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An Elementary Proof of the Prime Number Theorem

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INTRODUCTION

Among the oldest and most intriguing problems of number theory is that of the distribution of prime numbers. The most important result yet obtained in this area is the Prime Number Theorem. In order to discuss it, we shall find the following introductory material particularly useful.

Definition. An integer $p > 1$ that is not the product of two other positive integers, both smaller than p, is called a prime number; an integer that is not prime is called a composite.

Thus, for example, the numbers 2,3,5,7,11, and 13 are prime, whereas 4,6,8,9,10, and 12 are composite.

It has been known since antiquity (Euclid, about 300 B.C.) that the number of primes is infinite. The proof of this fact is quite short: Let $2,3,5,\ldots, p$ be the series of primes up to p. Now form the number $q=(2\cdot 3\cdot 5\cdot \ldots \cdot p)+1$. Since q is clearly not divisible by any of the numbers $2,3,5,\ldots p$, it is therefore either itself prime or is divisible by a prime between p and q. In either case there is a prime greater than p.

However, there are also arbitrarily large gaps in the series of primes: Let k be any positive integer. Now consider the integers $(k+1)!+2$, $(k+1)!+3, \ldots,$ $(k+1)!+k$, $(k+1)!+k+1$, where $n!$ is defined to be $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$. Each of these is composite since j divides $(k+1)!+j$ if $2 \leq j \leq k+1$. Thus given any positive integer k, there exist k consecutive composite integers.

It is obviously important to know something about the occurrence of prime numbers among the natural numbers. Let $\pi(x)$ denote the number of primes that do not exceed x. Because of the irregular occurrence of the primes, we cannot expect a simple formula for $\pi(x)$.

In a notebook published posthumously, Gauss (1777-1855) conjectures that

$$\lim_{x\to\infty} \frac{\pi(x)}{Li\ x} = 1,$$

where Li x= $\int_2^x dt/\log t$. He arrived at this conjecture by observing that the primes seem to have an asymptotic density which is 1/log x at x . Legendre (1752-1833) conjectured that

$$L = \lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1 \text{ ; that is, } \pi(x) \sim \frac{x}{\log x}$$

This conjecture is today known as the <u>Prime Number Theorem</u> (henceforth denoted PNT). Gauss's conjecture is the more profound since it has since been shown that

$$Li\ x = \frac{x}{\log x} + \frac{1!\,x}{(\log x)^2} + \dots + \frac{g!\,x}{(\log x)^{g+1}}[1+\varepsilon(x)],$$

where $\varepsilon(x) \longrightarrow 0$ as x becomes infinite. In an attempt to prove the PNT, Tchebycheff (1821-1894) showed that

$$(0.92\cdots)\frac{x}{\log x} \leq 1 \leq (1.105\cdots)\frac{x}{\log x}.$$

He also proved that if the limit L in Legendre's conjecture exists, then L=1.

In 1859, Riemann (1826-1866) approached the problem indirectly and connected it with the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

He was not completely successful; his proof had several serious gaps. The most important of these could not be filled until the properties of entire functions had been established. In 1876, using Hadamard's theory of entire functions, J. Hadamard (1865-1963) and de la Vallée Poussin (1866-1962) succeeded in proving the PNT. Several of the gaps which still remained have since been taken care of. However, the so-called Riemann Hypothesis (if $\zeta(x+iy)=0$, then $x=\frac{1}{2}$), which is most important to a more nearly precise formulation of the PNT, has so far defied all attempts at a proof or refutation.

In 1948, Atle Selberg and P. Erdös succeeded in finding an "elementary" proof of the PNT. Here "elementary" is used in the sense of avoiding the use of complex variables, Fourrier analysis, and similar non-elementary methods employed previously. That it should not be construed to mean "easy" will soon become apparent. The rest of this paper is devoted to such a proof of the PNT.*

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*Specifically, the following proof follows closely that given by J.G. van der Corput (see Bibliography) which is based on notes from the conference Erdös gave at Amsterdam for the "Wiskundig Genootschap" in October, 1948.

PROOF OF THE PRIME NUMBER THEOREM

In the theory of numbers we usually consider, not the function $\pi(x)$, but the function

$$\vartheta(x) = \sum_{p\le x} \log p ,$$

which is much easier to work with. The above sum is extended over all prime numbers $p \le x$. It is sufficient to show that $\frac{\vartheta(x)}{x}$ approaches 1 as x becomes infinite.

Now $\vartheta(x) = \sum_{p\le x} \log p \le \log x \sum_{p \le x} 1 = \pi(x) \log x.$

For $1 \le y \le x$, one has

$$\pi(x) - y \le \pi(x) - \pi(y) \le \frac{1}{\log y}\left(\vartheta(x) - \vartheta(y)\right) \le \frac{\vartheta(x)}{\log y}.$$

Thus

(1)
$$\frac{\vartheta(x)}{x} \le \frac{\pi(x)\log x}{x} \le \frac{y \log x}{x} + \frac{\log x}{\log y} \cdot \frac{\vartheta(x)}{x}$$

Let us choose $y = x^{\rho(x)}$, where $\rho(x) < 1$ and $\rho(x) \longrightarrow 1$ slowly enough so that

$$\frac{\log x}{x^{1-\rho(x)}} \longrightarrow 0 \text{ as } x \to \infty .$$

Such a $\rho(x)$ exists: Let $\rho(x) = 1 - \frac{1}{\log \log x}$.

Then $\lim_{x\to\infty}\left(1 - \frac{1}{\log\log x}\right) = 1.$

Now show

$$\lim_{x\to\infty} x^{1-\rho(x)} = \lim_{x\to\infty} x^{\frac{1}{\log \log x}} = \infty \; :$$

$$\lim_{x\to\infty} x^{\frac{1}{\log\log x}} = e^{\lim_{x\to\infty}\left(\frac{1}{\log\log x}\cdot \log x\right)} = e^{\lim_{x\to\infty}\left(\frac{1/x}{1/x\log x}\right)} = e^{\lim_{x\to\infty} \log x} = \infty$$

Now we can use L'Hopital's Rule to show $\lim_{x\to\infty}\left(\frac{\log x}{x^{1-\rho(x)}}\right) = \infty$:

$$\lim_{x\to\infty}\left(\frac{\log x}{x^{\frac{1}{\log\log x}}}\right) = \lim_{x\to\infty} \frac{\frac{1}{x}}{\left[\frac{1}{\log\log x}\, x^{-\left(1-\frac{1}{\log\log x}\right)}\right] + \left[x^{\frac{1}{\log\log x}}\left(\frac{1}{x\log x}\right)\left(\frac{-1}{[\log\log x]^2}\right)\right]}$$

$$= \lim_{x\to\infty} \frac{1}{x^{\frac{1}{\log\log x}}\left[\frac{1}{\log\log x} - \frac{1}{(\log\log x)^2}\right]}$$

$$= \lim_{x\to\infty} \frac{\frac{(\log\log x)^2}{(\log\log x)-1}}{x^{\frac{1}{\log\log x}}}$$

$$= \lim_{x\to\infty} \frac{\frac{\left[(\log\log x - 1)\, 2(\log\log x)\, \frac{1}{x\log x}\right] - (\log\log x)^2\, \frac{1}{x\log x}}{(\log\log x - 1)^2}}{\left(\frac{1}{x}\right)\left(x^{\frac{1}{\log\log x}}\right)(\log\log x - 1)\, \frac{1}{(\log\log x)^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{\log\log x}{x\log x}\, (\log\log x - 2)\, \frac{1}{(\log\log x - 1)^2}}{x^{\frac{1}{\log\log x}}\, (\log\log x - 1)\, \frac{1}{x(\log\log x)^2}}$$

$$= \lim_{x\to\infty} \frac{(\log\log x)^3\, (\log\log x - 2)}{(\log\log x - 1)^3\, x^{\frac{1}{\log\log x}} \cdot x\log x}$$

$$= \lim_{x\to\infty} \frac{\log\log x}{x^{\frac{1}{\log\log x}} \cdot x\log x} = 0, \text{ since } \lim_{x\to\infty} \frac{\log\log x}{\log x} = 0.$$

Equation (1) now becomes

$$\frac{\vartheta(x)}{x} \leqslant \frac{\pi(x)\log x}{x} \leqslant \frac{\log x}{x^{1-\rho(x)}} + \frac{1}{\rho(x)} \cdot \frac{\vartheta(x)}{x}$$

Thus $\frac{\vartheta(x)}{x} \longrightarrow 1$ implies that $\frac{\pi(x)\log x}{x} \longrightarrow 1$ (i.e., PNT).

Our proof is divided into two distinct parts. The first is devoted almost exclusively to the proof of Selberg's formula:

$$\frac{\vartheta(x)}{x} + \frac{2}{x\log x}\sum_{p\leqslant\sqrt{x}} \vartheta\left(\frac{x}{p}\right)\log p \longrightarrow 2 \text{ as } x\longrightarrow\infty .$$

In the second part we will deduce the PNT from this formula.

I <u>Proof of Selberg's Formula</u>.

Let $\mu(m)$ denote the Möbius function, which is defined as follows:

i) $\mu(1)=1$;

ii) $\mu(m)=0$, if p^2 divides m for some p;

iii) $\mu(m)=(-1)^r$, where $m=p_1p_2 \ldots p_r$ is the decomposition of m into prime factors.

Thus we have $\mu(1)=1$, $\mu(2)=-1$, $\mu(3)=-1$, $\mu(4)=0$, $\mu(5)=-1$, $\mu(6)=1$, etc.

The function $\mu(m)$ is multiplicative; that is, $\mu(ab)=\mu(a)\cdot\mu(b)$, if a and b are relatively prime natural numbers. This assertion results immediately from the way in which we have defined $\mu(m)$.

<u>Lemma 1</u>. For each integer $h\geqslant 0$, the function

$$\varphi_h(m) = \sum_{d|m} \mu(d)\log^h d$$

(where the sum is extended over all divisors of m, including 1 and m) equals zero, if the natural number m contains more than h different prime factors.

Remark: We will use this lemma only for h=0,1,2.

Proof: For h=0 the function becomes

$$\sum_{d|m} \mu(d) = 0 \tag{2}$$

for all integers $m > 1$ (the sum equals 1 for m=1).
This formula is evident. Let $m=p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the decomposition of m into prime factors. Since $\mu(d)=0$ if p_i^2 divides d for some p_i, only $1 + \binom{r}{1} + \binom{r}{2} + \dots + \binom{r}{r} = 2^r$ divisors of m have to be considered in the sum. Now the divisor d=1 contributes 1, the $\binom{r}{1}$ divisors $p_1, p_2, \dots, p_r$ each contribute -1, the $\binom{r}{2}$ divisors $p_1p_2, p_1p_3, \dots, p_{r-1}p_r$ each contribute + 1, etc. Therefore the total contribution is $1 - \binom{r}{1} + \binom{r}{2} - \dots \pm \binom{r}{r} = (1-1)^r = 0$. Thus the lemma is shown for h=0.

We now use finite induction on h. Assume the lemma has been shown for $h \leq k-1$. Let h=k. Also let $m=p^{\alpha}b$, where $\alpha \geq 1$ and where the integer b is not divisible by the prime number p. We then have

$$\begin{aligned}
\varphi_k(m) &= \sum_{d|bp^{\alpha}} \mu(d) \log^k d \\
&= \sum_{d_1|b,\, d_2|p^{\alpha}} \mu(d_1 d_2)(\log d_1 + \log d_2)^k \\
&= \sum_{n=0}^{k} \binom{k}{n} \sum_{d_1|b} (\mu(d_1) \log^n d_1) \sum_{d_2|p^{\alpha}} (\mu(d_2) \log^{k-n} d_2) \\
&= \sum_{n=0}^{k} \binom{k}{n} \varphi_n(b)\, \varphi_{k-n}(p^{\alpha}).
\end{aligned}$$

Since m contains more than k different prime factors, b contains more than k-1 different prime factors. Therefore by our inductive hypothesis $\varphi_n(b)=0$ for $n=1,2,\ldots,k-1$. The remaining term $\varphi_k(b)\,\varphi_o(p^\alpha)=0$ since $\varphi_o(p^\alpha)=0$, as we have already shown. Therefore the lemma is true for all h.

<u>Lemma 2</u>. Let $x > 0$ and set

$$\lambda(d) = \mu(d)\log^2\frac{x}{d} \qquad \text{and} \qquad f(m) = \sum_{d|m}\lambda(d)$$

(these are clearly also functions of x). Then we have

$f(1)=\log^2 x$;

$f(p^\alpha)=-\log^2 p + 2(\log x)(\log p)$, p prime, $\alpha \geq 1$;

$f(p^\alpha q^\beta)=2(\log p)(\log q)$, p,q different primes, $\alpha,\beta \geq 1$;

$f(m)=0$, if m contains more than two different prime factors.

Proof. $f(1) = \sum_{d|1}\lambda(d) = \lambda(1) = \mu(1)\log^2 x = \log^2 x$.

$$\begin{aligned} f(p^\alpha) = \sum_{d|p^\alpha}\lambda(d) &= \lambda(1) + \lambda(p) + \cdots + \lambda(p^\alpha) \\ &= \mu(1)\log^2 x + \mu(p)\log^2\frac{x}{p} + \cdots + \mu(p^\alpha)\log^2\frac{x}{p^\alpha} \\ &= \log^2 x - (\log^2 x - 2\log x\log p + \log^2 p) + 0 \\ &= -\log^2 p + 2\log x\log p. \end{aligned}$$

$$\begin{aligned} f(p^\alpha q^\beta) &= \mu(1)\log^2 x + \mu(p)\log^2\frac{x}{p} + \mu(q)\log^2\frac{x}{q} + \mu(pq)\log^2\frac{x}{pq} + 0 \\ &= \log^2 x - (\log x - \log p)^2 - (\log x - \log q)^2 + (\log x - \log pq)^2 \\ &= 2(\log p)(\log q). \end{aligned}$$

$$\begin{aligned} f(m) = \sum_{d|m}\lambda(d) &= \sum_{d|m}\left(\log^2\frac{x}{d}\right)\mu(d) \\ &= \sum_{d|m}\mu(d)\log^2 x - 2\log x\sum_{d|m}\mu(d)\log d + \sum_{d|m}\mu(d)\log^2 d \\ &= \sum \end{aligned}$$

$$= \log^2 x \cdot 0 \quad - 2 \log x \cdot 0 + 0$$
$$= 0$$

by the previous lemma.

Lemma 3. For $x \geq 2$ the quotient $\frac{\vartheta(x)}{x}$ is situated between the two fixed positive bounds. (Tchebycheff's Theorem, 1851-1852)

Remark: For the proof of Selberg's formula it suffices to know that the upper limit of $\frac{\vartheta(x)}{x}$ is finite. We prove here that the lower limit is positive because the proof is quite similar and because we shall need this result in applying Selberg's formula to the proof of the PNT.

Proof:1. Let us first show that the greatest limit of $\frac{\vartheta(x)}{x}$ is finite. Consider the natural number $P = \frac{(2n)!}{n!\,n!}$. Since the prime factors $p > n$ and $\leq 2n$ appear in the numerator and not in the denominator of P, the product of these primes is a divisor of P, and thus is at most equal to $1 + \binom{2n}{1} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{1} + 1 = (1+1)^{2n} = 2^{2n}$. The logarithm of this product is equal to $\sum_{n<p\leq 2n} \log p = \vartheta(2n) - \vartheta(n)$. It follows that $\vartheta(2n) - \vartheta(n) \leq \log 2^{2n} = 2n \log 2$. For $x = 2^m$ (m an integer ≥ 1), we obtain

$$\begin{aligned} \vartheta(x) &= \{\vartheta(2^m) - \vartheta(2^{m-1})\} + \{\vartheta(2^{m-1}) - \vartheta(2^{m-2})\} + \cdots \\ &< (2^m + 2^{m-1} + \cdots + 2) \log 2 \\ &< 2^{m+1} \log 2 \\ &= 2x \log 2 . \end{aligned}$$

For $2^{m-1} < x \leq 2^m$, we therefore have

$$\vartheta(x) \leq \vartheta(2^m) < 2^{m+1}\log 2 < 4x \log 2,$$

so that we find that for each number $x > 1$

$$\frac{\vartheta(x)}{x} < 4 \log 2.$$

2. Now let us show that the least limit of $\frac{\vartheta(x)}{x}$ is positive. The series $1,2,\ldots,n$ contains $\left[\frac{n}{p}\right]$ multiples of the prime number p (where $[x]$ designates the greatest integer $\leq x$), $\left[\frac{n}{p^2}\right]$ multiples of p^2, etc. Therefore $n!$ contains precisely $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots$ factors of p. Consequently, the number of factors p appearing in P is exactly equal to

$$Q = \sum\left\{\left[\frac{2n}{p^\alpha}\right] - 2\left[\frac{n}{p^\alpha}\right]\right\},$$

where the sum is extended over all natural numbers α such that $p^\alpha \leq 2n$. The number of terms of Q is $\leq \left[\frac{\log 2n}{\log p}\right]$.

$[2y]-2[y]$ is a function of y with period one which equals zero in the interval $0 \leq y < \frac{1}{2}$ and equals one in the interval $\frac{1}{2} \leq y < 1$. Therefore $[2y]-2[y]$ is always ≤ 1, from which it follows that $Q \leq \left[\frac{\log 2n}{\log p}\right]$, so that P divides the integer U defined by

$$\log U = \sum_{p \leq 2n}\left[\frac{\log 2n}{\log p}\right]\log p.$$

Since

$$P = \frac{(n+1)(n+2)\cdots(2n)}{1\cdot 2\cdot \cdots \cdot n} \geq 2^n,$$

we have

$$n \log 2 \leq \log P \leq \log U = \sum_{p \leq 2n}\left[\frac{\log 2n}{\log p}\right]\log p.$$

We also have

$$\left[\frac{\log 2n}{\log p}\right] \leq \frac{\log 2n}{\log p}, \text{ for all } p$$

and $\left[\frac{\log 2n}{\log p}\right] = 1$, for $p > \sqrt{2n}$.

Thus $$n \log 2 \le \sum_{p \le \sqrt{2n}} \log 2n + \sum_{p \le 2n} \log p$$

$$\le \sqrt{2n} \log 2n + \vartheta(2n).$$

We therefore find that $\vartheta(2n) \geq (n \log 2) - (\log 2n)\sqrt{2n}$. But by l'Hopital's Rule, we know $\lim_{n\to\infty} \frac{(\log 2n)\sqrt{2n}}{\frac{1}{2}(n-1)\log 2}$

$$= \lim_{n\to\infty} \frac{\frac{\log 2n}{\sqrt{2n}} + \frac{\sqrt{2n}}{2n}}{\frac{1}{2}\log 2} = \frac{1}{2}\log 2 \lim_{n\to\infty} \frac{\log 2n + 1}{\sqrt{2n}} = 0.$$

Therefore for $\varepsilon = 1$, $\exists$ (there exists) $N > 0 \ni$ (such that) $n > N \implies$ (implies)

$$\left| \frac{(\log 2n)\sqrt{2n}}{\frac{1}{2}(n-1)\log 2} \right| = \frac{(\log 2n)\sqrt{2n}}{\frac{1}{2}(n-1)\log 2} \le \varepsilon = 1.$$

$$\Rightarrow (\log 2n)\sqrt{2n} \le \frac{(n-1)}{2}\log 2 = n\log 2 - \tfrac{1}{2}(n+1)\log 2, \text{ for all } n > N.$$

Thus $\vartheta(2n) \geq \frac{1}{2}(n+1)\log 2$ for sufficiently large n. If x is sufficiently large, and if we let $2n \le x < 2n+2$, we obtain

$$\vartheta(x) \ge \vartheta(2n) \ge \tfrac{1}{2}(n+1)\log 2 > \tfrac{1}{4} x \log 2$$

from which it follows that for $x > 2$, $\frac{\vartheta(x)}{x}$ has a positive lower bound.

Remark: The above reasoning leads us to another formula which we shall not need for the proof of Selberg's formula, but which we shall use in applying this formula. We have stated that n! contains the prime factor p exactly

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots$$

times, thus $\log n! = \sum_{p \leq n} \left(\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots \right) \log p$.
Since this sum $< \sum_{p \leq n} \left(\left[\frac{n}{p}\right] + \frac{n}{p^2} + \cdots \right) \log p = \sum_{p \leq n} \left[\frac{n}{p}\right] \log p + n \sum_{p} \frac{\log p}{p(p-1)}$, we find that $\log n!$ is approximately equal to $\sum_{p \leq n} \left[\frac{n}{p}\right] \log p$ and their difference is at most of the order of n since $n \sum_{p} \frac{\log p}{p(p-1)}$ converges. Now $\sum_{p \leq n} \left[\frac{n}{p}\right] \log p$ is approximately equal to $\sum_{p \leq n} \frac{n}{p} \log p$ and their difference is at most equal to $\sum_{p \leq n} \log p = \vartheta(n)$, which by Lemma 3 (1) is at most the same order as n. In this manner we obtain that $\sum_{p \leq n} \frac{\log p}{p}$ is equal to

$$\frac{1}{n} \log n! = \frac{1}{n} \sum_{h=2}^{n} \log h,$$

to within a bounded term. We have that for every integer $h \geq 2$,

$$\log h = \log \left(\frac{h^h}{(h-1)^{h-1}} \cdot \left(\frac{h-1}{h}\right)^{h-1} \right)$$

$$= h \log h - (h-1) \log (h-1) - (h-1) \log \left(1 + \frac{1}{h-1}\right).$$

But for any natural number n we know from the Taylor series expansion that $\log(1+\frac{1}{n}) = \frac{1}{n} - \frac{\theta_n}{n^2}$, where $0 < \theta_n < 1$.
Thus $n \log(1+\frac{1}{n}) = 1 - \frac{\theta_n}{n}$ is bounded. Consequently,

$$\sum_{h=2}^{n} \log h = \sum_{h=2}^{n} \{ h \log h - (h-1) \log (h-1) + O(1) \},$$

(where "f(x) is O(g(x))" denotes that $|f(x)| < M \cdot g(x)$, for some positive constant M and for sufficiently large x.)

$$= n \log n - 0 + O(n).$$

Thus

$$\sum_{p \leq n} \frac{\log p}{p} = \frac{1}{n} \sum_{h=2}^{n} \log h + O(1) = \log n + O(1).$$

Lemma 4: $\frac{1}{x \log x} \sum_{p \le x} (\log p)(\log \frac{x}{p}) \to 0$ as $x \to \infty$.

Proof. Let $\varepsilon > 0$. We have $\log \frac{x}{p} < \log \frac{1}{\varepsilon}$ for $p > \varepsilon x$, so that the expression in question is at most equal to

$$\frac{1}{x \log x}\left(\sum_{p \le \varepsilon x} (\log p)(\log x) + \sum_{p \le x} (\log p)(\log \tfrac{1}{\varepsilon}) \right)$$

$$= \frac{\vartheta(\varepsilon x)}{x} + \frac{\vartheta(x)}{x} \cdot \frac{\log 1/\varepsilon}{\log x} < c\left\{ \varepsilon + \frac{\log 1/\varepsilon}{\log x} \right\},$$

where c is the upper bound of Lemma 3(1). When $\varepsilon \to 0$ slowly enough so that $\frac{\log 1/\varepsilon}{\log x} \to 0$

($\varepsilon(x) = \frac{1}{\log x}$ satisfies these conditions), we obtain the desired result.

Lemma 5:* $\frac{q^\beta}{x \log x} \sum_{p^\alpha \le \frac{x}{q^\beta}} \log p \to 0$ as $x \to \infty$,

the sum extending over all primes p and integers α such that $p^\alpha \le \frac{x}{q^\beta}$.

Proof. $\sum_{p^\alpha \le \frac{x}{q^\beta}} \log p = \vartheta(\frac{x}{q^\beta}) + \vartheta(\sqrt{\frac{x}{q^\beta}}) + \cdots + \vartheta(\sqrt[k]{\frac{x}{q^\beta}})$,

where k is the greatest integer such that $2^k \le \frac{x}{q^\beta}$; i.e.,

$$k \le \frac{\log \frac{x}{q^\beta}}{\log 2}.$$

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*This lemma is not as it originally appears in van der Corput's proof. I have altered it slightly so that it more nearly directly applies to the proof of Lemma 6.

Thus $$\sum_{p^\alpha \le \frac{x}{q^\beta}} \log p \le \vartheta\left(\frac{x}{q^\beta}\right) + k\,\vartheta\left(\sqrt{\frac{x}{q^\beta}}\right)$$
$$\le \vartheta\left(\frac{x}{q^\beta}\right) + \frac{\log x/q^\beta}{\log 2}\,\vartheta\left(\sqrt{\frac{x}{q^\beta}}\right).$$

We know that $\frac{\vartheta(y)}{y}$ is bounded above $\forall$ (for all) $y \geq 2$ and equals 0 for $1 \leq y < 2$.

Thus $$0 \le \lim_{x\to\infty} \frac{\vartheta\left(\frac{x}{q^\beta}\right)}{\frac{x}{q^\beta}\log x} \le \lim_{x\to\infty} \frac{M}{\log x} = 0, \text{ for some } M \ge 0;$$

and $$0 \le \lim_{x\to\infty} \frac{\frac{\log x/q^\beta}{\log 2}\,\vartheta(\sqrt{x/q^\beta})}{x/q^\beta \,\log x} = \lim_{x\to\infty} \frac{\log x/q^\beta \;\vartheta(\sqrt{x/q^\beta})}{\log 2\sqrt{\frac{x}{q^\beta}}\cdot\sqrt{\frac{x}{q^\beta}}\,\log x}$$

$$\le \lim_{x\to\infty} \frac{\left(\log \frac{x}{q^\beta}\right)M'}{\sqrt{\frac{x}{q^\beta}}\,\log x}, \text{ where } M' = \frac{M}{\log 2}.$$

(But $\frac{\log y}{\sqrt{y}} \leq 1$ for all $y \geq 1$.) Thus the given limit

$$\le \lim_{x\to\infty} \frac{M'}{\log x} = 0.$$

The lemma is thus proved.

<u>Lemma 6</u>: $\sum_{m\le x} f(m) = \log x\cdot\vartheta(x) + 2\sum_{p\le\sqrt{x}} \vartheta\left(\frac{x}{p}\right)\log p$ + $o(x \log x)$. (In general, $o(g(x))$ designates a function of x such that $\frac{o(g(x))}{g(x)} \longrightarrow 0$ as $x \longrightarrow \infty$.)

Proof. From Lemma 2, we have

(3) $$\sum_{m\le x} f(m) = \log^2 x + \sum_{p^\alpha\le x}(-\log^2 p + 2\log x\log p) + 2\sum_{\substack{p^\alpha q^\beta \le x\\ p<q}} \log p \log q$$

Let us first consider the second term on the right hand side (RHS). The contribution of the terms with $\alpha \geq 2$ is at most

$$2\log^2 x \sum_{\substack{p^\alpha \le x\\ \alpha\ge 2}} 1 = 2\log^2 x\left\{\sqrt{x} + \sqrt[3]{x} + \cdots + \sqrt[k]{x}\right\},$$

where k designates the greatest integer such that $2^k \leq x$. So that the contribution is at most $2\sqrt{x}\, k \log^2 x \leq \frac{2\sqrt{x}}{\log 2} \log^3 x$ equals $o(x \log x)$, since

$$\lim_{x\to\infty} \frac{\sqrt{x}\,\log^3 x}{x \log x} = \lim_{x\to\infty} \frac{\log^2 x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\,\frac{\log x}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 4 \lim_{x\to\infty} \frac{\log x}{\sqrt{x}} = 0 .$$

The contribution of the terms with $\alpha = 1$ is equal to

$$\sum_{p\leq x} \{-\log^2 p + 2\log x \log p\} = \log x \sum_{p\leq x} \log p + \sum_{p\leq x} (\log p) \log \frac{x}{p}$$

$= (\log x)\,\vartheta(x) + o(x \log x)$, by Lemma 4.

Let's consider finally the third term of the RHS. The contribution of terms with $\beta \geq 2$ and $\alpha \geq 1$ is by Lemma 5 equal to

$$2 \sum_{\substack{p^\alpha q^\beta \leq x \\ p<q}} (\log p)(\log q) = 2 \sum_{\substack{q^\beta \leq x \\ \beta \geq 2}} \log q \left(\sum_{p^\alpha \leq \frac{x}{q^\beta}} \log p\right)$$

$$= \sum_{\substack{q^\beta \leq x \\ \beta \geq 2}} \left(\log q \cdot o\left(\frac{x}{q^\beta} \log x\right)\right) = o\left(\sum_{\substack{q^\beta \leq x \\ \beta \geq 2}} (\log q) \frac{x}{q^\beta} \log x\right):$$

We show

$$\lim_{x\to\infty} \left(\frac{\sum_{\substack{q^\beta \leq x \\ \beta \geq 2}} \left(\log q \;\; o\left(\frac{x}{q^\beta} \log x\right)\right)}{\sum_{\substack{q^\beta \leq x \\ \beta \geq 2}} \left(\log q \;\; \frac{x}{q^\beta} \log x\right)} \right) = 0 .$$

Now

$$\lim_{x\to\infty} \left(\frac{o\left(\frac{x}{q^\beta} \log x\right)}{\frac{x}{q^\beta} \log x} \right) = 0$$

by definition $\Rightarrow \forall\, \varepsilon > 0$

$\exists N \ni n > N \Rightarrow \left| o\left(\frac{n}{q^\beta} \log n\right)\right| < \varepsilon \left|\frac{n}{q^\beta} \log n\right| = \varepsilon\left(\frac{n}{q^\beta} \log n\right).$

Thus $\forall \varepsilon > 0 \ \exists N \ni n > N \Rightarrow$

$$\sum_{N < q^\beta \le n}\left[\log q \; o\left(\frac{n}{q^\beta} \log n\right)\right] < \varepsilon \sum_{N < q^\beta \le n}\left(\log q \cdot \frac{n}{q^\beta} \log n\right)$$

$$\le \varepsilon \sum_{q^\beta \le n}\left(\log q \cdot \frac{n}{q^\beta} \log n\right).$$

Thus $$\lim_{x\to\infty}\left(\frac{\sum_{N<q^\beta\le x}\left(\log q \; o\left(\frac{x}{q^\beta} \log x\right)\right)}{\sum_{q^\beta \le x} \log q \left(\frac{x}{q^\beta} \log x\right)}\right) = 0 \; ;$$

that is ; $\sum_{N<q^\beta\le x}\left(\log q \; o\left(\frac{x}{q^\beta} \log x\right)\right) = o\left(\sum_{q^\beta \le x} \log q \; \frac{x}{q^\beta} \log x\right).$

Now $$\lim_{x\to\infty}\left(\frac{\sum_{q^\beta \le N \le x} \log q \cdot o\left(\frac{x}{q^\beta} \log x\right)}{\sum_{q^\beta \le x}\left(\log q \; \frac{x}{q^\beta} \log x\right)}\right)$$

$$= \lim_{x\to\infty} \sum_{q^\beta \le N < x} \log q \left(\frac{o\left(\frac{x}{q^\beta} \log x\right)}{\sum_{q^\beta \le x}\left(\log q \; x/q^\beta \; \log x\right)}\right)$$

since N is finite,

$$= \sum_{q^\beta \le N < x} \lim_{x \to \infty} \left(\log q \frac{o\left(\frac{x}{q^\beta} \log x\right)}{\sum_{q^\beta \le x} (\log q \, x/q^\beta \, \log x)} \right)$$

$$= \sum_{q^\beta \le N \le x} \log q \cdot 0 = 0 \quad , \text{ since } \forall\, \varepsilon > 0 \; \exists\, N > 0 \ni n > N$$

$$\Rightarrow o\left(\frac{x}{q^\beta} \log x\right) < \left(\frac{x}{q^\beta} \log x\right)\frac{\varepsilon}{2} < \varepsilon \left(\log q \frac{x}{q^\beta} \log x\right), \forall\, q \text{ prime}$$

$$< \varepsilon \sum_{q^\beta \le x} (\log q \, x/q^\beta \, \log x).$$

Now we have that the third term with $\alpha \geqslant 1$, $\beta \geqslant 2$

$$= o\left(\sum_{\substack{q^\beta \le x \\ \beta \ge 2}} (\log q \, x/q^\beta \, \log x) \right) = o\left(\sum_{\substack{\beta \ge 2 \\ q \text{ prime}}} \log q \, x/q^\beta \, \log x \right)$$

$= o(x \log x) \sum \frac{\log q}{q^\beta}$, where this sum is extended over all primes q and all integers $\beta \geqslant 2$ and is convergent since $\sum_{\beta=2}^{\infty} \frac{1}{q^\beta} = \frac{1}{q(q-1)} \leqslant \frac{2}{q^\beta}$. Consequently the contribution of terms with $\beta \geqslant 2$ (and equally that of terms with $\alpha \geqslant 2$) is equal to $o(x \log x)$, so that the last term of the RHS of (3) is equal to

$$2 \sum_{\substack{pq \le x \\ p < q}} \log p \log q + o(x \log x).$$

The first term is equal to

$$\sum_{pq \le x} (\log p)(\log q) - \sum_{p \le \sqrt{x}} \log^2 p$$

$$= \sum_{\substack{p \le \sqrt{x} \\ pq \le x}} \log p \log q + \sum_{\substack{q \le \sqrt{x} \\ pq \le x}} \log p \log q - \sum_{\substack{p \le \sqrt{x} \\ q \le \sqrt{x}}} \log p \log q - \sum_{p \le \sqrt{x}} \log^2 p$$

$$= \sum_{p \le \sqrt{x}} \log p \, \vartheta(\tfrac{x}{p}) + \sum_{q \le \sqrt{x}} \log q \, \vartheta(\tfrac{x}{q}) - \vartheta^2(\sqrt{x}) + O(\sqrt{x} \log^2 x)$$

$$= 2 \sum_{p \le \sqrt{x}} \log p \, \vartheta(\tfrac{x}{p}) + o(x \log x),$$

because we know that $\frac{\vartheta(\sqrt{x})}{\sqrt{x}}$ is bounded as $x \to \infty$. Therefore $\frac{\vartheta^2(\sqrt{x})}{x} = O(1) \Rightarrow \vartheta^2(\sqrt{x}) = O(x) \Rightarrow \lim_{x \to \infty} \frac{\vartheta^2(\sqrt{x})}{x \log x}$

$< k \lim_{x \to \infty} \frac{x}{x \log x} = 0$. Therefore $\vartheta^2(\sqrt{x}) = o(x \log x)$.

The term $O(\sqrt{x} \log^2 x)$ clearly is $o(x \log x)$.

Thus we find that the last two terms in the RHS of (3) are equal respectively to $(\log x)\vartheta(x) + o(x \log x)$ and $2 \sum_{p \le \sqrt{x}} \left(\vartheta(\frac{x}{p}) \log p\right) + o(x \log x)$, from which we have the lemma.

<u>Lemma 7</u>: For each natural number x, $\left| \sum_{d=1}^{x} \frac{\mu(d)}{d} \right| \le 1$.

Proof. Because of (2), we obtain

$1 = \sum_{m=1}^{x} \sum_{d \mid m} \mu(d) = \sum_{d=1}^{x} \mu(d) \sum_{h} 1$,where h designates the positive multiples $\le$ x of d. Thus $\sum_{h} 1 = \left[\frac{x}{d}\right]$, from which $1 = \sum_{d=1}^{x} \mu(d) \left[\frac{x}{d}\right]$.

Consequently,

$$\left|x \sum_{d=1}^{x} \frac{\mu(d)}{d} - 1\right| = \left|\sum_{d=1}^{x} \mu(d)\left\{\frac{x}{d} - \left[\frac{x}{d}\right]\right\}\right|$$

$$\le \sum_{d=1}^{x} \left|\frac{x}{d} - \left[\frac{x}{d}\right]\right| \le x - 1$$

since each term of the last sum is ≤ 1 and the last term equals zero. Thus we find $x\left|\sum_{d=1}^{x} \frac{\mu(d)}{d}\right| \le 1 + (x-1) = x$, from which the lemma follows.

Lemma 8: If $g(t)$ is a monotonically decreasing function, $g(t) > 0$ for all $t > 0$, then

$$\sum_{1 \le n \le X} g(n) = \int_1^X g(t)\,dt + C + O(g(X)).$$

Here n runs through integers only; X can be any real number, $X \ge 1$; and C is a constant depending only on the function $g(t)$.*

Proof. Since $g(t)$ is decreasing in the interval $[n, n+1]$, we have $g(n+1) \le \int_n^{n+1} g(t)\,dt \le g(n)$,

and thus $0 \le d_n = g_n - \int_n^{n+1} g(t)\,dt \le g(n) - g(n+1)$.

Therefore we have for any positive integers $M < N$,

$$\sum_{n=M}^{N} d_n \le \sum_{n=M}^{N} \{g(n) - g(n+1)\} = g(M) - g(N+1) < g(M)$$

This shows that the series $\sum_{n=1}^{\infty} d_n$ converges. In particular, we have $\sum_{n=1}^{\infty} d_n \le g(M)$. If we put $C = \sum_{n=1}^{\infty} d_n$, we have

$$C = \sum_{n=1}^{N} d_n + \sum_{n=N+1}^{\infty} d_n = \sum_{n=1}^{N} \left\{ g_n - \int_n^{n+1} g(t)\,dt \right\} + O(g(N+1)).$$

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*Hans Rademacher, Lectures on Elementary Number Theory (New York, 1964), pp. 98-99.
The material found in van der Corput's Lemma 8 is given below in Corollaries 8.1 and 8.2.

It follows that $\sum_{n=1}^{N} g(n) = \int_1^{N+1} g(t)\,dt + C + O(g(N+1))$.

For N=[X] this becomes

$$\sum_{1\le n\le X} g(n) = \int_1^{[X]+1} g(t)\,dt + C + O(g([X]+1))$$

$$= \int_1^{X} g(t)\,dt + C + O(g(X))$$

since $\int_X^{[X]+1} g(t)\,dt \le g(X)$ and $0 < g([X]+1) \le g(X)$.

This proves the lemma.

Corollary 8.1 $\sum_{n\le y} \frac{1}{n} = \log y + C + O(\frac{1}{y})$.

Furthermore if we let $\mathcal{E}(y) = \sum_{n\le y} \frac{1}{n} - \log y - C$, then $\mathcal{E}(y) \longrightarrow 0$ and $\mathcal{E}(y) \log y \longrightarrow 0$ as $y \longrightarrow \infty$.

Proof. The first part follows directly from the lemma if we simply note that $\int_1^y \frac{1}{t}\,dt = [\log t]_1^y = \log y$.

For the second part, since $\mathcal{E}(y) = O(\frac{1}{y})$, the two limits follow.

Corollary 8.2 $\sum_{n\le y} \frac{\log n}{n} - \frac{1}{2}\log^2 y + C \longrightarrow 0$ as $y \to \infty$.

Proof. Since $\int_1^y \frac{\log t}{t}\,dt = [\frac{1}{2}(\log t)^2]_1^y = \frac{1}{2}\log^2 y$.

we have $\sum_{n\le y} \frac{\log n}{n} - \frac{1}{2}\log^2 y + C = O\left(\frac{\log y}{y}\right) \longrightarrow 0$ as $y \to \infty$.

Lemma 9. If $\tau(n)$ designates the number of divisors of n, then $\sum_{n\le y} \frac{\tau(n)}{n} = \log^2 y + c_3 \log y + c_4 + o(1)$, where c_3 and c_4 are suitably chosen constants.

Proof. Since $\tau(n)$ is equal to the number of pairs of natural numbers a and b such that ab = n, the sum $\sum_{n\le y} \frac{\tau(n)}{n}$ is equal to the sum $\sum \frac{1}{ab}$ extended over all natural numbers a and b such that $ab \le y$.

First evaluate the contribution of pairs a and b with $a \leq \sqrt{y}$. This contribution is increased by the equal contribution of pairs a and b with $b \leq \sqrt{y}$, and the result must be diminished by the contribution of pairs a and b such that $a,b \leq \sqrt{y}$.

Now apply the preceding corollaries. The contribution of pairs a and b such that $a \leq \sqrt{y}$ can be written in the form

$$\sum_{a \leq \sqrt{y}} \frac{1}{a} \sum_{b \leq \frac{y}{a}} \frac{1}{b} = \sum_{a \leq \sqrt{y}} \frac{1}{a} \left\{ \log y/a + c_1 + o\left(\frac{1}{\log y}\right) \right\}$$

$$= (\log y) \sum_{a \leq \sqrt{y}} 1/a - \sum_{a \leq \sqrt{y}} \frac{\log a}{a} + c_1 \sum_{a \leq \sqrt{y}} 1/a + o\left(\frac{1}{\log y}\right) \sum_{a \leq \sqrt{y}} 1/a$$

$$= (\log y)\left\{ \log \sqrt{y} + c_1 + o\left(\frac{1}{\log y}\right) \right\} - \left\{ \tfrac{1}{2} \log^2 \sqrt{y} + c_5 + o(1) \right\}$$
$$+ c_1 \left\{ \log \sqrt{y} + c_1 + o(1) \right\} + o\left(\frac{1}{\log y}\right) \cdot \left\{ \log \sqrt{y} + c_1 + o(1) \right\}$$
$$= \frac{3}{8} \log^2 y + c_6 \log y + c_7 + o(1),$$

in this reasoning c_5, c_6, c_7, designate constants conveniently chosen.

Finally the contribution of terms with $a,b \leq \sqrt{y}$ is equal to

$$\left(\sum_{a \leq \sqrt{y}} \frac{1}{a} \right)^2 = \left(\log \sqrt{y} + c_1 + \varepsilon(\sqrt{y}) \right)^2$$

$$= 1/4 \log^2 y + c_1 \log y + c_1^2 + \varepsilon(\sqrt{y}) \left\{ 2 \log \sqrt{y} + 2c_1 + \varepsilon(\sqrt{y}) \right\}$$

$$= 1/4 \log^2 y + c_1 \log y + c_1^2 + o(1).$$

Thus we have shown that the lemma is true since $3/8 + 3/8 - 1/4 = 1/2$.

Lemma 10 $\sum_{d\le x} \frac{\mu(d)}{d} \log \frac{x}{d} = o(\log x).$

Proof. From Corollary 8.1, the left-side member of the assertion is equal to $\sum_{d\le x} \frac{\mu(d)}{d} \left\{ \sum_{n\le \frac{x}{d}} 1/n + c_1 + \varepsilon(\tfrac{x}{d}) \right\}$

$$= \sum_{m\le x} \frac{1}{m} \sum_{d|m} \mu(d) + c_1 \sum_{d\le x} \frac{\mu(d)}{d} + \sum_{d\le x} \frac{\mu(d)}{d}\, \varepsilon(\tfrac{x}{d})$$

(we have set dn=m). The first term is by (2) equal to 1 and the second is from lemma 7 in absolute value $\le |c_1|$, so that it suffices to demonstrate

$$\sum_{d\le x} \frac{1}{d} \left| \varepsilon(\tfrac{x}{d}) \right| = o(\log x).$$

Let $y = x^{\rho(x)}$, where $\rho(x) < 1$ and $\rho(x) \longrightarrow 1$ slowly enough so that $x^{1-\rho(x)} \longrightarrow \infty$ as $x \longrightarrow \infty$. (Such a $\rho(x)$ exists since we have seen on pages 4-5 that $\rho(x) = 1 - \frac{1}{\log\log x}$ satisfies these conditions.) For the numbers $d \le y$ we have $\frac{x}{d} \ge x^{1-\rho}$ so that $\varepsilon(\frac{x}{d}) \longrightarrow 0$ and the contribution of these numbers d equals

$$o\Big(\sum_{d\le x} 1/d\Big) = o(\log x):$$

To prove this last statement, we show that

$$\lim_{x\to\infty} \left(\frac{\sum_{d\le x^\rho} \left[\frac{1}{d}\cdot o(1)\right]}{\sum_{d\le x^\rho} 1/d} \right) = 0$$

Now $\lim_{x\to\infty} o(1) = 0$ means that $\forall\, \varepsilon > 0\ \exists\, N \ni d \ge N \Rightarrow |o(1)| < \varepsilon.$

Thus $\forall\, \varepsilon > 0\ \exists\, N \ni d \ge N \Rightarrow \sum_{N\le d\le x^\rho} \frac{1}{d}\cdot o(1) < \sum_{N\le d\le x^\rho} 1/d \cdot \varepsilon < \varepsilon \sum_{d\le x^\rho} 1/d$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{\sum_{N\le d\le x^\rho}\frac{1}{d}\cdot o(1)}{\sum_{d\le x^\rho}\frac{1}{d}}\right)=0.$$

Now

$$\lim_{x\to\infty}\frac{\sum_{d\le N\le x^\rho}\frac{1}{d}\cdot o(1)}{\sum_{d\le x^\rho}\frac{1}{d}} = \lim_{x\to\infty}\sum_{d<N}\frac{\frac{1}{d}\cdot o(1)}{\sum_{d\le x^\rho}\frac{1}{d}} = \sum_{d<N}\lim_{x\to\infty}\frac{\frac{1}{d}\cdot o(1)}{\sum_{d\le x^\rho}1/d}=0.$$

Therefore we have shown the given limit equals 0 as $x\to\infty$.

So that $\sum_{d\le x^\rho}\frac{1}{d}\left|\varepsilon(\frac{x}{d})\right| = \sum_{d\le x^\rho}\frac{1}{d}\cdot o(1) = o\left(\sum_{d\le x^\rho}1/d\right) = o\left(\sum_{d\le x}\frac{1}{d}\right)$

$= o(\,\varepsilon(x) + \log x + C) = o(\log x)$ since $\varepsilon(x)\to 0$ as $x\to\infty$.

The contribution of numbers d such that $y<d\le x$ possesses at most the order of magnitude of $\sum_{y<d\le x}\frac{1}{d}$, since $\varepsilon(\frac{x}{d}) = O(\frac{d}{x})$ $= O(1)$. This sum is, according to Corollary 8.1, (applied twice, the second time with x instead of y) approximately equal to $\log x - \log y = (1-\rho)\log x$ in such a way that

$\sum_{y<d\le x} 1/d - (1-\rho)x \longrightarrow 0$. Thus we obtain the desired result since $1-\rho(x)\to 0$.

<u>Lemma 11</u>: For each natural number k,

$$\sum_{d|k}\mu(d)\tau(k/d) = 1.$$

Proof. Since $\tau(m) = \sum_{d_1|m} 1$, the left member of the assertion can be written in the form

$$\sum_{d|k}\mu(d)\sum_{d_1|\frac{k}{d}} 1 = \sum_{d_1|k}\sum_{d|\frac{k}{d_1}}\mu(d).$$

The contribution of divisors $d_1 < k$ is zero according to (2), and the contribution of the divisor k is equal to 1.

<u>Lemma 12</u>: $\sum_{d\le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} = 2 \log x + o(\log x)$.

Proof: From lemma 9 (applied with $y = \frac{x}{d}$), the left member of the formula can be written in the form

$$2 \sum_{d\le x} \frac{\mu(d)}{d} \left\{ \sum_{m\le x/d} \frac{\tau(m)}{m} - c_3 \log \frac{x}{d} - c_4 \right\} + o(\log x),$$

since $\sum_{d\le x} \left(\frac{\mu(d)}{d} o_{\frac{x}{d}}(1) \right) = o(\log x)$, where $g(\frac{x}{d}) = o_{\frac{x}{d}}(1)$ means that $\lim_{\frac{x}{d}\to\infty} g(\frac{x}{d}) = 0$. Apply Corollary 8.1

The contribution of values of $d \le \frac{x}{\log x}$ is equal to $o(\sum_{d\le x} 1/d) = o(\log x)$, since $d \le x/\log x \Rightarrow x/d > \log x$.

Thus $x \to \infty \Rightarrow \log x \to \infty \Rightarrow \frac{x}{d} \to \infty \Rightarrow o_{\frac{x}{d}}(1) \to 0$.

so that $o_{\frac{x}{d}}(1)$ is in this case $o(1)$.

$$\text{Thus } \lim_{x\to\infty} \sum_{d\le x/\log x} \frac{1}{d} o_{\frac{x}{d}}(1) = \lim_{x\to\infty} \sum_{d\le x/\log x} \frac{1}{d} \cdot o(1) = o\Big(\sum_{d\le x/\log x} 1/d \Big)$$

$= o(\sum_{d\le x} 1/d) = o(\log x)$. The contribution of the other values of d is at most of the order of

$$\sum_{\frac{x}{\log x}<d\le x} \frac{1}{d} = \log x - \log\left(\frac{x}{\log x}\right) + o(\log x) = o(\log x).$$

From Lemmas 10 and 7 the first term if equal to $U + o(\log x)$, where

$$U = 2 \sum_{d\le x} \frac{\mu(d)}{d} \sum_{m\le x/d} \frac{\tau(m)}{m},$$

thus (letting $k = md$):

$$U = 2 \sum_{k\le x} \frac{1}{k} \sum_{d|k} \mu(d)\, \tau(k/d)$$

$$= 2 \sum_{k\le x} 1/k \quad \text{, from Lemma 11}$$

$$= 2 \log x + o(\log x) \quad \text{, from Corollary 8.1.}$$

After these preliminary consideration, we can now prove Selberg's formula:

$$\frac{\vartheta(x)}{x} + \frac{2}{x\log x}\sum_{p\le\sqrt{x}} \vartheta\left(\frac{x}{p}\right)\log p \to 2 \text{ as } x\to\infty.$$

From Lemma 6, the left side has, to a term which $\to 0$ as $x \to \infty$, the value $\frac{1}{x\log x}\sum_{m\le x} f(m)$.

We must now show that this expression approaches 2. From the definition of the function $f(m)$,

$$\sum_{m\le x} f(m) = \sum_{m\le x}\sum_{d|m}\lambda(d) = \sum_{d\le x}\lambda(d)\sum_{h} 1,$$

where h designated the multiples $\le$ x of d. Consequently $\sum_h 1$ is equal to $\frac{x}{d}$, to a term which has absolute value ≤ 1. The sum $\sum_{m\le x} f(m)$ is therefore approximately equal to

$$x\sum_{d\le x}\frac{\lambda(d)}{d} = x\sum_{d\le x}\frac{\mu(d)}{d}\cdot\log^2\frac{x}{d} = 2x\log x + o(x\log x)$$

from Lemma 12, and the error is at most of the order of magnitude of $\sum_{d\le x}|\lambda(d)| = \sum_{d\le x}\log^2\frac{x}{d}$, so it suffices to show that this sum is equal to $o(x \log x)$.

The contribution of the terms such that $\frac{x}{2^{k+1}} < d \le \frac{x}{2^k}$, where k designates an integer ≥ 0, is at most $(k+1)^2\frac{x}{2^k}$, since the number of terms is at most $\frac{x}{2^k}$ and each term $\le (k+1)^2$.

Consequently the sum is less than

$$\sum_{k=0}^{\infty}\frac{(k+1)^2 x}{2^k} = x\sum_{k=0}^{\infty}\frac{(k+1)^2}{2^k},$$

thus at most of the order of x and therefore $o(x \log x)$.

II Proof of PNT Using Selberg's Formula.

In this part we will employ the three following facts:

(1) $\frac{\vartheta(x)}{x} + \frac{2}{x\log x}\sum_{p\leq\sqrt{x}}\vartheta\left(\frac{x}{p}\right)\log p \longrightarrow 2 \quad \text{as } x\longrightarrow\infty$

(Selberg's Formula);

(2) $\liminf_{x\to\infty}\frac{\vartheta(x)}{x}$ 0, demonstrated in Lemma 3;

(3) $\frac{1}{\log x}\sum_{p\leq x}\frac{\log p}{p}\to 1 \quad \text{as } x\longrightarrow\infty$, shown in the remark following Lemma 3.

From these we shall prove that $\frac{\vartheta(x)}{x}\longrightarrow 1$ as $x\longrightarrow\infty$, which, as we have shown previously, implies PNT.

The formula (1) shows that $\frac{\vartheta(x)}{x}$ is bounded, so that the limits

$$A = \limsup_{x\to\infty}\frac{\vartheta(x)}{x} \quad\text{and}\quad a = \liminf_{x\to\infty}\frac{\vartheta(x)}{x}$$

exist. We have $0\leq a\leq A$, and $\frac{\vartheta(x)}{x}\to 1$ means that $a = 1 = A$.

<u>Lemma 13</u>: $A + a = 2$.

Proof. It is possible to make x become infinite in such a way that $\frac{\vartheta(x)}{x}\longrightarrow A$. If δ designates a fixed positive number, $\vartheta\left(\frac{x}{p}\right) > (a-\delta)\frac{x}{p}$ for each x sufficiently large and for each number $p\leq\sqrt{x}$, thus

$$\frac{2}{x\log x}\sum_{p\leq\sqrt{x}}\vartheta\left(\frac{x}{p}\right)\log p \geq \frac{2(a-\delta)}{\log x}\sum_{p\leq\sqrt{x}}\frac{\log p}{p}$$

The last term approaches $a-\delta$ in view of (3), so that Selberg's formula shows that $A + a - \delta \leq 2$. This is true for each positive fixed δ. Thus $A + a \leq 2$.

If in the above reasoning we replace A, a, δ, and $>$ by $a, A, -\delta$, and $<$ respectively, we find that $A + a \geq 2$. Thus $A + a = 2$.

In the rest of this argument $x \to \infty$ in such a way that $\frac{\vartheta(x)}{x} \to A$, and the number δ is positive and fixed.

Lemma 14: For each fixed $\lambda > a$,

$$\frac{1}{\log x} \sum_1 \frac{\log p}{p} \to 0,$$

if $\sum_1$ is extended over numbers $p \leq x$ such that $\vartheta(\frac{x}{p}) \geqslant \frac{\lambda x}{p}$.

Remark: This lemma is a theorem of compensation: if $\vartheta(x)$ is large, $\vartheta(\frac{x}{p})$ is small for "almost" each number $p \leq x$.

Proof: $\sum_{p \leq x} \vartheta(\frac{x}{p}) \log p = \sum_{pq \leq x} \log p \log q$

$$= \sum_{p \leq \sqrt{x}} \vartheta(\tfrac{x}{p}) \log p + \sum_{q \leq \sqrt{x}} \vartheta(\tfrac{x}{q}) \log q - \Big(\sum_{p \leq \sqrt{x}} \log p\Big)^2$$

$$= 2 \sum_{p \leq \sqrt{x}} \vartheta(\tfrac{x}{p}) \log p - (\vartheta(\sqrt{x}))^2,$$

where p and q are prime. The last term is at most of the order of x in view of (1)— applied with $\sqrt{x}$ instead of x.

We can therefore write the formula of Selberg also in the form

$$\frac{\vartheta(x)}{x} + \frac{1}{x \log x} \sum_{p \leq x} \vartheta(\tfrac{x}{p}) \log p \to 2.$$

If $\frac{x}{p}$ is greater than the value u conveniently chosen and dependent on δ, $\vartheta(\frac{x}{p}) > (a - \delta)\frac{x}{p}$. There exists a positive number b(u) such that $\vartheta(\frac{x}{p}) > (a - \delta)\frac{x}{p} - b$ for all p's such that $\frac{x}{p} \leq u$. Thus the inequality is valid for all $p \leq x$.

If we divide the sum $\sum_{p \le x}$ into $\sum_1$ and $\sum_1'$ we obtain

$$\sum_{p \le x} \vartheta\left(\frac{x}{p}\right) \log p \geq \lambda x \sum_1 \frac{\log p}{p} + x(a-\delta) \sum_1' \frac{\log p}{p} - b \sum_{p \le x} \log p$$

$$\geq (a-\delta) x \sum_{p \le x} \frac{\log p}{p} + (\lambda - a) x \sum_1 \frac{\log p}{p} - b\,\vartheta(x).$$

If we substitute this result in Selberg's formula we obtain:

$$A + a - \delta + (\lambda - a) \limsup \frac{1}{\log x} \sum_1 \frac{\log p}{p} \leq 2$$

which implies, since A+a = 2,

$$(\lambda - a) \limsup \frac{1}{\log x} \sum_1 \frac{\log p}{p} \leq \delta,$$

from which follows the lemma since $\lambda - a > 0$ and δ is an arbitrary positive number.

<u>Lemma 15</u>: For every fixed $\mu < A$,

$$\frac{1}{\log^2 x} \sum_2 \frac{\log p}{p} \cdot \frac{\log q}{q} \longrightarrow 0,$$

if $\sum_2$ is taken over pairs of numbers p and q such that

$$p \leq \sqrt{x}, \quad q \leq \sqrt{\frac{x}{p}}, \quad \vartheta\left(\frac{x}{pq}\right) \leq \frac{\mu x}{pq}.$$

Remark: Lemma 15 is a theorem of double compensation. If $\vartheta(x)$ is large, $\vartheta\left(\frac{x}{p}\right)$ is small for "almost" every number $p \leq x$, and $\vartheta\left(\frac{x}{pq}\right)$ is large for "almost" each pair of numbers $p \leq \sqrt{x}$ and $q \leq \sqrt{\frac{x}{p}}$.

Proof: If we substitute $\frac{x}{p}$ for x in Selberg's formula, we obtain

$$\vartheta\left(\frac{x}{p}\right) = \frac{2x}{p} + o\left(\frac{x}{p}\right) - \frac{2}{\log \frac{x}{p}} \sum_{q \le \sqrt{\frac{x}{p}}} \vartheta\left(\frac{x}{pq}\right) \log q.$$

Substituting this result in Selberg's formula gives

$$\vartheta(x) = 2x + o(x) - \frac{2x}{\log x} \sum_{p \le \sqrt{x}} \{ 2 + o(1) \} \frac{\log p}{p} + \frac{4V}{\log x},$$

where $$V = \sum_{p \le \sqrt{x},\, q \le x/p} \frac{1}{\log x/p} \vartheta\left(\frac{x}{pq}\right) \log p \log q.$$

By virtue of (3) we find

$$\sum_{p \le \sqrt{x}} \frac{\log p}{p} = \frac{1}{2} \log x + o(\log x), \text{ thus } \vartheta(x) = \frac{4V}{\log x} + o(x).$$

In each term of V, $p \le \sqrt{x}$ and $q \le \sqrt{\frac{x}{p}}$. Thus $pq = p^{\frac{1}{2}} (pq^2)^{\frac{1}{2}} \le x^{\frac{3}{4}}$,

so that $\vartheta\left(\frac{x}{pq}\right) < (A+\delta)\frac{x}{pq}$, if x is sufficiently large.

If we divide V into two sums $\sum_2$ and $\sum_2'$, we have

$$V \le \mu x \sum_2 \left(\frac{1}{\log x/p} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right) + (A+\delta)x \sum_2' \left(\frac{1}{\log x/p} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right)$$

$$= (A+\delta)xW - (A+\delta-\mu)x \sum_2 \left(\frac{1}{\log x/p} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right),$$

where $$W = \sum_{p \le \sqrt{x},\, q \le \sqrt{\frac{x}{p}}} \left(\frac{1}{\log x/p} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right)$$

$$= \sum_{p \le \sqrt{x}} \frac{1}{\log x/p} \cdot \frac{\log p}{p} \sum_{q \le \sqrt{x/p}} \frac{\log q}{q}$$

$$= \sum_{p \le \sqrt{x}} \frac{\log p}{p} \left\{ \frac{1}{2} + o(1) \right\} \quad \text{because of (3)}$$

$$= \frac{1}{4} \log x + o(\log x).$$

We thus have

$$\vartheta(x) \le (A+\delta)x - \frac{4}{\log x}(A+\delta-\mu)x \sum_2 \left(\frac{1}{\log \frac{x}{p}} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right) + o(x),$$

from which

$$\frac{4}{\log^2 x}(A-\mu) \sum_2 \left(\frac{\log p}{p} \cdot \frac{\log q}{q}\right) \le \frac{4}{\log x}(A+\delta-\mu) \sum_2 \left(\frac{1}{\log x/p} \cdot \frac{\log p}{p} \cdot \frac{\log q}{q}\right)$$

$$\le A + \delta - \frac{\vartheta(x)}{x} + o(1).$$

Consequently,

$$4(A-\mu)\limsup \frac{1}{\log^2 x}\sum_2\left(\frac{\log p}{p}\cdot\frac{\log q}{q}\right)\leq \delta,$$

from which results the lemma, since $A-\mu > 0$ and δ is an arbitrary positive number.

<u>End of the Proof</u>: Let σ be any positive number such that $\sigma a < A$ and let the number δ be so small that

(4) $$A - a\sigma \geqslant \delta\sigma + 2\delta.$$

Consider the sum

$$S = \sum_3 \frac{\log p}{p}\cdot\frac{\log q}{q}\sum_4 \frac{\log r}{r}.$$

$\sum_3$ is taken over pairs of prime numbers p and q such that

$$p \leq \sqrt{x}\ ,\ q \leq \sqrt{\frac{x}{p}}\ ,\ pq \geqslant N\ ,\ \vartheta\left(\frac{x}{pq}\right) \geqslant (A-\delta)\frac{x}{pq},$$

where N designates any fixed natural number. $\sum_4$ is taken over prime numbers r such that $\frac{pq}{\sigma} < r \leq \sigma pq$. If $\sigma \leqslant 1$, the sum is naturally equal to zero.

For each term of $\sum_4$,

$$r \leqslant \sigma pq = \sigma p^{1/2}(pq^2)^{1/2} \leqslant \sigma x^{1/4}x^{1/2} = \sigma x^{3/4} \leq x,$$

if x is sufficiently large.

If x is sufficiently large, all the terms of $\sum_4$ satisfy the inequality

(5) $$\vartheta\left(\frac{x}{r}\right) \geqslant (a+\delta)\frac{x}{r}.$$

This inequality is evident for the terms with $r \leqslant pq$, because then

$$\vartheta\left(\frac{x}{r}\right) \geqslant \vartheta\left(\frac{x}{pq}\right) \geqslant (A-\delta)\frac{x}{pq} \geqslant (A-\delta)\frac{x}{\sigma r} \geqslant (a+\delta)\frac{x}{r}$$

because of (4).

Consider now the terms with $r > pq$. If we let $\frac{x}{r} = u$ and $\frac{x}{pq} = v$, we obtain $u < v \leq \sigma u$. If we replace x by v and by u in Selberg's formula,

$$(\log x)\vartheta(x) + 2\sum_{p\leq\sqrt{x}} \vartheta\left(\frac{x}{p}\right)\log p = 2x\log x + o(x\log x),$$

we obtain by subtraction:

$$\log v \;\vartheta(v) - \log u \;\vartheta(u) \leq 2v\log v - 2u\log u + o(u\log u),$$

thus

$$\vartheta(u) \geq \frac{\log v}{\log u}\,\vartheta(v) - 2(v-u) - 2v\left(\frac{\log v - \log u}{\log u}\right) + o(u).$$

In the second member the first term is at least $\vartheta(v) \geq (A-\delta)v$ and the third term is $o(u)$, thus

$$\begin{aligned}\vartheta(u) &\geq (A-\delta)v - 2(v-u) + o(u) = 2u - (2-A+\delta)v + o(u)\\ &\geq 2u - (a+\delta)\sigma u + o(u)\\ &\geq (a+2\delta)u + o(u),\end{aligned}$$

because of (4) and since $a + A = 2$. Thus we find, if x is large enough, that

$$\vartheta\left(\frac{x}{r}\right) = \vartheta(u) \geq (a+\delta)u = (a+\delta)\frac{x}{r}$$

so that (5) is valid for each term of $\sum_4$. Consequently,

$$(6) \qquad S \leq \sum_{\substack{r\leq x\\ \vartheta(\frac{x}{r})\geq(a+\delta)\frac{x}{r}}} \frac{\log r}{r} \sum\nolimits_5 \frac{\log p}{p}\cdot\frac{\log q}{q},$$

where $\sum_5$ is taken over pairs p and q such that

$$p \leq \sqrt{x}, \quad q \leq \sqrt{\frac{x}{p}}, \quad \frac{r}{\sigma} \leq pq < \sigma r.$$

We thus obtain

$$\sum_5 \log p \log q \le \frac{\sigma}{r} \sum_{p \le \sqrt{x}} \log p \sum_{q < \frac{\sigma r}{p}} \log q$$

$$= \frac{\sigma}{r} \sum_{p \le \sqrt{x}} \log p \, \vartheta\left(\frac{\sigma r}{p}\right)$$

$$< c_1 \sum_{p \le \sqrt{x}} \frac{\log p}{p} < c_2 \log x ,$$

where c_1, c_2 are positive constants conveniently chosen.

Consequently,

$$S \le c_2 \log x \sum_{\substack{r \le x \\ \vartheta(\frac{x}{r}) > (a+\delta)\frac{x}{r}}} \frac{\log r}{r} = o(\log^2 x),$$

from Lemma 14. We now introduce the sum

$$T = \sum_{p \le \sqrt{x},\; q \le \sqrt{x/p},\; pq \le N} \frac{\log p}{p} \cdot \frac{\log q}{q} ,$$

which is at least equal to

$$\sum_{\sqrt{N} \le p \le \sqrt{x}} \frac{\log p}{p} \cdot \sum_{\sqrt{N} \le q < \sqrt[4]{x}} \frac{\log q}{q} .$$

Each of these two factors has, from (3), at most the order of magnitude of $\log x$, thus $T > c_3 \log^2 x$, where c_3 is a positive number independent of x. If we let

$$T = \sum_3 \frac{\log p}{p} \cdot \frac{\log q}{q} + \sum_3{}' \frac{\log p}{p} \cdot \frac{\log q}{q} ,$$

where each term of the latter sum satisfies

$$p \le \sqrt{x} , \; q \le \sqrt{\frac{x}{p}} , \; \vartheta\left(\frac{x}{pq}\right) < (A-\delta)\frac{x}{pq} ,$$

Lemma 15 shows that this sum equals $o(\log^2 x)$. Thus

$$\sum_3 \frac{\log p}{p} \cdot \frac{\log q}{q} > \frac{1}{2} c_3 \log^2 x ,$$

if x is large enough.

For a fixed value of x, we consider the prime pairs p and q in $\sum_3$ for which $\sum_4 \frac{\log r}{r}$ takes the minimum value μ, where μ depends on x alone.

Thus $S \geqslant \mu \sum_3 \frac{\log p}{p} \cdot \frac{\log q}{q} > \frac{1}{2} \mu c_3 \log^2 x$.

Comparing this result with (6), we obtain

(7) $$\mu = \sum_4 \frac{\log r}{r} \longrightarrow 0 .$$

Consequently to each positive ε and each natural number N there corresponds a number $t = pq \geqslant N$ such that

$$\sum_{\frac{t}{\sigma} < r \leqslant \sigma t} \frac{\log r}{r} < \varepsilon ,$$

thus

$$\frac{1}{\sigma t} \sum_{t/\sigma < r \leqslant \sigma t} \log r < \varepsilon ,$$

from which $\vartheta(\sigma t) - \vartheta(t/\sigma) < \varepsilon \sigma t$.

If N, thus also t, is large enough, we have

$$\vartheta(\sigma t) > (a - \varepsilon) \sigma t \quad \text{and} \quad \vartheta(t/\sigma) < (A + \varepsilon) t/\sigma.$$

Thus $$(a - \varepsilon)\sigma - \frac{A + \varepsilon}{\sigma} < \varepsilon \sigma .$$

This inequality holds for each positive number ε, so that $a\sigma^2 - A \quad 0$.

Thus each positive number σ such that $a\sigma < A$ satisfies also the inequality $a\sigma^2 \leqslant A$. This result gives nothing new if $a = 0$, but let us apply the fact that lim inf $\frac{\vartheta(x)}{x}$ is positive; that is, a is positive. Each number $\sigma < \frac{A}{a}$ possesses the property $\sigma^2 < \frac{A}{a}$. If $\sigma \to A/a$, we obtain $\left(\frac{A}{a}\right)^2 < \frac{A}{a} \Rightarrow \frac{A}{a} < 1$. Thus $A = a = 1$ since $a \leqslant A$ and $a + A = 2$.

Thus $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1$ and therefore

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \log x} = 1 .$$

Q.E.D.

It has lately been possible to improve the original proof given by Selberg and Erdös. The older versions gave just the asymptotic equality

$$\pi(x) \sim \frac{x}{\log x}$$

without an estimate of the error term

$$E(x) = \pi(x) - \frac{x}{\log x} .$$

S.A. Amitsur, E. Bombieri, W.B. Jurkat, R. Breusch, and E. Wirsing have since given estimates for E(x) which approach the accuracy obtained by transcendental arguments. However, these recent elementary proofs become extremely sophisticated and the sharpest results are still obtainable by transcendental methods.

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