)M^{2n}\varphi(2\beta kt)}{2\pi i k^{2n+1}t^{2n}}\left(\frac{1}{e^{2\pi kt}-1}+1\right)\left((-1)\sum_{j=0}^{\infty}\left(\frac{M}{kt}\right)^{j+1} z^j\right)\\
&\quad\left.+\sum_{k=1}^{\infty}\frac{(-1)M^{2n}\varphi(-2\beta kt)}{2\pi i k^{2n+1}t^{2n}(e^{2\pi kt}-1)}\left(\sum_{j=0}^{\infty}(-1)^j\left(\frac{M}{kt}\right)^{j+1} z^j\right)\right)\\
&= \frac{(-1)^{n+1}M^{2n+p+1}}{2\pi i^{p+2}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left[\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2\pi k/t}-1}\right)+\varphi(2\beta ik)\right]\\
&\quad-\frac{M^{2n+p+1}}{2\pi i t^{2n+p+1}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left[\left(\frac{\varphi(2\beta kt)+(-1)^{p-1}\varphi(-2\beta kt)}{e^{2\pi kt}-1}\right)+\varphi(2\beta kt)\right]
\end{aligned}
\tag{3.1.10}
$$

Equating (3.1.9) and (3.1.10) gives

$$
\begin{aligned}
&-\frac{M^{2n+p+1}}{2\pi i t^{2n+p+1}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left[\left(\frac{\varphi(2\alpha k)+(-1)^{p-1}\varphi(-2\alpha k)}{e^{2\pi kt}-1}\right)+\varphi(2\alpha k)\right]\\
&= -\frac{(-1)^{n+1}M^{2n+p+1}}{2\pi i^{p+2}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left[\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2\pi k/t}-1}\right)+\varphi(2\beta ik)\right]\\
&\quad+\frac{it(2M)^{p+2n+1}}{\pi^2}\sum_{k+j+l=p+2n+3}\beta^k\frac{\varphi^{(k)}(0)}{k!}\frac{(-\pi)^j}{j!}B_j\frac{\left(\frac{\pi i}{t}\right)^l}{l!}B_l.
\end{aligned}
\tag{3.1.11}
$$

Now subtracting the lefthand side of (3.1.11) from both sides, we have

$$0 = -\frac{(-1)^{n+1}M^{2n+p+1}}{2\pi i^{p+2}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2\pi k/t}-1}+\varphi(2\beta ik)\right)$$
$$+\frac{M^{2n+p+1}}{2\pi it^{2n+p+1}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left(\frac{\varphi(2\alpha k)+(-1)^{p-1}\varphi(-2\alpha k)}{e^{2\pi kt}-1}+\varphi(2\alpha k)\right)$$
$$+\frac{it(2M)^{p+2n+1}}{\pi^2}\sum_{k+j+l=p+2n+3}\beta^k\frac{\varphi^{(k)}(0)}{k!}\frac{(-\pi)^j}{j!}B_j\frac{\left(\frac{\pi i}{t}\right)^l}{l!}B_l$$

and multiplying both sides by $\pi/(i(2M)^{2n+p+1}\alpha^{2n+p+1})$

$$0 = -\frac{1}{2i^{2n+1+p}(2\beta)^{2n+1+p}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2k\beta^2}-1}+\varphi(2\beta ik)\right)$$
$$+\frac{1}{2(2\alpha)^{2n+1+p}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2+p}}\left(\frac{\varphi(2\alpha k)+(-1)^{p-1}\varphi(-2\alpha k)}{e^{2k\alpha^2}-1}+\varphi(2\alpha k)\right)$$
$$+\frac{1}{\beta^{2n+3+p}}\sum_{k+j+l=p+2n+3}\beta^k\frac{\varphi^{(k)}(0)}{k!}(-1)^j(\pi)^j\frac{B_j}{j!}\beta^{2l}i^l\frac{B_l}{l!} \tag{3.1.12}$$

In (3.1.12) let $2n+2+p=q$, where $p,q,n\in\mathbb{Z}^+$.

$$0 = -\frac{1}{2i^{q-1}(2\beta)^{q-1}}\sum_{k=1}^{\infty}\frac{1}{k^q}\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2k\beta^2}-1}+\varphi(2\beta ik)\right)$$
$$+\frac{1}{2(2\alpha)^{q-1}}\sum_{k=1}^{\infty}\frac{1}{k^q}\left(\frac{\varphi(2\alpha k)+(-1)^{p-1}\varphi(-2\alpha k)}{e^{2k\alpha^2}-1}+\varphi(2\alpha k)\right)$$
$$+\frac{1}{\beta^{q+1}}\sum_{k+j+l=q+1}\beta^k\frac{\varphi^{(k)}(0)}{k!}(-1)^j\pi^j\frac{B_j}{j!}\beta^{2l}i^l\frac{B_l}{l!} \tag{3.1.13}$$

In (3.1.13), let $q=2n+1$, which implies that p is odd in (3.1.12).

$$0 = -\frac{(-1)^n}{2(2\beta)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{\varphi(2\beta ik)+(-1)^{p-1}\varphi(-2\beta ik)}{e^{2k\beta^2}-1}+\varphi(2\beta ik)\right)$$
$$+\frac{1}{2(2\alpha)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{\varphi(2\alpha k)+(-1)^{p-1}\varphi(-2\alpha k)}{e^{2k\alpha^2}-1}+\varphi(2\alpha k)\right)$$
$$+\sum_{k=0}^{2n+2}\frac{\varphi^{(k)}(0)}{k!}\sum_{j=0}^{2n+2-k}(-1)^j\alpha^j\beta^{2n+2-k-j}i^{2n+2-k-j}\frac{B_j}{j!}\frac{B_{2n+2-k-j}}{(2n+2-k-j)!}$$

Therefore,

$$\frac{(-1)^n}{(2\beta)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ki)+\varphi(-2\beta ki)}{k^{2n+1}(e^{2k\beta^2}-1)}+\frac{(-1)^n}{(2\beta)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\beta ki)}{k^{2n+1}}$$

$$-\frac{1}{(2\alpha)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\alpha k)+\varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2}-1)}-\frac{1}{(2\alpha)^{2n}}\sum_{k=1}^{\infty}\frac{\varphi(2\alpha k)}{k^{2n+1}}$$

$$=-\frac{\pi i}{2}\frac{\varphi^{(2n)}(0)}{(2n)!}+\alpha\sum_{k=0}^{n}(-1)^{n+k}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\beta^{2n-2k}$$

$$-i\beta\sum_{k=0}^{n}\frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\frac{B_{2n-2k}}{(2n-2k)!}\alpha^{2n-2k}$$

$$+2\sum_{k=0}^{n+1}\frac{\varphi^{(2k)}(0)}{(2k)!}\sum_{j=0}^{n+1-k}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!}\alpha^{2n+2-2k-2j}\beta^{2j},$$

as claimed. This completes the proof of Theorem 3.1.1.

3.2 Ramanujan's Formula for $\zeta(2n+1)$

For $\alpha\beta=\pi^2$ and any positive integer n:

$$\alpha^{-n}\left[\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{1}{e^{2k\alpha}-1}\right)+\frac{1}{2}\zeta(2n+1)\right]$$

$$=(-\beta)^{-n}\left[\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{1}{e^{2k\beta}-1}\right)+\frac{1}{2}\zeta(2n+1)\right]$$

$$-2^{2n}\sum_{j=0}^{n+1}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2j}}{(2n+2-2j)!}\alpha^j\beta^{n+1-j} \quad (3.2.1)$$

Proof:

In Theorem (3.1.1), replace α by $\sqrt{\alpha}$, replace β by $\sqrt{\beta}$, let $\varphi\equiv 1$, and n be any positive integer. Since $\varphi\equiv 1$, a constant, any derivative in (3.1.2) is equal to zero.

Therefore, (3.1.2) reduces to:

$$\frac{(-1)^n}{2^{2n}\beta^n}\sum_{k=1}^{\infty}\frac{2}{k^{2n+1}(e^{2k\beta}-1)}+\frac{(-1)^n}{2^{2n}\beta^n}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}$$
$$+\frac{1}{2^{2n}\alpha^n}\sum_{k=1}^{\infty}\frac{2}{k^{2n+1}(e^{2k\alpha}-1)}+\frac{1}{2^{2n}\alpha^n}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}$$
$$=2\sum_{j=0}^{n+1}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2j}}{(2n+2-2j)!}\alpha^{n+1-j}\beta^j$$

Multiplying both sides by $2^{(2n-1)}$ and rearranging, we get

$$\alpha^{-n}\left[\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{1}{e^{2k\alpha}-1}\right)+\frac{1}{2}\zeta(2n+1)\right]$$
$$=(-\beta)^{-n}\left[\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\left(\frac{1}{e^{2k\beta}-1}\right)+\frac{1}{2}\zeta(2n+1)\right]$$
$$-2^{2n}\sum_{j=0}^{n+1}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2j}}{2n+2-2j!}\alpha^j\beta^{n+1-j}. \tag{3.2.2}$$

3.3 Euler's Formula for $\zeta(2n)$

For N any positive integer,

$$\zeta(2N)=\frac{-B_{2N}(2\pi i)^{2N}}{2(2N)!} \tag{3.3.1}$$

In (3.1.13) replace α by $\sqrt{\alpha}$, replace β by $\sqrt{\beta}$, let $\varphi \equiv 1$ and $q = 2n + 2 + p$ be even and denote it as $q = 2N$. Then

$$\begin{aligned}
0 &= -\frac{1}{2i^{2N-1}(2\beta)^{2N-1}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}} \varphi(2\beta i k) + \frac{1}{2(2\alpha)^{2N-1}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}} \varphi(2\alpha k) \\
&+ \frac{1}{\beta^{2N+1}} \sum_{k+j+l=q+1} \beta^k \frac{\varphi^{(k)}(0)}{k!} (-1)^j (\pi)^j \frac{B_j}{j!} \beta^{2l} i^l \frac{B_l}{l!} \\
&= \left(\frac{1}{i^{2N+1} \sqrt{\beta}^{2N-1}} + \frac{1}{\sqrt{\alpha}^{2N-1}} \right) \zeta(2N) \\
&+ 2^{2N} \left((-1)^{2N} (-\frac{1}{2}) \frac{B_{2N}}{(2N)!} \sqrt{\alpha}^{2N} \sqrt{\beta} i + (-1)(-\frac{1}{2}) \frac{B_{2N}}{(2N)!} \sqrt{\alpha} \sqrt{\beta}^{2N} i^{2N} \right) \\
&= \left(\frac{1}{i^{2N+1} \sqrt{\beta}^{2N-1}} + \frac{1}{\sqrt{\alpha}^{2N-1}} \right) \zeta(2N) + \frac{2^{2N} B_{2N}}{2(2N)!} (-\sqrt{\alpha}^{2N} \sqrt{\beta} i + \sqrt{\alpha} \sqrt{\beta}^{2N} i^{2N}) \\
&= \left(\frac{1}{(\pi i)^{2N}} + \frac{2^{2N} B_{2N}}{2(2N)! \zeta(2N)} \right) (-\sqrt{\alpha}^{2N} \sqrt{\beta} i + \sqrt{\alpha} \sqrt{\beta}^{2N} i^{2N})
\end{aligned}$$

Therefore for $N \in \mathbb{Z}^+$,

$$\zeta(2N) = \frac{-B_{2N}(2\pi i)^{2N}}{2(2N)!}. \tag{3.3.2}$$

CHAPTER 4

COMPARING THE TWO METHODS

This chapter will show that the Mittag-Leffler partial fraction expansion proof, which has a weaker hypothesis, has stronger bounded convergence than the contour integration proof.

4.1 Contour Integration Method

Let

$$f_m(z) = \frac{\varphi(2\beta Mz)}{z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}.$$

By the contour integration method, we have

$$\lim_{M\to+\infty} \frac{\varphi(2\beta Mz)}{M^{2n}z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)} = 0 \quad \text{boundedly on } C\setminus\{\pm i, \pm t\}.$$

Therefore, we have

$$\begin{aligned} 0 &= \lim_{M\to+\infty} \frac{\varphi(2\beta Mz)}{M^{2n}z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)} \\ &= \lim_{M\to+\infty} \frac{\varphi(2\beta Mz)}{z(Mz)^{2n}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)} \end{aligned}$$

Since z is a constant in the contour integration method, we can remove it from the denominator and get

$$\lim_{M\to+\infty}\frac{\varphi(2\beta Mz)}{(Mz)^{2n}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}=0,\ \ z\in\mathbb{C}\setminus\{\pm i,\pm t\}.$$

Let $w=Mz$. Then

$$\lim_{\substack{|w|\to+\infty\\ w\in\mathbb{C}}}\frac{\varphi(2\beta w)}{w^{2n}(e^{-2\pi w}-1)(e^{2\pi iw/t}-1)}=0.$$

Define

$$h(w)=\frac{\varphi(2\beta w)}{w^{2n}(e^{-2\pi w}-1)(e^{2\pi iw/t}-1)},$$

then

$$\lim_{\substack{|w|\to+\infty\\ w\in\mathbb{C}}} h(w)=0 \text{ boundedly.} \tag{4.1.1}$$

4.2 Partial Fraction Expansion Method

By the Mittag-Leffler method, we have

$$\lim_{\substack{|z|\to\infty\\ z\in\mathbb{C}\setminus D_\varepsilon}}\frac{\varphi(2\beta Mz)}{z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi iMz/t}-1)}=0.$$

For $\varepsilon > 0$, let $U_\varepsilon = \bigcup_{k\in\mathbb{Z}} \{D_\varepsilon(kt) \cup D_\varepsilon(ik)\}$. Since M is a constant by the Mittag-Leffler method we can put M^{2n+1} as a factor in the denominator and have

$$\begin{aligned}
0 &= \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \frac{\varphi(2\beta Mz)}{M^{2n+1}z^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi i Mz/t}-1)} \\
&= \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \frac{\varphi(2\beta Mz)}{(Mz)^{2n+1}(e^{-2\pi Mz}-1)(e^{2\pi i Mz/t}-1)} \\
&= \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \frac{\varphi(2\beta Mz)}{(Mz)(Mz)^{2n}(e^{-2\pi Mz}-1)(e^{2\pi i Mz/t}-1)} \\
&= \lim_{\substack{|w|\to\infty \\ w\in\mathbb{C}\setminus U_\varepsilon}} \frac{\varphi(2\beta w)}{w^{2n+1}(e^{-2\pi w}-1)(e^{2\pi i w/t}-1)} \\
&= \lim_{\substack{|w|\to\infty \\ w\in\mathbb{C}\setminus U_\varepsilon}} \frac{h(w)}{w}.
\end{aligned}$$

Thus

$$\lim_{\substack{|w|\to\infty \\ w\in\mathbb{C}\setminus U_\varepsilon}} \frac{h(w)}{w} = 0. \tag{4.2.1}$$

4.3 Comparison

Figure 4.1 geometrically shows the comparison of the contour integration method to the Mittag-Leffler partial fraction expansion method. In the contour integration proof, a parallelogram is constructed and the growth of the function is based on the limit of M approaching infinity for a fixed z on the contour. Then as M approaches infinity, so does $|w| = M|z|$. Whereas, for the partial fraction expansion method, z can approach infinity in any direction except along the real and imaginary axes, which is a more natural growth hypothesis of the function $f_m(z)$. For this method,

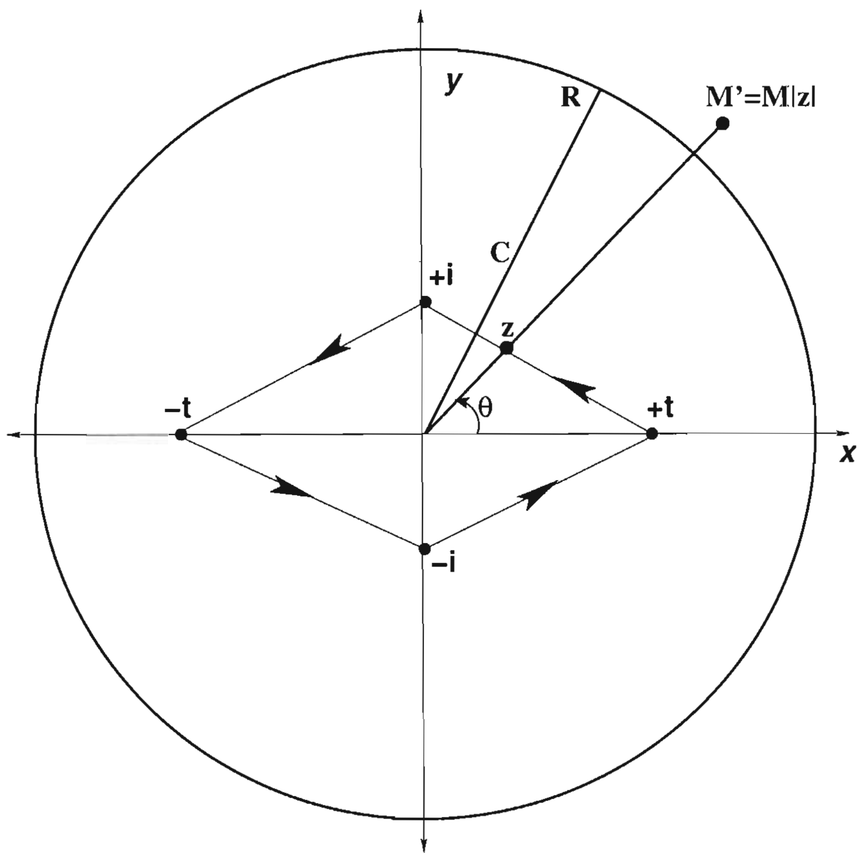

Figure 4.1: Comparison of Contour Integration to Mittag-Leffler

we find a disc of radius R and show convergence inside the disc.

We claim that (4.2.1) is a weaker hypothesis than (4.1.1).

Theorem 4.3.1. *Let* $h : \mathbb{C} \to \mathbb{C}$ *be meromorphic, with no poles off the real and imaginary axes. Then for* $0 < \theta < 2\pi$, $\theta \neq \frac{\pi}{2}$, π, $\frac{3\pi}{2}$,

$$\lim_{\substack{M \to \infty \\ \frac{1}{2}+M \in \mathbb{Z}^+}} h(Mz) = 0$$

boundedly for z *on* C*, implies*

$$\lim_{\substack{|w| \to \infty \\ w \in \mathbb{C} \setminus U_\epsilon}} \frac{h(w)}{w} = 0.$$

Proof: Let

$$\lim_{\substack{M \to \infty \\ \frac{1}{2}+M \in \mathbb{Z}^+}} h(Mz) = 0$$

boundedly on C. Then by the bounded convergence theorem, $\left|h(Mz)\right| \leq K$ for all $M > 0$ and z on C.

Given $\epsilon > 0$, we need to find $R > 0$ such that $\left|\frac{h(w)}{w}\right| < \epsilon$ for $|w| > R$ and $w \notin U_\epsilon$. Let $\epsilon > 0$ and $\left|h(w)\right| \leq K$. So take $R > \frac{K}{\epsilon}$. Then if $|w| > R$, $\left|\frac{h(w)}{w}\right| < \left|\frac{h(w)}{R}\right| \leq K/\frac{K}{\epsilon} = \epsilon$.

Therefore,

$$\lim_{\substack{M \to \infty \\ \frac{1}{2}+M \in \mathbb{Z}^+}} h(Mz) = 0$$

boundedly on C, implies

$$\lim_{\substack{|w| \to \infty \\ w \in \mathbb{C} \setminus U_\epsilon}} \frac{h(w)}{w} = 0.$$

CHAPTER 5

CHARACTER ANALOG

In this chapter a character analog of Entry 20 is given. Bradley's acceleration formula [8] for Dirichlet series with periodic coefficients can be obtained as a special case. Also, letting the modulus $h = 4$ yields Corollary 4 [8, p. 336], and letting $h = 1$ yields Ramanujan's formula for $\zeta(2n+1)$.

5.1 Odd Characters

T. M. Apostol [1] defines Dirichlet characters as follows:

> ***Definition*** *Dirichlet characters. Let G be the group of reduced residue classes modulo h. Corresponding to each character $\tilde{g}$ of G we define an arithmetical function as follows:*
>
> $$g(j) = \tilde{g}(\hat{j}) \qquad if\ gcd(j,h) = 1,\ where\ \hat{j}\ is\ the\ residue\ class\ of\ j,$$
>
> $$g(j) = 0 \qquad if\ gcd(j,h) > 1.$$
>
> *The function g is called a Dirichlet character modulo h. The principal character g_1 is defined by*
>
> $$g_1(j) = \begin{cases} 1 & \text{if } gcd(j,h) = 1, \\ 0 & \text{if } gcd(j,h) > 1. \end{cases}$$

Theorem 5.1.1. *Let α, β, $t > 0$ with $\alpha\beta = \pi$ and $t = \alpha/\beta$. Let $h \in \mathbb{Z}^+$ and $0 \le j \le h-1$. Let m be a positive integer and let $M = m + \frac{1}{2}$. For $\varepsilon > 0$, let $D_\varepsilon = \bigcup_{k\in\mathbb{Z}} \{\bigcup_{j=0}^{h-1} D_\varepsilon\left(\frac{t(hk-j)}{(hM)}\right) \cup D_\varepsilon(ik/M)\}$. Let $\varphi_j(z)$ be an entire function, where $\eta \equiv j \bmod h$, $0 \le j \le h-1$ and further for all $\eta \in \mathbb{Z}$ define $\varphi_\eta = \varphi_j$. Let $g(j)$ be an odd Dirichlet character mod h, and define for each positive integer n*

$$f_m(g,z) = \frac{1}{z^{2n+1}(e^{-2\pi Mz}-1)} \sum_{j=0}^{h-1} \frac{\varphi_j(2\beta Mz)\, g(j)}{e^{2\pi ij/h}e^{2\pi iMz/t}-1} \tag{5.1.1}$$

and assume that

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(g,z) = 0 \tag{5.1.2}$$

for some $0 < \varepsilon < \min\left(\frac{t}{4hM}, \frac{1}{4M}\right)$.
Also, assume that for every j, where $j = 0, ..., h-1$

$$\sum_{0\neq k\in\mathbb{Z}} \left|\frac{\varphi_j(2\beta ik)}{k^{2n+1}}\right| < \infty$$

and

$$\sum_{0\neq k\in\mathbb{Z}} \left|\frac{\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+1}}\right| < \infty.$$

Let B_κ, $0 \le \kappa < \infty$, denote the κth Bernoulli number and $B_\kappa(x)$ denote the κth Bernoulli polynomial evaluated at x. The Bernoulli polynomials are defined by

$$\frac{te^{tx}}{e^t-1} \equiv \sum_{\kappa=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Then for $q \in \mathbb{Z}^+$,

$$\frac{\alpha^{(1-q)/2}}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k) + (-1)^q\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1} + \varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$

$$= \sum_{k=1}^{\infty}\frac{\beta^{(1-q)/2}i}{2hi^q}\sum_{j=0}^{h-1}\frac{g(j)}{k^q}\left(\frac{\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik) + (-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1} - \varphi_j(2\beta^{1/2}h^{-1/2}ik)\right)$$

$$+ \frac{1}{h}\sum_{r=0}^{q}(-1)^r\frac{B_r}{r!}\sum_{s=0}^{q-r}\alpha^{r/2}\beta^{(-q+r+2s+1)/2}2^{r+s-1}M^{-q+r+s}\frac{i^{s+1}}{(s+1)!}$$

$$\times \sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right) \tag{5.1.3}$$

Proof. Similar to the work in Chapter 3 for $f_m(z)$, the partial fraction expansion and the Laurent expansion will be derived for $f_m(g,z)$. The proof of Theorem 5.1.1 depends on the following partial fraction expansion. The partial fraction expansion of $f_m(g,z)$ is given by

Theorem 5.1.2.

$$
\begin{aligned}
f_m(g,z) = {} & \frac{M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\left(\frac{1}{z-(-ik/M)}\right) \\
& + \frac{M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1}g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi j/h}e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-ik/M}\right) \\
& + \frac{M^{2n}h^{2n+1}}{2\pi it^{2n}}\sum_{k=0}^{\infty}\sum_{\substack{j=0 \\ (j,k)\neq(0,0)}}^{h-1}\frac{g(j)\varphi_j(2\beta(hk-j)t/h)}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)}\left(\frac{1}{z-(hk-j)t/(hM)}\right) \\
& - \frac{M^{2n}h^{2n+1}}{2\pi it^{2n}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{-g(j)\varphi_j(-2\beta(hk+j)t/h)}{(hk+j)^{2n+1}(e^{2\pi t(j+hk)/h}-1)}\left(\frac{1}{z-(-hk-j)t/(hM)}\right) \\
& + \sum_{l=1}^{2n+2}\frac{1}{z^l}\sum_{r=0}^{2n+2-l}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{2n+2-l-r}(2\pi M)^{r-1+s}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \\
& \times\sum_{j=0}^{h-1}\frac{g(j)\varphi_j^{(2n+2-l-r-s)}(0)}{(2n+2-l-r-s)!}(2\beta M)^{2n+2-l-r-s}\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}
\tag{5.1.4}
$$

Proof. The function, $f_m(g,z)$, has simple poles at $z=\pm ik/M$ for $k=1,2,\ldots$, and at $z=(\pm hk-j)t/(hM)$ for $j=0,\ldots,h-1$ and $k=0,1,2,\ldots$, except for $j=k=0$. There is also a pole at the origin of order $2n+2$.

Let $R_a :=$ the residue of f_m at a, then for every j where $j = 0, ..., h-1$ and $k \in \mathbb{Z}^+$, the residues for the simple poles are:

$$R_{ik/M} = \frac{M^{2n}}{2\pi(ik)^{2n+1}} \sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right),$$

$$R_{-ik/M} = \frac{M^{2n}}{2\pi(ik)^{2n+1}} \sum_{j=0}^{h-1} \frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1},$$

$$R_{(hk-j)t/(hM)} = \frac{M^{2n}h^{2n+1}}{2\pi it^{2n}} \frac{g(j)\varphi_j(2\beta(hk-j)t/h)}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)},$$

and for $k = 0, 1, 2, ...$ and $(j, k) \neq (0, 0)$

$$R_{(-hk-j)t/(hM)} = \frac{-M^{2n}h^{2n+1}}{2\pi it^{2n}} \frac{g(j)\varphi_j(-2\beta(j+hk)t/h)}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)}.$$

For $k = 1, 2, ...$, the corresponding principal parts are:

$$Q_{ik/M} = \frac{M^{2n}}{2\pi(ik)^{2n+1}} \sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right) \times \left(\frac{1}{z-ik/M}\right),$$

$$Q_{-ik/M} = \frac{M^{2n}}{2\pi(ik)^{2n+1}} \sum_{j=0}^{h-1} \frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\left(\frac{1}{z+ik/M}\right),$$

$$Q_{(hk-j)t/(hM)} = \frac{M^{2n}h^{2n+1}}{2\pi it^{2n}} \frac{g(j)\varphi_j(2\beta(hk-j)t/h)}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \times \left(\frac{1}{z-(hk-j)t/(hM)}\right),$$

and for $k = 0, 1, 2, ...$ and $(j, k) \neq (0, 0)$

$$Q_{(-hk-j)t/(hM)} = \frac{-M^{2n}h^{2n+1}}{2\pi it^{2n}} \frac{g(j)\varphi_j(-2\beta(hk+j)t/h)}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \times \left(\frac{1}{z-(-hk-j)t/(hM)}\right).$$

To find the principal part for the pole at the origin of order $2n+2$, we use (2.2.1), Lemma (5.1.3), and expand the entire function φ_j as a Taylor series expansion. This gives the Laurent series expansion for $f_m(g,z)$:

$$f_m(g,z) = \sum_{r=0}^{\infty}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{\infty}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!}z^{r-2n-2+s}$$
$$\times\sum_{j=0}^{h-1} g(j)\sum_{d=0}^{\infty}\frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right) \tag{5.1.5}$$

Lemma 5.1.3.

$$\sum_{j=0}^{h-1}\frac{g(j)}{e^{2\pi ij/h}e^{2\pi iMz/t}-1}\sum_{d=0}^{\infty}\frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d$$
$$=\sum_{s=0}^{\infty}\frac{(2\pi iMzh/t)^s}{(s+1)!}\sum_{j=0}^{h-1}g(j)\sum_{d=0}^{\infty}\frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right) \tag{5.1.6}$$

Proof:

$$\frac{1}{e^{2\pi ij/h}e^{2\pi iMz/t}-1} = \frac{1}{e^{2\pi ij/h}-1}\frac{1-e^{-2\pi ij/h}}{e^{2\pi iMz/t}-e^{-2\pi ij/h}}$$

Now, from [9], let $a = w^r$ a primitive hth root of unity, where $w = e^{2\pi i/h}$ and $gcd(h,r)=1$. Then

$$\frac{1-a}{e^z-a} = \sum_{k=0}^{\infty}H_k(a)\frac{z^k}{k!},$$

where $H_k(a)$ is the kth Eulerian number and

$$H_{k-1}(a) = \frac{1-a}{ka}h^{k-1}\sum_{l=0}^{h-1}a^{-l}B_k\left(\frac{l}{h}\right).$$

Changing $\sum_{k=0}^{\infty}$ to $\sum_{k=1}^{\infty}$ and $\frac{z^k}{k!}$ to $\frac{z^{k-1}}{(k-1)!}$ we get

$$\begin{aligned}\frac{1-a}{e^z-a} &= \sum_{k=1}^{\infty} H_{k-1}(a)\frac{z^{k-1}}{(k-1)!} \\ &= \sum_{k=1}^{\infty} \frac{1-a}{ka} h^{k-1} \frac{z^{k-1}}{(k-1)!} \sum_{l=0}^{h-1} a^{-l} B_k\left(\frac{l}{h}\right). \end{aligned} \tag{5.1.7}$$

Now, substituting (5.1.7) into the lefthand side of (5.1.6) we get:

$$\begin{aligned}
&\sum_{j=0}^{h-1} \frac{g(j)}{e^{2\pi ij/h}e^{2\pi iMz/t}-1} \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \\
&= \sum_{j=0}^{h-1} g(j) \frac{1}{e^{2\pi ij/h}-1} \frac{1-e^{-2\pi ij/h}}{e^{2\pi iMz/t}-e^{-2\pi ij/h}} \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \\
&= \sum_{j=0}^{h-1} g(j) \frac{1}{e^{2\pi ij/h}-1} \sum_{k=1}^{\infty} \frac{1-e^{-2\pi ij/h}}{ke^{-2\pi ij/h}} \frac{h^{k-1}(2\pi iMz/t)^{k-1}}{(k-1)!} \sum_{l=0}^{h-1} \left(e^{-2\pi ij/h}\right)^{-l} \\
&\qquad \times B_k\left(\frac{l}{h}\right) \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \\
&= \sum_{j=0}^{h-1} g(j) \sum_{k=1}^{\infty} \frac{1}{k!}(2\pi iMzh/t)^{k-1} \sum_{l=0}^{h-1} e^{2\pi ijl/h} B_k\left(\frac{l}{h}\right) \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \\
&= \sum_{j=0}^{h-1} g(j) \sum_{k=0}^{\infty} \frac{(2\pi iMzh/t)^k}{(k+1)!} \sum_{l=0}^{h-1} e^{2\pi ijl/h} B_{k+1}\left(\frac{l}{h}\right) \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \\
&= \sum_{s=0}^{\infty} \frac{(2\pi iMzh/t)^s}{(s+1)!} \sum_{j=0}^{h-1} g(j) \sum_{d=0}^{\infty} \frac{\varphi_j^{(d)}(0)}{d!}(2\beta Mz)^d \sum_{k=0}^{h-1} e^{2\pi ijk/h} B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}$$

This completes the proof of Lemma 5.1.3 and (5.1.5) now follows.

Therefore the principal part of $f_m(g,z)$ at the origin is

$$\begin{aligned}
Q_0 &= \sum_{l=1}^{2n+2} \frac{1}{z^l} \sum_{r=0}^{2n+2-l} (-1)^{r-1} \frac{B_r}{r!} \sum_{s=0}^{2n+2-l-r} (2\pi M)^{r+s-1} \frac{i^s}{t^s} \frac{h^s}{(s+1)!} \\
&\quad \times \sum_{j=0}^{h-1} \frac{g(j)\varphi_j^{(2n+2-l-r-s)}(0)}{(2n+2-l-r-s)!} (2\beta M)^{2n+2-l-r-s} \sum_{k=0}^{h-1} e^{2\pi ikj/h} B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}$$

It will now be shown that the sum of the principal parts, $\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)$, converges uniformly on compact subsets by the Weierstrass M-test.

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4hM}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$ and $|z| \leq N$,

$$\begin{aligned}
&\left|\sum_{k=2NM}^{\infty} Q_{-ik/M}\right| \\
&= \left|\sum_{k=2NM}^{\infty} \frac{1}{2\pi i k\,(ik/M)^{2n}} \sum_{j=0}^{h-1} \frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\left(\frac{1}{z+ik/M}\right)\right| \\
&\leq \left|\frac{(-1)^n M^{2n}}{2\pi i} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\left(\frac{1}{z+ik/M}\right)\right| \\
&\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|g(j)\varphi_j(-2\beta ik)|}{|e^{2\pi ij/h}e^{2\pi k/t}-1|}\left|\frac{1}{z+ik/M}\right|
\end{aligned}$$

Replace $\left|\dfrac{1}{z+ik/M}\right|$ by $\max\{1/(-N+k/M), 1/\varepsilon\}$.

So in the case when $\max\{1/(-N+k/M), 1/\varepsilon\} = 1/(-N+k/M)$,

substitute this into the inequality and we have

$$\begin{aligned}
&\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(-2\beta ik)|}{(e^{2\pi k/t}-1)} \frac{1}{-N+k/M} \\
&\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(-2\beta ik)|}{(e^{2\pi k/t}-1)} \frac{1}{k/M\,(1-NM/k)} \\
&\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(-2\beta ik)|}{(e^{2\pi k/t}-1)} \frac{2M}{k} < \infty, \text{ since } \frac{k}{M} \geq 2N.
\end{aligned}$$

For the case of $1/\varepsilon$, we obtain similar results. Therefore, $\sum_{k=1}^{\infty} Q_{-ik/M}$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4hM}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$ and $|z| \le N$,

$$\begin{aligned}
&\left|\sum_{k=2NM}^{\infty} Q_{ik/M}\right| \\
&= \left|\sum_{k=2NM}^{\infty} \frac{1}{2\pi ik(ik/M)^{2n}} \sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-ik/M}\right)\right| \\
&\le \left|\frac{(-1)^n M^{2n}}{2\pi i} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right)\left(\frac{1}{z-ik/M}\right)\right| \\
&\le \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{|e^{-2\pi ij/h}e^{2\pi k/t}-1|}\left|\frac{1}{z-ik/M}\right| \\
&\quad + \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)|\left|\frac{1}{z-ik/M}\right|
\end{aligned}$$

Replace $\left|\frac{1}{z+ik/M}\right|$ by $\max\{1/(-N+k/M), 1/\varepsilon\}$.

So in the case when $\max\{1/(-N+k/M), 1/\varepsilon\} = 1/(-N+k/M)$,

substitute this into the inequality and we have

$$\begin{aligned}
&\le \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{e^{2\pi k/t}-1}\left|\frac{1}{Ni-ik/M}\right| \\
&\quad + \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)|\left|\frac{1}{Ni-ik/M}\right| \\
&\le \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{e^{2\pi k/t}-1}\frac{1}{k/M-N} \\
&\quad + \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)|\frac{1}{k/M-N} \\
&\le \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{e^{2\pi k/t}-1}\frac{1}{(k/M)(1-NM/k)} \\
&\quad + \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)|\frac{1}{(k/M)(1-NM/k)}
\end{aligned}$$

$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{e^{2\pi k/t}-1} \frac{M}{k\,(1-1/2)}$$
$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)| \frac{M}{k\,(1-1/2)}$$
$$\leq \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} \frac{|\varphi_j(2\beta ik)|}{e^{2\pi k/t}-1} \left(\frac{2M}{k}\right)$$
$$+ \frac{M^{2n}}{2\pi} \sum_{k=2NM}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{h-1} |\varphi_j(2\beta ik)| \frac{2M}{k} < \infty.$$

For the case of $1/\varepsilon$, we obtain similar results. Therefore, $\sum_{k=1}^{\infty} Q_{ik/M}$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4hM}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$, $|z| \leq N$, and for every j where $j = 0, ..., h-1$,

$$\left| \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} Q_{(hk-j)t/(hM)} \right|$$

$$= \left| \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{g(j)\varphi_j(2\beta(hk-j)t/h)}{2\pi i(\frac{t}{M})^{2n}(\frac{1}{h})^{2n+1}(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \right.$$
$$\left. \times \left(\frac{1}{z-(hk-j)t/(hM)}\right) \right|$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(2\beta(hk-j)t/h)|}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \left| \frac{1}{z-(hk-j)t/(hM)} \right|$$

Replace $\left|\frac{1}{z+ik/M}\right|$ by $\max\{1/((N-(hk-j)t)/(hM)), 1/\varepsilon\}$.

So in the case when $\max\{1/((N-(hk-j)t)/(hM)), 1/\varepsilon\} = 1/((N-(hk-j)t)/(hM))$, substitute this into the inequality and we have

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(2\beta(hk-j)t/h)|}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \left| \frac{1}{N-(hk-j)t/(hM)} \right|$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(2\beta(hk-j)t/h)|}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \frac{1}{(hk-j)t/(hM)-N}$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(2\beta(hk-j)t/h)|}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)}$$
$$\times \frac{1}{((hk-j)t/(hM))(1-NhM/(hk-j)t)}$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} + \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(2\beta(hk-j)t/h)|}{(hk-j)^{2n+1}(e^{-2\pi t(hk-j)/h}-1)} \frac{2hM}{t(hk-j)} < \infty.$$

For the case of $1/\varepsilon$, we obtain similar results. Therefore, $\sum_{k=1}^{\infty} Q_{(hk-j)t/(hM)}$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Let $N \in \mathbb{Z}^+$, $0 < \varepsilon < \min\left(\frac{t}{4hM}, \frac{1}{4M}\right)$. Now, for $z \in \mathbb{C} \setminus D_\varepsilon$, $|z| \leq N$, and for every j where $j = 0, ..., h-1$,

$$\left| \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} Q_{(-hk-j)t/(hM)} \right|$$

$$= \left| \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} \frac{-g(j)\varphi_j(-2\beta(hk+j)t/h)}{2\pi i(\frac{t}{M})^{2n}(\frac{1}{h})^{2n+1}(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \times \left(\frac{1}{z+(hk+j)t/hM}\right) \right|$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(-2\beta(hk+j)t/h)|}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \left| \left(\frac{1}{z+(hk+j)t/(hM)}\right) \right|$$

Replace $\left|\dfrac{1}{z+ik/M}\right|$ by $\max\{1/((N-(hk+j)t)/(hM)), 1/\varepsilon\}$.

So in the case when $\max\{1/((N-(hk+j)t)/(hM)), 1/\varepsilon\} = 1/((N-(hk+j)t)/(hM))$, substitute this into the inequality and we have

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(-2\beta(hk+j)t/h)|}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \left(\frac{1}{N-(hk+j)t/(hM)}\right)$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(-2\beta(hk+j)t/h)|}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \times \left(\frac{hM}{(hk+j)t\,(1-NhM/((hk+j)t))}\right)$$

$$\leq \frac{M^{2n}h^{2n+1}}{2\pi t^{2n}} \sum_{k=\lfloor \frac{2NM}{t} - \frac{j}{h} \rfloor + 1}^{\infty} \frac{|\varphi_j(-2\beta(hk+j)t/h)|}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)} \frac{2hM}{t(hk+j)} < \infty.$$

For the case of $1/\varepsilon$, we obtain similar results. Therefore, $\sum_{k=1}^{\infty} Q_{(-hk-j)t/(hM)}$ converges uniformly on compact subsets of $\mathbb{C} \setminus D_\varepsilon$ by the Weierstrass M-test.

Clearly,

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) = 0 \tag{5.1.8}$$

because uniform convergence allows us to interchange the limit and the summation.

By Mittag-Leffler (Theorem 8.5.2), there exists an entire function G such that

$$f_m(g,z) = \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) + G(z).$$

It will be shown that $G(z) = 0$, by the assumption on $f_m(g,z)$ and the fact that $\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)$ converges uniformly on compact sets. First, we will show that G is bounded. Hence, G is constant by Liouville's theorem. Then taking the limit as $|z|$ approaches infinity, it will be shown that $G = 0$.

Now, it will be shown that G is bounded on $\mathbb{C} \setminus D_\varepsilon$. We first look at the strips $t(-h-j)/(2hM) \leq \operatorname{Re} z \leq t(h-j)/(2hM)$ for every j where $j = 0, ..., h-1$, and at the strip $-1/(2M) \leq \operatorname{Im} z \leq 1/(2M)$. We will show that G is bounded in each strip. Since G is entire, G is analytic and continuous in these strips. Also since G is entire, G assumes it maximum on the boundary of any connected compact set by the Maximum Modulus Principle.

For all the vertical strips in Figure 5.1, we have for each j, $t(-h-j)/(2hM) \leq \operatorname{Re} z \leq t(h-j)/(2hM)$. For $k \in \mathbb{Z}$, we examine the E_k's, where each E_k is a compact region defined for some j by $t(-h-j)/(2hM) \leq \operatorname{Re} z \leq t(h-j)/(2hM)$ and $(k-\frac{1}{2})/M \leq y \leq (k+\frac{1}{2})/M$. Since the boundary of E_k's avoids the poles, $|f_m(g,z)|$ is uniformly bounded by (5.2.2). Also, $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ is bounded on the boundary of

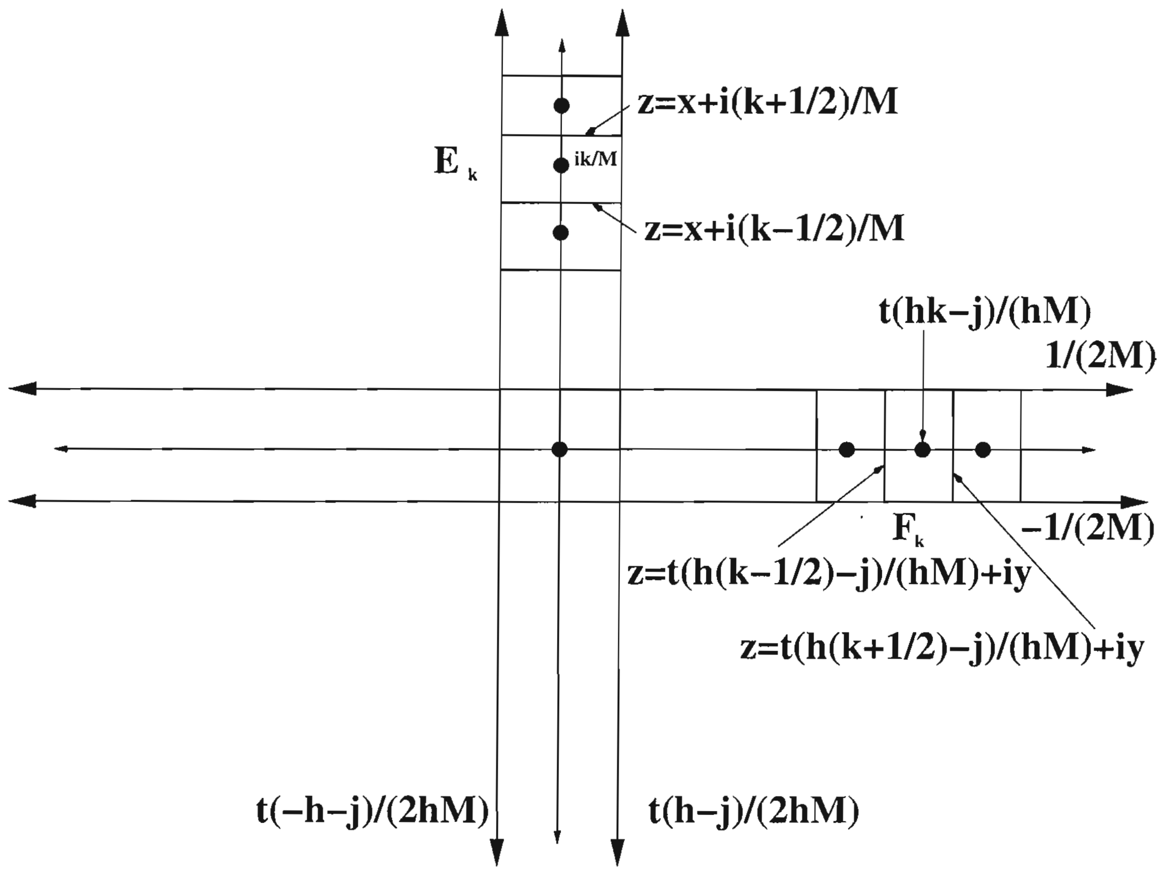

Figure 5.1: Contour Strips: $\frac{t(-h-j)}{2hM} \leq \operatorname{Re} z \leq \frac{t(h-j)}{2hM}$ and $-\frac{1}{2M} \leq \operatorname{Im} z \leq \frac{1}{2M}$

each E_k by (5.1.8). Since k is arbitrary, $|f_m(g,z)|$ and $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ are bounded on the entire vertical strip. Therefore G, their difference, is likewise bounded on the entire vertical strip. This holds for each vertical strip depending on j.

For the horizontal strip in Figure 5.1, we have $-1/(2M) \leq \operatorname{Im} z \leq 1/(2M)$. For $k \in \mathbb{Z}^+$, we examine the $F_k's$, where there is for each j an F_k, a compact region with the boundary $t(h(k-\frac{1}{2})-j)/(hM) \leq \operatorname{Re} z \leq t(h(k+\frac{1}{2})-j)/(hM)$ and $-1/(2M) \leq y \leq 1/(2M)$. Since the boundary of the F_k's avoids the poles, $|f_m(g,z)|$ is uniformly bounded by (5.2.2) . Also, $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ is bounded on the boundary of the F_k's by (5.1.8). Since k is arbitrary, $|f_m(g,z)|$ and $\left|\sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right)\right|$ are bounded on the entire horizontal strip. Therefore G, their difference, is likewise bounded on the entire horizontal strip.

Now inside any disc of radius R, minus the strips, G is obviously bounded. Since G is bounded on the strips, G is bounded inside the disc and strips. Outside the disc and strips, G is bounded by (5.2.2) and (5.1.8). Therefore, G is bounded on $\mathbb{C}$. Hence, by Liouville's theorem, G is constant.

Now take the limit as $|z|$ approaches infinity,

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(g,z) = \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) + \lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} G(z).$$

By the hypothesis (5.2.2), $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(g,z) = 0$. By uniform convergence and (5.1.8), $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} \sum_p Q_{z_p}\left(\frac{1}{z-z_p}\right) = 0$. Therefore, $\lim\limits_{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus D_\varepsilon}} G(z) = 0$. Since G is constant, it follows that $G \equiv 0$. Hence, $f_m(g, z)$ has the partial fraction expansion (5.1.4). This completes the proof of Theorem 5.1.2.

Proof of Theorem 5.1.1. To obtain (5.1.3), the partial fraction expansion will be expanded in powers of z. Each of the principal parts are expanded into powers of z as follows:

$$\begin{aligned}
Q_{ik/M} &= \frac{-M^{2n}}{2\pi} \frac{1}{(ik)^{2n+1}} \sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik) \left(\frac{1}{e^{-2\pi ij/h} e^{2\pi k/t} - 1} + 1 \right) \\
&\quad \times \sum_{l=0}^{\infty} \left(\frac{M}{ik} \right)^{l+1} z^l, \\
Q_{-ik/M} &= \frac{M^{2n}}{2\pi} \frac{1}{(ik)^{2n+1}} \sum_{j=0}^{h-1} \frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h} e^{2\pi k/t} - 1} \sum_{l=0}^{\infty} (-1)^l \left(\frac{M}{ik} \right)^{l+1} z^l, \\
Q_{(hk-j)t/(hM)} &= \frac{-M^{2n} h^{2n+1}}{2\pi i t^{2n}} \frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+1} (e^{-2\pi(hk-j)t/h} - 1)} \\
&\quad \times \sum_{l=0}^{\infty} \left(\frac{hM}{t(hk-j)} \right)^{l+1} z^l, \\
Q_{(-hk-j)t/(hM)} &= \frac{M^{2n} h^{2n+1}}{2\pi i t^{2n}} \frac{-g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^{2n+1} (e^{2\pi(hk+j)t/h} - 1)} \\
&\quad \times \sum_{l=0}^{\infty} (-1)^l \left(\frac{hM}{t(hk+j)} \right)^{l+1} z^l, \quad \text{and} \\
Q_0 &= \sum_{l=1}^{2n+2} \frac{1}{z^l} \sum_{r=0}^{2n+2-l} (-1)^{r-1} \frac{B_r}{r!} \sum_{s=0}^{2n+2-l-r} (2\pi M)^{r+s-1} \frac{i^s}{t^s} \frac{h^s}{(s+1)!} \\
&\quad \times \sum_{j=0}^{h-1} \frac{g(j)\varphi_j^{(2n+2-l-r-s)}(0)}{(2n+2-l-r-s)!} (2\beta M)^{2n+2-l-r-s} \sum_{k=0}^{h-1} e^{2\pi ikj/h} B_{s+1}\left(\frac{k}{h} \right).
\end{aligned}$$

We now expand the partial fraction decomposition (5.1.4) in powers of z as follows:

$$
\begin{aligned}
f_m(g,z) = {} & \frac{-M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi j/h}e^{2\pi k/t}-1}+1\right)\\
& \times \sum_{l=0}^{\infty}\left(\frac{M}{ik}\right)^{l+1} z^l\\
& + \frac{M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\sum_{l=0}^{\infty}(-1)^l\left(\frac{M}{ik}\right)^{l+1} z^l\\
& - \frac{M^{2n}h^{2n+1}}{2\pi i t^{2n}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+1}(e^{-2\pi(hk-j)t/h}-1)}\\
& \times \sum_{l=0}^{\infty}\left(\frac{hM}{t(hk-j)}\right)^{l+1} z^l\\
& + \frac{M^{2n}h^{2n+1}}{2\pi i t^{2n}}\sum_{k=0}^{\infty}\sum_{\substack{j=0\\(j,k)\neq(0,0)}}^{h-1}\frac{-g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)}\\
& \times \sum_{l=0}^{\infty}(-1)^l\left(\frac{hM}{t(hk+j)}\right)^{l+1} z^l\\
& + \sum_{l=1}^{2n+2}\frac{1}{z^l}\sum_{r=0}^{2n+2-l}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{2n+2-l-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!}\\
& \times \sum_{j=0}^{h-1}\frac{g(j)\varphi_j^{(2n+2-l-r-s)}(0)}{(2n+2-l-r-s)!}(2\beta M)^{2n+2-l-r-s}\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}
$$

Now, we find the z^p coefficient of the partial fraction expansion for $p \in \mathbb{Z}^+$:

$$
\begin{aligned}
[z^p]f_m(g,z) = &\frac{-M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1}g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right)\\
&\times\left(\frac{M}{ik}\right)^{p+1}\\
&+\frac{M^{2n}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+1}}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}(-1)^p\left(\frac{M}{ik}\right)^{p+1}\\
&+\frac{-M^{2n}h^{2n+1}}{2\pi it^{2n}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+1}(e^{-2\pi(hk-j)t/h}-1)}\\
&\times\left(\frac{hM}{t(hk-j)}\right)^{p+1}\\
&-\frac{M^{2n}h^{2n+1}}{2\pi it^{2n}}\sum_{k=0}^{\infty}\sum_{\substack{j=0\\(j,k)\neq(0,0)}}^{h-1}\frac{g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^{2n+1}(e^{2\pi(hk+j)t/h}-1)}\\
&\times(-1)^p\left(\frac{hM}{t(hk+j)}\right)^{p+1}\\
=&\frac{-M^{2n+p+1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+p+2}}\sum_{j=0}^{h-1}g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right)\\
&+\frac{(-1)^pM^{2n+p+1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^{2n+p+2}}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}\\
&-\frac{M^{2n+p+1}h^{2n+p+2}}{2\pi it^{2n+p+1}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+p+2}(e^{-2\pi(hk-j)t/h}-1)}\\
&+\frac{(-1)^{p+2}M^{2n+p+1}h^{2n+p+2}}{2\pi it^{2n+p+1}}\sum_{k=0}^{\infty}\sum_{\substack{j=0\\(j,k)\neq(0,0)}}^{h-1}\frac{g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^{2n+p+2}(e^{2\pi(hk+j)t/h}-1)}
\end{aligned}
$$

Let $q = 2n + 2 + p$.

$$\begin{aligned}
[z^p]f_m(g,z) = & \frac{-M^{q-1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^q}\sum_{j=0}^{h-1} g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right) \\
& + \frac{(-1)^q M^{q-1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^q}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1} \\
& - \frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^q(e^{-2\pi(hk-j)t/h}-1)} \\
& + \frac{(-1)^{q+2}M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=0}^{\infty}\sum_{\substack{j=0 \\ (j,k)\neq(0,0)}}^{h-1}\frac{g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^q(e^{2\pi(hk+j)t/h}-1)} \qquad (5.1.9)
\end{aligned}$$

In (5.1.9) line 3, replace $hk-j$, by k i.e., $k \equiv -j \bmod h$. And in (5.1.9) line 4, replace $hk+j$ by k, i.e., $k \equiv j \bmod h$. Since g is odd,

$$\begin{aligned}
[z^p]f_m(g,z) &= \frac{M^{q-1}}{2\pi i^q}\sum_{k=1}^{\infty}\frac{1}{k^q} \\
&\quad\times\sum_{j=0}^{h-1}\left(\frac{g(j)((-1)^q\varphi_j(-2\beta ik)+\varphi_{h-j}(2\beta ik))}{e^{2\pi ij/h}e^{2\beta^2k}-1}-g(j)\varphi_j(2\beta ik)\right) \\
&\quad+\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q} \\
&\quad\times\left(\frac{(-1)\varphi_{-k}(2\alpha k/h)}{e^{-2\alpha^2k/h}-1}+\frac{(-1)^q\varphi_k(-2\alpha k/h)}{e^{2\alpha^2k/h}-1}\right) \\
&= \frac{M^{q-1}}{2\pi i^q}\sum_{k=1}^{\infty}\frac{1}{k^q} \\
&\quad\times\sum_{j=0}^{h-1}\left(\frac{g(j)((-1)^q\varphi_j(-2\beta ik)+\varphi_{h-j}(2\beta ik))}{e^{2\pi ij/h}e^{2\beta^2k}-1}-g(j)\varphi_j(2\beta ik)\right) \\
&\quad+\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q} \\
&\quad\times\left(\frac{\varphi_{-k}(2\alpha k/h)}{e^{2\alpha^2k/h}-1}+\frac{(-1)^q\varphi_k(-2\alpha k/h)}{e^{2\alpha^2k/h}-1}+\varphi_{-k}(2\alpha k/h)\right) \\
&= \frac{M^{q-1}}{2\pi i^q}\sum_{k=1}^{\infty}\frac{1}{k^q} \\
&\quad\times\sum_{j=0}^{h-1}\left(\frac{g(j)((-1)^q\varphi_j(-2\beta ik)+\varphi_{h-j}(2\beta ik))}{e^{2\pi ij/h}e^{2\beta^2k}-1}-g(j)\varphi_j(2\beta ik)\right) \\
&\quad+\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q} \\
&\quad\times\left(\frac{\varphi_{-k}(2\alpha k/h)}{e^{2\alpha^2k/h}-1}+\frac{(-1)^q\varphi_k(-2\alpha k/h)}{e^{2\alpha^2k/h}-1}+\varphi_{-k}(2\alpha k/h)\right). \qquad (5.1.10)
\end{aligned}$$

The z^p coefficient of the Laurent expansion (5.1.5), is:

$$[z^p]f_m(g,z) = \sum_{r=0}^{q}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{q-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \times \sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ijk/h} B_{s+1}\left(\frac{k}{h}\right). \tag{5.1.11}$$

Equating (5.1.10) and (5.1.5) gives

$$-\frac{M^{q-1}h^q}{2\pi i t^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{(-1)\varphi_{-k}(2\alpha k/h)}{e^{2\alpha^2 k/h}-1}+\frac{(-1)^{q+1}\varphi_k(-2\alpha k/h)}{e^{2\alpha^2 k/h}-1}-\varphi_{-k}(2\alpha k/h)\right)$$
$$= -\frac{M^{q-1}}{2\pi i^q}\sum_{k=1}^{\infty}\frac{1}{k^q} \times \sum_{j=0}^{h-1}\left(\frac{g(j)((-1)^q\varphi_j(-2\beta ik)+\varphi_{h-j}(2\beta ik))}{e^{2\pi ij/h}e^{2\beta^2 k}-1} - g(j)\varphi_j(2\beta ik)\right)$$
$$+\sum_{r=0}^{q}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{q-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \times \sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ijk/h} B_{s+1}\left(\frac{k}{h}\right).$$

Subtracting the lefthand side from both sides and then replacing α with $\sqrt{\alpha h}$ and β with $\sqrt{\beta/h}$, we get

$$
\begin{aligned}
0 = {} & \frac{M^{q-1}h^q}{2\pi i t^{q-1}} \sum_{k=1}^{\infty} \frac{g(k)}{k^q} \\
& \times \left(\frac{(-1)\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1} + \frac{(-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1} - \varphi_{-k}(2\alpha^{1/2}h^{-1/2}k) \right) \\
& + \frac{M^{q-1}}{2\pi i^q} \sum_{k=1}^{\infty} \frac{1}{k^q} \\
& \times \sum_{j=0}^{h-1} \left(\frac{g(j)((-1)^q \varphi_j(-2\beta^{1/2}h^{-1/2}ik) + \varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik))}{e^{2\pi i j/h}e^{2\beta k/h}-1} - g(j)\varphi_j(2\beta^{1/2}h^{-1/2}ik) \right) \\
& + \sum_{r=0}^{q} (-1)^{r-1} \frac{B_r}{r!} \sum_{s=0}^{q-r} (2\pi M)^{r+s-1} \frac{i^s}{t^s} \frac{h^s}{(s+1)!} \\
& \times \sum_{j=0}^{h-1} g(j) \frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!} (2\beta^{1/2}h^{-1/2}M)^{q-r-s} \sum_{k=0}^{h-1} e^{2\pi i j k/h} B_{s+1}\left(\frac{k}{h}\right) \\
= {} & \frac{\alpha^{(-q+1)/2}}{2} \sum_{k=1}^{\infty} \frac{g(k)}{k^q} \\
& \times \left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k) + (-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1} + \varphi_{-k}(2\alpha^{1/2}h^{-1/2}k) \right) \\
& + \sum_{k=1}^{\infty} \frac{\beta^{(-q+1)/2} i}{2hi^q} \frac{1}{k^q} \\
& \times \sum_{j=0}^{h-1} \left(\frac{g(j)((-1)^q \varphi_j(-2\beta^{1/2}h^{-1/2}ik) + \varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik))}{e^{2\pi i j/h}e^{2\beta k/h}-1} - g(j)\varphi_j(2\beta^{1/2}h^{-1/2}ik) \right) \\
& + \sum_{r=0}^{q} (-1)^r \frac{B_r}{r!} \sum_{s=0}^{q-r} \alpha^{r/2}\beta^{(-q+r+2s+1)/2} h^{-1} 2^{r-1+s} M^{-q+r+s} \frac{i^{s+1}}{(s+1)!} \\
& \times \sum_{j=0}^{h-1} g(j) \frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!} (2\beta^{1/2}h^{-1/2}M)^{q-r-s} \sum_{k=0}^{h-1} e^{2\pi i j k/h} B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}
$$

Rearranging the terms we get

$$\frac{\alpha^{(-q+1)/2}}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^q\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}+\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$

$$=\sum_{k=1}^{\infty}\frac{\beta^{(-q+1)/2}i}{2hi^q}\sum_{j=0}^{h-1}\frac{g(j)}{k^q}\left(\frac{\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)+(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(2\beta^{1/2}h^{-1/2}ik)\right)$$

$$+\frac{1}{h}\sum_{r=0}^{q}(-1)^r\frac{B_r}{r!}\sum_{s=0}^{q-r}\alpha^{r/2}\beta^{(-q+r+2s+1)/2}2^{r-1+s}M^{-q+r+s}\frac{i^{s+1}}{(s+1)!}$$

$$\times\sum_{j=0}^{h-1}g(j)\frac{\varphi^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right),$$

as claimed. This completes the proof of Theorem 5.1.1.

5.2 Even Characters

Theorem 5.2.1. *Let $\alpha,\ \beta,\ t\ >0$ with $\alpha\beta=\pi$ and $t=\alpha/\beta$. Let $h\in\mathbb{Z}^+$ and $1\le j\le h$ and $h\ne 1$. Let m be a positive integer and let $M=m+\frac{1}{2}$. For $\varepsilon>0$, let $D_\varepsilon=\bigcup_{k\in\mathbb{Z}}\{\bigcup_{j=0}^{h-1}D_\varepsilon\left((hk-j)t/(hM)\right)\cup D_\varepsilon\left(ik/M\right)\}$. Let $\varphi_j(z)$ be an entire function, where $\eta\equiv j \bmod h$, $0\le j\le h-1$ and further for all $\eta\in\mathbb{Z}$ define $\varphi_\eta=\varphi_j$. Let $g(j)$ be an even Dirichlet character mod h, and define for each positive integer n*

$$f_m(g,z)=\frac{1}{z^{2n+1}(e^{-2\pi Mz}-1)}\sum_{j=0}^{h-1}\frac{\varphi_j(2\beta Mz)\ g(j)}{e^{2\pi ij/h}e^{2\pi iMz/t}-1}\tag{5.2.1}$$

and assume that

$$\lim_{\substack{|z|\to\infty\\ z\in\mathbb{C}\setminus D_\varepsilon}} f_m(g,z)=0\tag{5.2.2}$$

for some $0<\varepsilon<\min\left(\frac{t}{4hM},\frac{1}{4M}\right)$.
Also, assume that for every j, $j=0,...,h-1$,

$$\sum_{0\ne k\in\mathbb{Z}}\left|\frac{\varphi_j(2\beta ik)}{k^{2n+1}}\right|<\infty$$

and

$$\sum_{0\ne k\in\mathbb{Z}}\left|\frac{\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^{2n+1}}\right|<\infty.$$

Then for $q \in \mathbb{Z}^+$,

$$-\frac{1}{2i}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)-\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(-2\beta^{1/2}h^{-1/2}ik)\right)$$
$$=-\frac{h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}-\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$
$$+\sum_{r=0}^{q}(-1)^{r-2}\frac{B_r}{r!}\sum_{s=0}^{q-1}\pi^{r+s}2^{r+s-1}M^{-q+r+s}\frac{i^s}{t^s}\frac{h^s}{(s+1)!}\sum_{j=0}^{h-1}g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}$$
$$\times\sum_{k=0}^{h-1}e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right). \qquad (5.2.3)$$

Proof. To find the formula for the character analog when g is even, we proceed again by determining the coefficient of the z^p term of the partial fraction expansion and compare it to the z^p term of the Laurent expansion.

From equation (5.1.9), we have:

$$[z^p]f_m(g,z)=\frac{-M^{q-1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^q}\sum_{j=0}^{h-1}g(j)\varphi_j(2\beta ik)\left(\frac{1}{e^{-2\pi ij/h}e^{2\pi k/t}-1}+1\right)$$
$$+\frac{(-1)^qM^{q-1}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(ik)^q}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j(-2\beta ik)}{e^{2\pi ij/h}e^{2\pi k/t}-1}$$
$$-\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\sum_{j=0}^{h-1}\frac{g(j)\varphi_j\left(2\beta(hk-j)t/h\right)}{(hk-j)^q(e^{-2\pi(hk-j)t/h}-1)}$$
$$+\frac{(-1)^{q+2}M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=0}^{\infty}\sum_{\substack{j=0\\(j,k)\neq(0,0)}}^{h-1}\frac{g(j)\varphi_j\left(-2\beta(hk+j)t/h\right)}{(hk+j)^q(e^{2\pi(hk+j)t/h}-1)}.$$

Since g is even,

$$[z^p]f_m(g,z)=\frac{M^{q-1}}{2\pi i^q}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q\varphi_j(-2\beta ik)-\varphi_{h-j}(2\beta ik)}{e^{2\pi ij/h}e^{2\beta^2k}-1}-\varphi_j(2\beta ik)\right)$$
$$+\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha k/h)+(-1)^q\varphi_k(-2\alpha k/h)}{e^{2\alpha^2k}-1}+\varphi_{-k}(2\alpha k/h)\right).$$

Now the z^p of the Laurent expansion is (5.1.11):

$$[z^p]f_m(g,z) = \sum_{r=0}^{q}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{q-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \times \sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right)$$

Now equating the two coefficients, we get:

$$-\frac{M^{q-1}}{2\pi i^q}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q\varphi_j(-2\beta ik)-\varphi_{h-j}(2\beta ik)}{e^{2\pi ij/h}e^{2\beta^2 k}-1}-\varphi_j(-2\beta ik)\right)$$
$$= -\frac{M^{q-1}h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha k/h)+(-1)^{q+1}\varphi_k(-2\alpha k/h)}{e^{2\alpha^2 k}-1}-\varphi_{-k}(2\alpha k/h)\right)$$
$$+\sum_{r=0}^{q}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{q-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \times\sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right).$$

Subtracting the lefthand side from both sides and then replacing α with $\sqrt{\alpha h}$ and β with $\sqrt{\beta/h}$, we get

$$0 = \frac{M^{q-1}}{2\pi i^q} \times\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left((-1)^q\frac{\varphi_j(-2\beta^{1/2}h^{-1/2}ik)-\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(-2\beta^{1/2}h^{-1/2}ik)\right)$$
$$-\frac{h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}-\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k\right)$$
$$+\sum_{r=0}^{q}(-1)^{r-1}\frac{B_r}{r!}\sum_{s=0}^{q-r}(2\pi M)^{r+s-1}\frac{i^s}{t^s}\frac{h^s}{(s+1)!} \times\sum_{j=0}^{h-1} g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1} e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right).$$

Rearranging the terms, the formula for g even is

$$-\frac{1}{2i}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)-\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(-2\beta^{1/2}h^{-1/2}ik)\right)$$
$$=-\frac{h^q}{2\pi it^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}-\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$
$$+\sum_{r=0}^{q}(-1)^{r-2}\frac{B_r}{r!}\sum_{s=0}^{q-1}\pi^{r+s}2^{r+s-1}M^{-q+r+s}\frac{i^s}{t^s}\frac{h^s}{(s+1)!}$$
$$\times\sum_{j=0}^{h-1}g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1}e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right).$$

This completes the proof of Theorem 5.2.1.

5.3 Recover Theorem 1 of [8] for g Odd

Recall (5.1.3)

$$\frac{\alpha^{(1-q)/2}}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^q\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}+\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$
$$=\sum_{k=1}^{\infty}\frac{\beta^{(1-q)/2}i}{2hi^q}\sum_{j=0}^{h-1}\frac{g(j)}{k^q}\left(\frac{\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)+(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(2\beta^{1/2}h^{-1/2}ik)\right)$$
$$+\frac{1}{h}\sum_{r=0}^{q}(-1)^r\frac{B_r}{r!}\sum_{s=0}^{q-r}\alpha^{r/2}\beta^{(-q+r+2s+1)/2}2^{r+s-1}M^{-q+r+s}\frac{i^{s+1}}{(s+1)!}$$
$$\times\sum_{j=0}^{h-1}g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right)$$

Corollary 5.3.1. *Let $L(s,g)$ be defined as in* [8, p. 331], $L(s,g)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$. *If we assume the hypotheses of Theorem 5.1.1 and let $\varphi_j\equiv 1$, then*

$$\alpha^{-q+\frac{1}{2}}\left\{\frac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\alpha n}-1}\right\}$$
$$=(-1)^q\beta^{-q+\frac{1}{2}}\frac{i}{2}h^{-1}\sum_{j=0}^{h-1}g(j)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi ij/h}e^{2n\beta/h}-1}$$
$$+\sum_{r=0}^{q}(-1)^{r+1}\alpha^{-q+r-\frac{1}{2}}\beta^{-r}\zeta(2r)L(2q-2r+1,g).$$

Proof. We begin with (5.1.3) and subtracting the lefthand side from both sides, we get

$$0=-\frac{\alpha^{(1-q)/2}}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^q\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}+\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)$$
$$+\sum_{k=1}^{\infty}\frac{\beta^{(1-q)/2}i}{2hi^q}\sum_{j=0}^{h-1}\frac{g(j)}{k^q}\left(\frac{\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)+(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(2\beta^{1/2}h^{-1/2}ik)\right)$$
$$+\frac{1}{h}\sum_{r=0}^{q}(-1)^r\frac{B_r}{r!}\sum_{s=0}^{q-r}\alpha^{r/2}\beta^{(-q+r+2s+1)/2}2^{r+s-1}M^{-q+r+s}\frac{i^{s+1}}{(s+1)!}\tag{5.3.1}$$
$$\times\sum_{j=0}^{h-1}g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{s+1}\left(\frac{k}{h}\right)\tag{5.3.2}$$

In (5.3.1), let $\varphi\equiv 1$. Then

$$0=-\frac{\alpha^{(-q+1)/2}}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{1+(-1)^q}{e^{2\alpha k}-1}+1\right)$$
$$+\sum_{k=1}^{\infty}\frac{\beta^{(-q+1)/2}i}{2hi^q}\sum_{j=0}^{h-1}\frac{g(j)}{k^q}\left(\frac{1+(-1)^q}{e^{2\pi ij/h}e^{2\beta k/h}-1}-1\right)$$
$$+\sum_{r=0}^{q}(-1)^r\frac{B_r}{r!}\alpha^{r/2}\beta^{(q-r+1)/2}h^{(q-3)/2}2^{q-1}\frac{i^{q-r+1}}{(q-r+1)!}$$
$$\times\sum_{j=0}^{h-1}g(j)\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{q-r+1}\left(\frac{k}{h}\right).$$

For the odd character analog, we have an odd L-function and an odd g function, so replace q by $2q$ and r by $2r$, then

$$\begin{aligned}
0 =& -\alpha^{-q+1/2}\frac{1}{2}\sum_{k=1}^{\infty}\frac{g(k)}{k^{2q}}\left(\frac{1+(-1)^{2q}}{e^{2\alpha k}-1}+1\right) \\
&+\sum_{k=1}^{\infty}\frac{\beta^{-q+1/2}i}{2hi^{2q}}\sum_{j=0}^{h-1}\frac{g(j)}{k^{2q}}\left(\frac{1+(-1)^{2q}}{e^{2\pi ij/h}e^{2\beta k/h}-1}\right) \\
&+\sum_{r=0}^{q}(-1)^{2r}\alpha^{r}\beta^{q-r+1/2}h^{q-3/2}2^{2q-1}\frac{B_{2r}}{(2r)!}\frac{i^{2q-2r+1}}{(2q-2r+1)!} \\
&\quad\times\sum_{j=0}^{h-1}g(j)\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{2q-2r+1}\left(\frac{k}{h}\right) \\
=& -\alpha^{-q+1/2}\left\{\frac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\alpha n}-1}\right\} \\
&+(-1)^{q}\beta^{-q+1/2}ih^{-1}\sum_{j=0}^{h-1}g(j)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi ij/h}e^{2n\beta/h}-1} \\
&+\sum_{r=0}^{q}\alpha^{r}\beta^{q-r+1/2}h^{q-3/2}2^{2q-1}\frac{B_{2r}}{(2r)!}\frac{(-1)^{q-r}i}{(2q-2r+1)!} \\
&\quad\times\sum_{j=0}^{h-1}g(j)\sum_{k=0}^{h-1}e^{2\pi ikj/h}B_{2q-2r+1}\left(\frac{k}{h}\right). \qquad (5.3.3)
\end{aligned}$$

Now, we substitute the following identity (5.3.4) as given by Bradley [8, pp. 333-335]

$$L(2q-2r+1,g)=\frac{1}{2}\frac{(2\pi i)^{2q-2r+1}}{(2q-2r+1)!}\sum_{k=0}^{h-1}\sum_{j=0}^{h-1}\frac{1}{h}g(j)e^{-2\pi ijk/h}B_{2q-2r+1}\left(\frac{k}{h}\right) \qquad (5.3.4)$$

into (5.3.3) and we get

$$\begin{aligned}0 &= -\alpha^{-q+1/2}\left\{\frac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\alpha n}-1}\right\}\\&+(-1)^q\beta^{-q+1/2}ih^{-1}\sum_{j=0}^{h-1}g(j)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi ij/h}e^{2n\beta/h}-1}\\&+\sum_{r=0}^{q}\alpha^{-q+r-1/2}2^{2r-1}\frac{B_{2r}}{(2r)!}(-1)^{q-r}i^{-2q+2r}L(2q-2r+1,g).\end{aligned} \tag{5.3.5}$$

Now, we substitute (3.3.2) into (5.3.5) and we get

$$\begin{aligned}0 &= -\alpha^{-q+1/2}\left\{\frac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\alpha n}-1}\right\}\\&+(-1)^q\beta^{-q+1/2}ih^{-1}\sum_{j=0}^{h-1}g(j)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi ij/h}e^{2n\beta/h}-1}\\&+\sum_{r=0}^{q}(-1)^{r+1}\alpha^{-q+r-1/2}\beta^{-r}\zeta(2r)L(2q-2r+1,g).\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha^{-q+1/2}&\left\{\frac{1}{2}L(2q,g)+\sum_{n=1}^{\infty}\frac{n^{-2q}g(n)}{e^{2\alpha n}-1}\right\}\\&=(-1)^q\beta^{-q+1/2}ih^{-1}\sum_{j=0}^{h-1}g(j)\sum_{n=1}^{\infty}\frac{n^{-2q}}{e^{2\pi ij/h}e^{2n\beta/h}-1}\\&+\sum_{r=0}^{q}(-1)^{r+1}\alpha^{-q+r-1/2}\beta^{-r}\zeta(2r)L(2q-2r+1,g),\end{aligned}$$

as claimed. This completes the proof of Corollary 5.3.1.

5.4 Recover Theorem 1 of [8] for g Even

Recall (5.1.3)

$$
\begin{aligned}
&-\frac{1}{2i}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q\varphi_j(-2\beta^{1/2}h^{-1/2}ik)-\varphi_{h-j}(2\beta^{1/2}h^{-1/2}ik)}{e^{2\pi ij/h}e^{2\beta k/h}-1}-\varphi_j(-2\beta^{1/2}h^{-1/2}ik)\right)\\
&=-\frac{h^q}{2\pi i t^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)+(-1)^{q+1}\varphi_k(-2\alpha^{1/2}h^{-1/2}k)}{e^{2\alpha k}-1}-\varphi_{-k}(2\alpha^{1/2}h^{-1/2}k)\right)\\
&+\sum_{r=0}^{q}(-1)^{r-2}\frac{B_r}{r!}\sum_{s=0}^{q-1}\pi^{r+s}2^{r+s-1}M^{-q+r+s}\frac{i^s}{t^s}\frac{h^s}{(s+1)!}\sum_{j=0}^{h-1}g(j)\frac{\varphi_j^{(q-r-s)}(0)}{(q-r-s)!}(2\beta^{1/2}h^{-1/2}M)^{q-r-s}\\
&\qquad\times\sum_{k=0}^{h-1}e^{2\pi ijk/h}B_{s+1}\left(\frac{k}{h}\right).
\end{aligned}
$$

Corollary 5.4.1. *Let $L(s,g)$ be defined as in* [8, p. 331]. *If $\varphi\equiv 1$, then*

$$
\begin{aligned}
&-\alpha^{-q}\left(\frac{1}{2}L(2q+1,g)+\sum_{n=1}^{\infty}\frac{g(n)n^{-2q-1}}{e^{2\alpha n}-1}\right)\\
&=(-\beta)^{-q}\left(h^{-1}\frac{1}{2}\sum_{n=1}^{\infty}g(n)\zeta(2q+1)+\frac{1}{h}\sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2\pi ij/h}e^{2\beta n/h}-1}\right)\\
&+\sum_{r=0}^{q+1}(-1)^{r+1}\alpha^{-q-1+r}\beta^{-r}h^{2q-r+2}\zeta(2n)L(2q-2r+2,g). \qquad (5.4.1)
\end{aligned}
$$

Proof. From (5.2.3), subtract the lefthand side from both sides and we get

$$
\begin{aligned}
0=&\frac{1}{2i}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^q}\left(\frac{(-1)^q-1}{e^{2\pi ij/h}e^{2\beta k/h}-1}-1\right)\\
&-\frac{h^q}{2\pi i t^{q-1}}\sum_{k=1}^{\infty}\frac{g(k)}{k^q}\left(\frac{1+(-1)^{q+1}}{e^{2\alpha k}-1}-1\right)\\
&+2^{q-1}\sum_{r=0}^{q}(-1)^r\pi^q\frac{B_r}{r!}\frac{i^{q-r}}{t^{q-r}}\frac{h^{q-r}}{(q-r+1)!}\sum_{j=0}^{h-1}g(j)\\
&\times\sum_{k=0}^{h-1}e^{2\pi ijk/h}B_{q-r+1}\left(\frac{k}{h}\right).
\end{aligned}
$$

For g even, $q = 2q+1$ and $r = 2r$, then

$$0 = \beta^{-2q}\frac{1}{2}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^{2q+1}}\left(\frac{2}{e^{2\pi ij/h}e^{2\beta k/h}-1}-1\right)$$
$$-\alpha^{-2q}\frac{h^q}{2}\sum_{j=0}^{h-1}\sum_{k=1}^{\infty}\frac{g(j)}{k^{2q+1}}\left(\frac{2}{e^{2\alpha k}-1}-1\right)$$
$$+\sum_{r=0}^{q}(-1)^{2r}2^{2r-1}i^{-1}\alpha^{(-2q-3+2r)/2}\beta^{(2q-1)/2}h^{2q-r+2}\frac{B_r}{r!}L(2q-2r+2,g).$$

Replace α by $h\alpha$ and β by β/h, then

$$0 = \beta^{-q}(-1)^{q+1}h^{-1}\frac{1}{2}\sum_{n=1}^{\infty}\frac{g(n)}{n^{2q+1}}\left(\frac{2}{e^{2\pi ij/h}e^{2\beta n/h}-1}+1\right)$$
$$-\alpha^{-q}\left(\frac{h}{t}\right)^{2q}\frac{1}{2}\sum_{n=1}^{\infty}\frac{g(n)}{n^{2q+1}}\left(\frac{2}{e^{2\alpha n}-1}-1\right)$$
$$+\sum_{r=0}^{q+1}(-1)^{r}\alpha^{-q-1+r}\beta^{-r}h^{2q-r+2}\zeta(2n)L(2q-2r+2,g)$$
$$= -(-\beta)^{-q}h^{-1}\frac{1}{2}\sum_{n=1}^{\infty}g(n)\zeta(2q+1)+\frac{1}{h}\sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2\pi ij/h}e^{2\beta n/h}-1}$$
$$-\alpha^{-q}\frac{1}{2}L(2q+1,g)+\sum_{n=1}^{\infty}\frac{g(n)n^{-2q-1}}{e^{2\alpha n}-1}$$
$$+\sum_{r=0}^{q+1}(-1)^{r}\alpha^{-q-1+r}\beta^{-r}h^{2q-r+2}\zeta(2n)L(2q-2r+2,g).$$

Rearranging the terms we get

$$-\alpha^{-q}\frac{1}{2}L(2q+1,g)+\sum_{n=1}^{\infty}\frac{g(n)n^{-2q-1}}{e^{2\alpha n}-1}$$
$$= (-\beta)^{-q}h^{-1}\frac{1}{2}\sum_{n=1}^{\infty}g(n)\zeta(2q+1)+\frac{1}{h}\sum_{n=1}^{\infty}\frac{n^{-2q-1}}{e^{2\pi ij/h}e^{2\beta n/h}-1}$$
$$+\sum_{r=0}^{q+1}(-1)^{r+1}\alpha^{-q-1+r}\beta^{-r}h^{2q-r+2}\zeta(2n)L(2q-2r+2,g),$$

as claimed. This completes the proof of Corollary 5.4.1.

5.5 Recover Ramanujan's Formula for $\zeta(2n+1)$

Ramanujan's formula for $\zeta(2n+1)$ can be recovered by letting $h = 1$ and by letting $g(0) = 1$, [8].

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Katherine Merrill was born in Ohio on June 1st, 1961. She received her B.S. in Mathematics from The Ohio State University in 1982, where she studied Statistics with Dr. D. Ransom Whitney.

Upon graduation, she was employed as a statistician by Western Electric at the Columbus Works. From 1982 until 1989, she worked in the field of Software Quality Analysis, implementing both a Software Quality Assurance Program for the OS Product Line in Columbus, Ohio and a Software Quality Control Program for the Transmission Product Line in North Andover, Massachusetts. Katherine worked with many from Bell Laboratories and AT&T, and with outside consultants such as Deming and Juran, to establish software quality standards as a corporate initiative.

Katherine received her M.A. in Mathematics (Statistics and Probability) from Boston University in 1988.

In 1989, Katherine and her husband Robert Merrill divested themselves of all their worldy possessions and travelled extensively for a year, backpacking through the South Pacific and Asia. Upon their return to New England, they settled in Camden, Maine. There, Robert is a Family Practice Physician, Katherine is adjunct faculty for The University of Maine at Augusta, and they have two daughters, Amelia and Francie.

Katherine is a candidate for the Master of Arts degree in Mathematics from The University of Maine in May, 2005.