5-2003

Discovering Properties of Complex Numbers by Starting with Known Properties of Real Numbers

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DISCOVERING PROPERTIES OF COMPLEX NUMBERS BY STARTING WITH KNOWN PROPERTIES OF REAL NUMBERS

by

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A Thesis Submitted in Partial Fulfillment
of the Requirements for a Degree with Honors
(Mathematics and Statistics)

The Honors College

University of Maine

May 2003

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Introduction to the Thesis

The following work completes the thesis requirement of the Honors College and fulfills the capstone requirement in the Mathematics and Statistics Department of the University of Maine. This thesis will record a process of discovery in which facts of arithmetic, algebra, and calculus are conjectured for complex numbers. This will be done by reviewing the real number system and attempting to extend known properties and uses of real numbers to complex numbers.

This thesis is written in the spirit of the "Moore Method." The "Moore Method" is a method of teaching used by R.L. Moore that depends on student discovery and original proof. His course is open only to those who have no previous instruction in a topic, and thus no preconceived notion of how to solve a problem.[1] Preconceived notions would make one too knowledgeable for new discovery and original proof. All the proofs in the following thesis, unless otherwise noted, are original proofs I thought up and wrote.

I have studied some of the topics in this thesis, so one might ask how I can write a thesis based on discovery. The focus of my previous mathematical education was on real numbers. In elementary and secondary school, mathematical education focused on computations of real numbers. Properties of real numbers were used, but not explored. For complex numbers, all I learned before college was $\sqrt{-1} = i$, and that there existed a graphical representation where real numbers are represented on the x-axis and imaginary ones on the y-axis. The following thesis illustrates that there is much more to know about complex numbers.

[1] Devlin, Keith. "The Greatest Math Teacher Ever, Part 2" *Mathematical Association of America*. June 1999. 12 May 2003 4:50pm <http://www.maa.org/devlin/devlin_6_99.html,>.

My former studies focused on computational mathematics and not theoretical mathematics, yet for this thesis, discovery begins with analyzing properties of real numbers. The thesis will help me better understand properties of real numbers, because the basic facts of arithmetic, algebra, and calculus must be identified for the real numbers before they can be studied and extended to the complex numbers. In college, I studied complex numbers in the undergraduate complex analysis course in the fall of my senior year—the same time I began this thesis. This complex analysis course proceeds at an accelerated rate; thus full exploration of complex numbers and comparison to the real numbers was not possible. This thesis is a slower study which builds a solid foundation for complex numbers.

This study gives me the opportunity to ask questions during my work and to guide my own discoveries. As I come upon different real number topics, I have the opportunity to ask myself questions and discover the answers. I could never rediscover all the properties of complex numbers; there are too many and some are beyond my scope of knowledge. However, since this thesis is an open-ended project, I have the opportunity to pause on certain subjects for more exploration and omit other subjects.

The deeper understanding of complex numbers will help me in two ways. First, by identifying properties of real numbers that I need to explore, I strengthen my knowledge of real numbers. Secondly, my solid understanding of the properties of real numbers and complex numbers will be an asset when teaching high school mathematics because I will understand the reasoning behind these and how they work in basic mathematical functions.

Introduction to Complex Numbers

The earliest appearance of a negative square root comes from *Stereometria* of Heron of Alexandria and calculations made by Diophantus. Heron of Alexandria was trying to calculate $\sqrt{81-144}$. Instead of correctly writing $\sqrt{-63}$ for this computation, $\sqrt{63}$ was recorded. Diophantus had the quadratic equation $172x = 336x^2 + 24$ for which $x = \frac{43 \pm \sqrt{-167}}{168}$. Diophantus did not calculate the roots of the quadratic equation because he believed that "the quadratic equation was not possible" since there were no rational solutions. While it is true that there are no rational roots to that quadratic formula, he did not explore the quadratic equation's irrational roots and discover the square root of a negative number.[2] Later mathematicians such as Descartes and Euler cast off square roots of negative numbers as being unimportant.

Before any information was known about its existence or properties, mathematicians formally manipulated the "quantity" $\sqrt{-1}$ and called it i. Then, in 1835, William Rowan Hamilton (1805-1865) considered ordered pairs or couples (a,b) and (c,d), for $a,b,c,d \in \mathbf{R}$. He defined addition and multiplication respectively as the following:

$$(a,b)+(c,d) = (a+c,b+d) \quad (a,b)(c,d) = (ac-bd,ad+bc).$$[3]

This definition of multiplication helped in discovering further properties of $\sqrt{-1}$. Some of these properties will be shown in this thesis.

Ordered pairs or couples are not unknown or a mystery to math students today, in fact ordered pairs are studied in depth as vectors in Calculus III. Vectors are one way to visualize and work with ordered pairs in two space, $\mathbf{R}^2$. Vectors have an origin at (0,0) and a

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[2] Nahin, Paul J. An Imaginary Tale. Princeton, New Jersey. Princeton University Press 1998. p. 4 .
[3] ibid, p.79-80.

directional arrow to an ordered pair. The additive properties of ordered pairs are known because they are properties of vectors, however, in Calculus III, no multiplication is given for vectors in $\mathbf{R}^2$—just addition. Furthermore, there is no good multiplication defined for $\mathbf{R}^n$, the study of ordered *n*-tuples of real numbers. The additive properties of **R** and ordered pairs of $\mathbf{R}^2$ are listed:

Associativity:

For real numbers, $(a+b)+c = a+(b+c)$.

For ordered pairs, $[(a,b)+(c,d)]+(e,f) = (a,b)+[(c,d)+(e,f)]$.

Commutativity:

For real numbers, $a+b=b+a$.

For ordered pairs, $(a,b)+(c,d)=(c,d)+(a,b)$.

Identity:

For real numbers, There exists $0 \in \mathbf{R}$ such that $a+0=a$ for all $a \in \mathbf{R}$.

For ordered pairs,

There exists $(0,0) \in \mathbf{R}^2$ such that $(a,b)+(0,0)=(a,b)$ for all $a,b \in \mathbf{R}$.

Inverse:

For real numbers, $a+(-a)=0$ for all $a \in \mathbf{R}$.

For ordered pairs, $(a,b)+(-a,-b)=(0,0)$ for all $a,b \in \mathbf{R}$.

Cancellation:

For real numbers, If $a+b=a+c$, then $b=c$ for all $a,b,c \in \mathbf{R}$.

For ordered pairs,

If $(a,b)+(c,d)=(a,b)+(e,f)$ then $(c,d)=(e,f)$ for all $a,b,c,d,e,f \in \mathbf{R}$.

Since addition and multiplication are defined for both real numbers and ordered pairs, there may be more properties of both real numbers and ordered pairs. To ensure a thorough examination of real number properties that might apply to ordered pairs, we refer to Ethan D. Bloch's "Properties of Numbers" found in his textbook Proofs and Fundamentals.[4] The multiplicative properties or real numbers in this list will be examined to see if the properties are still true for ordered pairs using Hamilton's multiplication.

[4] Bloch, Ethan D. Proofs and Fundamentals. Boston. Birkhauser. 2000. p.375-376.

Multiplicative associativity for real numbers is given as $(ab)c = a(bc)$ for all $a,b,c \in \mathbf{R}$. To see if multiplicative associativity is true for coordinate pairs, consider arbitrary pairs (a,b), (c,d), and (e,f).

First $[(a,b)(c,d)](e,f) = [(ac-bd, ad+bc)](e,f)$

$$= [(ace-bde)-(adf+bcf),(acf-bdf)+(ade+bce)]$$
$$= [ace-bde-adf-bcf, acf-bdf+ade+bce]$$

and by rearranging terms,

$$= [(ace-adf)-(bde+bcf),(bce-bdf)+(ade+acf)]$$
$$= (a,b)(ce-df, de+cf)$$
$$= (a,b)[(c,d)(e,f)].$$

Thus there is multiplicative associativity for coordinate pairs.

Multiplicative commutativity for real numbers is given as $ab = ba$. To see if multiplicative commutativity is true for coordinate pairs, consider arbitrary pairs (a,b) and (c,d).

$$(a,b)(c,d) = (ac-bd, ad+bc)$$
$$= (ca-db, da+cb)$$
$$= (c,d)(a,b).$$

The multiplicative identity for real numbers is 1. That is, $a1 = a$ for all $a \in \mathbf{R}$. In order to find if there is a multiplicative identity, we ask if there exists (x,y) such that for all $a,b \in \mathbf{R}$, $(a,b)(x,y) = (a,b)$.

Assume $(a,b)(x,y) = (a,b)$ for all $a,b \in \mathbf{R}$. Therefore $ax-by = a$ and $bx+ay = b$. From these, we set up the two equations $b(ax-by) = b(a)$ and $-a(bx+ay) = -a(b)$. Adding these two gives us $-b^2y - a^2y = ab - ba$. $y(-b^2-a^2) = 0$. Then $y = 0$ or $-b^2 - a^2 = 0$. If $-b^2 - a^2 = 0$ then $a^2 = -b^2$ and $a = b = 0$. This cannot hold for all a,b thus $y = 0$. Therefore

$ax = a$, $bx = b$, and $x = 1$. Therefore, if an identity exists, it must be $(1,0)$. By definition of complex multiplication, $(a,b)(1,0) = (a,b)$. Thus $(1,0)$ is the identity.

The multiplicative inverse for a non-zero real number a, is a number denoted by a^{-1} such that $aa^{-1} = 1$, where 1 is the multiplicative identity. If a multiplicative inverse (x, y) exists for some coordinate pair (a,b), it must satisfy the equation $(a,b)(x,y) = (1,0)$ since $(1,0)$ is the multiplicative identity. Given $(a,b)(x,y) = (1,0)$ it follows that $(ax - by, ay + bx) = (1,0)$ so $ax - by = 1$ and $ay + bx = 0$. For $a \neq 0$, $ay = -bx$, and $y = \frac{-bx}{a}$. By substitution, $ax - b\frac{-bx}{a} = 1$ and $x(a + \frac{b^2}{a}) = 1$. Therefore $x = \frac{a}{a^2+b^2}$. Through substitution, $y = \frac{-b}{a^2+b^2}$. If an inverse exists, it must be $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. Since $(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0)$ by definition of complex multiplication, $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the inverse of (a,b). In fact, $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is seen to be the inverse of (a,b) even if only one of a or b is 0. Thus $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the multiplicative inverse for all $a,b \in \mathbf{R}$, $(a,b) \neq 0$.

The multiplicative cancellation law for real numbers is given by: if $c \neq 0$, then $ac = bc$ only if $a = b$. To prove the cancellation law is true for coordinate pairs, show that if $(a,b)(c,d) = (e,f)(c,d)$, then $(a,b) = (e,f)$ for arbitrary pairs (a,b), (c,d), and (e,f).

Assume $(a,b)(c,d) = (e,f)(c,d)$. That is, $(ac - bd, ad + bc) = (ec - fd, ed + fc)$
Thus $ac - bd = ec - fd$ and $ad + bc = ed + fc$.
Solve the system of equations:
$c(ac - bd) = c(ec - fd)$ and $d(ad + bc) = d(ed + fc)$.
This becomes $ac^2 - bcd = ec^2 - fcd$ and $ad^2 + bcd = ed^2 + fcd$.
Simplifying, $ac^2 + ad^2 = ec^2 + ed^2$.
Factoring, $a(c^2 + d^2) = e(c^2 + d^2)$.
Thus $a = e$, if $(c,d) \neq (0,0)$.
Substutiting $a = e$ into $ac - bd = ec - fd$ gives us $ac - bd = ac - fd$.
Thus $-bd = -fd$ and $b = f$, if $d \neq 0$. Similarly, $b = f$ if $c \neq 0$.
Therefore, $(a,b) = (e,f)$.

The reverse implication is true merely by substitution.

The distributive law for real numbers is given by: $a(b+c) = (ab) + (ac)$. For coordinate pairs, consider arbitrary pairs (a,b), (c,d), and (e,f) and show $(e,f)[(a,b) + (c,d)] = (e,f)(a,b) + (e,f)(c,d)$. We have

$$\begin{aligned}(e,f)[(a,b) + (c,d)] &= (e,f)[a + c, b + d] \\ &= [e(a+c) - f(b+d), e(b+d) + f(a+c)] \\ &= (ea + ec - fb - fd, eb + ed + fa + fc) \\ &= (ea - fb, fa + eb) + (ec - fd, ed + fc) \\ &= (e,f)(a,b) + (e,f)(c,d).\end{aligned}$$

Bloch lists a double negation law, that for a real number x, $-(-x) = x$. This is an elementary property of vector spaces. The double negation property is the point on Bloch's list where this examination will pause.

According to The Language of Mathematics by Keith Devlin, a field is a system that satisfies the following conditions.

1. For all *m*, *n*: *m+n=n+m* and *nm=mn* (the commutative laws for addition and multiplication.)

2. For all m,n, k: $m+(n+k)=(m+n)+k$ and $m(nk)=(mn)k$ (the associative laws for addition and multiplication.
3. For all m,n, k: $k(m+n)=(km)+(kn)$ (the distributive law).
4. For all n: $n+0=n$ (the additive identity law).
5. For all n: $n1=n$ (the multiplicative identity law).
6. For all n, there is a number k such that $n+k=0$ (the additive inverse law).
7. For all m,n,k where $k \neq 0$: if $mk=nk$, then $m=n$ (the cancellation law).
8. For all n other than 0, there is a k such that $nk=1$.[5] [6]

These axioms were shown true for coordinate pairs during the review of Bloch's list, and thus the set of ordered pairs with the operations of addition and coordinate pair multiplication is a field. This field is called the complex field and denoted as **C**. So **C** is $\mathbf{R}^2$ with Hamilton's additional multiplication.

The real numbers can be considered an extension of the rational numbers. Likewise the rational numbers are an extension of integers and the integers are an extension of the set of natural numbers. This raises the question of whether the complex field can be considered an extension of the real numbers.

Let $R = \{(a,0) | a \in \mathbf{R}\}$. Thus $R \subseteq \mathbf{C}$. Define a function $f : \mathbf{R} \to R$ by $f(a) = (a,0)$. If $f(a) = f(b)$, then $(a,0) = (b,0)$ hence $a = b$. Thus f is one-to-one. Also, for a $(a,0) \in R$, $f(a) = (a,0)$ for $(a,0) \in \mathbf{R}$. Therefore f is onto. Note the following:

$f(a+b) = (a+b,0) = (a,0) + (b,0) = f(a) + f(b)$ and

[5] Devlin, Keith The Language of Mathematics. New York. W.H. Freeman and Company. 1998. p.72 & 73
[6] Some notation has been changed for consistency.

$f(ab) = (ab,0) = (a \times b - 0 \times 0,\ a \times 0 + b \times 0) = (a,0)(b,0) = f(a)f(b)$. We have just shown f preserves addition and multiplication, thus **R** is isomorphic to R, denoted by $\mathbf{R} \cong R$.

Define an ordered pair of the form $(a,0)$ for $a \in \mathbf{R}$ as "real." Define an ordered pair of the form $(0,a)$ for $a \in \mathbf{R}$ as "imaginary." Lastly, define $i = (0,1)$. Note that $i^2 = -1$ since $(0,1)(0,1) = (-1,0) = -(1,0)$. The simplistic definition of $i^2 = -1$ is what mathematicians worked with before they had $i = (0,1)$ and this simplistic definition is what most high school students are familiar with.

We have shown **C** is at least a vector space over **R**, and using scalar multiplication, where a is a scalar and (b,c) is a vector, we have $a(b,c) = (ab,ac)$. Identifying a with $(a,0)$ we get $(a,0)(b,c) = (ab - 0, ac + 0) = (ab,ac)$. Therefore scalar multiplication is given by complex multiplication by elements in R.

Furthermore, consider (a,b). First, $(a,b) = (a + 0, 0 + b) = (a,0) + (0,b)$. By coordinate pair multiplication $(0,b) = (b,0)(0,1)$. Thus $(a,b) = (a,0) + (b,0)(0,1)$. Due to the isomorphism between **R** and R, we write a for $(a,0)$ and b for $(b,0)$, so we can write $(a,b) = a + bi$; $a + bi$ is used as an alternate notation.

From this we can consider that $\mathbf{R} \subseteq \mathbf{C}$.

Geometric Representations of Complex Arithmetic

We will now look at the geometric interpretations of the most basic mathematical properties of complex numbers—arithmetic. In the real number system, there are four operations: addition, subtraction, multiplication, and division. Addition and multiplication for complex numbers have already been discussed. A visual representation can be given for these arithmetic operations by using a vector drawing in real two space.

The addition of two coordinate pairs (a,b) and (c,d) where $(a,b),(c,d) \in \mathbf{C}$ can be seen in Figures 1 and 2 as vector addition.

$(a,b)+(c,d)=(a+c,b+d)$

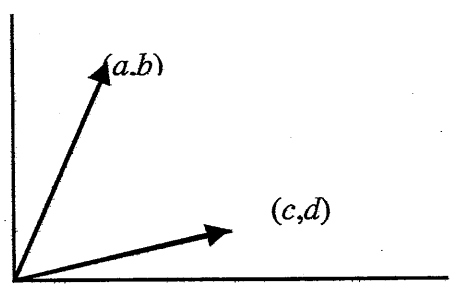

Figure 1

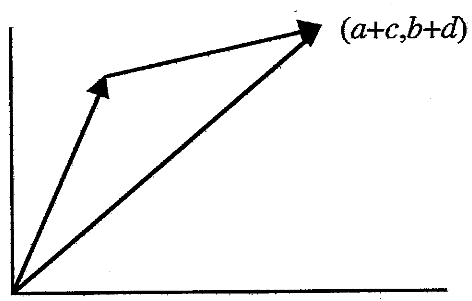

Figure 2

Subtraction is the same as adding the inverse of a coordinate pair. Therefore $(a,b)-(c,d)=(a,b)+(-1)(c,d)=(a,b)+(-c,-d)=(a-c,b-d)$. See Figures 3 and 4.

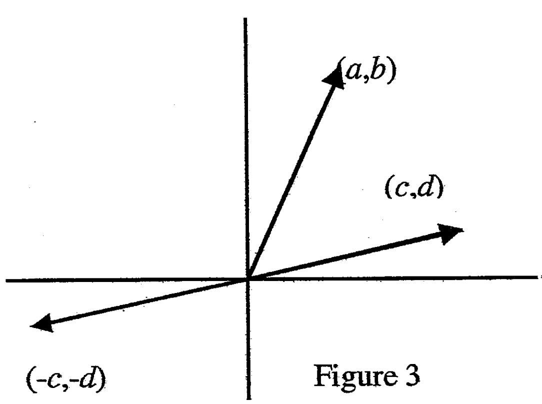

Figure 3

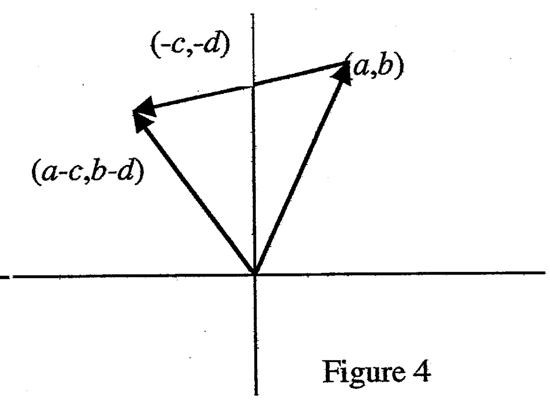

Figure 4

Polar coordinates provide an alternate way of denoting points in $\mathbf{R}^2$. From Calculus III, $x = r\cos\theta$ and $y = r\sin\theta$, where r is the modulus of the vector (x,y) and θ is the argument. Therefore $(x,y) = (r\cos\theta, r\sin\theta)$. Using complex multiplication:

$$\begin{aligned}(x_1,y_1)(x_2,y_2) &= (r_1\cos\theta_1, r_1\sin\theta_1)(r_2\cos\theta_2, r_2\sin\theta_2)\\ &= (r_1r_2\cos\theta_1\cos\theta_2 - r_1r_2\sin\theta_1\sin\theta_2, r_1r_2\cos\theta_1\sin\theta_2 + r_1r_2\sin\theta_1\cos\theta_2)\\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\\ &= r_1r_2(\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2)) \text{ by trigonometric identities.}\end{aligned}$$

A graphical representation of this shows that the moduli are multiplied and the arguments are added. This is pictured in Figures 5 and 6.

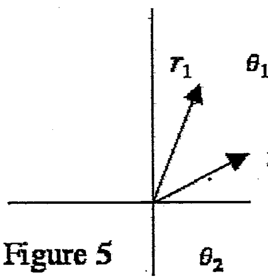

Figure 5

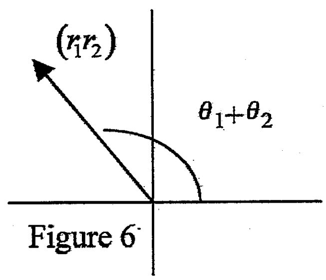

Figure 6

Polar coordinates can help visualize complex division:

$$\begin{aligned}\frac{(x_1,y_1)}{(x_2,y_2)} &= (x_1,y_1)(x_2,y_2)^{-1}\\ &= (x_1,y_1)\left(\frac{x_2}{x_2^2+y_2^2}, \frac{-y_2}{x_2^2+y_2^2}\right)\\ &= (r_1\cos\theta_1, r_1\sin\theta_1)\left(\frac{r_2\cos\theta_2}{(r_2\cos\theta_2)^2+(r_2\sin\theta_2)^2}, \frac{-r_2\sin\theta_2}{(r_2\cos\theta_2)^2+(r_2\sin\theta_2)^2}\right)\\ &= (r_1\cos\theta_1, r_1\sin\theta_1)\left(\frac{r_2\cos\theta_2}{r_2^2(\cos^2\theta_2+\sin^2\theta_2)}, \frac{-r_2\sin\theta_2}{r_2^2(\cos^2\theta_2+\sin^2\theta_2)}\right)\\ &= (r_1\cos\theta_1, r_1\sin\theta_1)\left(\frac{r_2\cos\theta_2}{r_2^2}, \frac{-r_2\sin\theta_2}{r_2^2}\right)\\ &= \left(\frac{r_1\cos\theta_1\cos\theta_2}{r_2} - \frac{-r_1\sin\theta_1\sin\theta_2}{r_2}, \frac{-r_1\cos\theta_1\sin\theta_2}{r_2} + \frac{r_1\sin\theta_1\cos\theta_2}{r_2}\right) \text{(by complex multiplication)}\\ &= \left(\frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2), \frac{r_1}{r_2}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right)\end{aligned}$$

$$= \left(\tfrac{r_1}{r_2}\cos(\theta_1 - \theta_2), \tfrac{r_1}{r_2}\sin(\theta_1 - \theta_2)\right) \text{ (by trigonometric identities)}$$
$$= \tfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2), \sin(\theta_1 - \theta_2)).$$

Graphically, this can be interpreted as dividing the moduli and subtracting the arguments, as shown in Figures 7 and 8.

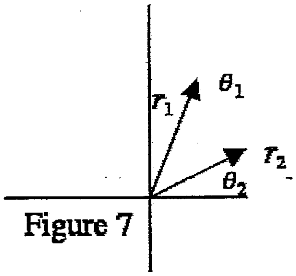

Figure 7

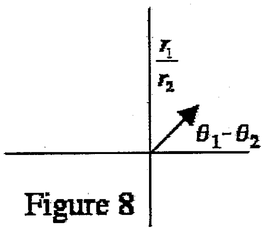

Figure 8

Trigonometric Functions:

Next we will examine complex exponential functions by starting with known real power series expansions of cosine and sine. We have $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots$ and $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots$. The expansion of their sum looks similar to the power series expansion of e^x, but is not equal to e^x:

$\cos y + \sin y = 1 + y - \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} - \frac{y^6}{6!} - \frac{y^7}{7!} + \ldots$, but

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$. In the expansion of cos y, substitute $\frac{(iy)^n}{n!}$ for $\frac{-y^n}{n!}$ for n=2,6,10,... and $\frac{(iy)^n}{n!}$ for $\frac{y^n}{n!}$ for n=4,8,12,... . Now $\cos y = 1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \frac{(iy)^6}{6!} + \ldots$. This gives us a power series expansion of cosine with all positive signs. The same substitution does not work for sin y. Rather, those substitutions give us $\sin y = y + \frac{i^2 y^3}{3!} + \frac{i^4 y^5}{5!} + \frac{i^6 y^7}{7!} + \ldots$.

We have calculated an expansion of sine that has all positive signs. However, the terms in each numerator are not raised to the same power. Therefore, we will multiply both sides by i: $i \sin y = iy + \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} + \frac{(iy)^7}{7!} + \ldots$. Thus

$\cos y + i \sin y = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \frac{(iy)^7}{7!} + \ldots$. This is of the same form as the power series expansion for e^x. Thus $\cos y + i \sin y = e^{iy}$. Multiplying both sides by e^x gives us $e^x(\cos y + i \sin y) = e^x e^{iy}$. Using desired properties of exponents would give us

$e^x e^{iy} = e^{x+iy} = e^z$.

Definition: For $z=x+iy$, $e^z = e^x(\cos y + i \sin y)$.

We look at e^z to see if it fulfills exponential properties and we see

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 \cos y_2 + i\sin y_2 \cos y_1 + i\sin y_1 \cos y_2 - \sin y_1 \sin y_2) \\
&= e^{x_1+x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) \\
&= e^{x_1+x_2+i(y_1+y_2)} \\
&= e^{x_1+x_2+iy_1+iy_2} \\
&= e^{z_1+z_2}
\end{aligned}$$

Also, $e^0 = e^0(\cos 0 + i\sin 0) = 1(1) = 1$.

The definition for e^{iy} can be used as an alternate way to show that all real numbers have a mapping to the complex numbers. First, $e^{i\pi} = \cos\pi + i\sin\pi = -1$. Then for any real number, a:

$$\begin{aligned}
a &= -a(-1) \\
a &= -ae^{i\pi} \\
a &= -a(\cos\pi + i\sin\pi) \\
a &= -a(\cos\pi, \sin\pi) \\
a &= -a(-1,0) \\
a &= (a,0).
\end{aligned}$$

Geometry: Triangles

Lastly, we will look at how complex numbers and complex multiplication relate to geometry. Only real numbers were used to discuss triangles in my high school geometry course. To start off, we will consider triangles with vertices at points of the complex plane.

Consider triangles with vertices *O*=(0,0), *A*=(*a*,0) and a third unknown vertex *B*. We wish to calculate the coordinates of *B*, when *B* satisfies various conditions. First we seek *B* such that *OAB* is a right isosceles triangle, where $\angle BOA$ is a right angle. It is clear that *B*=(0,*a*) but let's see how to compute *B* using complex multiplication. We know that anything multiplied by *i* is rotated 90° about the origin in the complex plane, with no change in magnitude of the modulus. That is, $ai = (0, a)$ and $|a| = |ai|$. A right isosceles triangle is created using the *i* rotation; it is the triangle with vertices at (0,0), (*a*,0) and (0,*a*) as illustrated in Figure 9.

i: 90° rotation
right, isosceles triangle

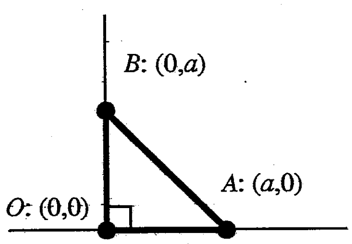

Figure 9

The next isosceles triangle to consider is the equilateral triangle. Instead of rotating (*a*,0) by *i* or 90°, rotate (*a*,0) by 60°. The corresponding trigonometric triangle for 60° has $\cos 60^\circ = \frac{1}{2}$ and $\sin 60^\circ = \frac{\sqrt{3}}{2}$. Thus, we will try a rotation of $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. $(a,0)(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (\frac{a}{2}, \frac{a\sqrt{3}}{2})$ as shown in Figure 10.

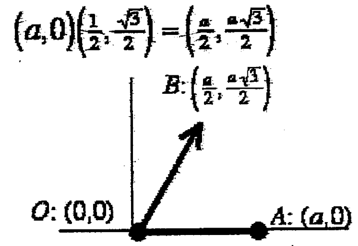

Figure 10

If this is correct, than the three vertices of the equilateral triangle will be (0,0), (a,0), and $(\frac{a}{2},\frac{a\sqrt{3}}{2})$. Let us find B by another route: rotate through 120° and then translate by adding (a,0). This rotation is accomplished by a multiplication of $(\frac{-1}{2},\frac{\sqrt{3}}{2})$. We have $(a,0)(\frac{-1}{2},\frac{\sqrt{3}}{2})=(\frac{-a}{2},\frac{a\sqrt{3}}{2})$ as drawn in Figure 11.

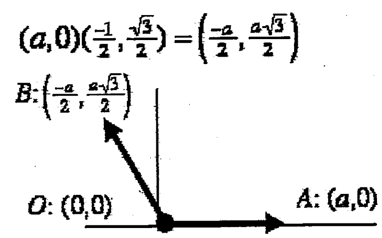

Figure 11

Next this point must be shifted. When calculated, $(\frac{-a}{2},\frac{a\sqrt{3}}{2})+(a,0)=(\frac{a}{2},\frac{a\sqrt{3}}{2})$, which is the same point calculated using the first rotation, as illustrated in Figure 12.

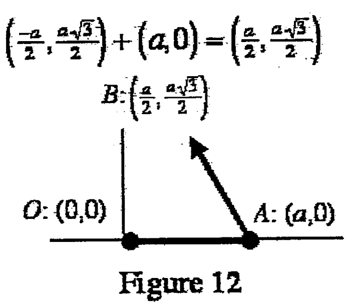

Figure 12

Lastly, to check that we truly have equilateral triangles, consider the lengths of the sides. Reconsider (a,0). The modulus of $(a,0)$ is $\sqrt{a^2}=a$. The modulus of $(\frac{a}{2},\frac{a\sqrt{3}}{2})$ is

$\sqrt{(\frac{a}{2})^2+(\frac{a\sqrt{3}}{2})^2}=\sqrt{\frac{a^2+3a^2}{4}}=\sqrt{\frac{4a^2}{4}}=\sqrt{a^2}=a$. Lastly, check the modulus of $(a,0)-(\frac{a}{2},\frac{a\sqrt{3}}{2})$.

We see that $\left|(a,0)-(\frac{a}{2},\frac{a\sqrt{3}}{2})\right|=\sqrt{(a-\frac{a}{2})^2+(0-\frac{a\sqrt{3}}{2})^2}=\sqrt{(\frac{2a}{2}-\frac{a}{2})^2+(\frac{-a\sqrt{3}}{2})^2}=$

$\sqrt{\frac{4a^2}{4}}=\sqrt{a^2}=a$. In conclusion, we see that an equilateral triangle with a base of (0,0) and (a,0), has a third vertex at $(\frac{a}{2},\frac{a\sqrt{3}}{2})$ and all sides have a length of a. (Figure 13)

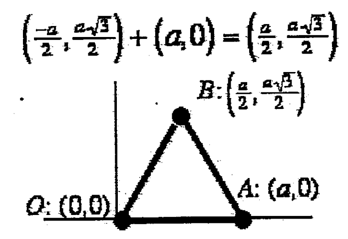

Figure 13

An equilateral triangle is a special kind of isosceles triangle. To generalize the findings on the third vertex of a triangle, consider a non-equilateral triangle. We seek B such that OAB is a 30°, 30°, 120° isosceles triangle (that is, angle B is 120°). Here (0,0) and (a,0) are the two base points. Using complex multiplication we will rotate that base by 30° to create the other two sides of the triangle. The corresponding trigonometric functions are $\cos 30^\circ=\frac{\sqrt{3}}{2}$ and $\sin 30^\circ=\frac{1}{2}$. Therefore we will look at $(a,0)(\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(a,0)(-\frac{\sqrt{3}}{2},\frac{1}{2})+(a,0)$. $(a,0)(\frac{\sqrt{3}}{2},\frac{1}{2})=(\frac{a\sqrt{3}}{2},\frac{1}{2})$ and $(a,0)(-\frac{\sqrt{3}}{2},\frac{1}{2})+(a,0)=(\frac{-a\sqrt{3}}{2},\frac{1}{2})+(a,0)=(\frac{2a-a\sqrt{3}}{2},\frac{1}{2})$. Figure 14 illustrates these rotations.

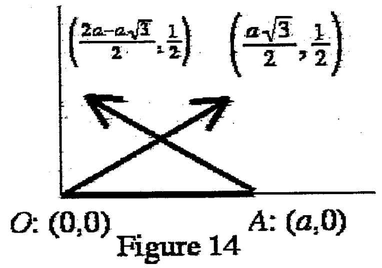

Figure 14

This time, unlike the 60° rotation, the same point has not been calculated, and we must scale the new sides accordingly. Our new triangle can be viewed as the following. Using the side lengths for a 30°, 60°, 90° right triangle, (Figure 15) the scaling factor is $\frac{1}{\sqrt{3}}$.

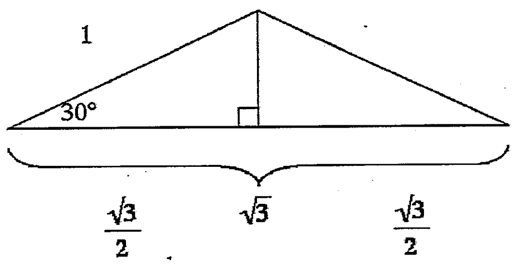

Figure 15

To verify this we must calculate $\frac{1}{\sqrt{3}}[(a,0)(\frac{\sqrt{3}}{2},\frac{1}{2})]$ and $\frac{1}{\sqrt{3}}[(a,0)(\frac{-\sqrt{3}}{2},\frac{1}{2})]+(a,0)$.

$\frac{1}{\sqrt{3}}[(a,0)(\frac{\sqrt{3}}{2},\frac{1}{2})]=\frac{1}{\sqrt{3}}(\frac{a\sqrt{3}}{2},\frac{a}{2})=(\frac{a}{2},\frac{a}{2\sqrt{3}})$ and

$\frac{1}{\sqrt{3}}[(a,0)(\frac{-\sqrt{3}}{2},\frac{1}{2})]+(a,0)=\frac{1}{\sqrt{3}}(\frac{-a\sqrt{3}}{2},\frac{a}{2})+(a,0)=(\frac{-a}{2},\frac{a}{2\sqrt{3}})+(a,0)=(\frac{a}{2},\frac{a}{2\sqrt{3}})$. Now we have obtained the same point which will be the vertex of the 120° angle of the triangle.

Another triangle of interest would be the 45°, 45°, 90° isosceles triangle, shown in Figure 16.

45° 45° 90° Triangle

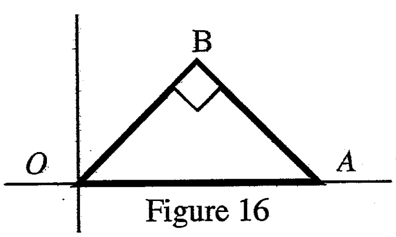

Figure 16

$\sin 45^\circ=\frac{\sqrt{2}}{2}$ and $\cos 45^\circ=\frac{\sqrt{2}}{2}$, therefore the rotation factor will be $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$. We will once again calculate B two ways. The first by a rotation of $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ as illustrated in Figure 17 and the other way by a rotation of $\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ followed by a translation of $(a,0)$. This is diagramed in Figures 18 and 19. We calculate $(a,0)\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=\left(\frac{a\sqrt{2}}{2},\frac{a\sqrt{2}}{2}\right)$ and

$(a,0)\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)+(a,0)=\left(\frac{a-a\sqrt{2}}{2},\frac{a\sqrt{2}}{2}\right)$.

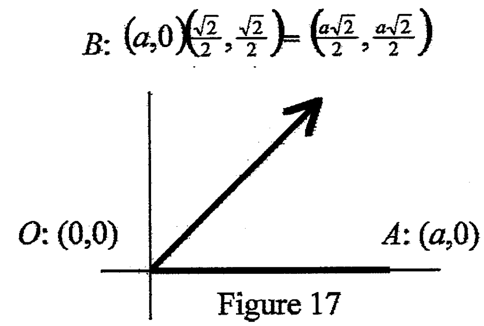

Figure 17

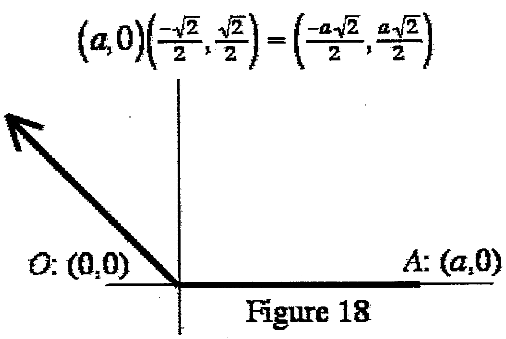

Figure 18

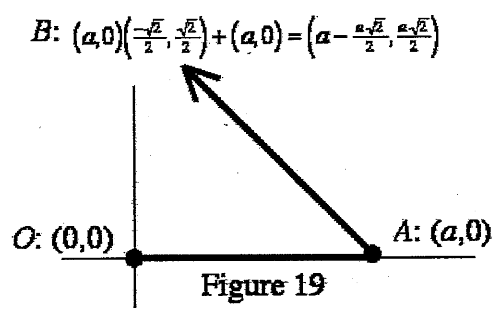

Figure 19

Once again we will need a scaling factor and will look at the 45°, 45°, 90° triangle. The ratio between the side and twice the base of the 90° triangle is $\frac{1}{\sqrt{2}}$. This triangle is shown in Figure 20.

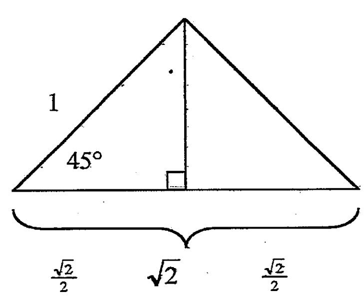

Figure 20

Recalculating the former rotations with the scaling factor gives us the following;

$\frac{1}{\sqrt{2}}\left[(a,0)\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right]=\left(\frac{a}{2},\frac{a}{2}\right)$ and $\frac{1}{\sqrt{2}}\left[(a,0)\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right]+(a,0)=\frac{1}{\sqrt{2}}\left(\frac{-a\sqrt{2}}{2},\frac{a\sqrt{2}}{2}\right)+(a,0)=\left(\frac{a}{2},\frac{a}{2}\right)$.

Now both rotations calculate the same point $B=\left(\frac{a}{2},\frac{a}{2}\right)$ for the third vertex of the triangle as shown in Figure 21.

45° 45° Isosceles Triangle

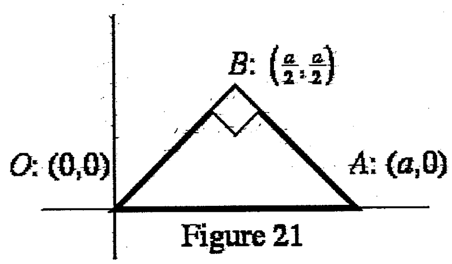

Figure 21

The previous results can be seen in the following table.

Angle of rotation	Complex Multiplication (cos x, sin x)	Scaling factor
60°	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	1
30°	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	$\frac{1}{\sqrt{3}}$
45°	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\frac{1}{\sqrt{2}}$

It would be interesting to look at a less common angle, 15°. Using the half angle formulas, $\cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}$ and $\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}$ we calculate that $\cos 15°=\frac{\sqrt{2+\sqrt{3}}}{2}$ and $\sin 15°=\frac{\sqrt{2-\sqrt{3}}}{2}$. We will calculate the third vertex B the same two ways that we have for previous triangles, by using complex multiplication to rotate the base by the cosines and sines. We have $(a,0)\left(\frac{\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)=\left(\frac{a\sqrt{2+\sqrt{3}}}{2},\frac{a\sqrt{2-\sqrt{3}}}{2}\right)$, and using the second method to calculate B, $(a,0)\left(\frac{-\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)+(a,0)=\left(\frac{a(2-\sqrt{2+\sqrt{3}})}{2},\frac{a\sqrt{2-\sqrt{3}}}{2}\right)$. Once again we need to find a scaling factor. Using the 15°, 75°, 90° triangle, the ratio of the side to twice the base of the right triangle is $\frac{1}{\sqrt{2+\sqrt{3}}}$, as the picture shows in Figure 22.

15° 75° 90° Triangles

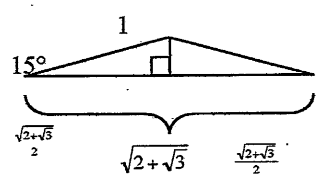

Figure 22

We calculate that $\frac{1}{\sqrt{2+\sqrt{3}}}\left[(a,0)\left(\frac{\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)\right]=\left(\frac{a}{2},\frac{a\sqrt{2-\sqrt{3}}}{2\sqrt{2+\sqrt{3}}}\right)$. Rationalizing the denominator gives us $\frac{a(2-\sqrt{3})}{2}$. Next we calculate $\frac{1}{\sqrt{2+\sqrt{3}}}\left[(a,0)\left(\frac{-\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)\right]+(a,0)=\left(\frac{a}{2},\frac{a\sqrt{2-\sqrt{3}}}{2\sqrt{2+\sqrt{3}}}\right)=\left(\frac{a}{2},\frac{a(2-\sqrt{3})}{2}\right)$.

It would be helpful to know the scaling factor before starting calculations so that one doesn't need to go back and re-calculate each rotation. Observe the following table, which shows a pattern for the general solution k, where k is the scaling factor.

Angle of Rotation	Complex Multiplication	Scaling Factor
x	$(\cos x, \sin x)$	k
60°	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	1
45°	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\frac{1}{\sqrt{2}}$
30°	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	$\frac{1}{\sqrt{3}}$
15°	$\left(\frac{\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)$	$\frac{1}{\sqrt{2+\sqrt{3}}}$

One can see that $k=\frac{1}{2\cos x}$. The general forms of the two equations to calculate the third vertex B of an isosceles triangle with a base of a and equal angles of x using complex multiplication are $\frac{1}{2\cos x}(a,0)(\cos x,\sin x)$ and $\frac{1}{2\cos x}(a,0)(-\cos x,\sin x)+(a,0)$. We will try the angle 22.5° to see if our equations calculate the same point *B*. We have

$$\frac{1}{2\cos x}(a,0)(\cos x,\sin x)$$
$$=\frac{1}{2\cos 22.5^\circ}(a,0)(\cos 22.5^\circ,\sin 22.5^\circ)$$
$$=\frac{1}{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)}(a,0)\left(\frac{\sqrt{2+\sqrt{2}}}{2},\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$
$$=\frac{1}{\sqrt{2+\sqrt{2}}}\left(\frac{a\sqrt{2+\sqrt{2}}}{2},\frac{a\sqrt{2-\sqrt{2}}}{2}\right)$$
$$=\left(\frac{a}{2},\frac{a\sqrt{2-\sqrt{2}}}{2\sqrt{2+\sqrt{2}}}\right)$$
$$=\left(\frac{a}{2},\frac{a(2-\sqrt{2})}{2}\right)$$

$$\frac{1}{2\cos x}(a,0)(-\cos x,\sin x)+(a,0)$$
$$=\frac{1}{2\cos 22.5^\circ}(a,0)(-\cos 22.5^\circ,\sin 22.5^\circ)+(a,0)$$
$$=\frac{1}{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)}(a,0)\left(-\frac{\sqrt{2+\sqrt{2}}}{2},\frac{\sqrt{2-\sqrt{2}}}{2}\right)+(a,0)$$
$$=\frac{1}{\sqrt{2+\sqrt{2}}}\left(-\frac{a\sqrt{2+\sqrt{2}}}{2},\frac{a\sqrt{2-\sqrt{2}}}{2}\right)+(a,0)$$
$$=\left(-\frac{a}{2},\frac{a\sqrt{2-\sqrt{2}}}{2\sqrt{2+\sqrt{2}}}\right)+(a,0)$$
$$=\left(\frac{a}{2},\frac{a(2-\sqrt{2})}{2}\right).$$

Thus we have discovered a general formula for finding the third vertex of an isosceles triangle given a base of (0,0) and (a,0); $\frac{1}{2\cos x}(a,0)(\cos x,\sin x)$ and $\frac{1}{2\cos x}(a,0)(-\cos x,\sin x)+(a,0)$. Both of these expressions yield $\left(\frac{a}{2},\frac{a\tan x}{2}\right)$.

Hatch's Theorem:

An isosceles triangle with base of (0,0) and (a,0) has a third vertex of $\left(\frac{a}{2},\frac{a\tan x}{2}\right)$ where angle x is a base angle.

To show triangle *AOB* is isosceles, we must show that it has two sides of equal lengths; that is, show $|OB|=|AB|$.[7]

[7] Ohmer, Merlin M. Elementary Geometry for Teachers. Reading, Massachusetts. Addison-Wesley. 1969. p.111.

Thus

$|OB|$

$$=\sqrt{\left(\tfrac{a}{2}-0\right)^2+\left(\tfrac{a\tan x}{2}-0\right)^2}$$

$$=\sqrt{\tfrac{a^2}{4}+\tfrac{a^2\tan^2 x}{4}}$$

$$=\sqrt{\tfrac{a^2+a^2\tan^2 x}{4}}$$

$$=\tfrac{a}{2}\sqrt{1+\tan^2 x}$$

$|AB|$

$$=\sqrt{\left(\tfrac{a}{2}-a\right)^2+\left(\tfrac{a\tan x}{2}-0\right)^2}$$

$$=\sqrt{\left(\tfrac{-a}{2}\right)^2+\tfrac{a^2\tan^2 x}{4}}$$

$$=\sqrt{\tfrac{a^2}{4}+\tfrac{a^2\tan^2 x}{4}}$$

$$=\tfrac{a}{2}\sqrt{1+\tan^2 x}$$

Thus triangle *OAB* where *O*=(0,0), *A*=(*a*,0) and *B*=$\left(\tfrac{a}{2},\tfrac{a\tan x}{2}\right)$ is an isosceles triangle.

Lastly, we will show that $\angle AOB = x$.

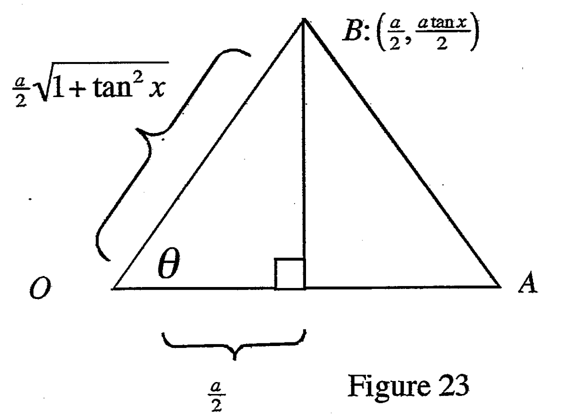

Figure 23

Let $\angle AOB = \theta$. From Figure 23,

$$\cos\theta = \frac{\frac{a}{2}}{\frac{a}{2}\sqrt{1+\tan^2 x}}$$
$$\cos\theta = \frac{1}{\sqrt{1+\tan^2 x}}$$
$$\cos^2\theta = \frac{1}{1+\tan^2 x}$$
$$\cos^2\theta = \frac{1}{\sec^2 x}$$
$$\cos^2\theta = \cos^2 x$$

$\cos\theta = \cos x$, since all quantities here are positive.

$\theta = x$.

Therefore $\angle AOB = x$.

Conclusion:

Complex numbers have many properties which can be discovered by looking at properties of real numbers. I had studied complex numbers in a college mathematics course, but the discovery process of this thesis was new to me. The proofs, which show these properties of complex numbers to be true, were written solely for this thesis. This thesis clearly shows the steps that relate the high school understanding of complex numbers to the deeper understanding of complex numbers. That is, $i^2 = -1$ is true because $(0,1)(0,1) = (-1,0) = -(1,0)$. The explanation of the equation $i^2 = -1$ and the mapping of real numbers to complex numbers make the complex numbers less mysterious, and more useable.

The Moore Method centers around student discovery and original proof. One original proof in this thesis was for trigonometric functions. There are many different proofs for the relationship between real trigonometric functions and complex trigonometric functions; this thesis shows one I discovered. However, more importantly to this thesis than showing that there exists a relationship is clearly showing and explaining the steps leading to my definition of e^z.

Hatch's Theorem is the climax of this thesis for original proof and discovery. When I began to write this thesis, it was not my intention to discover a theorem concerning isosceles triangles. However, while I was searching to discover how complex multiplication could be applied to geometric ideas, I realized that I was close to finding a general formula concerning the three vertexes of a triangle, and pursued that challenge. Using the properties shown in the beginning of the theorem, I had the tools to apply complex multiplication to geometry. Hatch's Theorem could not have been proved using complex multiplication without first showing the multiplication's basic properties.

Throughout this thesis I also caught a glimpse of how mathematicians can discover new mathematical ideas. By starting with what one knows, and curiously wandering through the mathematical unknown, one may discover something new or discover an area to pursue further. Centering learning around discovery on one's own is an idea that may be worked into a high school mathematics class, where the students are asked to discover new properties by starting with what they already know. The Moore Method of original proof and discovery does work, even when one has already studied the subject.

Author's Biography

Esther D. Hatch grew up in Palermo, Maine. She attended high school at Kents Hill School, where she was actively involved in sports, music, and academics. Upon high school graduation she attended the University of Maine where she continued to be actively involved in many campus activities. She majored in mathematics, which led her to teach math at the Maine State GEAR-UP summer camps, tutor students at the university's math lab, and become a member of the math honor society, Pi Mu Epsilon. As a minor in music, Esther studied pipe organ with Dr. Kevin Birch for 3.5 years after having studied the piano for 11 years. During the spring semester of her junior year, Esther joined the University of Maine Jazz combo as the pianist. She has also accompanied soloists and church congregations on both the pipe organ and piano.

Esther joined the University of Maine's Swimming and Diving team her freshman year of college, and continued to compete for the team for four years. As a result of this, Esther joined Athletes in Action, a group on campus designed to help athletes better know each other and God. She has also worked at the Laboratory for Surface Science and Technology for four years as a student administrative assistant.

Upon graduation from college, she hopes to teach high school mathematics and music in New England, continue her independent study of astronomy, tour the roller coasters of Pennsylvania, and continue to write musical compositions.

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